THE SIGMA ORIENTATION FOR ANALYTIC CIRCLE-EQUIVARIANT ELLIPTIC COHOMOLOGY

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ABSTRACT. We construct a *canonical* Thom isomorphism for $\mathbb{T}$-oriented $BO\langle 8\rangle$-bundles in $\mathbb{T}$-equivariant analytic elliptic cohomology, which is natural under pull-back of vector bundles and exponential under Whitney sum. It extends in the rational case the non-equivariant sigma orientation of Hopkins, Strickland, and the author. The construction relates the sigma orientation to the representation theory of loop groups and Looijenga's weighted projective space, and sheds light even on the non-equivariant case. Rigidity theorems of Witten-Bott-Taubes including generalizations by Kefeng Liu follow.

CONTENTS

1. INTRODUCTION

1.1. **Summary.** Let E be an even periodic, homotopy commutative ring spectrum, let C be an elliptic curve over $S_E = \operatorname{spec} \pi_0 E$, and let t be an isomorphism of formal groups

$$t : \widehat{C} \cong \operatorname{spf} E^0(\mathbb{C}P^\infty),$$

so that (E, C, t) is an elliptic spectrum. In [AHS01], to which we refer the reader for the terminology in this paragraph, Hopkins, Strickland, and the author construct a canonical map of homotopy commutative ring spectra

$$MU\langle 6\rangle \xrightarrow{\sigma(E,C,t)} E$$

(or even, conjecturally, $MO\langle 8\rangle \to E$) called the *sigma orientation.*

Let $\mathbb{T}$ be the circle group. We expect that the sigma orientation has an equivariant extension

$$(\mathbb{T}\text{-equivariant } MO\langle 8\rangle) \xrightarrow{\sigma_{\mathbb{T}}(E_{\mathbb{T}},C,t)} E_{\mathbb{T}}, \tag{1.1}$$

to a multiplicative map of $\mathbb{T}$-equivariant spectra. Note however that the construction of $\sigma_{\mathbb{T}}$ would require us among other things to say what $\mathbb{T}$-spectra we have in mind for the domain and codomain.

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Geometrically, a map $\sigma_{\mathbb{T}}$ as in (1.1) would be equivalent to specifying, for each virtual $\mathbb{T}$-$BO\langle 8\rangle$ vector bundle V (whatever that means) over a $\mathbb{T}$-space X, a trivialization $\gamma(V)$ of $E_{\mathbb{T}}(X^V)$ as an $E_{\mathbb{T}}(X)$-module. The trivialization would be natural, in the sense that

$$\gamma(f^*V) = f^*\gamma(V)$$

if $f : X' \to X$ is a map of $\mathbb{T}$-spaces, and exponential, in the sense that

$$\gamma(V \oplus V') = \gamma(V) \otimes \gamma(V')$$

under the isomorphism

$$E_{\mathbb{T}}(X^{V\oplus V'}) \cong E_{\mathbb{T}}(X^V) \underset{E_{\mathbb{T}}(X)}{\otimes} E_{\mathbb{T}}(X^{V'}).$$

One expects that equivariant elliptic cohomology comes with a completion isomorphism

$$E_{\mathbb{T}}(X^V)^\wedge \cong E((X_{\mathbb{T}})^{V_{\mathbb{T}}}), \tag{1.2}$$

and that this isomorphism carries $\gamma(V)$ to the Borel-equivariant sigma orientation of V. (The notation $X_{\mathbb{T}}$ will denote the Borel construction $E\mathbb{T} \times_{\mathbb{T}} X$.)

In this paper we carry out this program for the complex-analytic equivariant elliptic cohomology of Grojnowski; along the way we gain some insight into what is meant by the domain and codomain in (1.1). Let $\Lambda \subset \mathbb{A}^1_{\mathrm{an}}$ be a lattice in the complex plane, and let C be the analytic variety $C = \mathbb{A}^1_{\mathrm{an}}/\Lambda$. Grojnowski defines a functor $E_{\mathbb{T}}$ from finite $\mathbb{T}$-CW complexes to sheaves of $\mathbb{Z}/2$-graded $\mathcal{O}_C$-algebras, equipped with a natural isomorphism (1.2) ([Gro94]; for a published account see [Ros01]). We recall that the Weierstrass sigma function $\sigma = \sigma(z, \Lambda)$ gives rise to a map of ring spectra

$$MSO \xrightarrow{\Sigma} H\mathbb{C}$$

whose restriction to $MU\langle 6\rangle$ *is* the sigma orientation for the elliptic spectrum associated to the elliptic curve C (see Example 5.7 and [AHS01, §2.7]).

If V is a $\mathbb{T}$-equivariant vector bundle over X, then we write $E_{\mathbb{T}}(V)$ for the reduced equivariant elliptic cohomology of its Thom space; it is an $E_{\mathbb{T}}(X)$-module. If V is a spin bundle, then $E_{\mathbb{T}}(V)$ is an invertible $E_{\mathbb{T}}(X)$-module. More generally, if $V = V_0 - V_1$ is a virtual spin bundle, then we may define

$$E_{\mathbb{T}}(V) = E_{\mathbb{T}}(V_0) \underset{E_{\mathbb{T}}(X)}{\otimes} E_{\mathbb{T}}(V_1)^{-1}.$$

([Ros01, AB00]; see §8).

Definition 1.3. Let V be a $\mathbb{T}$-vector bundle over X. A $\mathbb{T}$-*orientation* on V is a choice of orientation on the fixed sub-bundle W^A for each closed subgroup A of $\mathbb{T}$. A $\mathbb{T}$-*oriented vector bundle* is a $\mathbb{T}$-vector bundle equipped with a $\mathbb{T}$-orientation. The $\mathbb{T}$-oriented vector bundles over X form a monoid under direct sum, whose Grothendieck group is the group of $\mathbb{T}$-*oriented virtual bundles* over X.

(It is a result of Bott and Samelson ([BS58] see [BT89], Lemma 3.7, and §6.4) that a $\mathbb{T}$-equivariant spin bundle is $\mathbb{T}$-orientable.)

Theorem 1.4 (10.1). *Let V be a $\mathbb{T}$-oriented virtual bundle over X, with the property that*

$$\begin{aligned} w_2(V_{\mathbb{T}}) &= 0 \\ c_2(V_{\mathbb{T}}) &= 0. \end{aligned} \tag{1.5}$$

(Here $c_2 \in H^4(BSpin, \mathbb{Z})$ is the generator whose restriction to BSU is the second Chern class.) Then there is a canonical trivialization $\gamma(V)$ of $E_{\mathbb{T}}(V)$, whose value in $E_{\mathbb{T}}(V)_0$ is $\Sigma(V_{\mathbb{T}})$. Moreover we have

$$\gamma(V \oplus V') = \gamma(V) \otimes \gamma(V')$$

under the isomorphism

$$E_{\mathbb{T}}(V \oplus V') \cong E_{\mathbb{T}}(V) \otimes_{E_{\mathbb{T}}(X)} E_{\mathbb{T}}(V').$$

It is natural in the sense that

$$\gamma(f^*V) = f^*\gamma(V),$$

if $f : X' \to X$ is a map of $\mathbb{T}$-spaces.

Already in [AB00], Maria Basterra and the author showed that, under the hypotheses of the Theorem, there is a global section $\gamma(V)$ of $E_{\mathbb{T}}(V)$, whose value in the stalk at the origin of C is the Borel-equivariant sigma orientation of V; earlier Rosu [Ros01] did the same for the orientation associated to the Euler formal group law. However, those papers do not address the trivialization, exponential, and naturality properties of the classes they construct.

The naturality is particularly hard to discern. Both of the papers [Ros01, AB00] closely follow [BT89] in their implementation of the "transfer" argument, and so all three papers depend on meticulous choices for the characters of the action of $\mathbb{T}$ on $V|_{X^{\mathbb{T}}}$ and of $\mathbb{T}[n]$ on $V|_{\mathbb{T}[n]}$ and for the orientations of $V^{\mathbb{T}}$ and $V^{\mathbb{T}[n]}$, along with surprising and not particularly intuitive results concerning the compatibility of these choices. In this paper we eliminate all of those choices, and show that *any* choice will do, with *no* effect on the resulting Thom class, which is completely determined by the choices of orientations of the bundles $V^{\mathbb{T}[n]}$ and $V^{\mathbb{T}}$. The argument is indifferent to the parity of n.

From a practical point of view there are two important new ingredients. The first is a careful account of the choice of characters (rotation numbers) for the action of $A \subseteq \mathbb{T}$ on the vector bundle $V|_{X^A}$. This makes it easy to study the effect on our constructions of varying the characters and for that matter the subgroup A. The second is a systematic use of the geometry of the affine Weyl group of G, and its associated theory of theta functions [Loo76].

More important than any single practical improvements was the conceptual progress in understanding the relationship between the sigma orientation and Looijenga's work on root systems and elliptic curves. Let $G = Spin(2d)$ with maximal torus T and Weyl group W. Let $\check{T} = \hom(\mathbb{T}, T)$ be the lattice of cocharacters. Let $\mathbb{V}C$ be the complex abelian variety

$$\mathbb{V}C = \check{T} \otimes C.$$

Looijenga constructs a line bundle $\mathcal{A}$ over $(\mathbb{V}C/W)$ (see also §6.2). The sigma function σ defines a global holomorphic section σ_d of $\mathcal{A}$, whose zeroes define an ideal sheaf $\mathcal{I}$ on $\mathbb{V}C$, such that σ_d is an *trivialization* of $\mathcal{A} \otimes \mathcal{I}$.

The idea, which we discuss in more detail as Principle X in §11, is that if V is a $\mathbb{T}$-equivariant G-vector bundle over a $\mathbb{T}$-space X, and if $w_2(V_{\mathbb{T}}) = 0$, then $\mathcal{A} \otimes \mathcal{I}$ defines an invertible sheaf of $E_{\mathbb{T}}(X)$-modules $\mathcal{A}(V) \otimes \mathcal{I}(V)$. This sheaf has the properties

(1) there is a canonical isomorphism
$$\mathcal{I}(V) \cong E_{\mathbb{T}}(V)$$
of sheaves of $E_{\mathbb{T}}(X)$-modules
(2) the $E_{\mathbb{T}}(X)$-module $\mathcal{A}(V)$ is trivial precisely when $c_2(V_{\mathbb{T}}) = 0$.

Thus in the case that $c_2(V_{\mathbb{T}}) = 0$, the section σ_d gives a trivialization of $E_{\mathbb{T}}(V)$ which is the sigma orientation. The letter $\mathcal{A}$ was chosen to stand for "anomaly".

Early versions of this paper were attempts to prove Principle X and then to deduce Theorem 1.4 from it. However, for reasons we explain in §11, we were only occasionally able to convince even ourselves of those proofs. Eventually, detailed consideration of the consequences of Principle X led us to the formulae in §6.4 and so to concrete proofs.

In §11 we do establish Principle X for a functor which captures the behavior of the stalks of Grojnowski's functor. It was also inspired by Greenlees's rational $\mathbb{T}$-equivariant elliptic spectra [Gre01] and by Hopkins's work on characters in elliptic cohomology [Hop89]. Indeed we suspect that Greenlees's rational $\mathbb{T}$-equivariant elliptic spectra will admit a very natural proof of Principle X, and so give an account of the *rational* circle-equivariant sigma orientation. Likely this will involve recasting the arguments in this paper about the sigma function and the affine Weyl group in terms of the algebraic theory of theta functions. We hope to turn to those problems in the near future.

1.2. **Plan.** The plan of the paper is the following. We begin in §3 with a study of the structure of principal G bundles with an action of the circle. In §4 we discuss complex-orientable cohomology theories in general and ordinary and elliptic cohomology theories in particular. In §5 we interpret the analysis of §3 in the presence of a (rational) complex-orientable cohomology theory. We begin §6 with an interlude (§6.1) on degree-four characteristic classes. In §6.2 we recall a result essentially due to [Loo76], that a degree-four characteristic class $\xi \in H^4(BG;\mathbb{Z})$ gives rise to a W-equivariant line bundle $\mathcal{L}(\xi)$ over $(\check{T} \otimes C)$. We define a *theta function* *of level* ξ for G to be a W-invariant holomorphic section of the line bundle $\mathcal{L}(\xi)$; the sigma function provides the most important examples for us, and so we discuss it in §6.3.

Section 6.4 is the heart of the paper. In it we use the results of §3—6.3 to construct some holomorphic characteristic classes for $\mathbb{T}$-equivariant principal G-bundles which are the building blocks of the Thom classes in §9 and §10.

In §7 we recall the construction of Grojnowski's analytic $\mathbb{T}$-equivariant elliptic cohomology associated to a lattice $\Lambda \subset \mathbb{A}^1_{\mathrm{an}}$. In §8 we review the equivariant elliptic cohomology of Thom complexes [Ros01, AB00], recalling what is involved in constructing a global section of $E_{\mathbb{T}}(V)$, where V is (virtual) $\mathbb{T}$-equivariant *Spin* vector bundle.

In §10 we construct the equivariant sigma orientation, proving Theorem 1.4. In §9 we prove the following related result. Let G be a spinor group, let G' be a connected compact Lie group. Let V be a $\mathbb{T}$-equivariant G-bundle over X, and let V' be a $\mathbb{T}$-equivariant G'-bundle. Suppose that $V_{\mathbb{T}}$ is a G-bundle, and that $V'_{\mathbb{T}}$ is a G'-bundle, over $X_{\mathbb{T}}$. Suppose that ξ' is a degree-four characteristic classes for G', with the property that

$$c_2(V_{\mathbb{T}}) = \xi'(V'_{\mathbb{T}}).$$

Suppose that θ' is a theta function for G' of level ξ'.

Theorem 1.6 (9.5). *A* $\mathbb{T}$-*orientation on* V *determines a canonical global section* γ *of* $E_{\mathbb{T}}(V)^{-1}$*, whose value in* $E_{\mathbb{T}}(V)_0^{-1}g$ *is* $\theta'(V'_{\mathbb{T}})\Sigma(V_{\mathbb{T}})^{-1}$.

In particular, suppose that V' is a $Spin(2d')$ $\mathbb{T}$-vector bundle over X, and V is a $Spin(2d)$ $\mathbb{T}$-vector bundle. If

$$w_2(V'_{\mathbb{T}}) = 0 = w_2(V_{\mathbb{T}}),$$

then the Borel constructions are again spinor bundles. Suppose that θ' is the character of a representation of $LSpin(2d')$ of level k: then it is a theta function of level kc_2 for $Spin(2d')$. If

$$c_2(V_{\mathbb{T}}) = kc_2(V'_{\mathbb{T}}),$$

then Theorem 1.6 gives a global section γ of $E_{\mathbb{T}}(V)^{-1}$. The Pontrjagin-Thom construction for the map $\pi : X \to *$ gives map

$$(E_{\mathbb{T}}\pi)_* E_{\mathbb{T}}(V)^{-1} \to E_{\mathbb{T}}(*) = \mathcal{O}_C$$

of $\mathcal{O}_C$-modules which takes γ to the equivariant Witten genus of V twisted by the characteristic class $\theta(V'_{\mathbb{T}})$. Since the global sections of $\mathcal{O}_C$ are the constants, we have the following result of [Liu96].

Corollary 1.7. *Under these conditions, the equivariant Witten genus of* V *twisted by* $\theta(V'_{\mathbb{T}})$ *is constant.*

If §6.4 is the heart of the paper, then §11 is the soul. There we discuss the conceptual framework which led to the results in this paper. We give a conceptual construction of the equivariant sigma orientation, which even illuminates the non-equivariant case. Because the characters of representations of the loop group LG are sections of the line bundle $\mathcal{A}$, it illuminates the relationship between the sigma orientation, equivariant elliptic cohomology, and representations of loop groups which was studied in [And00].

1.3. **Acknowledgments.** My first debt is to Maria Basterra. Our work on [AB00] led directly to the results in this paper, and I have very much enjoyed and benefitted from our collaboration. Maria was to have been an author of this paper as well, and I have reluctantly accepted her request to withdraw her name from it. I thank Alejandro Adem for inviting me to visit Madison; it was during the eight-hour round-trip drive that the the function F (6.15) was discovered, along with its properties as described in §6.4. John Greenlees, Haynes Miller, and Jack Morava offered useful comments and advice. I thank Haynes Miller, Jack Morava, and Amnon Neeman for encouraging me to record the ideas in §11; I hope they do not regret the result. My work on elliptic cohomology has been profoundly influenced by Mike Hopkins, and I am grateful to him for his work on the subject and his generosity to me in particular.

2. Notation

2.1. **Abelian groups.** Let $\mathcal{C}$ be a category with finite products. The category $A\mathcal{C}$ of abelian groups in $\mathcal{C}$ is an additive category. In fact $A\mathcal{C}$ is tensored over the subcategory of finitely generated free abelian groups.

That is, a finitely generated free abelian group F and an abelian group X of $\mathcal{C}$ determine (naturally in F and X) an object $F \otimes X$ of $A\mathcal{C}$, with a natural isomorphism

$$A\mathcal{C}[F \otimes X, Y] \cong (\text{abelian groups})[F, A\mathcal{C}[X, Y]].$$

If A is an abelian group written additively, and M is an abelian group written multiplicatively, then we write m^a for the element $a \otimes m$ of $A \otimes M$. Similarly, if M' is an abelian group, $u \in M'$, and $m \in \hom(M', M)$ then we may write u^m for $m(u)$.

2.2. **Lie groups and the group $\mathbb{V}X$.** In general, the letter G will stand for a compact Lie group with maximal torus T with Weyl group W. We define

$$\check{T} \stackrel{\text{def}}{=} \hom[\mathbb{T}, T]$$
$$\hat{T} \stackrel{\text{def}}{=} \hom[T, \mathbb{T}]$$

to be the lattices of cocharacters and characters. We write

$$c : G \to \operatorname{Aut}(G)$$

for the action of G on itself by *conjugation*:

$$c_g h = ghg^{-1}.$$

If X is an abelian group in any category, then we write $\mathbb{V}X$ for the tensor product

$$\mathbb{V}X \stackrel{\text{def}}{=} \check{T} \otimes X,$$

which carries an action of the Weyl group W. If r is the rank of G, then $\mathbb{V}X$ is isomorphic to X^r.

2.3. **Elliptic curves.** Fix τ in the complex upper half plane, and let Λ be the lattice

$$\Lambda = 2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}.$$

We write $\mathbb{G}_a^{\text{an}}$ for the analytic affine plane $\mathbb{A}^1_{\text{an}}$ with its additive structure of abelian topological group, and we write z for the standard coordinate on $\mathbb{G}_a$ and also on $\mathbb{A}^1$, $\mathbb{A}^1_{\text{an}}$, $\mathbb{G}_a^{\text{an}}$, etc. We write $\mathbb{G}_m^{\text{an}}$ for the punctured plane $(\mathbb{A}^1_{\text{an}})^\times$ with its multiplicative group structure. For $r \in \mathbb{Q}$ we write $u^r = e^{rz}$ and $q^r = e^{2\pi i r\tau}$.

Let C be the elliptic curve

$$C = \mathbb{G}_a^{\text{an}}/\Lambda \cong \mathbb{G}_m^{\text{an}}/q^{\mathbb{Z}}.$$

We write $\wp$ for the covering map

$$\mathbb{G}_a^{\text{an}} \xrightarrow{\wp} C.$$

If V is an open set in a complex analytic variety, then we write $\mathcal{O}_V$ for the sheaf of holomorphic functions on V.

If A is an abelian topological group and $a \in A$, when we write τ_a for the translation map; and if $V \subset G$ is an open set, then we write

$$V - g \stackrel{\text{def}}{=} \tau_{-g}(V).$$

Definition 2.1. An open set U of C is *small* if it is connected and $\wp^{-1}U$ is a union of connected components V with the property that

$$\wp|_V : V \to U$$

is an isomorphism.

If U is small and V is a component of $\wp^{-1}U$, then the covering map induces an isomorphism

$$\mathcal{O}_U \cong \mathcal{O}_V.$$

In particular, if U contains the origin of C, then there is a unique component V of $\wp^{-1}U$ containing 0. This determines a $\mathbb{C}[z]$-algebra structure on $\mathcal{O}_U$, and a $\mathbb{C}[z, z^{-1}]$ structure on $\mathcal{O}_U|_{U\setminus 0}$.

2.4. **Ringed spaces.** Ginzburg-Kapranov-Vasserot [GKV95] have proposed that the $\mathbb{T}$-equivariant elliptic cohomology associated to an elliptic curve C should be a covariant functor

$$E_{gkv} : (\mathbb{T}\text{-spaces}) \to (\text{schemes})/C.$$

Grojnowski's $\mathbb{T}$-equivariant elliptic cohomology is a contravariant functor

$$E_{\mathbb{T}} : (\text{finite } \mathbb{T}\text{-CW complexes}) \to (\mathbb{Z}/2\text{-graded } \mathcal{O}_C\text{-algebras}) \tag{2.2}$$

(see §7 and [Gro94, Ros01, AB00]). These are meant to be related by the formula

$$E_{\mathbb{T}}(X) = f_* \mathcal{O}_{E_{gkv}(X)}, \tag{2.3}$$

where

$$f : E_{gkv}(X) \to C$$

is the structural map. However, Grojnowksi's functor can not quite be of the form (2.3), since in (2.2) $\mathcal{O}_C$ is the sheaf of *holomorphic* functions on the *analytic* space $C = \mathbb{G}_a^{\mathrm{an}}/\Lambda$. However, it does give a covariant functor

$$(\mathbb{T}\text{-spaces}) \to (\text{ringed spaces})/C.$$

Precisely, we have the following.

Definition 2.4. By a *(super, or $\mathbb{Z}/2$-graded) ringed space* we shall mean a pair $(X, \mathcal{O}_X)$ consisting of a space X and a sheaf $\mathcal{O}_X$ of $\mathbb{Z}/2$-graded rings on X. A *map of ringed spaces*

$$f = (f_1, f_2) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

consists of a map of spaces $f_1 : X \to Y$ and a map of sheaves of $\mathbb{Z}/2$-graded algebras over Y

$$f_2 : \mathcal{O}_Y \to (f_1)_* \mathcal{O}_X.$$

The resulting category of ringed spaces will be denoted $\mathcal{R}$. If $\mathcal{X} = (X, \mathcal{O}_X)$ is a ringed space and U is an open set of X, then we may write $\mathcal{X}(U)$ in place of $\mathcal{O}_X(U)$.

If X is a finite $\mathbb{T}$-CW complex, then

$$X_{E\mathbb{T}} = (C, E_{\mathbb{T}}(X))$$

is a ringed space, and this defines a covariant functor

$$(-)_{E\mathbb{T}} : (\mathbb{T}\text{-CW complexes}) \to \mathcal{R}/C.$$

We have found this point of view to be extremely helpful, and so we have adopted it in writing this paper.

3. Principal bundles with an action of the circle

3.1. **A-bundles over trivial A-spaces.** Let A be a closed subgroup of the circle $\mathbb{T}$. Let G be connected compact Lie group, and let $\pi : Q \to Y$ be a principal G-bundle over a connected space Y. Suppose that A acts on Q/Y, fixing Y. The group of automorphisms of Q/Y is the $\Gamma((Q \times_{G,c} G)/Y)$, and an action of A on Q/Y is equivalent to a section $a \in \Gamma((Q \times_{G,c} \hom[A, G])/Y)$.

Every G-orbit in $\hom[A, G]$ intersects $\hom[A, T]$ nontrivially, and so for $x \in Y$ we may choose a homomorphism $m_x \in \hom[A, T]$ such that

$$a(x) = [p, m_x]$$

for some $p \in Q$; the square brackets indicate the class in the Borel construction of the element $(p, m_x) \in Q \times \hom[A, G]$. The choice of m_x determines p only up to the centralizer $Z(m_x)$ in G of the homomorphism m_x. Since $\hom[A, T]$ is discrete, we may choose m_x to be constant on Y.

Definition 3.1. A *reduction* of the action of A on Q is a homomorphism

$$m : A \to T$$

such that, for all $x \in Y$, there is a $p \in Q$ such that

$$a(x) = [p, m].$$

The terminology is justified by the following observation. Let

$$Q(m) = \{p \in Q | [p, m] = a(\pi(p))\}.$$

Then $\pi|_{Q(m)} : Q(m) \to Y$ is a principal $Z(m)$ bundle over Y, the reduction of the structure group of Q to $Z(m)$. In other words, we have given a factorization

$$\begin{array}{ccc} & & BZ(m) \\ & \overset{Q(m)}{\nearrow} & \downarrow \\ Y & \xrightarrow[Q]{} & BG. \end{array} \tag{3.2}$$

Let

$$W(m) = \{w \in W | wm = m\}$$

be the stabilizer of m. One sees that T is a maximal torus of $Z(m)$, with Weyl group $W(m)$.

Example 3.3. Suppose that $G = U(n)$ is the unitary group with its maximal torus $T = \Delta(z_1, \dots, z_n)$ of diagonal matrices, and suppose that $A = \mathbb{T}$. Every homomorphism

$$m : \mathbb{T} \to T$$

is conjugate in $U(n)$ to one of the form

$$m(z) = \Delta(z^{m_1}, \dots, z^{m_1}, z^{m_2}, \dots, z^{m_2}, \dots, z^{m_r}, \dots, z^{m_r}),$$

where $m_i \in \mathbb{Z}$. Let d_j be the multiplicity of m_j; then the centralizer is the block-diagonal matrix

$$Z(m) = \begin{bmatrix} U(d_1) & & \\ & \cdots & \\ & & U(d_r) \end{bmatrix},$$

with Weyl group

$$W(m) = \Sigma_{d_1} \times \cdots \times \Sigma_{d_r} \subset \Sigma_n = W.$$

If V is a $\mathbb{T}$-equivariant complex vector bundle over a connected trivial $\mathbb{T}$-space Y, and if this m is a reduction of the action of $\mathbb{T}$ on the principal bundle of V, then the reduction of the structure group to $Z(m)$ corresponds to the decomposition of V as the direct sum

$$V \cong V(m_1) \oplus \cdots \oplus V(m_r),$$

where $\mathbb{T}$ acts on the fiber of $V(m_j)$ by the character z^{m_j}. □

The composition

$$g(m) : A \times Z(m) \xrightarrow{m \times Z(m)} T \times Z(m) \to Z(m)$$

is a group homomorphism, and so we have a map

$$BA \times BZ(m) \xrightarrow{Bg(m)} BZ(m).$$

The following Lemma will be used directly to prove Lemma 6.5, a calculation of degree-four characteristic classes. Moreover, the algebro-geometric form of this diagram after applying a complex-orientable cohomology theory (see Lemma 5.1) captures the essential point of the "transfer argument" of [BT89], as we explain in Remark 11.7.

Lemma 3.4. (1) *The diagram*

$$\begin{array}{ccccc} & & BT \times BT & & \\ & \overset{Bm \times BT}{\nearrow} & & \searrow & \\ BA \times BT & & & & BT \\ \downarrow & & & & \downarrow \\ BA \times BZ(m) & & \xrightarrow{Bg(m)} & & BZ(m) \end{array}$$

commutes.

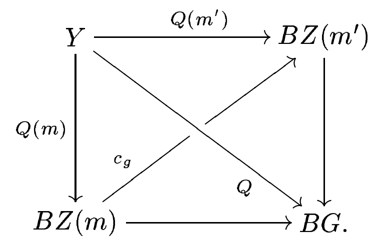
(2) *The map $Bg(m)$ classifies the principal $Z(m)$-bundle $EZ(m)_A = EA \times_{A,m} EZ(m)$ over $BA \times BZ(m)$.* □

Proof. The first part is easy. For the second part, it suffices to construct a map of principal $Z(m)$-bundles over $BA \times BZ(m)$. The map

$$Eg(m) : EA \times EZ(m) \to EZ(m)$$

factors through $EA \times_{A,m} EZ(m)$, and gives the desired map. (Note that the map $Eg(m)$ is obtained by constructing EA and $EZ(m)$ functorially as spaces with actions on the same side, say the left. In forming $EA \times_{A,m} EZ(m)$, one makes A act on the right of EA by the inverse.) □

If $m' : A \to T$ is another reduction of the action of A on Q, then m and m' differ by conjugation in G, and we have for some g in G a commutative diagram



(3.5)

3.2. The case of a connected centralizer.

Lemma 3.6. *If the centralizer $Z(m)$ is connected, then the element g in the diagram (3.5) may be taken to be in the normalizer N_GT of T in G.*

Proof. Let $g \in G$ be such that

$$c_g m = m'$$

Then $m'(A) \subset c_g(T) \cap T$, so both T and $c_g(T)$ are maximal tori in $Z(m')$. Since $Z(m')$ is connected, there is an element $h \in Z(m')$ such that

$$c_h c_g(T) = T,$$

so $hg \in N_GT$. Since $h \in Z(m')$, we have

$$m' = c_h m' = c_h c_g m.$$

□

Example 3.3 shows that $Z(m)$ is connected if G is a unitary group. Bott and Samelson have shown ([BS58]; see Proposition 10.2 of [BT89]) that $Z(m)$ is connected if G is a spinor group.

Lemma 3.7. *If G is simple and simply connected then the centralizer $Z(m)$ is connected.* □

3.3. Nested fixed-point sets.

Now suppose that $B \supseteq A$ is a larger closed subgroup of $\mathbb{T}$ (primarily, we shall be interested in the case that $B = \mathbb{T}$), and that the bundle Q/Y is actually an equivariant B-bundle over Y.

Lemma 3.8. *If*

$$m : A \to T$$

is a reduction of the action of A on Y, then the action of B on Q induces on $Q(m)$ the structure of a B-equivariant $Z(m)$-bundle over Y. If moreover the Borel construction Q_B is a principal G-bundle over Y_B, then the Borel construction $Q(m)_B$ is a principal $Z(m)$-bundle over Y_B.

Proof. This follows from the fact that B is abelian. □

If $F \subset Y^B$ is a connected component of the subspace of Y fixed by the action of B, then we may choose a reduction

$$m_F : B \to T$$

of the action of B on $Q|_F$.

Lemma 3.9. *The restriction*

$$m_F|_A : A \to T$$

is a reduction of the action of A on Y.

Proof. The action of A on Q/Y is a section a of $(Q \times_c \hom[A, G])/Y$. The restriction $a|_F$ records the action of A on $Q|_F$. The action of B on $Q|_F$ is a section b of $(Q|_F \times_c \hom[B, G])/F$, and with the obvious notation we have

$$b|_A = a|_F.$$

□

4. Cohomology

4.1. **Ordinary cohomology.** If R is a commutative ring, let HR denote ordinary cohomology with coefficients in R. If X is a space, then we write

$$X_{HR} \stackrel{\text{def}}{=} \operatorname{spec}(HR^{2*}(X))$$

for scheme over $\operatorname{spec} R$ associated to the cohomology of X. We may also view X_{HR} as the ringed space with underlying space $\operatorname{spec}(HR^{2*}(X))$ and structure sheaf associated to

$$HR^{\text{even}}(X) \oplus HR^{\text{odd}}(X).$$

We shall write H for $H\mathbb{C}$, cohomology with complex coefficients.

4.2. **Equivariant cohomology.** If X is a space with a circle action, then $(X_{\mathbb{T}})_{HR}$ is a scheme over $B\mathbb{T}_{HR}$, which we denote $X_{\mathbb{T}HR}$.. We choose a generator of the character group of $\mathbb{T}$, and write z for the resulting generator of $H\mathbb{Z}^2(B\mathbb{T})$; this gives an isomorphism

$$B\mathbb{T}_{HR} \cong (\mathbb{G}_a)_R \tag{4.1}$$

of group schemes over $\operatorname{spec} R$. We shall use (4.1) to view $X_{\mathbb{T}HR}$ as a scheme over $\mathbb{A}^1_R$.

We recall [Qui71] that equivariant cohomology satisfies a localization theorem.

Theorem 4.2. *If X has the homotopy type of a finite $\mathbb{T}$-CW complex (e.g. if X is a compact $\mathbb{T}$-manifold), then the natural map*

$$X^{\mathbb{T}}_{\mathbb{T}H} \to X_{\mathbb{T}H}$$

induces an isomorphism over $\operatorname{spec} \mathbb{C}[z, z^{-1}] \subset B\mathbb{T}_H$. □

Holomorphic cohomology. Let $\mathbb{A}^1_{\text{an}}$ be the analytic complex plane, so $\mathcal{O}_{\mathbb{A}^1_{\text{an}}}$ is the sheaf of holomorphic functions on $\mathbb{C}$. Because of the natural maps

$$\mathbb{A}^1_{\text{an}} \to \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1 \cong B\mathbb{T}_{H\mathbb{Z}}$$

we may view z as a function on $\mathbb{A}^1_{\text{an}}$. Given a $\mathbb{T}$-space X we define the *holomorphic* cohomology of X to be the sheaf of super $\mathcal{O}_{\mathbb{A}^1_{\text{an}}}$-algebras given by

$$\mathcal{H}(X; U) \stackrel{\text{def}}{=} H\mathbb{C}(X_{\mathbb{T}}) \underset{\mathbb{C}[z]}{\otimes} \mathcal{O}_{\mathbb{A}^1_{\text{an}}}(U).$$

We view $\mathcal{H}(X)$ as the structure sheaf of a ringed space (2.4) $X_{\mathcal{H}}$ over $\mathbb{A}^1_{\text{an}}$, namely the the pull-back in the diagram

$$\begin{array}{ccccc} X_{\mathcal{H}} & \longrightarrow & X_{\mathbb{T}H} & \longrightarrow & X_{\mathbb{T}H\mathbb{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^1_{\text{an}} & \longrightarrow & \mathbb{A}^1_{\mathbb{C}} & \longrightarrow & \mathbb{A}^1_{\mathbb{Z}}. \end{array}$$

The stalk of holomorphic cohomology. Let

$$\mathbb{A}^1_{\text{an},0} = \operatorname{spec}(\mathcal{O}_{\mathbb{A}^1_{\text{an}},0})$$

be the local scheme associated to the stalk of $\mathcal{O}_{\mathbb{A}^1_{\text{an}}}$ at the origin. The stalk of $\mathcal{H}(X)$ at the origin is

$$\mathcal{H}(X)_0 \cong H(X_{\mathbb{T}}) \underset{\mathbb{C}[z]}{\otimes} (\mathcal{O}_{\mathbb{A}^1_{\text{an}},0}).$$

We write $X_{\mathbb{T}H,0}$ for the resulting scheme over $\mathbb{A}^1_{\text{an},0}$.

Periodic Borel cohomology. Let $\hat{H}$ denote *periodic* ordinary cohomology with complex coefficients: that is,

$$\hat{H} = \bigvee_{k\in\mathbb{Z}} \Sigma^{2k} H$$

so

$$\pi_* \hat{H} = \mathbb{C}[v, v^{-1}]$$

with $v \in \pi_2 \hat{H}$. Then

$$\hat{H}^0(B\mathbb{T}) \cong \mathbb{C}[\![z]\!]$$

so $\operatorname{spf} \hat{H}^0(B\mathbb{T}) = (\mathbb{A}^1_{\mathrm{an}})^\wedge_0 = (\mathbb{A}^1_{\mathbb{C}})^\wedge_0$, and $\hat{H}^0(X_{\mathbb{T}})$ is the ring of formal functions on the pull-back $X_{\mathbb{T}\hat{H}}$ in the diagram of formal schemes

$$\begin{array}{ccc} X_{\mathbb{T}\hat{H}} & \longrightarrow & X_{\mathbb{T}H,0} \\ \downarrow & & \downarrow \\ (\mathbb{A}^1_{\mathrm{an}})^\wedge_0 & \longrightarrow & \mathbb{A}^1_{\mathrm{an},0}. \end{array}$$

Combining all these, we have a collection of forms of ordinary cohomology

$$\begin{array}{ccccccccc} X_{\mathbb{T}\hat{H}} & \longrightarrow & X_{\mathbb{T}H,0} & \longrightarrow & X_{\mathcal{H}} & \longrightarrow & X_{\mathbb{T}H} & \longrightarrow & X_{\mathbb{T}H\mathbb{Z}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\mathbb{A}^1_{\mathbb{C}})^\wedge_0 & \longrightarrow & \mathbb{A}^1_{\mathrm{an},0} & \longrightarrow & \mathbb{A}^1_{\mathrm{an}} & \longrightarrow & \mathbb{A}^1_{\mathbb{C}} & \longrightarrow & \mathbb{A}^1_{\mathbb{Z}}. \end{array} \tag{4.3}$$

4.3. Generalized cohomology.

Even periodic ring spectra. A ring spectrum E will be called "even periodic" if $\pi_{\mathrm{odd}} E = 0$ and $\pi_2 E$ contains a unit of $\pi_* E$.

If E is an even periodic ring spectrum, and if X is a space, then we shall write $E^* X$ for the unreduced cohomology of X. As in [AHS01], we write X_E for the formal scheme

$$X_E = \operatorname{colim}_{F\subset X} \operatorname{spec} E^0 F$$

over $S_E = \operatorname{spec} E^0(*)$; the colimit is over the compact subsets of X.

An even periodic ring spectrum is always complex-orientable. In particular

$$P_E \overset{\mathrm{def}}{=} B\mathbb{T}_E$$

is a (commutative, one-dimensional) formal group over S_E.

Borel cohomology. If E is an even periodic ring spectrum, and X is a space with a circle action, then we write

$$X_{\mathbb{T}E} \overset{\mathrm{def}}{=} (X_{\mathbb{T}})_{\hat{E}}$$

for the formal scheme associated to the even Borel $\hat{E}$-theory of X. The natural map

$$X_{\mathbb{T}} \to B\mathbb{T}$$

induces a map

$$X_{\mathbb{T}E} \to P_E.$$

For example, let $HP\mathbb{Z}$ denote periodic ordinary cohomology with integer coefficients. Then $P_{HP\mathbb{Z}} = \widehat{\mathbb{G}}_a$.

Elliptic spectra. We recall [AHS01] that an *elliptic spectrum* is a triple (E, C, t) consisting of

(1) an even periodic ring spectrum E,
(2) a (generalized) elliptic curve C over S_E, and
(3) an isomorphism of formal groups

$$t : \widehat{C} \cong P_E.$$

Rational elliptic spectra. Let C be an elliptic curve over a $\mathbb{Q}$-algebra R, and let $\underline{\omega} = 0^*\Omega^1_{C/R}$. If ω is a trivialization of $\underline{\omega}$, then there is a canonical isomorphism of formal groups over R

$$\widehat{C} \xrightarrow[\cong]{\log_\omega} \widehat{\mathbb{G}}_a$$

with the property that $0^* \log_\omega^* dz = 0^* dz$.

Let $\Gamma^\times(\underline{\omega})$ be the functor from rings to sets which given by

$$\Gamma^\times(\underline{\omega})(T) = \{(j,\omega) \mid j : \operatorname{spec} T \to \operatorname{spec} R, \omega \text{ a trivializaton of } j^*\underline{\omega}\}.$$

If ω is a trivialization of $\underline{\omega}$, then the trivialization of $j^*\underline{\omega}$ are of the form $vj^*\omega$ for $u \in T^\times$, so

$$\Gamma^\times(\underline{\omega}) \cong \operatorname{spec} R[u, u^{-1}].$$

Let $S = \mathcal{O}(\Gamma^\times(\underline{\omega}))$.

The logarithm then gives a canonical isomorphism of formal groups

$$\widehat{C}_S \xrightarrow{\log_{\widehat{C}}} (\widehat{\mathbb{G}}_a)_S$$

by the formula

$$(c,\omega) \mapsto (\log_\omega c, \omega).$$

If $HS[v, v^{-1}]$ is the even periodic ring spectrum such that

$$(HS[v, v^{-1}])^* X \overset{\text{def}}{=} H^*(X; S[v, v^{-1}])$$

then

$$P_{HS[v,v^{-1}]} = (\widehat{\mathbb{G}}_a)_S,$$

and we have an elliptic spectrum

$$(HS[v, v^{-1}], C, \log_{\widehat{C}}).$$

Alternatively, given over R a trivialization ω of $\underline{\omega}$ we have the elliptic spectrum $(HR[v, v^{-1}], C, \log_\omega)$.

Complex elliptic spectra. The projection $\wp : \mathbb{G}_a^{\text{an}} \to C$ induces an isomorphism of formal groups

$$\widehat{\wp} : \widehat{\mathbb{G}}_a \to \widehat{C}.$$

There is a unique cotangent vector ω such that

$$\wp^*\omega = 0^* dz.$$

We have

$$\log_\omega = (\widehat{\wp})^{-1},$$

and so an elliptic spectrum

$$(\hat{H}, C, \widehat{\wp}^{-1}) = (\hat{H}, C, \log_\omega). \tag{4.4}$$

5. Rational cohomology of principal bundles with compact connected structure group

Let E be an *rational* even periodic ring spectrum. Let G be a connected compact Lie group. Let T be a maximal torus of G, with Weyl group W. The natural isomorphism

$$\check{T} \otimes \mathbb{T} \to T$$

induces a W-equivariant isomorphism

$$\mathbb{V}P_E \cong BT_E$$

of formal groups over S_E. Moreover [Bor55] the natural map

$$BT_E/W \to BG_E$$

is an isomorphism. We shall repeatedly use the resulting isomorphism

$$BG_E \cong \mathbb{V}P_E/W.$$

For example, a principal G-bundle Q over X is classified by a map

$$X \xrightarrow{Q} BG$$

whose effect in E-theory

$$X_E \xrightarrow{Q_E} \mathbb{V}P_E/W$$

is an X_E-valued point of $\mathbb{V}P_E/W$.

5.1. **Periodic cohomology of circle-equivariant principal bundles.** We interpret the analysis of §3 in E-theory. Suppose that G is a connected compact Lie group, and that Q is a $\mathbb{T}$-equivariant principal G-bundle over a connected $\mathbb{T}$-space Y. Suppose that a closed subgroup A of the circle acts trivially on Y. Suppose that

$$m : A \to T$$

is a reduction of the action of A on Q/Y with *connected* centralizer $Z(m)$.

Applying E-cohomology to the diagram (3.2) yields

$$\begin{array}{ccc} & & \mathbb{V}P_E/W(m) \\ & \overset{Q(m)_E}{\nearrow} & \downarrow \\ Y_E & \xrightarrow[Q_E]{} & \mathbb{V}P_E/W \end{array}$$

The multiplication

$$BT \times BT \to BT$$

induces the addition map

$$\mathbb{V}P_E \times \mathbb{V}P_E \xrightarrow{+} \mathbb{V}P_E$$

whose restriction to

$$(\mathbb{V}P_E)^{W(m)} \times \mathbb{V}P_E \xrightarrow{+} \mathbb{V}P_E$$

factors to give a translation map

$$(\mathbb{V}P_E)^{W(m)} \times (\mathbb{V}P_E)/W(m) \xrightarrow{+} (\mathbb{V}P_E)/W(m).$$

In E cohomology, Lemma 3.4 implies the following result. It is used in the case $A = \mathbb{T}$ to construct the commutative diagram (9.2) and so prove Lemma 9.4. It is also implies the commutativity of the diagram (11.6), which captures the essence of the "transfer formula" of Bott-Taubes [BT89]; see Remark 11.7.

Lemma 5.1. *The diagram*

$$\begin{array}{ccc} BA_E \times BZ(m)_E & \xrightarrow{\cong} & (\mathbb{V}P_E)^{W(m)} \times (\mathbb{V}P_E)/W(m) \\ {\scriptstyle BA_E \times Q(m)_E}\uparrow & & \downarrow{\scriptstyle +} \\ BA_E \times Y_E & \xrightarrow{Q(m)_{AE}} & (\mathbb{V}P_E)/W(m) \end{array}$$

commutes. If moreover the Borel construction $Q_{\mathbb{T}}$ is a principal G-bundle over $Y_{\mathbb{T}}$, then the diagram

$$\begin{array}{ccc} BA_{\hat{E}} \times Y_{\hat{E}} & \xrightarrow{Q(m)_{AE}} & (\mathbb{V}P_E)/W(m) \\ \downarrow & \overset{Q(m)_{\mathbb{T}E}}{\nearrow} & \\ Y_{\mathbb{T}E} & & \end{array}$$

commutes. □

5.2. **Holomorphic characteristic classes.** Let f be a W-invariant holomorphic function on $\mathbb{V}\mathbb{G}_a^{\mathrm{an}}$. By the splitting principle, the Taylor expansion of f at the origin defines a class in $\hat{H}(BG)$. Suppose that Q is a principal G-bundle over X, with the property that $Q_{\mathbb{T}}$ is a principal G-bundle over $X_{\mathbb{T}}$. Then we get a class

$$f(Q_{\mathbb{T}}) \in \hat{H}(X_{\mathbb{T}}).$$

The following result is due to Rosu.

Lemma 5.2. *The class $f(Q_{\mathbb{T}})$ is in fact an element of $\mathcal{H}(X; \mathbb{A}^1_{\mathrm{an}})$. Similarly, if $f \in (\mathcal{O}_{\mathbb{V}\mathbb{G}_a^{\mathrm{an}}})_0$, then*

$$f(Q_{\mathbb{T}}) \in \mathcal{H}(X)_0.$$

Proof. Proposition A.6 of [Ros01] proves this result in the case that $G = U(n)$ and

$$f(z) = \prod_j g(z_j),$$

where $g \in \mathcal{O}_{\mathbb{A}^1_{\text{an}},0}$ and $z = (z_1, \dots, z_n) \in \mathbb{A}^n_{\text{an}} \cong \mathbb{V}\mathbb{G}_a^{\text{an}}$. The same argument works in the indicated generality. □

An important example of a holomorphic characteristic class is the euler class associated to a "multiplicative analytic orientation". A power series

$$f(z) = z + \text{ higher terms} \in \hat{H}(B\mathbb{T}) \cong \mathbb{C}[\![z]\!]$$

satisfying

$$f(-z) = -f(z)$$

determines a multiplicative orientation (map of homotopy commutative ring spectra)

$$\phi : MSO \to \hat{H},$$

characterized by the property that if

$$V = L_1 + \cdots + L_d$$

is a sum of complex line bundles, then its euler class is

$$e_\phi(V) = \prod_j f(c_1 L_j).$$

If V is an oriented vector bundle, we write $\phi(V) \in \hat{H}(V) = \hat{H}(X^V)$ for the resulting Thom class. It is multiplicative in the sense that

$$\phi(V \oplus W) = \phi(V) \wedge \phi(W) \tag{5.3}$$

under the isomorphism

$$(X \times Y)^{V \oplus W} \cong X^V \wedge Y^W.$$

Definition 5.4. The orientation ϕ is *analytic* if f is contained in the subring $\mathcal{O}_{\mathbb{A}^1_{\text{an}},0} \subset \mathbb{C}[\![z]\!]$ of germs of holomorphic functions at 0; equivalently, if there is a neighborhood U of 0 in $\mathbb{C}$ on which the power series f converges to a holomorphic function.

Lemma 5.2 implies the following.

Corollary 5.5 ([Ros01]). *If ϕ is analytic and V is an oriented $\mathbb{T}$-vector bundle over a compact $\mathbb{T}$-space X, then the euler class e_ϕ associated to ϕ satisfies*

$$e_\phi(V_{\mathbb{T}}) \in \mathcal{H}(X)_0.$$

□

If Φ denotes the standard Thom isomorphism, then

$$\phi(V_{\mathbb{T}}) = \frac{e_\phi(V_{\mathbb{T}})}{e_\Phi(V_{\mathbb{T}})} \Phi(V_{\mathbb{T}}),$$

and the ratio of euler classes is a unit in $\mathcal{H}(X)_0$. Of course multiplication by $\Phi(V_{\mathbb{T}})$ induces an isomorphism

$$H(X_{\mathbb{T}}) \cong H(V_{\mathbb{T}}),$$

and so we have the following.

Corollary 5.6. *There is a neighborhood U of the origin in $\mathbb{A}^1_{\text{an}}$ such that*

$$\phi(V_{\mathbb{T}}) \in \mathcal{H}(V; U),$$

and such that multiplication by this class induces an isomorphism of sheaves

$$\mathcal{H}(X)|_U \xrightarrow[\cong]{\phi} \mathcal{H}(V)|_U.$$

In other words, for every open set $U' \subseteq U$, multiplication by $\phi(V_{\mathbb{T}})$ induces an isomorphism

$$\mathcal{H}(X; U') \xrightarrow[\cong]{\phi} \mathcal{H}(V; U').$$

□

Example 5.7. For example, let $\sigma = \sigma(u,q)$ denote the expression

$$\sigma = (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \prod_{n\geq 1} \frac{(1-q^n u)(1-q^n u^{-1})}{(1-q^n)^2}. \tag{5.8}$$

This may be considered as an element of $\mathbb{Z}[\![q]\!][u^{\pm\frac{1}{2}}]$ which is a holomorphic function of $(u^{\frac{1}{2}}, q) \in \mathbb{C}^\times \times D$, where $D = \{q \in \mathbb{C} | 0 < |q| < 1\}$. Let $\mathfrak{H} = \{\tau \in \mathbb{C} | \Im\tau > 0\}$ be the open upper half plane. We may consider σ as a holomorphic function of $(z,\tau) \in \mathbb{A}^1_{\mathrm{an}} \times \mathfrak{H}$ by setting

$$u^r = e^{rz}$$
$$q^r = e^{2\pi i r\tau}$$

for $r \in \mathbb{Q}$. It is easy to check using (5.8) that

$$\sigma(-z) = -\sigma(z) \tag{5.9a}$$
$$\sigma(z) = z + o(z^2) \tag{5.9b}$$
$$\sigma(uq^n) = (-1)^n u^{-n} q^{-\frac{n^2}{2}} \sigma(u). \tag{5.9c}$$

The equations (5.9) imply that the Taylor expansion of σ at the origin defines a multiplicative analytic orientation

$$MSO \xrightarrow{\Sigma} \hat{H}. \tag{5.10}$$

Definition 5.11. If V is an oriented vector bundle, we write $\Sigma(V)$ for the Thom class associated to the orientation (5.10), and $\sigma(V)$ for the associated euler class.

In [AHS01, §2.7] is it shown that Σ is the sigma orientation associated to the elliptic curve C: that is, the diagram

$$\begin{array}{ccc} MSO & \xrightarrow{\Sigma} & \hat{H} \\ \uparrow & \nearrow_{\sigma(\hat{H},C,\hat{\wp}^{-1})} & \\ MU\langle 6\rangle & & \end{array}$$

commutes.

6. Degree-four characteristic classes and theta functions

6.1. **Degree-four characteristic classes.** If G is a connected compact Lie group, then by the splitting principle the natural maps

$$\begin{aligned} H^2(BG,\mathbb{Z}) &\to H^2(BT,\mathbb{Z})^W \cong \hat{T}^W \\ H^4(BG,\mathbb{Z}) &\to H^4(BT,\mathbb{Z})^W \cong (S^2\hat{T})^W \cong \hom(\Gamma_2\check{T},\mathbb{Z})^W \end{aligned} \tag{6.1}$$

are rational isomorphisms. Here, if M is an abelian group, then S^2M and $\Gamma_2 M$ denote degree-two parts of the symmetric and divided power algebras on M.

Without rationalizing, a degree-four characteristic class $\xi \in H^4(BG,\mathbb{Z})$ gives rise to a homomorphism

$$\Gamma_2\check{T} \xrightarrow{I} \mathbb{Z}.$$

We shall abuse notation and also write I for the bilinear map

$$\check{T} \times \check{T} \xrightarrow{\gamma_1\times\gamma_1} \Gamma_2\check{T} \xrightarrow{I} \mathbb{Z}. \tag{6.2}$$

We shall say that the characteristic class ξ is *positive definite* if the pairing I is so. We also write ϕ for the quadratic function given by the composition

$$\begin{array}{ccc} \Gamma_2\check{T} & \xrightarrow{I} & \mathbb{Z} \\ {\scriptstyle\gamma_2}\uparrow & \nearrow_{\phi} & \\ \check{T} & & \end{array}$$

(If A and B are abelian groups, then a function

$$f : A \to B$$

is *quadratic* if

$$\begin{aligned} 0 &= f(0) \\ 0 &= f(x+y+z) - f(x+y) - f(x+z) - f(y+z) + f(x) + f(y) + f(z) \\ f(-x) &= f(x). \end{aligned}$$

The function

$$A \xrightarrow{\gamma_2} \Gamma_2 A$$

is the universal quadratic function out of A).

From the definitions it follows that

$$\begin{aligned} \phi(a+b) &= \phi(a) + I(a,b) + \phi(b) \\ \phi(na) &= n^2\phi(a) \\ \phi(wa) &= \phi(a) \\ I(wa, wb) &= I(a,b) \end{aligned} \tag{6.3}$$

for $a, b \in \check{T}$, $n \in \mathbb{Z}$, and $w \in W$.

There are a variety of ways to express the relationship between the characteristic class ξ and the map I. For example, suppose that Q_0 and Q_1 are two principal G-bundles over X, given as maps

$$X \xrightarrow{Q_i} BT.$$

Then we get a new principal G-bundle as the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{Q_0 \times Q_1} BT \times BT \xrightarrow{\mu} BT.$$

The effect of Q_i in cohomology

$$X_{H\mathbb{Z}} \xrightarrow{(Q_i)_{H\mathbb{Z}}} BT_{H\mathbb{Z}} \cong \mathbb{V}\mathbb{G}_a.$$

is an $X_{H\mathbb{Z}}$-valued point of $\mathbb{V}\mathbb{G}_a$. Equation (6.3) implies that

$$\xi(\mu(Q_0 \times Q_1)) = \xi(Q_0) + I((Q_0)_{H\mathbb{Z}}, (Q_1)_{H\mathbb{Z}}) + \xi(Q_1),$$

where we have extended I to a bilinear map

$$\mathbb{V}\mathbb{G}_a \times \mathbb{V}\mathbb{G}_a \to \mathbb{G}_a.$$

As another example, suppose that $A \subseteq \mathbb{T}$ is a closed subgroup with character group $\mathbb{Z}/n\mathbb{Z}$ ($n \geq 0$), and suppose that $m : A \to T$ is a homomorphism. If

$$\overline{m}, \overline{m}' \in \hom(\mathbb{T}, T) = \check{T}$$

satisfy

$$\overline{m}|_A = \overline{m}'|_A = m,$$

then

$$\overline{m}' = \overline{m} + n\delta$$

for some $\delta \in \check{T}$, and equation (6.3) implies that

$$\phi(\overline{m}') \equiv \phi(\overline{m}) \mod n$$

We write $\phi(m)$ for the class of $\phi(\overline{m})$ in $\mathbb{Z}/n$. If z is the chosen generator of $H\mathbb{Z}^2 B\mathbb{T}$, write also z for the induced generator of $H\mathbb{Z}^* BA \cong (\mathbb{Z}/n\mathbb{Z})[z]$. With these conventions

$$(Bm)^*\xi = \phi(m) z^2. \tag{6.4}$$

We write $\hat{I}$ for the map

$$\hat{I} : \check{T} \to \hat{T}$$

which is the adjoint of (6.2). Note that we have

$$\hat{I}(wa)(wb) = I(wa, wb) = I(a,b) = \hat{I}(a)(b).$$

It follows that if $a \in \check{T}^W$ then

$$\hat{I}(a) \in \hat{T}^W \to H^2(BG, \mathbb{Q}).$$

defines a rational characteristic class of principal G-bundles, which we also denote $\hat{I}(a)$.

Now suppose that Q is a $\mathbb{T}$-equivariant principal G-bundle over a connected trivial $\mathbb{T}$-space Y. Suppose as usual that $Q_{\mathbb{T}}$ is a principal G-bundle over $Y_{\mathbb{T}} = B\mathbb{T} \times Y$. Let $m \in \check{T}$ be a reduction of the action of $\mathbb{T}$ on Q/Y. Then $Q(m)$ is a principal $Z(m)$-bundle over Y, and we have the following.

Lemma 6.5. *In* $H^4(Y_{\mathbb{T}}, \mathbb{Q}) = H^4(B\mathbb{T} \times Y, \mathbb{Q})$ *we have*

$$\xi(Q_{\mathbb{T}}) = \phi(m)z^2 + \hat{I}(m)(Q(m))z + \xi(Q).$$

Proof. By the splitting principle, we may suppose that we have a factorization

$$\begin{array}{ccc} & & BT \\ & \overset{Q}{\nearrow} & \downarrow \\ Y & \xrightarrow[Q]{} & BG. \end{array}$$

Lemma 3.4 implies that the map

$$B\mathbb{T} \times Y \xrightarrow{Bm \times Q} BT \times BT \xrightarrow{\mu} BT$$

classifies $Q_{\mathbb{T}}$. It follows that

$$\begin{aligned} \xi(Q_{\mathbb{T}}) &= \xi(\mu(Bm \times Q)_{H\mathbb{Q}}) \\ &= \xi(Bm) + I((Bm)_{H\mathbb{Q}}, Q_{H\mathbb{Q}}) + \xi(Q) \\ &= \phi(m)z^2 + \hat{I}(m)(Q)z + \xi(Q). \end{aligned}$$

□

Example 6.6. Let $T_{SO(2d)} \cong \mathbb{T}^d$ be the standard maximal torus in $SO(2d)$ (the image under the map $U(d) \to SO(2d)$ of the torus of diagonal matrices), and let T be its preimage in $Spin(2d)$. If $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \cong (T_{SO(2d)})^{\vee}$, then there is a lift in the diagram

$$\begin{array}{ccc} & & T \\ & \nearrow & \downarrow \\ \mathbb{T} & \xrightarrow{m} & T_{SO(2d)} \end{array}$$

precisely when $\sum m_i$ is even, that is

$$\check{T} \cong \{(m_1, \ldots, m_d) \in \mathbb{Z}^d | \sum m_i \text{ even}\}.$$

The function

$$\phi(m_1, \ldots, m_d) = \frac{1}{2} \sum m_i^2 \tag{6.7}$$

therefore defines a quadratic map

$$\phi : \check{T} \to \mathbb{Z}$$

with associated bilinear form

$$I(m, m') = \sum m_i m_i'.$$

It is not hard to check using (6.7) that it is the quadratic form associated to $c_2 \in H\mathbb{Z}^4(BSpin(2d))$.

Now suppose that

$$V = L_1 \oplus \cdots \oplus L_d$$

is a $\mathbb{T}$-equivariant $Spin(2d)$ bundle, written as a sum of $\mathbb{T}$-equivariant complex line bundles, over a trivial $\mathbb{T}$-space Y. Let $x_i = c_1 L_i$, and suppose that $\mathbb{T}$ acts on L_i by the character m_i. In order that $V_{\mathbb{T}}$ be a spin bundle, we must have

$$0 = w_2(V_{\mathbb{T}}) = \sum_i (m_i z + x_i) \pmod 2.$$

In that case, Lemma 6.5 says that

$$c_2(V_{\mathbb{T}}) = \left(\frac{1}{2}\sum m_i^2\right) z^2 + (\sum m_i x_i) z + c_2 V$$

in $H^4(Y \times B\mathbb{T})$.

6.2. **Theta functions.** Recall that $q = e^{2\pi i \tau}$, and that C is the elliptic curve

$$C = \mathbb{G}_a^{\mathrm{an}}/\Lambda \cong \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}$$

Following [Loo76], we define a line bundle over $\mathbb{V}C$ by the formula

$$\mathcal{L} = \mathcal{L}(\xi) \overset{\mathrm{def}}{=} \frac{\mathbb{V}\mathbb{G}_m^{\mathrm{an}} \times \mathbb{C}}{(u, \lambda) \sim (uq^m, u^{\hat{I}(m)} q^{\phi(m)} \lambda)}. \tag{6.8}$$

for $m \in \check{T}$.

Remark 6.9. The identity map of $\mathbb{V}\mathbb{G}_m^{\mathrm{an}} \times \mathbb{C}$ induces for $w \in W$ an isomorphism of line bundles

$$\mathcal{L} \cong w^* \mathcal{L}$$

over $\mathbb{V}C$, which is certainly compatible with the multiplication in W, so $\mathcal{L}$ descends to a line bundle $\mathcal{A}(\xi)$ on $\mathbb{V}C/W$.

A theta function for G of level ξ is a W-invariant holomorphic section of $\mathcal{L}(\xi)$.

Definition 6.10. A *theta function* for G of level ξ is a function

$$\theta = \sum_{n>>-\infty} a_n(u) q^n \in (\mathbb{Z}[\hat{T}])((q)) \tag{6.11}$$

which for $u = e^z$ and $q = e^{2\pi i \tau}$ is a holomorphic function of $(z, \tau) \in \mathbb{V}\mathbb{A}^1_{\mathrm{an}} \times \mathfrak{H}$, and which satisfies

$$\theta(uq^m) = u^{-\hat{I}(m)} q^{-\phi(m)} \theta(u) \tag{6.12a}$$

$$\theta(u^w) = \theta(u) \tag{6.12b}$$

for $m \in \check{T}$ and $w \in W$, where ϕ and I are the quadratic form and bilinear map associated to the characteristic class ξ.

Remark 6.13. There is a good deal of redundancy in the definition. Looijenga studies *formal* series of the form (6.11) which transform according to (6.12). One has to be careful to identify a group formal series which is closed under the operations implied by (6.12). If ξ is positive definite, then every such formal theta function defines a holomorphic function of (z, τ) (see [Loo76]).

6.3. **The sigma function and the basic representation of** $LSpin(2d)$**.** If G is a simple and simply connected Lie group, then there is a unique generator ξ of $H^4(BG; \mathbb{Z}) \cong \mathbb{Z}$ such that the associated pairing I is positive definite. If $\mathcal{V}$ is a representation of LG of level k in the sense of [PS86], then its character χ is a theta function for G of level $k\xi$ [Kac85]. The most important example for us is the *basic* representation of $LSpin(2d)$, whose character is a theta function associated to the characteristic class c_2 of $Spin(2d)$. Up to a factor, it is the euler class of the sigma orientation (5.11).

It is useful to be more explicit about this euler class. As in Example 6.6, let $T_{SO(2d)}$ be the image under the map $U(d) \to SO(2d)$ of the torus of diagonal matrices. For $u = (u_1, \dots, u_d) \in T$ write

$$\sigma_d(u) = \prod_{i=1}^{d} \sigma(u_i). \tag{6.14}$$

The product expression (6.14) and the fact (5.9a) that σ is odd imply that $\sigma_d(u^w) = \sigma_d(u)$ for $w \in W$, and so σ_d is a W-invariant function on $\mathbb{V}\mathbb{G}_a^{\mathrm{an}}$ (with zeroes precisely at the points $\mathbb{V}\Lambda$) and so by Lemma 5.2 defines a holomorphic characteristic class for oriented vector bundles of rank $2d$. If V is such a vector bundle, then $\sigma(V) = \sigma_d(V)$.

The fractional powers of u in the expression (5.8) for σ prevent σ_d from being a theta function for $SO(2d)$, but if $G = Spin(2d)$ then the formula

$$\sigma_d(u) = (-1)^d (\prod_i u_i)^{\frac{1}{2}} \prod_i \left((1-u_i) \prod_{n\geq 1} \frac{(1-q^n u_i)(1-q^n u_i^{-1})}{(1-q^n)^2} \right)$$

shows that $\sigma_d \in (\mathbb{Z}[\hat{T}])[\![q]\!]$. The formula (5.9c) implies that, if I and ϕ are the pairing and quadratic form associated to the generator $c_2 \in H^4(BG;\mathbb{Z})$ as in Example 6.6, then

$$\sigma_d(uq^m) = u^{-\hat{I}(m)} q^{-\phi(m)} \sigma_d(u),$$

so σ_d is a theta function for $Spin(2d)$ of level c_2. Up to the factor $\prod_n (1-q^n)^{2d}$, it is the character of the so-called "basic" representation of $LSpin(2d)$ [Kac85, PS86, Liu96].

6.4. **A useful holomorphic characteristic class.** Suppose that Q is a $\mathbb{T}$-equivariant principal G-bundle over a connected $\mathbb{T}$-space Y. Suppose that $\mathbb{T}[n] \subset \mathbb{T}$ acts trivially on Y. Let

$$m : \mathbb{T}[n] \to T$$

be a reduction of the action of $\mathbb{T}[n]$ on Q/Y.

Let $\xi \in H^4(BG;\mathbb{Z})$ be a positive definite class, with associated quadratic form ϕ and bilinear map I. Let θ be a theta function of level ξ for G (Definition 6.10). Let C be the elliptic curve $\mathbb{G}_a^{\mathrm{an}}/(2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z})$. Let a be a point of C of order n. Choose a point $\bar{a} \in \mathbb{G}_a^{\mathrm{an}}$ such that $\wp(\bar{a}) = a$.

We are going to define a holomorphic function

$$F = F(\theta, m, \bar{a}) \in \mathcal{O}(\mathbb{V}\mathbb{G}_a^{\mathrm{an}})^{W(m)},$$

and so a holomorphic characteristic class of principal $Z(m)$-bundles (see §5.2). To give a formula for F it is convenient to define

$$A = e^{2\pi i \bar{a}},$$

and recall that we have set

$$\begin{aligned} q^r &= e^{2\pi i r\tau} \\ u^r &= e^{rz} \end{aligned}$$

for $r \in \mathbb{Q}$.

Since $na = 0$ in C, there are unique integers ℓ and k such that

$$n\bar{a} = 2\pi i\ell + 2\pi i\tau k.$$

We choose an extension $\overline{m}$ making the diagram

$$\begin{array}{ccc} \mathbb{T} & \overset{\overline{m}}{\dashrightarrow} & T \\ \uparrow & \nearrow_{m} & \\ \mathbb{T}[n] & & \end{array}$$

commute. With these choices, the formula for F is

$$F(z) = u^{\frac{k}{n}\hat{I}(\overline{m})} A^{\frac{k}{n}\phi(\overline{m})} \theta(uA^{\overline{m}}). \tag{6.15}$$

Remark 6.16. The factors preceding the θ are closely related to the line bundle $\mathcal{V}^{\frac{1}{n}}$ which appears in [BT89] and [AB00].

Lemma 6.17. *F is independent of the choice of lift $\overline{m}$.*

Proof. Let $\overline{m}'$ be another choice. Let F' be the function defined using $\overline{m}'$. Since $\overline{m}'$ and $\overline{m}$ both restrict to m on A, there is a $\Delta \in \check{T}$ such that

$$\overline{m}' = \overline{m} + n\Delta.$$

We have

$$\begin{aligned} F'(z) &= u^{\frac{k}{n}\hat{I}(\overline{m}')}A^{\frac{k}{n}\phi(\overline{m}')}\theta(uA^{\overline{m}'}) \\ &= u^{\frac{k}{n}\hat{I}(\overline{m}+n\Delta)}A^{\frac{k}{n}\phi(\overline{m}+n\Delta)}\theta(uA^{\overline{m}}q^{k\Delta}) \\ &= u^{\frac{k}{n}\hat{I}(\overline{m})}u^{k\hat{I}(\Delta)}A^{\frac{k}{n}\phi(\overline{m})}A^{kI(\overline{m},\Delta)}q^{k^2\phi(\Delta)}u^{-k\hat{I}(\Delta)}A^{-kI(\Delta,\overline{m})}q^{-k^2\phi(\Delta)}\theta(uA^{\overline{m}}) \\ &= F(z). \end{aligned}$$

□

Lemma 6.18. *F is invariant under $W(m)$.*

Proof. Suppose $w \in W(m)$. We have

$$wm = m$$

so

$$\overline{m} = w\overline{m} + n\Delta$$

for some $\Delta \in \check{T}$. The proof is now similar to the proof of Lemma 6.17. □

The dependence of $F(\theta, m, \bar{a})$ on the lift $\overline{a}$ can be calculated as follows. Let $\overline{a}'$ be another lift. Then

$$\overline{a}' = \bar{a} + \lambda$$

for some $\lambda \in 2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}$, so letting

$$B = e^{\overline{a}'}$$

we have

$$B = Aq^{\delta}$$

for some $\delta \in \mathbb{Z}$. Thus

$$B^n = q^{k'}$$

with $k' = k + n\delta$, and so the quantity

$$w(a, q^{\frac{1}{n}}) \stackrel{\text{def}}{=} A^{-1}q^{\frac{k}{n}} = B^{-1}q^{\frac{k'}{n}}$$

is an n^{th} root of unity which is independent of the choice of lift of a; in fact it is the Weil pairing of a and $q^{\frac{1}{n}}$ in the curve C. Because it is an n^{th} root of unity, the quantity

$$w(a, q^{\frac{1}{n}})^{\phi(m)} \stackrel{\text{def}}{=} w(a, q^{\frac{1}{n}})^{\phi(\overline{m})}$$

is independent of the lift $\overline{m}$ of m.

Lemma 6.19.

$$F(\theta, m, \overline{a}') = w(a, q^{\frac{1}{n}})^{\delta\phi(m)}F(\theta, m, \overline{a}).$$

Proof. If $F' = F(\theta, m, \overline{a}')$ then

$$\begin{aligned} F'(z) &= u^{\frac{k'}{n}\hat{I}(\overline{m})}B^{\frac{k'}{n}\phi(\overline{m})}\theta(uB^{\overline{m}}) \\ &= u^{\frac{k}{n}\hat{I}(\overline{m})}u^{\delta\hat{I}(\overline{m})}A^{\frac{k}{n}\phi(\overline{m})}A^{\delta\phi(\overline{m})}q^{\delta\frac{k}{n}\phi(\overline{m})}q^{\delta^2\phi(\overline{m})}\theta(uA^{\overline{m}}q^{\delta\overline{m}}) \\ &= u^{\frac{k}{n}\hat{I}(\overline{m})}u^{\delta\hat{I}(\overline{m})}A^{\frac{k}{n}\phi(\overline{m})}A^{\delta\phi(\overline{m})}q^{\delta\frac{k}{n}\phi(\overline{m})}q^{\delta^2\phi(\overline{m})}u^{-\delta\hat{I}(\overline{m})}A^{-2\phi(\overline{m})}q^{-\delta^2\phi(\overline{m})}\theta(uA^{\overline{m}}) \\ &= u^{\frac{k}{n}\hat{I}(\overline{m})}A^{\frac{k}{n}\phi(\overline{m})}(A^{-1}q^{\frac{k}{n}})^{\delta\phi(\overline{m})}\theta(uA^{\overline{m}}) \\ &= w(a, q^{\frac{1}{n}})^{\delta\phi(\overline{m})}F(\theta, m, \overline{a}). \end{aligned}$$

□

Lemma 6.18 implies that the Taylor series expansion of $F(\theta, m, \overline{a})$ defines a class in $\hat{H}(BZ(m)) = \hat{H}(BT)^{W(m)}$, which we also denote $F(\theta, m, \overline{a})$. Let $\theta(Q, m, \overline{a}) \in \mathcal{H}(Y; \mathbb{A}^1_{\text{an}})$ be the holomorphic cohomology class given by the formula

$$\theta(Q, m, \overline{a}) = Q(m)^*F(\theta, m, \overline{a}) \tag{6.20}$$

(using Lemma 5.2 to conclude that the class $\theta(Q, m, \overline{a})$ is in fact holomorphic).

Lemma 6.21. *If the centralizer $Z(m)$ is connected, for example if G is unitary or spin, then the class $\theta(Q, m, \overline{a})$ is independent of the reduction m of the action of $\mathbb{T}[n]$ on Q/Y.*

Proof. Let m' be another reduction of the action of $\mathbb{T}[n]$ on Q/Y. By Lemma 3.6, there is an element $w \in W$ such that

$$m' = wm.$$

If $\overline{m} : \mathbb{T} \to T$ is a lift of m, then $w\overline{m}$ is a lift of m'. Using this lift to define $F(\theta, m', \overline{a})$, we have

$$\begin{aligned} w^* F(\theta, m', \overline{a})(z) &= F(\theta, m', \overline{a})(w(z)) \\ &= (u^w)^{\frac{k}{n}\hat{I}(w\overline{m})} A^{\frac{k}{n}\phi(w\overline{m})} \theta(u^w A^{w\overline{m}}) \\ &= u^{\frac{k}{n}\hat{I}(\overline{m})} A^{\frac{k}{n}\phi(\overline{m})} \theta(u A^{\overline{m}}) \\ &= F(\theta, m, \overline{a})(z). \end{aligned}$$

Since the diagram

$$\begin{array}{ccc} Y_{\mathbb{T}} & \xrightarrow{Q(m)} & BZ(m) \\ & \underset{Q(m')}{\searrow} & \downarrow{\scriptstyle w} \\ & & BZ(m') \end{array}$$

commutes, we have

$$\begin{aligned} \theta(Q, m', \overline{a}) &= Q(m')^* F(\theta, m', \overline{a}) \\ &= Q(m)^* w^* F(\theta, m', \overline{a}) \\ &= Q(m)^* F(\theta, m, \overline{a}) \\ &= \theta(Q, m, \overline{a}). \end{aligned}$$

□

The results of this section justify the following.

Definition 6.22. Suppose that Q is a $\mathbb{T}$-equivariant principal G-bundle over a connected $\mathbb{T}$-space Y on which $\mathbb{T}[n]$ acts trivially. Suppose that $Q_{\mathbb{T}}$ is a principal G-bundle over $Y_{\mathbb{T}}$, and suppose also that for some (equivalently every) reduction

$$m : \mathbb{T}[n] \to T$$

of the action of $\mathbb{T}[n]$ on Q/Y, the centralizer $Z(m)$ is connected. Let C be the elliptic curve $\mathbb{G}_a^{\text{an}}/2\pi i\mathbb{Z}+2\pi i\tau\mathbb{Z}$. Let a be a point of C of order n. Let θ be a theta function for G of level ξ, and let $\overline{a}$ be a point of $\mathbb{A}^1_{\text{an}}$ whose image in C is a. We define $\theta(Q, \overline{a}) \in \mathcal{H}(Y; \mathbb{A}^1_{\text{an}})$ to be the holomorphic cohomology class

$$\theta(Q, \overline{a}) = \theta(Q, m, \overline{a}),$$

where m is any reduction of the action of $\mathbb{T}[n]$ on Q/Y.

Lemma 6.23. *If $\overline{a}' = \overline{a} + 2\pi i s + 2\pi i \delta\tau$ is another lift of a, then*

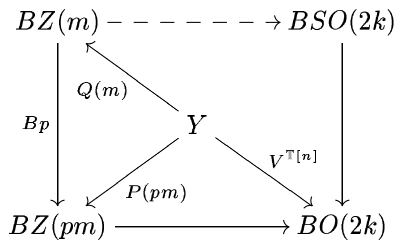
$$\theta(Q, \overline{a}') = w(a, q^{\frac{1}{n}})^{\delta\phi(m)} \theta(Q, \overline{a}),$$

where again m is any reduction of the action of $\mathbb{T}[n]$ on Q/Y. □

An important case of the preceding constructions is that $G = Spin(2d)$, and σ_d is the character of the basic representation of LG as in §6.3. Let $p : G \to SO(2d)$ be standard the double cover. Let P/Y be the resulting $\mathbb{T}$-equivariant $SO(2d)$-bundle over Y, and let V be the associated vector bundle. We recall from [BT89] that Lemma 3.7 implies that the sub-vector bundle $V^{\mathbb{T}[n]}$ is orientable. Explicitly, if m is a reduction of the action of $\mathbb{T}[n]$ on Q/Y, then pm is a reduction of the action of $\mathbb{T}[n]$ on P/Y. If $V^{\mathbb{T}[n]}$ has rank $2k$, then the map

$$Y \to BO(2k)$$

classifying $V^{\mathbb{T}[n]}$ factors through the map $Y \to BZ(pm)$ classifying $P(pm)$, and we have the solid diagram



Since (by Lemma 3.7) the centralizer $Z(m)$ is connected, there is a dotted arrow making the diagram commute. In other words, $V^{\mathbb{T}[n]}$ is orientable, and a choice of orientation $\Omega(V^{\mathbb{T}[n]})$ determines a map

$$f : BZ(m) \to BSO(k)$$

such that $fQ(m)$ classifies $(V^{\mathbb{T}[n]}, \Omega(V^{\mathbb{T}[n]}))$.

Let $\sigma_k \in \hat{H}(BSO(2k))$ be the characteristic class associated to the sigma function as in § 6.3. We then have two $W(m)$-invariant holomorphic functions on $\mathbb{V}\mathbb{G}_a^{\mathrm{an}}$. One is $f^*\sigma_k$, and the other is $F(\theta, m, \overline{a})$.

Lemma 6.24. *The ratio*

$$R = \frac{f^*\sigma_k}{F(\sigma_d, m, \overline{a})}$$

is a $W(m)$-invariant unit of $\mathcal{O}_{\mathbb{V}\mathbb{G}_a^{\mathrm{an}}{}_0}$.

Proof. The poles of R occur at zeroes of F. Using the standard maximal torus of $SO(2d)$ then we may write a typical element of $\mathbb{V}\mathbb{G}_m^{\mathrm{an}}$ as

$$(u_1, \ldots, u_d) \in (\mathbb{G}_m^{\mathrm{an}})^d \cong \mathbb{V}\mathbb{G}_m^{\mathrm{an}}.$$

In these terms, a lift $\overline{m}$ of m is of the form

$$u^{\overline{m}} = (u^{m_1}, \ldots, u^{m_d})$$

for some integers $m_1, \ldots, m_d$. We have

$$F(\sigma_d, m, \overline{a}) = u^{\frac{k}{n}\hat{I}(\overline{m})} A^{\frac{k}{n}\phi(\overline{m})} \prod_{j=1}^{d} \sigma(u_j A^{m_j}).$$

The product of sigma functions contributes a zero near $z = 0$ if and only if $m_j a = 0$ in C. Let $j_1, \ldots, j_k \in \{1, \ldots, d\}$ be the indices such that $m_{j_i} a = 0$ in C; then

$$f^*\sigma_k = \prod_{i=1}^{k} \sigma(u_{j_i} A^{m_{j_i}}).$$

So the zeroes of $f^*\sigma_k$ precisely cancel those of $F(\sigma_d, m, \overline{a})$. □

Lemmas 5.2 and 6.24 together imply that R defines a holomorphic characteristic class for $Z(m)$-bundles.

Corollary 6.25. *The holomorphic characteristic class*

$$R(V, V^{\mathbb{T}[n]}, \Omega(V^{\mathbb{T}[n]}), \overline{a}) \stackrel{def}{=} Q(m)^* R \in (\mathcal{H}(Y)_0)^\times$$

is independent of the reduction m, and satisfies

$$\sigma(V^{\mathbb{T}[n]}, \Omega(V^{\mathbb{T}[n]})) = R(V, V^{\mathbb{T}[n]}, \Omega(V^{\mathbb{T}[n]}), \overline{a})\sigma(V, \overline{a}).$$

□

7. Equivariant elliptic cohomology

7.1. **Adapted open cover of an elliptic curve.** If X is a $\mathbb{T}$-space and if a is a point of C, then we define

$$X^a = \begin{cases} X^{\mathbb{T}[k]} & a \text{ is of order exactly } k \text{ in } C \\ X^{\mathbb{T}} & \text{otherwise.} \end{cases}$$

Let $N \geq 1$ be an integer.

Definition 7.1. A point $a \in C$ is *special* for X if $X^a \neq X^{\mathbb{T}}$.

If V is a $\mathbb{T}$-bundle over a $\mathbb{T}$-space X, then it is convenient to consider a few additional points to be special. Suppose that F is a component of $X^{\mathbb{T}}$ and

$$m : \mathbb{T} \to T$$

is a reduction of the action of $\mathbb{T}$ on the principal bundle associated to V. If we choose an isomorphism

$$\check{T} \cong \mathbb{Z}^r,$$

then we may view m as an array of integers $(m_1, m_2, \ldots, m_r)$. These integers are called the *exponents* or *rotation numbers* of V at F. Let V^+ denote the one-point compactification of V.

Definition 7.2. A point a in C is *special* for V if it is special for V^+ or if for some component F of $X^{\mathbb{T}}$ there is a rotation number m_j of V such that $m_j a = 0$.

In either case, if X is a finite $\mathbb{T}$-CW complex, then the set of special points is a finite subset of the torsion subgroup of C.

Definition 7.3. An indexed open cover $\{U_a\}_{a \in C}$ of C is *adapted to* X or V if it satisfies the following.

1) a is contained in U_a for all $a \in C$.
2) If a is special and $a \neq b$, then $a \notin U_a \cap U_b$.
3) If a and b are both special and $a \neq b$, then the intersection $U_a \cap U_b$ is empty.
4) If b is ordinary, then $U_a \cap U_b$ is non-empty for at most one special a.
5) Each U_a is small (2.1).

Lemma 7.4. *Let X be a finite $\mathbb{T}$-CW complex. Then C has an adapted open cover, and any two adapted open covers have a common refinement.* □

7.2. **Complex elliptic cohomology.** Let

$$\hat{E} \stackrel{\text{def}}{=} (\hat{H}, C, \widehat{\wp^{-1}}) = (\hat{H}, C, \log_\omega)$$

be the equivariant elliptic spectrum associated to the elliptic curve C (see (4.4)). Since this is just a form of ordinary cohomology, we write

$$X_E = X_H.$$

Suppose that $U \subset C$ is a small open neighborhood of the identity in C. Suppose that $V \subset \mathbb{A}^1_{\text{an}}$ is the component of $\wp^{-1}U$ containing the origin. We let $X_{\mathbb{T}E}|_U$ be the ringed space defined as the pull-back in the diagram

$$\begin{array}{ccc} X_{\mathbb{T}E}|_U & \longrightarrow & X_{\mathcal{H}}|_V \\ \downarrow & & \downarrow \\ U & \xrightarrow{(\wp|_V)^{-1}} & V. \end{array} \tag{7.5}$$

The diagram (4.3) shows that $X_{\mathbb{T}\hat{E}}$ and $X_{\mathbb{T}E}|_U$ are related by the formula

$$X_{\mathbb{T}\hat{E}} \cong (X_{\mathbb{T}E}|_U)^\wedge_0.$$

7.3. **Equivariant elliptic cohomology.** Grojnowski's circle-equivariant extension of $\hat{E}$ is a contravariant functor associating to a compact $\mathbb{T}$-manifold X a $\mathbb{Z}/2$-graded $\mathcal{O}_C$-algebra $E_{\mathbb{T}}(X)$, with the property that

$$E_{\mathbb{T}}(*) = \mathcal{O}_C.$$

It is equipped with a natural isomorphism

$$E(X_{\mathbb{T}}) \xrightarrow[\cong]{A(X)} (E_{\mathbb{T}}(X))_0^\wedge, \tag{7.6}$$

such that

$$A(*) = \log_\omega : (\mathcal{O}_C)_0^\wedge \cong \hat{E}(B\mathbb{T}).$$

We shall write $X_{E\mathbb{T}}$ for the ringed space $(C, E_{\mathbb{T}}(X))$ (see (2.4)). We take this opportunity to phrase the account in [AB00] of the construction of $E_{\mathbb{T}}(X)$ as the construction of a covariant functor

$$X \mapsto X_{E\mathbb{T}}$$

from finite $\mathbb{T}$-CW complexes to ringed spaces (2.4) over C, equipped with an identification

$$(*)_{E\mathbb{T}} = C$$

and a natural isomorphism of formal schemes

$$X_{\mathbb{T}\hat{E}} \xrightarrow[\cong]{A(X)} (X_{E\mathbb{T}})_0^\wedge$$

such that

$$A(*) = \log_\omega : P_{\hat{E}} = \widehat{\mathbb{G}}_a \cong \widehat{C}.$$

If $\mathcal{X} = (X, \mathcal{O}_X)$ is a ringed space and U is an open set of X, then we may write $\mathcal{X}(U)$ in place of $\mathcal{O}_X(U)$.

Let $\{U_a\}_{a\in C}$ be an adapted open cover of C. For each $a \in C$, we make a ringed space $X_{E\mathbb{T},a}$ over U_a as the pull back in the diagram

$$\begin{array}{ccccc} X_{E\mathbb{T},a} & \longrightarrow & X_{\mathbb{T}E}|_{U_a-a} & \longrightarrow & X_{\mathcal{H}}|_V \\ \downarrow & & \downarrow & & \downarrow \\ U_a & \xrightarrow{\tau_{-a}} & U_a - a & \xrightarrow{(\wp|_V)^{-1}} & V. \end{array} \tag{7.7}$$

As in (7.5), V is the component of $\wp^{-1}(U_a - a)$ containing the origin. In other words, let $V_a \subset \wp^{-1}(U_a)$ be the component containing the origin. For $U \subset U_a$ let $V = V_a \cap \wp^{-1}(U - a)$, and let

$$X_{E\mathbb{T},a}(U) = \mathcal{H}(X^a; V),$$

considered as an $\mathcal{O}_C(U)$-algebra via the isomorphism

$$U \xrightarrow{\tau_{-a}} U - a \xrightarrow{(\wp|_V)^{-1}} V.$$

If $a \neq b$ and $U_a \cap U_b$ is not empty, then by the definition (7.3) of an adapted cover, at least one of U_a and U_b, suppose U_b, contains no special point. In particular we have $X^b = X^{\mathbb{T}}$ and so an isomorphism

$$X^b_{\mathbb{T}E}|_U \cong X^b_E \times U$$

i.e.

$$E(X^b_{\mathbb{T}}) \underset{\mathbb{C}[z]}{\otimes} \mathcal{O}_U \cong E(X^b) \underset{\mathbb{C}}{\otimes} \mathcal{O}_U$$

for any small neighborhood U of the origin.

Lemma 7.8. *If $a \neq b$, $U \subset U_a \cap U_b$, and b is not special, then the inclusion*

$$i : X^b \to X^a$$

induces an isomorphism

$$X^b_{\mathbb{T}E}|_{U-a} \cong X^a_{\mathbb{T}E}|_{U-a}.$$

Proof. If a is not special, then $X^a = X^b$ and the result is obvious. If a is special, then it is not contained in U (by the definition of an adapted cover), and so 0 is not contained in $U - a$. The localization theorem (4.2) gives the result. □

Let $U = U_a \cap U_b$. We define

$$\psi_{ab} = \psi^X_{ab} : X_{E\mathbb{T},a}|_U \xrightarrow{\cong} X_{E\mathbb{T},b}|_U$$

as the arrow making the diagram

$$\begin{array}{ccccccc}
X_{E\mathbb{T},a}|_U & \longrightarrow & & & X^a_{\mathbb{T}E}|_{U-a} & \xleftarrow[\cong]{} & X^b_{\mathbb{T}E}|_{U-a} \\
\big\downarrow & \searrow & & & \big\downarrow & \swarrow & \big\downarrow \cong \\
& & U & \xrightarrow{\tau_{-a}} & U-a & \longleftarrow & X^b_{\hat{E}} \times (U-a) \\
\psi_{ab} & & \Big\| & & \big\uparrow \tau_{b-a} & & \big\uparrow X^b_{\hat{E}} \times \tau_{b-a} \\
& & U & \xrightarrow{\tau_{-b}} & U-b & \longleftarrow & X^b_{\hat{E}} \times (U-b) \\
\big\downarrow & \nearrow & & & & \nwarrow & \big\uparrow \\
X_{E\mathbb{T},b}|_U & \longrightarrow & & & & & X^b_{\mathbb{T}E}|_{U-b}
\end{array} \tag{7.9}$$

commutes. The cocycle condition

$$\psi_{bc}\psi_{ab} = \psi_{ac}$$

needs to be checked only when two of a, b, c are not special; and in that case it follows easily from the equation

$$\tau_{c-b}\tau_{b-a} = \tau_{c-a}.$$

We shall write $X_{E\mathbb{T}}$ for the ringed space over C, and $E_{\mathbb{T}}(X)$ for its structure sheaf. One then has the following ([Gro94]; for a published account see [Ros01]).

Proposition 7.10. *$X_{E\mathbb{T}}$ is a ringed space over C, which is independent up to canonical isomorphism of the choice of adapted open cover.* □

8. Equivariant elliptic cohomology of Thom spaces

Suppose that V is a $\mathbb{T}$-equivariant vector bundle over X, and let V_0 be the complement of the zero section of V. We abbreviate as

$$V_{E\mathbb{T}} = E_{\mathbb{T}}(V, V_0) \tag{8.1}$$

the $E_{\mathbb{T}}(X)$-module associated to the reduced $E\mathbb{T}$-cohomology of the Thom space of V. Explicitly, for each point $a \in C$ we define a sheaf $V_{E\mathbb{T},a}$ of $\mathcal{O}_{X_{E\mathbb{T},a}}$-modules as the pull-back

$$\begin{array}{ccc}
V_{E\mathbb{T},a} & \longrightarrow & (V^a)_{\mathbb{T}E}|_{U_a - a} \\
\big\downarrow & & \big\downarrow \\
U_a & \longrightarrow & U_a - a.
\end{array} \tag{8.2}$$

For a special and b not with $U = U_a \cap U_b$ non-empty, the isomorphism

$$\psi^V_{ab} : V_{E\mathbb{T},a}|_U \cong V_{E\mathbb{T},b}|_V$$

is given by

$$(V^a)_{\mathbb{T}E}|_{U-a} \xrightarrow[\cong]{} (V^b)_{\mathbb{T}E}|_{U-a} \cong V^b_{\hat{E}} \times (U-a) \xrightarrow{\tau_{a-b}} V^b_{\hat{E}} \times (U-b) = (V^b)_{\mathbb{T}E}|_{U-b};$$

we have omitted a pullback (8.2) at either end.

Definition 8.3. The vector bundle V is $\mathbb{T}$-*orientable* if for each closed subgroup $A \subseteq \mathbb{T}$, the fixed bundle V^A over X^A is orientable. A $\mathbb{T}$-*orientation* on V is a choice $\Omega(V^A)$ of orientation on V^A for each A.

If V is $\mathbb{T}$-orientable, then the Thom isomorphism implies that each $V_{E\mathbb{T},a}$ is a line bundle (invertible sheaf) over $X_{E\mathbb{T},a}$, and so $V_{E\mathbb{T}}$ is a line bundle over $X_{E\mathbb{T}}$. We recall from [Ros01, AB00] the construction of an explicit cocycle for this line bundle. Let ϕ be a multiplicative analytic orientation (5.4), and let Ω be a $\mathbb{T}$-orientation on V.

Definition 8.4. An indexed open cover $\{U_a\}_{a\in C}$ of C is *adapted to* (V,ϕ,Ω) if it is adapted to V (see Definition 7.3), and if for every point $a\in C$, the equivariant Thom class $\phi(V^a_{\mathbb{T}},\Omega)$ induces an isomorphism

$$\mathcal{H}(X;U_a-a)\xrightarrow[\cong]{\phi}\mathcal{H}(V^a;U_a-a).$$

Corollary 5.6 implies that C has an indexed open cover adapted to (V,ϕ,Ω). Choose such a cover. Suppose that a,b are two points of C, such that $U=U_a\cap U_b$ is non-empty: we may suppose that $a=b$ or that b is not special. Consider the case that b is not special. Let $\overline{U}_a\subset\mathbb{A}^1_{\text{an}}$ be the component of the preimage of U_a-a containing the origin, and let

$$\wp|_{\overline{U}_a}:\overline{U}_a\to U_a-a$$

be the induced isomorphism. Let $\overline{U}\subset\overline{U}_a$ be the preimage of U. Let

$$j:(V^b,V^b_0)\to(V^a,V^a_0)$$

be the natural map. The Localization Theorem 4.2 implies that the ratio of euler classes

$$e_\phi(V^a,V^b,\Omega)=\frac{j^*\phi((V^a)_{\mathbb{T}},\Omega(V^a))}{\phi((V^b)_{\mathbb{T}},\Omega(V^b))}$$

is a unit of $\mathcal{H}(X^{\mathbb{T}};\overline{U})$.

Recall that there are tautological isomorphisms

$$X_{E\mathbb{T},a}(U)\xrightarrow[\cong]{\tau_a^*}X^a_{\mathbb{T}E}|_{U-a}(U-a)\xrightarrow[\cong]{\wp|_{\overline{U}}^*}\mathcal{H}(X^a;\overline{U}).$$

Let

$$e_\phi(a,b,\Omega)\in X_{E\mathbb{T},a}(U)^\times$$

be given by the formula

$$(\wp|_{\overline{U}}^{-1})^*\tau_a^*e_\phi(a,b,\Omega)=e_\phi(V^a,V^b,\Omega).\tag{8.5}$$

Note that $e_\phi(a,b,\Omega)=1$ if neither a nor b is special; we also set

$$e_\phi(a,a,\Omega)=1$$

for all a. It is easy to check that

$$\psi^X_{bc}(\psi^X_{ab}(e_\phi(a,b,\Omega))e_\phi(b,c,\Omega))=\psi^X_{ac}e_\phi(a,c,\Omega)\tag{8.6}$$

if $U_a\cap U_b\cap U_c$ is non-empty (since in that case at least two of a,b,c are ordinary). Thus the $e_\phi(a,b)$ define a cohomology class $[\phi,V,\Omega]\in H^1(C;\mathcal{O}^\times_{X_{E\mathbb{T}}})$. Let $X^{[\phi,V,\Omega]}_{E\mathbb{T}}$ be the resulting invertible sheaf of $\mathcal{O}_{X_{E\mathbb{T}}}$-modules over C. By construction we have the

Proposition 8.7. *The Thom isomorphism ϕ induces an isomorphism*

$$V_{E\mathbb{T}}\cong X^{[\phi,V,\Omega]}_{E\mathbb{T}}$$

of $E_{\mathbb{T}}(X)$-modules. □

In the case of the orientation Σ associated to the sigma function (5.11), we can be explicit about the open set on which

$$\sigma(V^a,V^b,\Omega)\overset{\text{def}}{=}e_\Sigma(V^a,V^b,\Omega)$$

is a unit.

Lemma 8.8. *Let $\bar{B}\subset\mathbb{A}^1_{\text{an}}$ be the preimage of the ordinary points of C: that is, the complement of the (closed) set of points $\overline{a}\in\mathbb{A}^1_{\text{an}}$ such that $X^{\wp(\overline{a})}\neq X^{\mathbb{T}}$ or $m\overline{a}\in\Lambda$ for m a character of the action of $\mathbb{T}$ on $V|_{X^{\mathbb{T}}}$. Then*

$$\sigma(V^{\mathbb{T}[n]},V^{\mathbb{T}},\Omega)\in\mathcal{H}(X^{\mathbb{T}};\bar{B})^\times.$$

Proof. Let T be the standard maximal torus in $SO(2d)$, giving an isomorphism

$$\check{T} \cong \mathbb{Z}^d.$$

The reduction m is then an array or integers $m = (m_1, \ldots, m_d)$. It suffices to consider the case that

$$V|_{X^{\mathbb{T}}} \cong L_1 + \cdots + L_d$$

is a sum of line bundles, with $\mathbb{T}$ acting on L_i by the character m_i. Let $x_i = c_1 L_i$. Then

$$\begin{aligned}\sigma(V^{\mathbb{T}}, V^{\mathbb{T}[n]}) &= \frac{\prod_{m_j \equiv 0 \pmod n} \sigma(m_j z + x_j)}{\prod_{m_j = 0} \sigma(m_j z + x_j)} \\ &= \prod_{0 \neq m_j \equiv 0 \pmod n} \sigma(m_j z + x_j).\end{aligned}$$

Since the x_j are nilpotent, this is a unit in a neighborhood of z provided that $\prod_{0 \neq m_j \equiv 0 \pmod n} \sigma(m_j z)$ is non-zero. This happens if and only if $m_j z \in \Lambda$. □

Now suppose that W is an oriented virtual $\mathbb{T}$-bundle. We may write

$$W = V_0 - V_1 \tag{8.9}$$

with each V_i a genuine oriented $\mathbb{T}$-bundle of even rank. We also clearly have

$$W = (V_0 + 3V_1) - 4V_1,$$

and it is easy to check that

$$\begin{aligned} w_1(4V_1)_{\mathbb{T}} &= 0 \\ w_2(4V_1)_{\mathbb{T}} &= 0 \\ w_1(V_0 + 3V_1)_{\mathbb{T}} &= w_1(W)_{\mathbb{T}} \\ w_2(V_0 + 3V_1)_{\mathbb{T}} &= w_2(W)_{\mathbb{T}}. \end{aligned}$$

Thus if $W_{\mathbb{T}}$ is a virtual spin bundle, then we may require that each $(V_i)_{\mathbb{T}}$ be a spin bundle of even rank. If

$$W = V_0 - V_1 = V_0' - V_1'$$

with V_i and V_i' spin bundles of even rank, then (assuming without loss of generality that the rank of V_0 is greater than or equal to the rank of V_0')

$$V_i = V_i' + (-1)^i D$$

where D is a spin bundle. It follows that we may extend the notation (8.1) by defining $W_{E\mathbb{T}}$ to be the line bundle

$$W_{E\mathbb{T}} = (V_1)_{E\mathbb{T}}^{-1} \otimes (V_0)_{E\mathbb{T}}$$

over $X_{E\mathbb{T}}$. As in Proposition 8.7, a choice of $\mathbb{T}$-orientation Ω on W gives rise to a class $[\sigma, W, \Omega] = [\sigma, V_0 - V_1, \Omega] \in H^1(C, \mathcal{O}_{X_{E\mathbb{T}}}^{\times})$, equipped with a canonical isomorphism

$$W_{E\mathbb{T}} \cong X_{E\mathbb{T}}^{[\sigma, W, \Omega]}. \tag{8.10}$$

9. A Thom class

9.1. **Equivariant elliptic cohomology of principal bundles.** Suppose that Q/X is a $\mathbb{T}$-equivariant principal G-bundle over X, and suppose that $Q_{\mathbb{T}}/X_{\mathbb{T}}$ is a principal G-bundle as well. Then we have a map

$$X_{\mathbb{T}\hat{E}} \xrightarrow{Q_{\mathbb{T}\hat{E}}} \mathbb{V}\widehat{C}/W. \tag{9.1}$$

If F is a connected component of $X^{\mathbb{T}}$, and if

$$m : \mathbb{T} \to T$$

is a reduction of the action of $\mathbb{T}$ on $Q|_F$, and if $U \subset C$ is a small open set, then Lemma 5.1 implies that the diagram

$$\begin{array}{ccccc} F_{\mathbb{T}\hat{E}} & = & F_{\hat{E}} \times \widehat{C} & \xrightarrow{Q_{\mathbb{T}\hat{E}}} & \mathbb{V}C/W \\ \downarrow & & \downarrow & & \uparrow \\ F_{E\mathbb{T}} & = & F_{\hat{E}} \times C & \xrightarrow{Q(m)_{\hat{E}}+m} & \mathbb{V}C/W(m) \end{array} \tag{9.2}$$

commutes. We therefore write

$$Q(m)_{E\mathbb{T}} \stackrel{\text{def}}{=} Q(m)_{\hat{E}} + m.$$

Thus we have the commutative solid diagram

$$\begin{array}{ccccccc} F_{\hat{E}} \times \widehat{C} & & & \xrightarrow{Q_{\mathbb{T}\hat{E}}} & & & \mathbb{V}\widehat{C}/W(m) \\ \downarrow & \searrow & & & & \swarrow & \downarrow \\ & & X_{\mathbb{T}\hat{E}} & \xrightarrow{Q_{\mathbb{T}\hat{E}}} & \mathbb{V}\widehat{C}/W & & \\ & & \downarrow & & \downarrow & & \\ F_{\hat{E}} \times C & & X_{E\mathbb{T}} & \underset{Q_{E\mathbb{T}}}{\dashrightarrow} & \mathbb{V}C/W & & \\ \| & \nearrow & & & & \nwarrow & \downarrow \\ F_{E\mathbb{T}} & & & \xrightarrow[Q(m)_{E\mathbb{T}}=Q(m)_{\hat{E}}+m]{} & & & \mathbb{V}C/W(m) \end{array} \tag{9.3}$$

In writing this paper, we were guided by the idea that there should be a canonical map $Q_{E\mathbb{T}}$ making the whole diagram (9.3) commute. We shall return to that question in §11, discussing both why such a map would be a good thing and why it is difficult to construct. For now we merely observe that the definition of $Q(m)_{E\mathbb{T}}$ implies the following.

As we have already observed before Lemma 5.1, the addition

$$\mathbb{V}C \times \mathbb{V}C \to \mathbb{V}C$$

induces a translation

$$(\mathbb{V}C)^{W(m)} \times \mathbb{V}C/W(m) \to \mathbb{V}C/W(m),$$

and so for $a \in C$ we get an operator

$$\tau_{a^m} : \mathbb{V}C/W(m) \to \mathbb{V}C/W(m).$$

Lemma 9.4. *For $a \in C$ the diagram*

$$\begin{array}{ccc} F_{\hat{E}} \times C & \xrightarrow{Q(m)_{E\mathbb{T}}} & \mathbb{V}C/W(m) \\ {\scriptstyle F_{\hat{E}} \times \tau_a}\downarrow & & \downarrow{\scriptstyle \tau_{a^m}} \\ F_{\hat{E}} \times C & \xrightarrow{Q(m)_{E\mathbb{T}}} & \mathbb{V}C/W(m) \end{array}$$

commutes. □

9.2. **The Thom class.** Let G be a spinor group, and let G' be a simple and simply connected compact Lie group, a unitary group, or indeed any compact connected Lie group with the property that the centralizer of any element is connected. Let V be a $\mathbb{T}$-equivariant G-vector bundle over a finite $\mathbb{T}$-CW complex X, and let V' be a $\mathbb{T}$-equivariant G'-bundle (by which we mean the vector bundle associated to a principal G' bundle via a linear representation of G'). Suppose that $V_{\mathbb{T}}$ is a G-vector bundle, and that $V'_{\mathbb{T}}$ is a G'-vector bundle, over $X_{\mathbb{T}}$. Suppose that ξ' is a degree-four characteristic classes for G', with the property that

$$c_2(V)_{\mathbb{T}} = \xi'(V')_{\mathbb{T}}.$$

Suppose finally that θ' is a theta function for G' of level ξ'. In this section we prove the following

Theorem 9.5. *A $\mathbb{T}$-orientation Ω on V determines a canonical global section $\gamma = \gamma(V, V', \Omega)$ of $(V_{E\mathbb{T}})^{-1}$, such that*

$$\gamma_0 = \theta'(V'_{\mathbb{T}})\Sigma(V_{\mathbb{T}})^{-1} \tag{9.6}$$

under the isomorphism (7.6)

$$((V_{E\mathbb{T}})^{-1})^{\wedge}_0 \cong (V_{\mathbb{T}\hat{E}})^{-1} = (V_{\mathbb{T}\hat{H}})^{-1}$$

of line bundles over $X_{\mathbb{T}\hat{H}}$. It is natural in the sense that if $f : Z \to X$ is a map of finite $\mathbb{T}$-CW complexes, then

$$\gamma(f^*V, f^*V', f^*\Omega) = f^*\gamma(V, V', \Omega). \tag{9.7}$$

Let $\{U_a\}_{a\in C}$ be an indexed open cover of C adapted to (V, Σ, Ω) (Definition 8.4). By Proposition 8.7, to give a section of $V_{E\mathbb{T}}^{-1}$ satisfying (9.6), it is equivalent to give a global section of the sheaf $X_{E\mathbb{T}}^{-[\sigma,V,\Omega]}$ whose value in

$$(X_{E\mathbb{T}}^{-[\sigma,V,\Omega]})_0 \cong X_{\mathbb{T}H,0}$$

is $\theta'(V'_{\mathbb{T}})$; this is what we shall do. The description of $X_{E\mathbb{T}}^{-[\sigma,V,\Omega]}$ in §8 shows that this amounts to sections $\gamma_a \in X_{E\mathbb{T},a}(U_a)$ for $a \in C$, such that $\gamma_0 = \theta'(V'_{\mathbb{T}})$ and

$$\psi_{ab}(\sigma(a, b, \Omega)\gamma_a) = \gamma_b$$

when a is special and b is ordinary.

Let $B \subset C$ be the set of ordinary points, and, as in Lemma 8.8, let $\bar{B} \subset \mathbb{A}^1_{\mathrm{an}}$ be the preimage of B in $\mathbb{A}^1_{\mathrm{an}}$. Lemma 8.8 tells us that the formula

$$\bar{\gamma}_B \overset{\mathrm{def}}{=} \theta'(V'_{\mathbb{T}})|_{X^{\mathbb{T}}} \frac{\sigma(V^{\mathbb{T}})}{\sigma(V_{\mathbb{T}}|_{X^{\mathbb{T}}})}$$

defines an element of $\mathcal{H}(X^{\mathbb{T}}; \bar{B})$.

Lemma 9.8. *For $\lambda \in \Lambda$,*

$$\tau_\lambda^* \bar{\gamma}_B = \bar{\gamma}_B.$$

Proof. There is a k such that

$$q^k = e^\lambda.$$

Let Q be the principal G-bundle associated to V, and let Q' be the principal bundle associated to G'. If F is a component of $X^{\mathbb{T}}$, let $m : \mathbb{T} \to T$ be a reduction of the action of $\mathbb{T}$ on $Q|_F$, and let $m' : \mathbb{T} \to T'$ be a reduction of the action of $\mathbb{T}$ on $Q'|_F$.

The principal bundle $Q(m)/F$ is classified by a map

$$F \xrightarrow{Q(m)} BZ(m);$$

in rational cohomology this becomes

$$F_{H\mathbb{Q}} \xrightarrow{Q(m)_{H\mathbb{Q}}} BZ(m)_{H\mathbb{Q}} \cong (\mathbb{V}(\widehat{\mathbb{G}}_a)_{\mathbb{Q}})/W(m),$$

an $F_{H\mathbb{Q}}$-valued point of $(\mathbb{V}(\widehat{\mathbb{G}}_a)_{\mathbb{Q}})/W(m)$. Since F has the homotopy type of a finite CW-complex, the reduced cohomology of F is nilpotent, and so we may consider $\exp(Q(m)_{H\mathbb{Q}})$ as an $F_{H\mathbb{Q}}$-valued point of $(\mathbb{V}(\mathbb{G}_m)_{\mathbb{Q}})/W(m)$.

Let

$$\begin{aligned} D &= \exp(Q(m)_{H\mathbb{Q}}) \in (\mathbb{V}\mathbb{G}_m/W(m))(F_{H\mathbb{Q}}) \\ D' &= \exp(Q'(m')_{H\mathbb{Q}}) \in (\mathbb{V}\mathbb{G}'_m/W'(m'))(F_{H\mathbb{Q}}) \\ u &= \exp(z). \end{aligned}$$

Then

$$\sigma(V_{\mathbb{T}}|_F)(z) = \sigma_d(Du^m) \tag{9.9}$$

and

$$\begin{aligned} \tau_\lambda^*\sigma(V_{\mathbb{T}})(z) &= \sigma_d(Du^m q^{km}) \\ &= (Du^m)^{-k\hat{I}(m)} q^{-k^2\phi(m)} \sigma(V_{\mathbb{T}})(z) \\ &= D^{-k\hat{I}(m)} u^{-kI(m,m)} q^{-k^2\phi(m)} \sigma(V_{\mathbb{T}})(z). \end{aligned}$$

Similarly

$$\tau_\lambda^* \theta'(V'_{\mathbb{T}})(z) = (D')^{-k\hat{I}'(m')} u^{-kI'(m',m')} q^{-k^2\phi'(m')} \theta'(V'_{\mathbb{T}})(z).$$

If $c_2(V)_{\mathbb{T}} = \xi'(V')_{\mathbb{T}}$, then Lemma 6.5 implies that

$$D^{\hat{I}(m)} = \exp(\hat{I}(m)(Q(m)_{H\mathbb{Q}})) = \exp(\hat{I}(m')(Q'(m')_{H\mathbb{Q}})) = (D')^{\hat{I}'(m')}$$
$$\phi(m) = \phi'(m')$$

which gives the result. □

Example 9.10. To illustrate the notation used in the proof, suppose we have chosen an isomorphism $T \cong \mathbb{T}^d$ and so also $\check{T} \cong \mathbb{Z}^d$. Then we have

$$\mathbb{V}\widehat{\mathbb{G}}_a \cong \widehat{\mathbb{A}}^d \cong \operatorname{spf} \mathbb{Z}[\![x_1, \dots, x_d]\!].$$

Suppose that the map $F \to BG$ classifying $V|_F$ factors through BT, i.e. that

$$V|_F = L_1 \oplus \cdots \oplus L_d$$

is written as a sum of complex line bundles. Then under the resulting map

$$F_{H\mathbb{Q}} \to (\mathbb{V}(\widehat{\mathbb{G}}_a)_{\mathbb{Q}}),$$

the coordinate function x_j pulls back to $c_1 L_j$. If $\mathbb{T}$ acts on L_j by the character m_j, then equation (9.9) becomes the more familiar equation

$$\sigma(V_{\mathbb{T}}|_F)(z) = \prod_j \sigma(e^{x_j + m_j z})$$

□

Lemma 9.8 says that $\bar{\gamma}_B$ descends to a function γ_B on

$$X_E^{\mathbb{T}} \times B \subset X_E^{\mathbb{T}} \times C.$$

For b an ordinary point of C, we define

$$\gamma_b \overset{\text{def}}{=} \gamma_B|_{U_b} \in \mathcal{O}(X_E^b \times U_b) = \mathcal{O}(X_{E\mathbb{T},b}).$$

Now suppose that a is a special point. Let $\overline{a}$ be a preimage of a in $\mathbb{A}^1_{\text{an}}$, and define $\gamma_a \in \mathcal{O}(X_{E\mathbb{T},a})$ by the formula

$$\gamma_a \overset{\text{def}}{=} \tau_{-a}^*((\wp|_W)^{-1})^*(R(V, V^a, \Omega(V^a), \overline{a})\theta'(V', \overline{a})),$$

where $W \subset \wp^{-1}U_a$ is the component containing the origin (see (7.7)). This is a definition in view of the

Lemma 9.11. *The class γ_a is independent of the lift $\overline{a}$.*

Proof. Suppose that $\overline{a}$ and $\overline{a}'$ are two lifts of a. Let A, B, and δ be given by

$$A = \exp(\overline{a})$$
$$B = \exp(\overline{a}')$$
$$B = Aq^\delta.$$

Let Y be a component of X^a. Let m be a reduction of the action on $\mathbb{T}$ on $Q|_Y$, and let m' be a reduction of the action of $\mathbb{T}$ on $Q'|_Y$. Lemma 6.23 implies that

$$\theta'(Q', \overline{a}') = w(a, q^{\frac{1}{n}})^{\delta\phi(m)}\theta'(Q', \overline{a})$$
$$R(V, V^a, \Omega(V^a), \overline{a}') = w(a, q^{\frac{1}{n}})^{-\delta\phi(m')}R(V, V^a, \Omega(V^a), \overline{a}).$$

Equation (6.3) implies that

$$\phi(m') \equiv \phi(m) \mod n$$

which gives the result. □

Lemma 9.12. *The various sections γ_a for $a \in C$ define a global section of $X_{E\mathbb{T}}^{-[\sigma,V]}$, whose value in $(X_{E\mathbb{T}}^{-[\sigma,V]})_0$ is $\theta'(V')$.*

Proof. The value at 0 follows from the fact that

$$R(V,V,0)=1,$$

as is easily checked. To see that the γ_a assemble into a global section of $X_{E\mathbb{T}}^{-[\sigma,V]}$, we must show that, if a is a special point of order n and b is ordinary with $U=U_a\cap U_b$ nonempty, that

$$\psi_{ab}(\sigma(a,b,\Omega)\gamma_{ab})=\gamma_b. \tag{9.13}$$

Let $i:X^b\to X^a$ be the inclusion. The diagram (7.9) defining ψ_{ab} together with the equation (8.5) for $\sigma(a,b,\Omega)$ reduces (9.13) to

$$\tau_{b-a}^*\left(\sigma(V^a,V^b,\Omega)^{-1}i^*\tau_a^*\gamma_a\right)=\tau_b^*\gamma_b$$

or equivalently

$$\sigma(V^a,V^b,\Omega)^{-1}i^*\tau_a^*\gamma_a=\tau_a^*\gamma_b.$$

Suppose that Y is a component of X^a, and suppose that F is a component of X^b contained in Y. Let $\overline{m}:\mathbb{T}\to T$ be a reduction of the action of $\mathbb{T}$ on Q/F. By Lemma 3.9, $\overline{m}|_{\mathbb{T}[n]}$ is a reduction of the action of $\mathbb{T}[n]$ on Y. By Corollary 6.25 and Lemma 6.17, we may use $\overline{m}$ to calculate $R(V,V^a,\Omega(V^a),\overline{a})$, and so the left side becomes

$$\begin{aligned}\frac{\sigma(V^b,\Omega)}{\sigma(V^a_{\mathbb{T}},\Omega)|_{X^b}}\theta(V'_{\mathbb{T}},\overline{a})|_{X^b}R(V,V^a,\Omega,\overline{a})&=\frac{\sigma(V^b,\Omega)}{(\tau_{\overline{a}^{\overline{m}}}\sigma)(V_{\mathbb{T}})}(\tau_{\overline{a}^{\overline{m}}}\theta')(V'_{\mathbb{T}})\\&=\tau_a^*\left(\frac{\sigma(V^b,\Omega)}{\sigma(V_{\mathbb{T}})}\theta'(V'_{\mathbb{T}})\right)\\&=\tau_a^*\gamma_b.\end{aligned}$$

In the second equation we have used Lemma 9.4 and the fact that $\sigma((V^b)_{\mathbb{T}})=\sigma(V^{\mathbb{T}})$ is invariant under translation. ☐

This completes the construction of the section γ promised in Theorem 9.5. The naturality (9.7) is straightforward, given the canonical nature of the sections γ_a.

10. The sigma orientation

Now suppose that W is an oriented virtual $\mathbb{T}$-bundle over a finite $\mathbb{T}$-CW complex X, with the property that

$$\begin{aligned}w_2(W_{\mathbb{T}})&=0\\c_2(W_{\mathbb{T}})&=0.\end{aligned}$$

Theorem 10.1. *A $\mathbb{T}$-orientation Ω on W determines a canonical trivialization $\gamma(W)=\gamma(W,\Omega)$ of $W_{E\mathbb{T}}$, whose value in $W_{E\mathbb{T},0}\cong\mathcal{H}(W)_0$ is $\Sigma(W_{\mathbb{T}})$. Moreover we have*

$$\gamma(W\oplus W')=\gamma(W)\otimes\gamma(W')$$

under the isomorphism

$$(W\oplus W')_{E\mathbb{T}}\cong W_{E\mathbb{T}}\otimes W'_{E\mathbb{T}},$$

and if

$$f:Z\to X$$

is a map of finite $\mathbb{T}$-CW complexes, then

$$\gamma(f^*W)=f^*\gamma(W).$$

Proof. As discussed in the end of §8, we write

$$W=V_0-V_1$$

with each V_i an $\mathbb{T}$-oriented spin bundle of even rank. We may assume that each $(V_i)_{\mathbb{T}}$ a spin bundle, and then we have

$$c_2((V_0)_{\mathbb{T}})=c_2((V_1)_{\mathbb{T}})$$

and

$$W_{E\mathbb{T}}=(V_0)_{E\mathbb{T}}\otimes((V_1)_{E\mathbb{T}})^{-1}.$$

Now the proof proceeds much as the proof of Theorem 9.5: to construct $\gamma(W)$ it is equivalent to give a section γ of $X_{E\mathbb{T}}^{[\sigma,V_0-V_1,\Omega]}$ whose value in

$$(X_{E\mathbb{T}}^{-[\sigma,V_0-V_1]})_0 \cong \mathcal{H}(X)_0$$

is 1.

Once again, let $B \subset C$ be the set of ordinary points, and, as in Lemma 8.8, let $\bar{B} \subset \mathbb{A}^1_{\mathrm{an}}$ be the preimage of B in $\mathbb{A}^1_{\mathrm{an}}$. Lemma 8.8 tells us that the formula

$$\bar{\gamma}_B \overset{\text{def}}{=} \frac{\sigma((V_0)_{\mathbb{T}})}{\sigma(V_0^{\mathbb{T}})} \frac{\sigma(V_1^{\mathbb{T}})}{\sigma((V_1)_{\mathbb{T}})}$$

defines an *unit* in $\mathcal{H}(X^{\mathbb{T}};\bar{B})^\times$. The same argument as in Lemma 9.8 shows once again that $\bar{\gamma}_B$ descends to a function γ_B on

$$X_E^{\mathbb{T}} \times B \subset X_E^{\mathbb{T}} \times C.$$

For b an ordinary point of C, we define

$$\gamma_b \overset{\text{def}}{=} \gamma_B|_{U_b} \in \mathcal{O}(X_E^b \times U_b)^\times = X_{E\mathbb{T},b}(U_b)^\times.$$

Now suppose that a is a special point. Let $\overline{a}$ be a preimage of a in $\mathbb{A}^1_{\mathrm{an}}$, and define γ_a by the formula

$$\tau_a^*\gamma_a \overset{\text{def}}{=} \frac{R(V_1, V_1^a, \Omega(V_1^a), \overline{a})}{R(V_0, V_0^a, \Omega(V_0^a), \overline{a})}$$

As in Lemma 9.11, γ_a is independent of the lift $\overline{a}$. In this case, however, Corollary 6.25 implies that γ_a is a *unit*, i.e. an element of $X_{E\mathbb{T},a}(U_a)^\times$.

The same argument as in the proof of Lemma 9.12 shows that the sections γ_a for $a \in C$ assemble into a global section of $X_{E\mathbb{T}}^{[\sigma,V_0-V_1]}$, which is a trivialization because it is so on each U_a.

The fact that the section $\gamma(W)$ is independent of the choice of V_i, as well as the fact that $\gamma(W \oplus W') = \gamma(W) \otimes \gamma(W')$ follows from definition of $\gamma(W)$ and the equation

$$\sigma(W \oplus W') = \sigma(W)\sigma(W').$$

The naturality under change of base is clear from the construction. □

11. A conceptual construction of the equivariant sigma orientation

This section is devoted to a discussion of the following.

Principle X. *Equivariant elliptic cohomology ought to have the following feature. Suppose that Q is a $\mathbb{T}$-equivariant principal G-bundle over a $\mathbb{T}$-space X, and suppose that $Q_{\mathbb{T}}$ is a principal G-bundle over $X_{\mathbb{T}}$. Then there is a canonical map*

$$X_{E\mathbb{T}} \xrightarrow{Q_{E\mathbb{T}}} \mathbb{V}C/W$$

making the diagram

$$\begin{array}{ccc} X_{\mathbb{T}\hat{E}} & \xrightarrow{Q_{\mathbb{T}\hat{E}}} & \mathbb{V}\widehat{C}/W \\ \downarrow & & \downarrow \\ X_{E\mathbb{T}} & \xrightarrow{Q_{E\mathbb{T}}} & \mathbb{V}C/W \end{array}$$

commute. Moreover, for all components F of $X^{\mathbb{T}}$ and all reductions

$$m : \mathbb{T} \to T$$

of the action of $\mathbb{T}$ on $Q|_F$, the diagram (9.3) should commute.

11.1. **What Principle X is good for.** Principle X gives an elegant description of the equivariant sigma orientation, which even illuminates the non-equivariant case.

Let $G = Spin(2d)$ with maximal torus T and Weyl group W. As we saw in §6, the characteristic class $c_2 \in H^4(BSpin(2d))$ gives rise to a line bundle $\mathcal{A}(c_2)$ over $\mathbb{V}C/W$. The theta function σ_d, which is the euler class of the sigma orientation for oriented vector bundles of rank $2d$, defines a holomorphic section of $\mathcal{A}(c_2)$. Now σ_d is not a trivialization of $\mathcal{A}(c_2)$, but its zeroes define an ideal sheaf $\mathcal{I}(\sigma_d)$ on $\mathbb{V}C/W$, and so σ_d is a *trivialization* of the line bundle $\mathcal{A}(c_2) \otimes \mathcal{I}(\sigma_d)$ on $\mathbb{V}C/W$.

Suppose now that V is a $\mathbb{T}$-equivariant $Spin(2d)$ vector bundle over a $\mathbb{T}$-space X, with $w_2(V_{\mathbb{T}}) = 0$. Let Q/X be the associated principle bundle. Principle X gives a map

$$X_{E\mathbb{T}} \xrightarrow{Q_{E\mathbb{T}}} \mathbb{V}C/W,$$

and so a trivialization $Q^*_{E\mathbb{T}}\sigma_d$ of the line bundle

$$Q^*_{E\mathbb{T}}\mathcal{A}(c_2) \otimes Q^*_{E\mathbb{T}}\mathcal{I}(\sigma_d)$$

over $X_{E\mathbb{T}}$.

Now Proposition 8.7 essentially says that there is a canonical isomorphism

$$V_{E\mathbb{T}} \cong Q^*_{E\mathbb{T}}\mathcal{I}(\sigma_d)$$

of line bundles over $X_{E\mathbb{T}}$, and certainly any realization of the Principle will come with such an isomorphism. Thus the failure of $Q^*_{E\mathbb{T}}\sigma_d$ to be a trivialization of $V_{E\mathbb{T}}$ is the bundle $Q^*_{E\mathbb{T}}\mathcal{A}(c_2)$. Suppose that V' is a $\mathbb{T}$-equivariant $Spin(2d')$ vector bundle with $w_2(V'_{\mathbb{T}}) = 0$: then there will be a canonical isomorphism of line bundles

$$Q^*_{E\mathbb{T}}\mathcal{A}(c_2) \cong (Q'_{E\mathbb{T}})^*\mathcal{A}(c_2)$$

precisely when

$$c_2(V_{\mathbb{T}}) = c_2(V'_{\mathbb{T}}). \tag{11.1}$$

(It not hard to check using the definition (6.8) of $\mathcal{A}(c_2)$ that the equation (11.1) implies that there is a canonical isomorphism

$$Q^*_{E\mathbb{T}}\mathcal{A}(c_2) \cong (Q'_{E\mathbb{T}})^*\mathcal{A}(c_2)$$

when $\mathbb{T}$ acts *trivially* on X so that (9.3) defines $Q_{E\mathbb{T}}$ and $Q'_{E\mathbb{T}}$; this is essentially the point of Lemma 9.8.)

In that case the ratio

$$\frac{Q^*_{E\mathbb{T}}\sigma_d}{(Q')^*_{E\mathbb{T}}\sigma_{d'}}$$

will be a trivialization of

$$\frac{Q^*_{E\mathbb{T}}\mathcal{A}(c_2) \otimes Q^*_{E\mathbb{T}}\mathcal{I}(\sigma_d)}{(Q'_{E\mathbb{T}})^*\mathcal{A}(c_2) \otimes (Q'_{E\mathbb{T}})^*\mathcal{I}(\sigma_d)} \cong \frac{Q^*_{E\mathbb{T}}\mathcal{I}(\sigma_d)}{(Q'_{E\mathbb{T}})^*\mathcal{I}(\sigma_d)} \cong \frac{V_{E\mathbb{T}}}{V'_{E\mathbb{T}}}.$$

11.2. **Why the Principle should be true, and why it is nevertheless difficult to prove.** We have used the term "Principle" rather than "Conjecture" to emphasize that it is a feature to be sought in *some* equivariant elliptic cohomology theory. How difficult it is to establish the principle depends on your ontology. Given a fully developed theory of equivariant elliptic cohomology as proposed by Ginzburg-Kapranov-Vasserot, the result is automatic, in its stated form, for any elliptic curve C.

To see this, recall that Ginzburg, Kapranov, and Vasserot have proposed that equivariant elliptic cohomology for the curve C and the (compact connected Lie) group G should be a covariant functor

$$(-)_{EG,gkv} : (G\text{–spaces}) \to (\text{schemes over } (\mathbb{V}C/W)).$$

If Q is a $\mathbb{T}$-equivariant principal G-bundle over X, then the equivariant elliptic cohomology of Q should be a scheme

$$Q_{E(\mathbb{T}\times G),gkv} \to C \times (\mathbb{V}C/W)$$

over both C (via the $\mathbb{T}$-action) and $\mathbb{V}C/W$ (via the G-action). Now G acts freely on Q with quotient X, so one expects that

$$Q_{E(\mathbb{T}\times G),gkv} \cong X_{E\mathbb{T},gkv}.$$

Combining these observations leads to the prediction that a $\mathbb{T}$-equivariant principal G-bundle over X should give rise to the map

$$X_{E\mathbb{T},gkv} \xrightarrow{Q_{E\mathbb{T}}} \mathbb{V}C/W.$$

There are two problems with this proposal. First, we have in this paper described only $\mathbb{T}$-equivariant elliptic cohomology. It may not be difficult to resolve this problem: it is not so difficult to imagine an analogous construction of G-equivariant elliptic cohomology, and indeed Grojnowski does so in [Gro94]. That leaves the second problem, that Grojnowski's functor takes values in sheaves of $\mathbb{Z}/2$-graded algebras over the sheaf $\mathcal{O}_C$ of *holomorphic* functions on C. That is, we do not have quite the right space to construct the map $Q_{E\mathbb{T}}$: if

$$\pi : X_{E\mathbb{T},gkv} \to C$$

is the structural map associated to an elliptic cohomology proposed by Ginzburg-Kapranov-Vasserot, then we have worked in this paper with the sheaf

$$X_{E\mathbb{T}} = \pi_* \mathcal{O}_{X_{E\mathbb{T},gkv}}.$$

To see why this is a problem, note that in order to construct a map

$$Q_{E\mathbb{T}} : X_{E\mathbb{T}} \to \mathbb{V}C/W$$

we must in particular construct a map of topological spaces

$$C \to \mathbb{V}C/W. \tag{11.2}$$

If $X^{\mathbb{T}}$ is non-empty, then $X_{E\mathbb{T}}^{T} = X^{\mathbb{T}} \times C$, and for each connected component F of $X^{\mathbb{T}}$, a reduction of the action of $\mathbb{T}$ on $Q|_F$ gives a map

$$m : C \to \mathbb{V}C$$

and so one has a place to start. But it is perfectly possible that $X^{\mathbb{T}}$ is empty, and then it is not clear how to proceed.

As evidence for the Principle, we construct a map $Q_{\mathcal{E}}$ for a stylized functor $\mathcal{E}$, which is not quite Grojnowski's elliptic cohomology, but captures its behavior on stalks. We simply throw away the points of C for which we have no instructions for constructing a map (11.2). The functor $\mathcal{E}$ is inspired by the *rational* $\mathbb{T}$-equivariant elliptic spectra of Greenlees [Gre01] and by Hopkins's study of characters and elliptic cohomology [Hop89].

Recall that $\mathcal{R}$ denotes the category of ringed spaces. Let $\mathcal{S}$ be the category in which the objects are ringed spaces $(X, \mathcal{O}_X)$, and in which a map

$$f = (f_1, f_2) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a map of spaces

$$f_1 : X \to Y$$

and a map of sheaves of algebras over X

$$f_2 : \mathcal{O}_X \to f_1^{-1}\mathcal{O}_Y.$$

Let $Sub(\mathbb{T})$ be the category of closed subgroups of $\mathbb{T}$ with morphisms given by inclusions. We shall define $\mathcal{E}$ to be a functor

$$(\ -\ )_{\mathcal{E}} : (\mathbb{T}\text{-spaces}) \to \mathcal{S}^{Sub(\mathbb{T})}$$

from $\mathbb{T}$-spaces to the category of $Sub(\mathbb{T})$-diagrams in $\mathcal{S}$.

Let X be a $\mathbb{T}$-space. The ringed space $X_{\mathcal{E}}(\mathbb{T})$ is just

$$X_{\mathcal{E}}(\mathbb{T}) = X_E^{\mathbb{T}} \times C$$

(which is empty if $X^{\mathbb{T}}$ is). If $\mathbb{T}[N] \subset \mathbb{T}$ is a finite subgroup and $X^{\mathbb{T}[n]}$ is empty, then

$$X_{\mathcal{E}}(\mathbb{T}[N]) = \emptyset.$$

Otherwise,

$$X_{\mathcal{E}}(\mathbb{T}[N]) = C[N],$$

with structure sheaf $j^{-1}E_{\mathbb{T}}(X)$, where j denotes the inclusion

$$j : C[N] \to C.$$

Explicitly, for $U \subseteq C[N]$, $X_{\mathcal{E}}(\mathbb{T}[N])(U)$ is the product

$$X_{\mathcal{E}}(\mathbb{T}[N])(U) = \prod_{a \in U} E_{\mathbb{T}}(X)_a = \prod_{a \in U} E_{\mathbb{T}}(X^A)_a,$$

over $a \in U$ of the stalks of Grojnowski's elliptic cohomology. For $A = \mathbb{T}[N] \subseteq B$ with X^B empty, the map

$$X_{\mathcal{E}}(A) \to X_{\mathcal{E}}(B)$$

is trivial; otherwise it is induced by the map of sheaves of algebras over C

$$E_{\mathbb{T}}(X^A) \to E_{\mathbb{T}}(X^B)$$

by restriction to $C[N]$.

Proposition 11.3. *Let G be a simple and simply connected Lie group, and let Q be a principal G-bundle over a $\mathbb{T}$-space X, with the property that $Q_{\mathbb{T}}$ is a principal G-bundle over $X_{\mathbb{T}}$. Then there is a canonical map*

$$Q_{\mathcal{E}} : X_{\mathcal{E}} \to \mathbb{V}C/W$$

in $\mathcal{S}^{Sub(\mathbb{T})}$, where $\mathbb{V}C/W$ is considered as a constant $Sub(\mathbb{T})$-diagram, such that the diagram

$$\begin{array}{ccc} X_{\mathbb{T}\hat{E}} & \xrightarrow{Q_{\mathbb{T}\hat{E}}} & \mathbb{V}\widehat{C}/W \\ \downarrow & & \downarrow \\ X_{\mathcal{E}}(0) & \xrightarrow{Q_{\mathcal{E}}(0)} & \mathbb{V}C/W \end{array}$$

commutes, and such that for all components F of $X^{\mathbb{T}}$ and all reductions

$$m : \mathbb{T} \to T$$

of the action of $\mathbb{T}$ on $Q|_F$,

$$(Q|_F)_{\mathcal{E}}(\mathbb{T}) = Q(m)_E + m.$$

The proof will occupy the rest of this section.

Let A be a closed subgroup of $\mathbb{T}$. Let Y be a connected component of X^A. A reduction

$$m : A \to T$$

of the action of A on $Q|_Y$ determines a map

$$C(A) \xrightarrow{C \otimes m} (\mathbb{V}C)^{W(m)};$$

as usual if $a \in C(A)$ we write a^m for the resulting element of $(\mathbb{V}C)^{W(m)}$. The addition

$$\mathbb{V}C \times \mathbb{V}C \xrightarrow{+} \mathbb{V}C$$

induces a translation

$$(\mathbb{V}C)^{W(m)} \times \mathbb{V}C/W(m) \xrightarrow{+} \mathbb{V}C/W(m),$$

and so we get an operator

$$\tau_{a^m} : \mathbb{V}C/W(m) \to \mathbb{V}C/W(m).$$

If $A = \mathbb{T}$, then we define

$$Q_{\mathcal{E}}(\mathbb{T})_{Y,m} = Q(m)_E + m,$$

as required by the Proposition. If A is finite, then we define the map of ringed spaces $Q_{\mathcal{E}}(A)_{Y,m,a}$ to be the composition

$$\begin{array}{ccccc} Y_{\mathcal{E}}(A)_a & & \xrightarrow{Q_{\mathcal{E}}(A)_{Y,m,a}} & & \mathbb{V}C/W(m) \\ \downarrow{\scriptstyle \tau_{-a}} & & & & \uparrow{\scriptstyle \tau_{a^m}} \\ Y_{\mathcal{E}}(A)_0 & \xrightarrow{\cong} & Y_{\mathcal{H},0} & \xrightarrow{(Q_{\mathcal{H}})_0} & \mathbb{V}C/W(m). \end{array}$$

Here we have used the fact that the $Y_{\mathcal{E}}(A)_0$ is just the origin in C, with ring

$$(\mathcal{O}_{Y_{E\mathbb{T}}})_0 \cong \mathcal{H}(Y)_0;$$

Lemma 5.2 provides the map $(Q_{\mathcal{H}})_0$. We define

$$Q_{\mathcal{E}}(A)_{Y,m} = \coprod_{a \in C[N]} Q_{\mathcal{E}}(A)_{Y,m,a} : Y_{\mathcal{E}}(A) \to \mathbb{V}C/W(m).$$

Lemma 11.4. *If m and m' are two reductions of the action of A on $Q|_Y$, then the diagram*

$$\begin{array}{ccc} Y_{\mathcal{E}}(A) & \xrightarrow{Q_{\mathcal{E}}(A)_{Y,m}} & \mathbb{V}C/W(m) \\ {\scriptstyle Q_{\mathcal{E}}(A)_{Y,m'}}\big\downarrow & & \big\downarrow \\ \mathbb{V}C/W(m') & \longrightarrow & \mathbb{V}C/W \end{array} \tag{11.5}$$

commutes.

Proof. This follows from the fact, proved in Lemma 3.6, that m and m' differ by an element of the Weyl group of G. ☐

The lemma permits us to write $Q_{\mathcal{E}}(A)_Y$ for the map

$$Y_{\mathcal{E}}(A) \to \mathbb{V}C/W$$

described by (11.5). We define

$$Q_{\mathcal{E}}(A) : X_{\mathcal{E}}(A) = \coprod_Y Y_{\mathcal{E}}(A) \xrightarrow{\coprod Q_{\mathcal{E}}(A)_Y} \mathbb{V}C/W,$$

where the coproduct is over the components Y of X^A. The maps $Q_{\mathcal{E}}(A)$ as A ranges over closed subgroups of $\mathbb{T}$ assemble to give the map $Q_{\mathcal{E}}$ of Proposition 11.3.

11.3. **Relationship to the theory $E_{\mathbb{T}}$.** The construction in Proposition 11.3 is closely related to the theory $E_{\mathbb{T}}$. We shall briefly explain how this works, as it illuminates the the explanation sheds light on the relationship between the "transfer argument" of Bott-Taubes and the geometry of the variety $\mathbb{V}C/W$.

Suppose that a is a special point of C of order N and let $A = \mathbb{T}[N]$. By definition we have a map

$$X_{\mathcal{E}}(A)_a \to X_{E\mathbb{T},a};$$

indeed it is the inclusion of the stalk at a. Suppose that b is an ordinary point, and let $U = U_a \cap U_b$. Suppose that F is a component of $Y^{\mathbb{T}}$. Let

$$m_F : \mathbb{T} \to T$$

be a reduction of the action of $\mathbb{T}$ on $Q|_F$; we write

$$m_Y = m_F|_A$$

for the resulting reduction of the action of A on $Q|_Y$ as in Lemma 3.9. Consider the diagram

$$\begin{array}{ccccccc} X_{E\mathbb{T},a} & \longleftarrow & Y_{\mathcal{E}}(A)_a & \xrightarrow{Q_{\mathcal{E}}(A)_{Y,m_Y,a}} & \mathbb{V}C/W(m_Y) & \longrightarrow & \\ \big\uparrow & & {\scriptstyle \tau_{-a}}\big\downarrow & & \big\uparrow{\scriptstyle \tau_a m_Y} & & \big\downarrow \\ X_{E\mathbb{T},a}|_U & & Y_{\mathcal{E}}(A)_0 & \xrightarrow{(Q_{\mathcal{H}})_0} & \mathbb{V}C/W(m_Y) & & \\ & & \big\uparrow & & \big\uparrow & & \\ {\scriptstyle \psi_{ab}}\big\downarrow & & F_{\mathcal{E}}(\mathbb{T})_0 & \xrightarrow{Q_{\mathcal{E}}(\mathbb{T})_0} & \mathbb{V}C/W(m_F) & & \mathbb{V}C/W. \\ & & \big\downarrow & & \big\uparrow & & \\ X_{E\mathbb{T},b}|_U & & F_{\mathcal{E}}(\mathbb{T}) & \xrightarrow{Q_{\mathcal{E}}(\mathbb{T})} & \mathbb{V}C/W(m_F) & & \big\uparrow \\ \big\downarrow & & {\scriptstyle \tau_{-a}}\big\uparrow & & \big\uparrow{\scriptstyle \tau_{(-a)} m_F} & & \\ X_{E\mathbb{T},b} & \longleftarrow & F_E \times C & \xrightarrow{Q(m_F)_E + m_F} & \mathbb{V}C/W(m_F) & \longrightarrow & \end{array} \tag{11.6}$$

The commutativity of the rectangle on the left is just the definition of ψ_{ab}. The commutativity of the rectangle on the right is evident. The commutativity of the top and bottom rectangles in the middle is the definition of $Q_{\mathcal{E}}$. The commutativity of the remaining rectangles in the middle follows from the group structure on C and $\mathbb{V}C$, together with the definitions of the maps involved.

Remark 11.7. We conclude this paper where the research for it began, with an explanation of the relationship between "transfer formula" of [BT89] and the diagram (11.6). Let $F \subseteq Y^{\mathbb{T}} \subseteq Y \subseteq X^{\mathbb{T}[N]}$ be as above. Let

$$m \in \check{T} = \hom(\mathbb{T}, T)$$

be a reduction of the action of $\mathbb{T}$ on $Q|_F$ (so $m_Y = m|_{\mathbb{T}[N]}$ is a reduction of the action of $\mathbb{T}[N]$ on $Q|_Y$). Let $\theta \in \mathcal{O}(\mathbb{V}\mathbb{G}_a^{\mathrm{an}})^W$ be a theta function for G; it determines a holomorphic characteristic class for principal G-bundles of the form $Q_{\mathbb{T}}$: that is, the characteristic class $\theta(Q_{\mathbb{T}})$ lies in $\mathcal{H}(X; \mathbb{A}^1_{\mathrm{an}})$.

The first point is that, for any $a \in \mathbb{G}_a^{\mathrm{an}}$, $\tau_{a^m}\theta \in \mathcal{O}(\mathbb{V}\mathbb{G}_a^{\mathrm{an}})^{W(m)}$, so it gives a holomorphic characteristic class for principal $Z(m)$-bundles. Moreover, the commutativity of the diagram

$$\begin{array}{ccc} F_H \times \mathbb{G}_a^{\mathrm{an}} & \xrightarrow{Q(m)_{\mathcal{H}}} & \mathbb{V}\mathbb{G}_a^{\mathrm{an}}/W(m) \\ {\scriptstyle F_E \times \tau_a}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{a^m}} \\ F_H \times \mathbb{G}_a^{\mathrm{an}} & \xrightarrow{Q(m)_{\mathcal{H}}} & \mathbb{V}\mathbb{G}_a^{\mathrm{an}}/W(m) \end{array}$$

implies that

$$\tau_a(\theta(Q|_F)) = (\tau_{a^{m_F}}\theta)(Q(m_F)) \in \mathcal{H}(F; \mathbb{A}^1_{\mathrm{an}}).$$

The second point is that, if $a \in C[N]$, and $\overline{a}$ is a lift of a to $\mathbb{A}^1_{\mathrm{an}}$, then $\tau_{\overline{a}^m}\theta$ is nearly invariant under the action of $W(m_Y)$. Precisely, if $w \in W(m_Y)$, then

$$\overline{a}^{wm} = \overline{a}^m + \lambda$$

for some $\lambda \in \mathbb{V}\Lambda$: that is, $\overline{a}^{wm}$ and $\overline{a}^m$ are related by the action of the *affine Weyl group* of G. Since θ is a theta function for G, the relationship between $\tau_{\overline{a}^m}\theta$ and $\tau_{\overline{a}^{wm}}\theta$ is controlled by $c_2(Q_{\mathbb{T}})$ When this class is zero, or when the second Chern class of another bundle cancels it, then we may suppose that we have a characteristic class

$$(\tau_{\overline{a}^{m_Y}}\theta)(Q(m_Y)) \in \mathcal{H}(Y; \mathbb{A}^1_{\mathrm{an}}).$$

We then have

$$(\tau_{a^{m_Y}}\theta)(Q(m_Y))|_F = (\tau_{a^m}\theta)(Q(m)) = \tau_a(\theta(Q|_F)),$$

which is a typical "transfer formula".

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