ON MORAVA K-THEORIES OF AN S-ARITHMETIC GROUP

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ABSTRACT. We completely describe the Morava K-theories with respect to the prime p for the étale model of the classifying space of $GL_m(\mathbb{Z}[\sqrt[p]{1}, 1/p])$ when p is an odd regular prime. For $p = 3$ and $m = 2$ (and conjecturally for $m = \infty$) these cohomologies are the same as those of the classifying space itself.

1. Introduction

By using an Eilenberg-Moore type spectral sequence, Tanabe calculated the Morava K-theories for the classifying spaces of certain Chevalley groups. In particular, if $K(n)$ is the n-th Morava K-theory with the ring of coefficients

$$K(n)^*(pt) = \mathbb{F}_p[v_n, v_n^{-1}]$$

where p is a prime and v_n has degree $2(p^n - 1)$, and if q is a power of a prime different from p, then [12]

$$K(n)^*BGL_m(\mathbb{F}_q) \approx K(n)^*BGL_m(\bar{\mathbb{F}}_q)_{\psi^q} \approx \frac{K(n)^*(pt)[[c_1, ..., c_m]]}{(c_1 - \psi^q c_1, ..., c_m - \psi^q c_m)} \tag{1.1}$$

i.e. a ring of formal power series in certain "Chern classes" c_1,...,c_m modulo an ideal given in terms of generators. Here ψ^q is the "Adams operation" induced from the Frobenius automorphism $x \mapsto x^q$ of the algebraic closure $\bar{\mathbb{F}}_q$ of the field $\mathbb{F}_q$ with q elements. The same formula (1.1) holds for the p-adic version $\hat{K}(n)$ of $K(n)$ obtained by replacing $K(n)^*(pt)$ with $\hat{K}(n)^*(pt) = \mathbb{Z}_p[v_n, v_n^{-1}]$ where $\mathbb{Z}_p$ denotes the ring of p-adic integers [12].

On the other hand, if $A = \mathbb{Z}[\sqrt[p]{1}, 1/p]$ and p is a *regular* prime in the sense of number theory, then Dwyer and Friedlander [5, 6] calculated the mod p cohomology of a space $BGL_m(A_{\text{ét}})$ which is naturally associated to the classifying space $BGL_m(A)$ of the S-arithmetic group $GL_m(A)$. We call the space $BGL_m(A_{\text{ét}})$ *the étale model at* p for the classifying space $BGL_m(A)$ and recall that it is endowed with a natural map [4, 2.5]

$$f_A : BGL_m(A) \to BGL_m(A_{\text{ét}}) \tag{1.2}$$

The goal of this article is to show how we can use these two calculations in order to completely describe the Morava K-theories with respect to the prime p of the étale model above. The main result is

Theorem 1.1. *If $A = \mathbb{Z}[\sqrt[p]{1}, 1/p]$ with p an odd regular prime, then the n-th Morava K-theory with respect to the prime p of the étale model $BGL_m(A_{\text{ét}})$ is an*

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exterior algebra given by the formula

$$K(n)^*BGL_m(A_{\text{ét}}) \approx K(n)^*BGL_m(\mathbb{F}_q)\langle \sigma_1, ..., \sigma_m\rangle^{\otimes (p-1)/2}$$

where q is a prime $\equiv 1 \mod p$ but $\not\equiv 1 \mod p^2$, the tensor product is over the ring $K(n)^*BGL_m(\mathbb{F}_q)$, and σ_i has degree $2i-3$ $(1 \le i \le m)$. Moreover, the same formula holds for the p-adic version $\hat{K}(n)$.

In particular, if $m = \infty$ and $n = 1$ (and conjecturally for $n > 1$) the above theorem and (1.1) give $K(n)$ and $\hat{K}(n)$ theories of the classifying space $BGL_\infty(A)$ itself for p odd and regular, according to [7]. Here GL_∞ denotes the union of all GL_n for $n \ge 1$ with respect to the block inclusions.

Also, if $p = 3$ and $m = 2$, then we showed that the natural map (1.2) is a mod p equivalence [1]. Hence we deduce the following

Corollary 1.2. *The n-th Morava K-theory at the prime 3 of the S-arithmetic group $GL_2(\mathbb{Z}[\sqrt[3]{1}, 1/3])$ is given by*

$$K(n)^*BGL_2(\mathbb{Z}[\sqrt[3]{1}, 1/3]) \approx \frac{\mathbb{F}_3[v_n, v_n^{-1}][[a, c_2]]}{(a^{7^n}, c_2^{(7^n+1)/2} \mod a)}\langle \sigma_1, \sigma_2\rangle$$

where the degrees of the generators are $|v_n| = 2(3^n-1)$, $|a| = 2$, $|c_2| = 4$, $|\sigma_1| = -1$, $|\sigma_2| = 1$, and the second generator of the ideal is up to an indeterminacy $\mod a$. Moreover, a similar formula holds for the 3-adic version $\hat{K}(n)$.

Notation 1.3. In what follows p is an odd regular prime when not otherwise stated and $A = \mathbb{Z}[\zeta_p, 1/p]$ where $\zeta_p = \exp(2\pi i/p)$ is a prescribed p-th root of unity in the field $\mathbb{C}$ of complex numbers.

2. Étale models for classifying spaces

2.1. **The original definition.** Let $R = \mathbb{Z}[1/p]$, G a group scheme over $Spec(R)$, and BG the classifying simplicial scheme obtained by a bar construction as in [8, 1.2]. Then the classifying space $BG(D)$ of the group $G(D)$ of the D-points of G where D is any finitely generated R-algebra can be thought of as the connected component of a simplicial function complex [6, 1.4]

$$BG(D) = Map^0(Spec(D), BG)_{Spec(R)} \tag{2.1}$$

containing the natural base point induced by the unit map $Spec(R) \to G$ of G over $Spec(R)$. We recall that $Map(X, Y)_Z$ is a simplicial set given in dimension i by the set of simplicial scheme maps $X \otimes \Delta[i] \to Y$ over Z where X and Y are simplicial schemes over Z (a scheme is regarded as a constant simplicial scheme) and $\Delta[i]$ is the standard simplicial i-simplex. The tensor product between a simplicial scheme and a simplicial set is defined in [8, 1.1].

Also, we recall that the étale topological type $X_{\text{ét}}$ in the sense of Friedlander [8, 4.4] is a pro-space (i.e. inverse system of simplicial sets) which is naturally associated to a noetherian simplicial scheme X and reflects the étale cohomology of X. For any finitely generated R-algebra D, let $D_{\text{ét}}$ denote the étale topological type $Spec(D)_{\text{ét}}$. By replacing $Spec(D)$, BG, and $Spec(R)$ in (2.1) by their étale topological types $D_{\text{ét}}$, $(BG)_{\text{ét}}$, and $R_{\text{ét}}$, the space $BG(D_{\text{ét}})$ is defined in [6, 1.2] as the connected component of the simplicial complex of p-adic functions over $R_{\text{ét}}$

$$BG(D_{\text{ét}}) = Hom_p^0(D_{\text{ét}}, (BG)_{\text{ét}})_{R_{\text{ét}}} \tag{2.2}$$

containing the corresponding natural base point. This construction is similar to (2.1) and we can associate with each i-simplex of $BG(D)$ an i-simplex of $BG(D_{\text{ét}})$ regarded by definition as a map of pro-spaces over $R_{\text{ét}}$ from $D_{\text{ét}} \times \Delta[i]$ to the fibrewise p-adic completion of $(BG)_{\text{ét}}$ over $R_{\text{ét}}$ denoted by $(\mathbb{Z}/p)^{\bullet}(BG)_{\text{ét}}$ [4, 2.4]. This assignment is natural in both G and D and gives a map [4, 2.5]

$$f_D^G : BG(D) \to BG(D_{\text{ét}})$$

from the classifying space of the group $G(D)$ to its étale model $BG(D_{\text{ét}})$ at p. In the case when $G = GL_m$ is the group scheme over $SpecR$ corresponding to the general linear group $G(R) = GL_m(R)$ and $D = A$, we obtain the map (1.2). These definitions actually hold for *any* prime p.

2.2. **A model structure definition.** For convenience we will give an alternative way of thinking of (2.2) pointed out by Isaksen and based on its model structure. Namely, if pro-$\mathcal{SS}$ is the category of pro-spaces then there is a proper simplicial model structure on pro-$\mathcal{SS}$ introduced in [9]. This means that there are three classes of morphisms in pro-$\mathcal{SS}$ called *weak equivalences*, *cofibrations*, and *fibrations* subject to various axioms. Also there is a notion of *simplicial function complex* i.e. a natural assignment to each two pro-spaces X and Y of a simplicial set $Map(X,Y)$ interacting appropriately with the model structure [9, 16.2].

For the purpose of this paper we will use the induced proper simplicial model structure on the over-category pro-$\mathcal{SS}_V$ of pro-spaces over a fixed pro-space V. With respect to this model structure there is a *relative* simplicial function complex $Map(X,Y)_V$ naturally associated with every pair of objects X, Y in pro-$\mathcal{SS}_V$. Keeping the same notations as in the previous subsection we have the following

Proposition 2.1. *For any finitely generated R-algebra D, the space $BG(D_{\text{ét}})$ is weakly equivalent to the connected component of the natural base point of the simplicial function complex $Map(D_{\text{ét}}, T_p(BG)_{\text{ét}})_{R_{\text{ét}}}$ in the over-category of pro-spaces over $R_{\text{ét}}$,*

$$BG(D_{\text{ét}}) \simeq Map^0(D_{\text{ét}}, T_p(BG)_{\text{ét}})_{R_{\text{ét}}}$$

Here $T_p(BG)_{\text{ét}}$ is a fibrant replacement of $(\mathbb{Z}/p)^{\bullet}(BG)_{\text{ét}}$ over $R_{\text{ét}}$ in the sense of the simplicial model structure of [9].

Proof. Let $X = D_{\text{ét}} = \{X_\alpha\}$, $Y = (BG)_{\text{ét}}$, and $V = R_{\text{ét}}$. Then $Y \to V$ is a (strict) map of pro-spaces and let $T'_p(Y)$ be the level-space Moore-Postnikov tower naturally associated to the fibrewise p-adic completion of Y over V. Then we can think of $T'_p(Y) = \{T'_p(Y)_\delta\}$ as a pro-space over $V = \{V_\delta\}$ and by definition [4, 2.3]

$$Hom_p(X,Y)_V = holim_\delta colim_\alpha Map(X_\alpha, T'_p(Y)_\delta)_{V_\delta} \tag{2.3}$$

where Map is the usual relative simplicial function complex of simplicial sets and *holim* denotes the homotopy inverse functor from pro-spaces to spaces [2, §6]. By [9, 10.6], the pro-space $T_p(Y)$ is the fibrant replacement of $T'_p(Y)$ in the model structure of Edwards-Hastings. By standard arguments, the space (2.3) is weakly equivalent to

$$Map(X, T_p(Y))_V = lim_\delta colim_\alpha Map(X_\alpha, T_p(Y)_\delta)_{V_\delta}$$

and the conclusion follows from (2.2). □

3. A homotopy fibre square

3.1. **Preliminaries.** We collect here a couple of known facts which will be used in the construction of a computable model for $BGL_m(A_{\text{ét}})$ given in the next subsection. This model is naturally associated to the action of $\pi_1(R_{\text{ét}})$ on the p-primary roots of unity.

Let D be a finitely generated normal R-algebra and $pt : Spec(k) \to Spec(D)$ a geometric point corresponding to a homomorphism from D to a separable closed field k. Then pt determines a base point of $D_{\text{ét}}$ and we recall that $\pi_1(D_{\text{ét}}, pt)$ is the pro-finite Grothendieck fundamental group of D pointed by pt [8, §5]. This group classifies finite étale covering spaces of $Spec(D)$.

Let μ_{p^ν} be the set of all complex numbers z such that $z^{p^\nu} = 1$ and μ_{p^∞} the union of all μ_{p^ν} for $\nu \geq 0$. Let R_∞ denote the ring obtained from R by adjoining the set μ_{p^∞} of all p-primary roots of unity,

$$R_\infty = R[\sqrt[p^\infty]{1}] = \mathbb{Z}[1/p, \mu_{p^\infty}],$$

and Γ the Galois (pro-)group

$$\Gamma = Gal(R_\infty, R) = \{Aut(\mu_{p^\nu}), \nu \geq 1\} \approx \{(\mathbb{Z}/p^\nu)^*, \nu \geq 1\}$$

In this context, observe that $\pi_1(R_{\text{ét}})$ is the Galois group of the maximal unramified extension of R and let

$$\theta : \pi_1(R_{\text{ét}}) \to \Gamma \tag{3.1}$$

be the homomorphism given by the action of this Galois group on the p-primary roots of unity. In other words, $R_{\text{ét}}$ is provided with the natural structure map $R_{\text{ét}} \to K(\Gamma, 1)$ which "classifies" the finite étale extensions $R \to R[\mu_{p^\nu}]$. Also, $A_{\text{ét}}$ is provided with a natural structure map

$$A_{\text{ét}} \to R_{\text{ét}} \to K(\Gamma, 1)$$

If k is a field, then $k_{\text{ét}}$ is a pro-space of type $K(\pi, 1)$, where π is the Galois group over k of the separable algebraic closure of k. In particular, $\mathbb{C}_{\text{ét}}$ is contractible and $(\mathbb{F}_q)_{\text{ét}}$ is equivalent to the pro-finite completion of a circle. If $R \to \mathbb{F}_q$ is a residue field map, then $(\mathbb{F}_q)_{\text{ét}}$ is provided with a natural structure map

$$(\mathbb{F}_q)_{\text{ét}} \to R_{\text{ét}} \to K(\Gamma, 1)$$

as well. This structure map sends the Frobenius element of the Galois group of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$ identified with $\pi_1((\mathbb{F}_q)_{\text{ét}})$ to $q \in Aut(\mu_{p^\nu}) \cong (\mathbb{Z}/p^\nu)^*$ in Γ [6, 3.2].

3.2. **A homotopy fibre square.** Let U_m be the Lie group of $m \times m$ unitary matrices and $\hat{B}U_m$ the p-completion of its classifying space. The following proposition is the unstable analogue of [5, 4.5] and its proof is almost the same. For convenience, we review here the main arguments.

Proposition 3.1. *Let p be an odd regular prime, $A = \mathbb{Z}[\zeta_p, 1/p]$, and q a rational prime $\equiv 1 \mod p$ but $\not\equiv 1 \mod p^2$. Then there is a homotopy fibre square*

$$\begin{array}{ccc} BGL_m(A_{\text{ét}}) & \longrightarrow & \hat{B}U_m^W \\ \downarrow & & \downarrow \\ \hat{B}GL_m(\mathbb{F}_q) & \longrightarrow & \hat{B}U_m \end{array}$$

where W is the wedge of $(p-1)/2$ circles, $\hat{B}U_m^W$ denotes the simplicial function complex of unpointed maps from W to $\hat{B}U_m$, and the right-hand vertical map is the evaluation at the base-point.

Proof. As in [5, p. 145] we construct a map

$$(\mathbb{F}_q)_{\text{ét}} \vee W \to K(\Gamma, 1)$$

by sending the first summand via the natural structure map and mapping the other summand trivially. By a class-field argument (assuming the properties of q from hypothesis), there exists a map

$$g : (\mathbb{F}_q)_{\text{ét}} \vee W \to A_{\text{ét}}$$

over $K(\Gamma, 1)$ which is a mod p cohomology equivalence [5, p. 145]. In other words g is a "good mod p model" for A in the sense of [6, 1.9]. This means that by a spectral sequence argument [4, 2.11] and using 2.1 for $G = GL_m$ the map g induces a homotopy equivalence

$$Map^0(A_{\text{ét}}, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \simeq Map^0((\mathbb{F}_q)_{\text{ét}} \vee W, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}}$$

which can be reformulated by saying that we get a homotopy fibre square

$$\begin{array}{ccc} BGL_m(A_{\text{ét}}) & \longrightarrow & Map^0(W, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \\ \downarrow & & \downarrow \\ BGL_m((\mathbb{F}_q)_{\text{ét}}) & \longrightarrow & Map^0(pt, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \end{array}$$

where the right-hand vertical map is the evaluation at the base-point (the map $pt \to R_{\text{ét}}$ is induced from $R \subset \mathbb{C}$ recalling that $\mathbb{C}_{\text{ét}}$ is contractible). To finish the proof, we need only to identify the appropriate corners of this square.

For the two right-hand corners, we start with the fibration sequence [3, 2.3]

$$\{(\mathbb{Z}/p)_s(BGL_{m,\bar{\mathbb{F}}_q})_{\text{ét}}\}_s \to (\mathbb{Z}/p)^{\bullet}(BGL_m)_{\text{ét}} \to R_{\text{ét}} \tag{3.2}$$

where $\{(\mathbb{Z}/p)_s(-)\}_s$ denotes the Bousfield-Kan p-completion tower and $BGL_{m,\bar{\mathbb{F}}_q}$ is the classifying object of GL_m over $\bar{\mathbb{F}}_q$. Hence we get that

$$Map^0(pt, T_p(BG)_{\text{ét}})_{R_{\text{ét}}} \simeq holim\{(\mathbb{Z}/p)_s BGL_{m,\bar{\mathbb{F}}_q})_{\text{ét}}\}_s \simeq \hat{B}U_m$$

where the last equivalence is proved in [8, 8.8]. Because the composite map

$$\pi_1(W) \to \pi_1(A_{\text{ét}}) \to \pi_1(R_{\text{ét}}) \xrightarrow{\theta} \Gamma$$

is trivial by construction, as in [4, p. 146] we get a homotopy equivalence

$$Map^0(W, T_p(BGL_m)_{\text{ét}})_{R_{\text{ét}}} \simeq \hat{B}U_m^W$$

where $\hat{B}U_m^W$ denotes the function complex of unpointed maps from W to $\hat{B}U_m$ (basically $\pi_1(R_{\text{ét}})$ acts on the fibre of (3.2) via θ).

Finally, for the lower left-hand corner, there is a homotopy equivalence

$$\hat{B}GL_m(\mathbb{F}_q) \simeq BGL_m((\mathbb{F}_q)_{\text{ét}})$$

given in [3, 2.11] by exploiting the action of the Frobenius element on the fibre of (3.2) via the composite

$$\pi_1((\mathbb{F}_q)_{\text{ét}}) \to \pi_1(R_{\text{ét}}) \xrightarrow{\theta} \Gamma$$

and Quillen's homotopy fix point description of $\hat{B}GL_m(\mathbb{F}_q)$ [10]. □

4. The proof of the main theorem and its corollary

4.1. **Proof of 1.1.** The proof of the main theorem is based on Strickland's analysis of unitary bundles in [11] applied to the homotopy fibre square 3.1.

Let V be a complex vector bundle over a space X and write PV for the associated bundle of projective spaces and $U(V)$ for the associated bundle of unitary groups

$$U(V) = \{(x,g)|x \in X \text{ and } g \in U(V_x)\}$$

Let $EU(V)$ denote the geometric realization of the simplicial space $\{U(V)^{n+1}\}_{n\geq 0}$ and put $BU(V) = EU(V)/U(V)$ the usual simplicial model for the classifying space of $U(V)$.

Let E^* be an even periodic cohomology theory with complex orientation $x \in \tilde{E}^0\mathbb{C}P^\infty$. We are interested in describing $E^*U(V)$ as a Hopf algebra over E^*X (using the group structure on $U(V)$). The main result involves the exterior algebra over the ring E^*X generated by the module E^*PV which we denote by $\lambda^*_{E^*X}E^{*-1}PV$ and which is a Hopf algebra over E^*X by declaring E^*PV to be primitive.

Proposition 4.1 ([11, 4.4])**.** *There is a natural isomorphism of Hopf algebras over E^*X*

$$\lambda^*_{E^*X}E^{*-1}PV \approx E^*U(V)$$

We apply this proposition to the tautological bundle

$$V = \gamma_m = EU_m \times_{U_m} \mathbb{C}^m$$

over $X = BU_m$. In this case, we have

$$E^*P\gamma_m \approx \frac{E^*BU_m[x]}{(x^m + c_1x^{m-1} + ... + c_m)} \approx E^*BU_m\{1, ..., x^{m-1}\}$$

where c_i is the i-th Chern class of γ_m and the last isomorphism indicates that $E^*P\gamma_m$ is a free module over E^*BU_m with basis $1, x, ..., x^{m-1}$. In particular,

$$E^*U(\gamma_m) \approx \lambda^*_{E^*BU_m}E^{*-1}P\gamma_m \approx E^*BU_m\langle\sigma(1), ..., \sigma(x^{m-1})\rangle \tag{4.1}$$

where σ lowers the degree by 1.

Going back to the homotopy fibre square 3.1, we observe that $\hat{B}U_m^W$ is the $(p-1)/2$-fold fibre product of $\hat{B}U_m^{S^1} = \hat{U}(\gamma_m)$ over $\hat{B}U_m$ and for $E^* = K(n)$ or $\hat{K}(n)$ and any space X we have $E^*\hat{X} = E^*X$. In this case, if we apply E^* to $\hat{B}U_m^W$ and use (4.1) we obtain

$$\begin{aligned} E^*(\hat{B}U_m^W) &\approx (\lambda^*_{E^*BU_m}E^{*-1}P\gamma_m)^{\otimes(p-1)/2} \\ &\approx E^*BU_m\langle\sigma(1), ..., \sigma(x^{m-1})\rangle^{\otimes(p-1)/2} \end{aligned}$$

where the tensor product is over E^*BU_m. In particular, $E^*(\hat{B}U_m^W)$ is a free module over $E^*\hat{B}U_m = E^*BU_m$ and therefore 1.1 follows from the above formula by a base-change induced from 3.1:

$$E^*BGL_m(A_{\text{ét}}) \approx E^*\hat{B}GL_m(\mathbb{F}_q) \otimes_{E^*\hat{B}U_m} E^*(\hat{B}U_m^W)$$

where q can be always chosen with the prescribed properties (by Dirichlet's density theorem for instance).

4.2. **Proof of 1.2.** This has been already explained in the Introduction, except for the analysis of the formula (1.1) in the case $p = 3$, $m = 2$, and $q = 7$. The goal of this subsection is to complete this analysis.

Proposition 4.2. *Let p be an odd rational prime and $\mathbb{F}_q$ a finite field with q elements such that $q \equiv 1 \mod p^r$ but $\not\equiv 1 \mod p^{r+1}$ for some integer $r > 0$. Then*

$$K(n)^* BGL_2(\mathbb{F}_q) \approx \frac{K(n)^*(pt)[[a, c_2]]}{(a^{p^{nr}}, c_2^{(p^{nr}+1)/2} \mod a)}$$

Proof. According to Tanabe's formula

$$K(n)^* BGL_2(\mathbb{F}_q) \approx (K(n)^* BU_2)_\psi \tag{4.2}$$

where the co-invariants are calculated with respect to the q-th Adams operation ψ [12]. Recall that

$$K(n)^* BU_2 \approx K(n)^*(pt)[[c_1, c_2]] \tag{4.3}$$

where $c_1 = x + y$ and $c_2 = xy$ are expressed in terms of the generators of the ring

$$K(n)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \approx K(n)^*(pt)[[x, y]]$$

which are induced by a complex orientation on $K(n)^*(\mathbb{C}P^\infty)$ [12, 2.12]. It is easy to see that we can replace c_1 in (4.3) by the formal group sum $a = x +^{K(n)} y$ of x and y (induced from the tensor product of complex line bundles). Then the proposition follows from (4.2) and the following lemmas

Lemma 4.3. $a - \psi(a) = (unit) \times a^{p^{nr}}$

Lemma 4.4. $c_2 - \psi(c_2) \equiv (unit) \times c_2^{(p^{nr}+1)/2} \mod a$

where "unit" means invertible element in (4.3). □

Proof of 4.3. Let us expand q in the ring $\mathbb{Z}_p$ of p-adic integers as

$$q = \sum_{k=0}^{\infty} \alpha_k p^k$$

where the coefficients $\alpha_k \in \mathbb{Z}$ are subject to $0 \leq \alpha_k < p$, $\alpha_0 = 1$, $\alpha_r \neq 0$, and $\alpha_k = 0$ for $0 < k < r$. Then for $t = x$ or y we have

$$\psi(t) = [q](t) = {\sum}^{K(n)} [\alpha_k](t^{p^{nk}}) = t + \alpha_r t^{p^{nr}} + ...$$

where $[q](t)$ means the formal group q-multiple of t. Hence,

$$\psi(a) = \psi(x) +^{K(n)} \psi(y) = {\sum}^{K(n)} [\alpha_k](a^{p^{nk}}) = a + \alpha_r a^{p^{nr}} + ...$$

and the conclusion follows. □

Proof of 4.4. With the same notations as in the previous proof, we have

$$x \equiv [-1](y) \equiv -y \mod a$$

and hence

$$c_2 - \psi(c_2) \equiv -x^2 + \psi(x)\psi(x) \equiv (unit) \times x^{p^{nr}+1} \equiv (unit) \times c_2^{(p^{nr}+1)/2} \mod a$$

□

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