ON THE HOMOTOPY INVARIANCE OF CONFIGURATION SPACES

MOKHTAR AOUINA AND JOHN R. KLEIN

ABSTRACT. For a closed PL manifold M, we consider the configuration space $F(M,k)$ of ordered k-tuples of distinct points in M. We show that a suitable iterated suspension of $F(M,k)$ is a homotopy invariant of M. The number of suspensions we require depends on three parameters: the number of points k, the dimension of M and the connectivity of M. Our proof uses a mixture of embedding theory and fiberwise algebraic topology.

1. INTRODUCTION

For a closed PL manifold M and an integer $k \geq 2$, we will consider the configuration space

$$F(M,k) := \{(x_1,...,x_k) \,|\, x_i \in M \text{ and } x_i \neq x_j \text{ for } i \neq j\}\,.$$

A fundamental unsolved problem about these spaces concerns their *homotopy invariance:* when M and N are homotopy equivalent, is it true that $F(M,k)$ and $F(N,k)$ are homotopy equivalent?

Here is some background. It is known that the based loop space $\Omega F(M,k)$, is a homotopy invariant (see Levitt [L]). When M is smooth, the cohomology of $F(M,k)$ with field coefficients has been intensively studied (see e.g., Bödigheimer-Cohen-Taylor [B-C-T]). When M is a smooth projective variety over $\mathbb{C}$, Kriz [Kr] has shown that the rational homotopy type of $F(M,k)$ depends only on the rational cohomology ring of M. These results indicate that if homotopy invariance fails, a counterexample will be difficult to come by.

When $k = 2$ we have $F(M,2) = M \times M - \Delta$ is the deleted product. Even in this instance, the homotopy invariance question is still not settled (although partial results are known; see Levitt [L]).

The purpose of this paper is to show that a suitable iterated suspension of $F(M,k)$ is a homotopy invariant. The bound on the number of suspensions we need to take depends on three parameters: the number of points, the dimension of M and the connectivity of M.

Date: October 30, 2003.

The second author is partially supported by NSF Grant DMS-0201695.

2000 MSC. Primary 55R80; Secondary 57Q35, 55R70.

For an unbased space Y, we define its j-fold suspension

$$\Sigma^j Y := (* \times S^{j-1}) \cup (Y \times D^j)\,,$$

where the union is amalgamated along $Y \times S^{j-1}$ (up to homotopy, $\Sigma^j Y$ is the *join* of Y and S^{j-1}).

For integers $d, k \geq 3$ and $r \geq 0$, define

$$\alpha(k,d,r) \;:=\; (k-2)d - r + 2\,.$$

When $k = 2$ and $r \geq 3$, or when $d \leq 2$, we set $\alpha(k,d,r) := 0$.

Our main result is

Theorem A. *Let M and N be homotopy equivalent closed* PL *manifolds of dimension d. Assume M is r-connected for some $r \geq 0$. Then there is a homotopy equivalence*

$$\Sigma^{\alpha(k,d,r)} F(M,k) \;\simeq\; \Sigma^{\alpha(k,d,r)} F(N,k)\,,$$

Remark. (1). Cohen and Taylor (unpublished manuscript) prove by very different methods that the configuration spaces of *smooth* manifolds are stable homotopy invariant. In their work the bound on the number suspensions required to achieve homotopy invariance is significantly weaker.

Nevertheless, an advantage of their approach is its applicability to other kinds of configuration spaces. For example, their results apply as well to the *unordered* configuration spaces of a smooth manifold. We are unable to analyse the latter using our methods.

We now single out two corollaries of our main result. Assume in what follows that M is a connected closed PL manifold.

Corollary B. *The suspension spectrum $\Sigma^\infty F(M,k)_+$ is a homotopy invariant of M.*

The second corollary extends the work of Levitt [L], who considered only the case $r = 2$.

Corollary C ($k = 2$)**.** *If M is r-connected for some $0 \leq r \leq 2$, then $\Sigma^{2-r} F(M,2)$ is a homotopy invariant of M.*

Conventions. We work in the category **Top** of compactly generated spaces. A non-empty space is always (-1)-connected. A non-empty space is 0-connected if it is path connected. It is r-connected for $r > 0$ if it is path connected and its homotopy groups (with respect to a choice of basepoint) vanish in degrees $\leq r$. A map $A \to B$ of spaces (with B non-empty) is r-connected if for any choice of basepoint in B, the homotopy fiber with respect to this choice of basepoint is an $(r-1)$-connected space. A weak (homotopy) equivalence is an ∞-connected map. If two spaces A and B are related by a chain of weak equivalences, we will often indicate it by writing $A \simeq B$.

2. Fiberwise suspension

Let $A \to X$ be a map of spaces. Define

$$\mathbf{Top}_{A\to X}$$

to be the category of spaces "between A and X." Specifically, an *object* is a space Y and a choice of factorization $A \to Y \to X$. A *morphism* is a map of spaces which is compatible with their given factorizations. Call a morphism a *weak equivalence* if it is a weak homotopy equivalence of underlying spaces.

We use the notation $\mathbf{Top}_{/X}$ for $\mathbf{Top}_{\emptyset\to X}$. If $Y \in \mathbf{Top}_{/X}$ is an object, define its *(unreduced) j-fold fiberwise suspension* by

$$\Sigma^j_X Y \;:=\; (Y \times D^j) \cup (X \times S^{j-1})\,,$$

where the union is amalgamated over $Y \times S^{j-1}$. With respect to the first factor projection map $X \times S^{j-1} \to X$, we get a functor

$$\Sigma^j_X \colon \mathbf{Top}_{/X} \to \mathbf{Top}_{X\times S^{j-1}\to X}\,.$$

Lemma 2.1. *Let Y and Z be objects of $\mathbf{Top}_{/X}$ whose underlying spaces are path connected and have the homotopy type of CW complexes.*

Assume for some $j \geq 0$ that $\Sigma^j_X Y$ and $\Sigma^j_X Z$ are weak equivalent objects. Then there is a weak equivalence of spaces

$$\Sigma^j Y \;\simeq\; \Sigma^j Z\,.$$

Proof. The statement is obviously true for $j = 0$, so we will assume that $j > 0$. Moreover, we may assume that we are given a weak equivalence $\Sigma^j_X Y \xrightarrow{\sim} \Sigma^j_X Z$.

For any object $T \in \mathbf{Top}_{/X}$, we have a cofibration sequence of spaces

$$X \times S^{j-1} \to \Sigma^j_X T \to \Sigma^j(T_+)\,,$$

where we use the fact that $\Sigma^j(T_+)$ means $T \times D^j$ with $T \times S^{j-1}$ collapsed to a point. Using this cofiber sequence for both Y and Z, we get a commutative diagram

$$\begin{tikzcd} \Sigma^j_X Y \arrow[r] \arrow[d, "\simeq"'] & \Sigma^j(Y_+) \arrow[d] \\ \Sigma^j_X Z \arrow[r] & \Sigma^j(Z_+) \end{tikzcd}$$

which is also homotopy pushout. It is well-known that cobase change preserves weak equivalences (see e.g., Hirschhorn [H]), so it follows that the map $\Sigma^j(Y_+) \to \Sigma^j(Z_+)$ is a weak equivalence.

Choose basepoints for Y and Z. Since $j > 0$, we have $\Sigma^j(Y_+) \simeq (\Sigma^j Y) \vee S^j$ and similarly $\Sigma^j(Z_+) \simeq (\Sigma^j Z) \vee S^j$. It follows that there is a weak equivalence

$$(\Sigma^j Y) \vee S^j \;\simeq\; (\Sigma^j Z) \vee S^j\,.$$

Because Y and Z are connected, we have that $\Sigma^j Y$ and $\Sigma^j Z$ are j-connected. Using Lemma 2.2 below, we conclude that the composite

$$\Sigma^j Y \xrightarrow{\text{include}} (\Sigma^j Y) \vee S^j \simeq (\Sigma^j Z) \vee S^j \xrightarrow{\text{project}} (\Sigma^j Z)$$

is a weak equivalence. □

Lemma 2.2. *Let U and V be j-connected spaces with $j \geq 0$. Assume U and V are equipped with non-degenerate basepoints. Assume $h\colon U \vee S^j \to V \vee S^j$ is a weak equivalence. Then the composition*

$$g\colon U \xrightarrow{\text{include}} U \vee S^j \xrightarrow{h} V \vee S^j \xrightarrow{\text{project}} V$$

is also a weak equivalence.

Proof. Without loss in generality we can assume that U and V are CW complexes with no cells in positive dimensions $\leq j$. By cellular approximation, we may also assume that h is a cellular map. Then h preserves j-skeleta, so there is a commutative diagram

$$\begin{array}{ccc} S^j & \xrightarrow{\subset} & U \vee S^j \\ {\scriptstyle h_{|S^j}}\downarrow & & \downarrow{\scriptstyle h} \\ S^j & \xrightarrow[\subset]{} & V \vee S^j \end{array},$$

and it is straightforward to check that the left vertical map is a homotopy equivalence. We infer that the map $U \to V$ obtained by taking cofibers horizontally is also a weak equivalence. But this map coincides with g. □

3. Embeddings up to homotopy

Let K be a space. Write $\dim K \leq k$ if K is up to homotopy a cell complex of dimension $\leq k$. We say that K is *homotopy finite* if it is homotopy equivalent to a finite cell complex.

Let M be a PL manifold of dimension d, possibly with boundary. Fix a map $f\colon K \to M$, in which K is a homotopy finite space.

Definition 3.1. An *embedding up to homotopy* of f is a pair

$$(N, h)$$

in which

- N denotes a compact codimension zero PL submanifold of the interior of M, and
- $h\colon K \to N$ is a homotopy equivalence such that composition

  $$K \xrightarrow{h} N \subset M$$

  is homotopic to f.

A *concordance* of embeddings up to homotopy (N_0, h_0) and (N_1, h_1) of f consists of

- an embedded PL h-cobordism
$$(W, N_0, N_1) \subset (M \times I, M \times 0, M \times 1)\,,$$
where $\partial W = N_0 \cup \partial_1 W \cup N_1$ and $\partial_1 W$ is the internal part of ∂W, (i.e., $W \cap M \times \{i\} = N_i$ for $i = 0, 1$);
- a homotopy equivalence
$$H\colon (K{\times}I, K{\times}0, K{\times}1) \xrightarrow{\sim} (W, N_0, N_1)$$
which factors the map $f{\times}\mathrm{id}$ up to homotopy.

We remark that our definition of embedding up to homotopy differs from the one of Stallings and Wall in that we do not work with simple homotopy equivalences. Our notion of concordance accounts for this distinction (Stallings and Wall use s-cobordisms in their notion of concordance); our set of concordance classes coincides with theirs when $\dim K \le d - 3$.

Theorem 3.2 (Stallings [St], Wall [Wa1])**.** *Assume* $\dim K \le k \le d{-}3$. *If* $f\colon K \to M$ *is* $(2k{-}d{+}1)$*-connected, then* f *embeds up to homotopy. Furthermore, any two embeddings up to homotopy of* f *are concordant whenever* f *is* $(2k{-}d{+}2)$*-connected.*

4. Decompression

Let (N, h) be an embedding up to homotopy of $f\colon K \to M$. If C denotes the closure of the complement of N inside M, then C is an object of $\mathbf{Top}_{\partial M \subset M}$.

Definition 4.1. The object
$$C \in \mathbf{Top}_{\partial M \subset M}$$
is called the *complement* of (N, h).

By considering the inclusion $M \times 0 \subset M \times D^j$, and taking a compact regular neighborhood of N in $M \times D^j$, we have an associated embedding up to homotopy of the composite
$$f_j\colon K \xrightarrow{f} M = M \times 0 \subset M \times D^j\,.$$
Denote this embedding up to homotopy by (N_j, h_j), where $N_j \cong N \times D^j$ and the homotopy equivalence h_j is identified the composite
$$K \xrightarrow{h} N \subset N \times D^j\,.$$
This new embedding up to homotopy is called the *j-fold decompression* of (N, h). Its complement has the structure of an object of $\mathbf{Top}_{M \times S^{j-1} \subset M \times D^j}$.

However, to avoid technical problems, we will henceforth regard the complement as a space over M by projecting away from the D^j factor. That is, we will think of the complement as an object of $\mathbf{Top}_{M\times S^{j-1}\to M}$.

Lemma 4.2 (Compare [Kl2, §2.3])**.** *Assume that M is closed. Then the complement of (N_j, h_j) is weak equivalent to the object*

$$\Sigma^j_M C\, .$$

Proof. The regular neighborhood N_j can be chosen as $N \times D_{1/2} \subset M \times D^j$, where $D_{1/2} \subset D^j$ is the disk of radius 1/2. The complement of (N_j, h_j) is then

$$(M \times D^j) - \operatorname{int}(N \times D_{1/2}) \;=\; C \times D_{1/2} \;\cup\; M \times D_{[1/2,1]}\, ,$$

where $D_{[1/2,1]}$ denotes the annulus consisting of points in D^j whose norm varies between 1/2 and 1. The above union is amalgamated over $C \times \partial D_{1/2}$.

The subspace of the complement given by $(C \times D_{1/2}) \cup (M \times D_{1/2})$ is evidently isomorphic to $\Sigma^j_M C$. The inclusion map of this subspace is, up to isomorphism, a morphism of $\mathbf{Top}_{M\times S^{j-1}\to M}$. Furthermore, this inclusion is a weak homotopy equivalence of underlying spaces. □

5. The suspended complement

Proposition 5.1. *Assume $f\colon K^k \to M^d$ is an r-connected map, where M is a closed connected PL manifold of dimension d, and $k \le d-3$. Suppose that f has two embeddings up to homotopy (N, h) and (N', h') with respective complements C and C'. Then there is a homotopy equivalence,*

$$\Sigma^j C \;\simeq\; \Sigma^j C'\, ,$$

where $j = \max(2k - d - r + 2, 0)$.

Proof. By the Stallings-Wall theorem, with $j = \max(2k - d - r + 2, 0)$, we see that the j-fold decompressions of (N, h) and (N', h') are concordant. Using Lemma 4.2 we infer that there is a weak equivalence of objects

$$\Sigma^j_M C \;\simeq\; \Sigma^j_M C'\, .$$

By Lemma 2.1, we conclude $\Sigma^j C \simeq \Sigma^j C'$. □

6. Proof of Theorem A

Suppose that M and N are homotopy equivalent r-connected ($r \ge 0$) closed PL manifolds of dimension d.

With appropriate modifications, we will argue along the lines of Levitt's strategy for showing $F(M, 2) \simeq F(N, 2)$ when M and N are 2-connected (see [L]).

Case 1: $d \le 2$. By the classification of low dimensional manifolds, M and N are PL homeomorphic. It follows that $F(M,k)$ and $F(N,k)$ are homeomorphic for all k.

Case 2: $d > 2$. Let

$$\Delta_k^{\mathrm{fat}}(M) \subset M^{\times k}$$

denote the *fat diagonal.* This subpolyhedron is the space of k-tuples of points of M such that at least two entries in the k-tuple coincide.

By choosing a regular neighborhood, we obtain an embedding up to homotopy of the inclusion $\Delta_k^{\mathrm{fat}}(M) \subset M^{\times k}$. Its complement C is weak equivalent to $F(M,k)$ when the latter is considered as an object of $\mathbf{Top}/M^{\times k}$. Denote this embedding up to homotopy by (V,h). Note that that have an associated codimension one manifold splitting given by

$$(M^{\times k}; V, C; \partial V)\,.$$

Repeat this procedure for the fat diagonal of N in $N^{\times k}$ to get an embedding up to homotopy of the inclusion $\Delta_k^{\mathrm{fat}}(N) \subset N^{\times k}$. Call the latter embedding up to homotopy (W,h'). Its complement C' is identified with $F(N,k) \in \mathbf{Top}/N^{\times k}$. Thus we have a codimension one splitting

$$(N^{\times k}; W, D; \partial W)\,.$$

The next step is to choose a homotopy equivalence $g\colon M \xrightarrow{\sim} N$. The k-fold product of g with itself gives another homotopy equivalence $g_k\colon M^{\times k} \xrightarrow{\sim} N^{\times k}$. We can therefore use g_k to form another triad

$$(W \cup C; W, C; \partial W)$$

together with a homotopy equivalence $\psi\colon N^{\times k} \to W \cup C$ (note that we are using C instead of D). The latter triad is codimension one *Poincaré duality splitting.* According to the Browder-Casson-Sullivan-Wall theorem [Wa2, Th. 12.1], such splittings can be made into manifold splittings: there exists codimension one manifold splitting $(N^{\times k}; W', C'; \partial W')$ and a homotopy equivalence of triads

$$\phi\colon (N^{\times k}; W', C'; \partial W') \xrightarrow{\sim} (W \cup C; W, C; \partial W)$$

such that $\phi\colon N^{\times k} \to W \cup C$ is homotopic to ψ. These data describe another embedding up to homotopy of the inclusion $\Delta_k^{\mathrm{fat}}(N) \to N^{\times k}$ with the property that its complement is identified with $F(M,k)$ up to homotopy equivalence.

Summarizing thus far, we have two embeddings up to homotopy of the inclusion $\Delta_k^{\mathrm{fat}}(N) \to N^{\times k}$, one whose complement is identified with $F(N,k)$ and the other whose complement is identified with $F(M,k)$.

The next step of the argument is to verify the hypotheses of Proposition 5.1. One checks by elementary means that $\dim \Delta_k^{\mathrm{fat}}(N) \le (k-1)d$. As $d > 2$, the hypothesis $(k-1)d \le kd - 3$ is satisfied. Furthermore, the inclusion map

$\Delta_k^{\text{fat}}(N) \to N^{\times k}$ is r-connected (recall that r is the connectivity of N). Hence, applying 5.1, we infer

$$\Sigma^j D \simeq \Sigma^j C',$$

where $j = \max(2(k-1)d - kd - r + 2, 0)$. It is then straightforward to check that $j = \alpha(k, d, r)$.

Finally, we recall that $D \simeq F(N, k)$ and $C' \simeq F(M, k)$. With respect to these identifications, we get

$$\Sigma^{\alpha(k,d,r)} F(M, k) \simeq \Sigma^{\alpha(k,d,r)} F(N, k).$$

This concludes the proof of Theorem A.

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Mokhtar Aouina, Dept. of Mathematics, Wayne State University, Detroit, MI 48202
E-mail address: `aouina@math.wayne.edu`

John R. Klein, Dept. of Mathematics, Wayne State University, Detroit, MI 48202
E-mail address: `klein@math.wayne.edu`