

SELF HOMOTOPY EQUIVALENCES WHICH INDUCE THE IDENTITY ON HOMOLOGY, COHOMOLOGY OR HOMOTOPY GROUPS

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ABSTRACT. For a based, 1-connected, finite CW-complex X , we study the following subgroups of the group of homotopy classes of self homotopy equivalences of X : $\mathcal{E}_*(X)$, the subgroup of homotopy classes which induce the identity on homology groups, $\mathcal{E}^*(X)$, the subgroup of homotopy classes which induce the identity on cohomology groups and $\mathcal{E}_\#^{\dim+r}(X)$, the subgroup of homotopy classes which induce the identity on homotopy groups in dimensions $\leq \dim X + r$. We investigate these groups when X is a Moore space and when X is a co-Moore space. We give the structure of the groups in these cases and provide examples of spaces for which the groups differ. We also consider conditions on X such that $\mathcal{E}_*(X) = \mathcal{E}^*(X)$ and obtain a class of spaces (including compact, oriented manifolds and H -spaces) for which this holds. Finally, we examine $\mathcal{E}_\#^{\dim+r}(X)$ for certain spaces X and completely determine the group when $X = S^m \times S^n$ and $X = CP^n \vee S^{2n}$.

§1. INTRODUCTION

If X is a based topological space, let $\mathcal{E}(X)$ denote the set of homotopy classes of self homotopy equivalences of X . Then $\mathcal{E}(X)$ is a group with group operation given by composition of homotopy classes. The group $\mathcal{E}(X)$ and certain natural subgroups are fundamental objects in homotopy theory and have been studied extensively. For a finite CW-complex X , these subgroups include $\mathcal{E}_*(X)$, the subgroup of homotopy classes which induce the identity on the homology groups of X , $\mathcal{E}^*(X)$, the subgroup of homotopy classes which induce the identity on the cohomology groups of X and $\mathcal{E}_\#^{\dim+r}(X)$, the subgroup of homotopy classes which induce the identity on the homotopy groups of X in dimensions $\leq \dim X + r$. For a survey of known results and applications of $\mathcal{E}(X)$, see [Ar], and for a list of references on the subgroups mentioned above, see [A-L, pp. 1-2].

In this paper we investigate the subgroups $\mathcal{E}_*(X)$, $\mathcal{E}^*(X)$ and $\mathcal{E}_\#^{\dim+r}(X)$. We give examples for which the groups are different and present some general results which show when they are the same. In addition, we completely determine these groups in several specific cases.

We now briefly summarize the contents of this paper. Section 2 contains some useful results which will be needed in later sections. We first state several classical theorems

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such as the universal coefficient theorem for homotopy groups with coefficients and the Blakers-Massey theorem. We then consider Moore spaces, i.e., spaces with a single non-vanishing homology group, and give some elementary results about them. In Section 3 we study $\mathcal{E}_*(X)$, $\mathcal{E}^*(X)$ and $\mathcal{E}_\#^{\dim+r}(X)$ when X is a Moore space and determine these groups completely. In Section 4 we examine co-Moore spaces Y , i.e., spaces with a single non-vanishing cohomology group, and calculate the groups $\mathcal{E}_*(Y)$, $\mathcal{E}^*(Y)$ and $\mathcal{E}_\#^{\dim+r}(Y)$ for $r = 0, 1$. The results in Sections 3 and 4 provide us with many examples which show the possibilities for the groups \mathcal{E}_* , \mathcal{E}^* and $\mathcal{E}_\#^{\dim+r}$. In Section 5 we consider spaces X such that $\mathcal{E}_*(X) = \mathcal{E}^*(X)$. We show that compact, oriented manifolds, H -spaces and spaces with certain homological restrictions all have this property. In Section 6 we examine the groups $\mathcal{E}_\#^{\dim+r}(X)$ for specific spaces X in order to illustrate the varied behavior of these groups. In particular, we completely determine $\mathcal{E}_\#^{\dim+r}(S^m \times S^n)$ and $\mathcal{E}_\#^{\dim+r}(CP^n \vee S^{2n})$.

For the remainder of this section we present our notation and conventions. All topological spaces will be based and have the based homotopy type of a finite, 1-connected CW-complex. All maps and homotopies will preserve base points. If $f : X \rightarrow Y$ is a map, then $f_{*n} : H_n(X) \rightarrow H_n(Y)$, $f_{\#n} : \pi_n(X) \rightarrow \pi_n(Y)$ and $f^{*n} : H^n(Y) \rightarrow H^n(X)$ denote respectively the induced homology, homotopy and cohomology homomorphism in dimension n . The subscript or superscript ‘ n ’ will often be omitted. *In this paper we do not distinguish notationally between a map $X \rightarrow Y$ and its homotopy class in $[X, Y]$.*

If G is an abelian group and $n \geq 3$ an integer, then the Moore space $M(G, n)$ is the space, unique up to homotopy type, characterized by

$$\tilde{H}_i(M(G, n)) = \begin{cases} G & i = n \\ 0 & i \neq n. \end{cases}$$

If G is free-abelian, $M(G, n)$ is just a wedge of n -spheres. In general, the construction of $M(G, n)$ is as follows: Let $G = \mathfrak{F}/\mathfrak{R}$, where \mathfrak{F} is free-abelian and $\mathfrak{R} \subseteq \mathfrak{F}$ is a subgroup. The inclusion $\mathfrak{R} \subseteq \mathfrak{F}$ induces a map $j : M(\mathfrak{R}, n) \rightarrow M(\mathfrak{F}, n)$, and $M(G, n)$ is the mapping cone of j . Thus we have the *defining cofibre sequence of $M(G, n)$* :

$$M(\mathfrak{F}, n) \xrightarrow{i} M(G, n) \xrightarrow{q} M(\mathfrak{R}, n+1).$$

Note that when G is finitely-generated, $M(G, n)$ is a finite CW-complex of dimension n if G is free-abelian and of dimension $n+1$ if G is not free-abelian. It is known [Ba, p. 268-269] that $\pi_{n+1}(M(G, n)) = G \otimes \mathbf{Z}_2$ for $n \geq 3$ and that $\pi_{n+2}(M(G, n))$ is an extension of $\text{Tor}(G, \mathbf{Z}_2)$ by $G \otimes \mathbf{Z}_2$ for $n \geq 4$. Since $M(G, n)$ is a double suspension, the set of homotopy classes $[M(G, n), X]$ can be given abelian group structure with binary operation ‘+’. Then $\pi_n(G; X) = [M(G, n), X]$ is called the n th homotopy group of X with coefficients in G . The group of self homotopy equivalences $\mathcal{E}(M(G, n))$ of a Moore space has been studied by Rutter in [Ru].

Finally, if A is an abelian group, we write

$$\bigoplus^r A = A \oplus \cdots \oplus A \quad (r \text{ summands}).$$

We also use ‘ \oplus ’ to denote cartesian product of sets.

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§2 PRELIMINARIES

We begin with some well-known results. The first is the universal coefficient theorem for homotopy with coefficients [Hi, p. 30].

2.1 Theorem. *There is a short exact sequence*

$$0 \rightarrow \text{Ext}(G, \pi_{n+1}(X)) \rightarrow \pi_n(G; X) \xrightarrow{\lambda} \text{Hom}(G, \pi_n(X)) \rightarrow 0,$$

where $\lambda(f) = f_{\#n} : G \approx \pi_n(M(G, n)) \rightarrow \pi_n(X)$.

We obtain the following corollary.

2.2 Corollary. *Let $f, g : M(G, n) \rightarrow M(G', n)$ be maps and let G be free-abelian. Then $f = g \iff f_{*n} = g_{*n}$.*

Next we have the Blakers-Massey theorem [Hi, p. 49].

2.3 Theorem. *Given a cofibration $X \xrightarrow{i} E \xrightarrow{q} Y$ with X $(k-1)$ -connected and Y $(\ell-1)$ -connected, $k, \ell \geq 2$. Then for any abelian group G and all $m < k + \ell - 2$, there is an exact sequence*

$$\pi_m(G; X) \xrightarrow{i_{\#}} \pi_m(G; E) \xrightarrow{q_{\#}} \pi_m(G; Y) \xrightarrow{\partial} \pi_{m-1}(G; X) \rightarrow \cdots$$

If G is free-abelian, then the sequence is exact for $m \leq k + \ell - 2$.

We also will need the following proposition.

2.4 Proposition. *If X is $(k-1)$ -connected and Y is $(\ell-1)$ -connected, $k, \ell \geq 2$, and $\dim P < k + \ell$, then the projections $X \vee Y \rightarrow X$ and $X \vee Y \rightarrow Y$ induce a bijection*

$$[P, X \vee Y] \xrightarrow{\cong} [P, X] \oplus [P, Y].$$

Proposition 2.4 is a consequence of [Sp₂, p. 405] since the inclusion $X \vee Y \rightarrow X \times Y$ is a $(k + \ell - 1)$ -equivalence.

Now let $M(G, n)$ and $M(G', n)$ be Moore spaces with defining cofibre sequences

$$\begin{aligned} M(\mathfrak{F}, n) &\xrightarrow{i} M(G, n) \xrightarrow{q} M(\mathfrak{R}, n+1) \quad \text{and} \\ M(\mathfrak{F}', n) &\xrightarrow{i'} M(G', n) \xrightarrow{q'} M(\mathfrak{R}', n+1), \end{aligned}$$

where \mathfrak{F} and \mathfrak{F}' are free-abelian, $\mathfrak{R} \subseteq \mathfrak{F}$, $\mathfrak{R}' \subseteq \mathfrak{F}'$, $G = \mathfrak{F}/\mathfrak{R}$ and $G' = \mathfrak{F}'/\mathfrak{R}'$.

2.5 Proposition. *Let $f : M(G, n) \rightarrow M(G', n)$. Then $f_{*n} = 0 \iff$ there exists $a : M(\mathfrak{R}, n+1) \rightarrow M(\mathfrak{F}', n)$ such that $f = i' a q$.*

Proof. The implication ' \Leftarrow ' is obvious, so we show ' \Rightarrow '. Since $(fi)_{*} = 0 : H_n(M(\mathfrak{F}, n)) \rightarrow H_n(M(G', n))$, $fi = 0$ by Corollary 2.2. Now consider the Barratt-Puppe exact sequence of the defining cofibration of $M(G, n)$

$$[M(\mathfrak{R}, n+1), M(G', n)] \xrightarrow{q^*} [M(G, n), M(G', n)] \xrightarrow{i^*} [M(\mathfrak{F}, n), M(G', n)].$$

Since $i^*(f) = 0$, there exists $b : M(\mathfrak{R}, n+1) \rightarrow M(G', n)$ such that $f = bq$. Therefore $(q'b)_* = 0$, and so $q'b = 0$ by Corollary 2.2. Then Theorem 2.3 applied to the defining cofibration of $M(G', n)$ yields the exact sequence

$$\pi_{n+1}(\mathfrak{R}; M(\mathfrak{F}', n)) \xrightarrow{i'_{\#}} \pi_{n+1}(\mathfrak{R}; M(G', n)) \xrightarrow{q'_{\#}} \pi_{n+1}(\mathfrak{R}; M(\mathfrak{R}', n+1)).$$

Since $q'_{\#}(b) = 0$, $b = i'_{\#}(a) = i'a$ for some $a \in \pi_{n+1}(\mathfrak{R}; M(\mathfrak{F}', n))$. Hence $f = bq = i'aq$. \square

Next we consider abelian groups G_1 and G_2 and Moore spaces $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$. Let $X = M_1 \vee M_2 = M(G_1, n_1) \vee M(G_2, n_2)$ and denote by $i_j : M_j \rightarrow X$ the inclusions and by $p_j : X \rightarrow M_j$ the projections, $j = 1, 2$. If $f : X \rightarrow X$, then define $f_{jk} : M_k \rightarrow M_j$ by $f_{jk} = p_j f i_k$ for $j, k = 1, 2$.

2.6 Proposition. *The function θ which assigns to each $f \in [X, X]$, the 2×2 matrix*

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [M_k, M_j]$, is a bijection. In addition,

- (1) $\theta(f+g) = \theta(f) + \theta(g)$, so θ is an isomorphism $[X, X] \rightarrow \bigoplus_{j,k=1,2} [M_k, M_j]$.
- (2) $\theta(fg) = \theta(f)\theta(g)$, where fg denotes composition in $[X, X]$ and $\theta(f)\theta(g)$ denotes matrix multiplication.
- (3) Under the identification $H_r(M_1 \vee M_2) = H_r(M_1) \oplus H_r(M_2)$, we have $f_{*r}(x, y) = (f_{11*r}(x) + f_{12*r}(y), f_{21*r}(x) + f_{22*r}(y))$, for $x \in H_r(M_1)$ and $y \in H_r(M_2)$.
- (4) Under the identification $H^r(M_1 \vee M_2) = H^r(M_1) \oplus H^r(M_2)$, we have $f^{*r}(x, y) = (f_{11}^{*r}(x) + f_{21}^{*r}(y), f_{12}^{*r}(x) + f_{22}^{*r}(y))$, for $x \in H^r(M_1)$ and $y \in H^r(M_2)$.
- (5) If $\alpha_r : \pi_r(M_1) \oplus \pi_r(M_2) \rightarrow \pi_r(M_1 \vee M_2)$ and $\beta_r : \pi_r(M_1 \vee M_2) \rightarrow \pi_r(M_1) \oplus \pi_r(M_2)$ are the homomorphisms induced by the inclusions and projections respectively, then $\beta_r f_{\#} \alpha_r(x, y) = (f_{11\#r}(x) + f_{12\#r}(y), f_{21\#r}(x) + f_{22\#r}(y))$ for $x \in \pi_r(M_1)$ and $y \in \pi_r(M_2)$.

Proof. Clearly $[X, X] \approx [M_1, X] \oplus [M_2, X]$. But $[M_j, X] \approx [M_j, M_1] \oplus [M_j, M_2]$ by Proposition 2.4 for $j = 1, 2$. The rest of the proof is straightforward and hence omitted. \square

Next let G be a finitely-generated abelian group and write $G = F \oplus T$, where F is a free-abelian group of finite rank and T is a finite group. Let i_1 and i_2 be the inclusions of F and T into G and let p_1 and p_2 be the projections of G onto F and T . If $\phi : G \rightarrow G$ is a homomorphism, set $\phi_{11} = p_1 \phi i_1 : F \rightarrow F$, $\phi_{21} = p_2 \phi i_1 : F \rightarrow T$, $\phi_{12} = p_1 \phi i_2 : T \rightarrow F$ and $\phi_{22} = p_2 \phi i_2 : T \rightarrow T$. Then for $x \in F$ and $y \in T$,

$$\phi(x, y) = (\phi_{11}(x), \phi_{21}(x) + \phi_{22}(y)),$$

since $\phi_{12} \in \text{Hom}(T, F) = 0$. The proof of the next proposition is clear.

2.7 Proposition. *With the above notation, $\phi : G \rightarrow G$ is an isomorphism $\iff \phi_{11}$ and ϕ_{22} are isomorphisms. Furthermore, $\phi = 1 \iff \phi_{11} = 1, \phi_{22} = 1$ and $\phi_{21} = 0$.*

§3 MOORE SPACES

In this section and the next we study the self homotopy equivalences of Moore and co-Moore spaces by means of 2×2 matrices. This method was used earlier by Sieradski to study the self homotopy equivalences of a cartesian product [Si]. Let G be a finitely-generated abelian group and write $G = F \oplus T$, where F is a free-abelian group of rank r and T is a finite abelian group. We consider the Moore space $X = M(G, n) = M(F, n) \vee M(T, n)$ for $n \geq 3$. We set $M_1 = M(F, n)$ which is a wedge of r n -spheres, and $M_2 = M(T, n)$ which is a wedge of spaces of the form $M(\mathbf{Z}_m, n) = S^n \cup_m e^{n+1}$. Throughout this section X will denote the Moore space $M(G, n) = M_1 \vee M_2$.

We now let $f \in [X, X]$ and use the notation of §2 so that $f_{jk} = p_j f i_k \in [M_k, M_j]$, for $j, k = 1, 2$. Then $f \in \mathcal{E}(X) \Leftrightarrow f_{*n}$ is an isomorphism. By Propositions 2.6 and 2.7, we can identify $f \in \mathcal{E}(X)$ with the 2×2 matrix

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

where $f_{11} \in \mathcal{E}(M_1)$, $f_{12} \in [M_2, M_1]$, $f_{21} \in [M_1, M_2]$ and $f_{22} \in \mathcal{E}(M_2)$. The group structure in $\mathcal{E}(X)$ is then given by matrix multiplication.

Now we investigate the subgroup $\mathcal{E}^*(X) = \{f \in \mathcal{E}(X) \mid f^{*i} = 1, \text{ for all } i\}$ of $\mathcal{E}(X)$.

3.1 Theorem. *Let $f \in \mathcal{E}(X)$ be given as*

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

with $f_{11} \in \mathcal{E}(M_1)$ and $f_{22} \in \mathcal{E}(M_2)$. Then $f \in \mathcal{E}^*(X) \iff f_{11} = 1$ and $f_{22} \in \mathcal{E}_*(M_2)$. If $f, g \in \mathcal{E}^*(X)$, then

$$fg = \begin{pmatrix} 1 & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} 1 & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & f_{12} + g_{12} \\ f_{21} + g_{21} & f_{21}g_{12} + f_{22}g_{22} \end{pmatrix}.$$

Proof. The cohomology of X occurs only in dimensions n and $n + 1$ and

$$\begin{aligned} H^n(X) &= H^n(M_1) \oplus H^n(M_2) = H^n(M_1) = F \quad \text{and} \\ H^{n+1}(X) &= H^{n+1}(M_1) \oplus H^{n+1}(M_2) = H^{n+1}(M_2) = T. \end{aligned}$$

By Proposition 2.6, $f^*(x, y) = (f_{11}^*(x), f_{22}^*(y))$ for $x \in H^k(M_1)$ and $y \in H^k(M_2)$. Thus $f^{*k} = 1$ for $k = n, n + 1 \Leftrightarrow f_{11}^{*n} = 1$ and $f_{22}^{*n+1} = 1$. But $\mathcal{E}^*(M_1) = 1$ by Corollary 2.2, so $f_{11}^{*n} = 1 \Leftrightarrow f_{11} = 1$. Also $f_{22}^{*n+1} = 1 \Leftrightarrow f_{22}^{*n} = 1$ by the universal coefficient theorem for cohomology. This proves the first assertion.

To establish the formula for the product fg it suffices to show (1) $f_{21} + f_{22}g_{21} = f_{21} + g_{21}$, i.e., $f_{22}g_{21} = g_{21}$, and (2) $g_{12} + f_{12}g_{22} = f_{12} + g_{12}$, i.e., $f_{12}g_{22} = f_{12}$.

For (1), note that $(f_{22}g_{21})_* = g_{21*}$ since $f_{22*} = 1$. But this implies $f_{22}g_{21} = g_{21}$ by Corollary 2.2.

For (2), let

$$M(\mathfrak{F}', n) \xrightarrow{i'} M_2 \xrightarrow{q'} M(\mathfrak{X}', n+1)$$

be the defining cofibre sequence for $M_2 = M(T, n)$, where $T = \mathfrak{F}'/\mathfrak{X}'$ and \mathfrak{F}' free-abelian. Since $(g_{22} - 1)_* = 0 : T \rightarrow T$, by Proposition 2.5 there exists $a : M(\mathfrak{X}', n+1) \rightarrow M(\mathfrak{F}', n)$ such that $g_{22} = 1 + i' a q'$. Then

$$\begin{aligned} f_{12} g_{22} &= f_{12} (1 + i' a q') \\ &= f_{12} + f_{12} i' a q' \\ &= f_{12} \end{aligned}$$

since $(f_{12} i')_* = 0$ and so $f_{12} i' = 0$. \square

Next we consider $\mathcal{E}_*(X) = \{f \in \mathcal{E}(X) \mid f_{*n} = 1\}$. Let $T = T' \oplus P$, where T' is the 2-primary torsion subgroup of G and P is the sum of p -primary torsion subgroups of G for all primes $p \neq 2$. Thus $T' = \mathbf{Z}_{2^{a_1}} \oplus \cdots \oplus \mathbf{Z}_{2^{a_s}}$, where $a_\beta \geq 1$. We write $M(T', n)$ as $M_{T'}$, $M(P, n)$ as M_P and $M(\mathbf{Z}_{2^{a_\beta}}, n)$ as N_β , for $\beta = 1, \dots, s$. Then

$$M_2 = M(T, n) = M(T', n) \vee M(P, n) = M_{T'} \vee M_P \quad \text{and}$$

$$M_{T'} = M(\mathbf{Z}_{2^{a_1}}, n) \vee \cdots \vee M(\mathbf{Z}_{2^{a_s}}, n) = N_1 \vee \cdots \vee N_s.$$

Let $j_\alpha : S^n \rightarrow M_1$ be the inclusion of S^n onto the α th sphere of M_1 and let $p_\alpha : M_1 \rightarrow S^n$ be the projection onto the α th sphere, $\alpha = 1, \dots, r$. Similarly let $k_\beta : N_\beta \rightarrow M_2$ and $r_\beta : M_2 \rightarrow N_\beta$ be the β th inclusion and projection, respectively, for $\beta = 1, \dots, s$. Let $S^n \xrightarrow{i_\beta} N_\beta \xrightarrow{q_\beta} S^{n+1}$ be the defining cofibration of N_β and $\eta_n \in \pi_{n+1}(S^n)$ the non-trivial class. Then we define $u_{\alpha\beta} : M_2 \rightarrow M_1$ to be the composition

$$M_2 \xrightarrow{r_\beta} N_\beta \xrightarrow{q_\beta} S^{n+1} \xrightarrow{\eta_n} S^n \xrightarrow{j_\alpha} M_1,$$

$\alpha = 1, \dots, r$ and $\beta = 1, \dots, s$ and we define $v_{\gamma\delta} : M_2 \rightarrow M_2$ to be the composition

$$M_2 \xrightarrow{r_\delta} N_\delta \xrightarrow{q_\delta} S^{n+1} \xrightarrow{\eta_n} S^n \xrightarrow{i_\gamma} N_\gamma \xrightarrow{k_\gamma} M_2,$$

$\gamma, \delta = 1, \dots, s$.

3.2 Theorem. *Let $f \in \mathcal{E}(X)$ be given by $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ with $f_{11} \in \mathcal{E}(M_1)$ and $f_{22} \in \mathcal{E}(M_2)$. Then $f \in \mathcal{E}_*(X) \iff f_{11} = 1, f_{21} = 0$ and $f_{22} \in \mathcal{E}_*(M_2)$. If $f, g \in \mathcal{E}_*(X)$, then*

$$fg = \begin{pmatrix} 1 & f_{12} \\ 0 & f_{22} \end{pmatrix} \begin{pmatrix} 1 & g_{12} \\ 0 & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & f_{12} + g_{12} \\ 0 & f_{22} g_{22} \end{pmatrix}.$$

Furthermore,

$$\mathcal{E}_*(X) \approx \bigoplus^{(r+s)s} \mathbf{Z}_2,$$

where r the rank of G and s is the number of 2-torsion summands in G . In addition, generators of $\mathcal{E}_*(X)$ consist of all

$$\tilde{u}_{\alpha\beta} = \begin{pmatrix} 1 & u_{\alpha\beta} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{v}_{\gamma\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + v_{\gamma\delta} \end{pmatrix},$$

where $\alpha = 1, \dots, r$ and $\beta, \gamma, \delta = 1, \dots, s$.

Proof. We have $f \in \mathcal{E}_*(X) \Leftrightarrow f_{*n} = 1$. By Proposition 2.7, this is equivalent to $f_{11} \in \mathcal{E}_*(M_1) = 1$, $f_{22} \in \mathcal{E}_*(M_2)$ and $f_{21*n} = 0 : F \rightarrow T$. But by Corollary 2.2 the latter implies $f_{21} = 0 : M_1 \rightarrow M_2$. This proves the first assertion. Thus $\mathcal{E}_*(X) \subseteq \mathcal{E}^*(X)$ (which is easily proven directly) and so the formula for fg follows from Theorem 3.1.

Next $\mathcal{E}_*(X) \approx [M_2, M_1] \oplus \mathcal{E}_*(M_2)$. But

$$\begin{aligned} [M_2, M_1] &\approx \pi_n(T; M_1) \\ &\approx \text{Ext}(T, \pi_{n+1}(M_1)) \\ &\approx \text{Ext}(T, F \otimes \mathbf{Z}_2) \\ &\approx \text{Ext}(\mathbf{Z}_{2^{a_1}} \oplus \cdots \oplus \mathbf{Z}_{2^{a_s}}, \bigoplus^r \mathbf{Z}_2) \\ &\approx \bigoplus^{rs} \mathbf{Z}_2. \end{aligned}$$

Now consider $\mathcal{E}_*(M_2)$. Let $\mathcal{Z}_*(M_2) \subseteq [M_2, M_2]$ consists of all homotopy classes which are zero on homology. Then by Theorem 2.1 we have the short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Z}_*(M_2) & \longrightarrow & [M_2, M_2] & \xrightarrow{H} & \text{Hom}(T, T) & \longrightarrow & 0 \\ & & & & \parallel & & \downarrow \approx & & \\ 0 & \longrightarrow & \text{Ext}(T, \pi_{n+1}(M_2)) & \longrightarrow & \pi_n(T; M_2) & \xrightarrow{\lambda} & \text{Hom}(T, \pi_n(M_2)) & \longrightarrow & 0, \end{array}$$

where $H(h) = h_{*n}$. Thus $\mathcal{Z}_*(M_2) \approx \text{Ext}(T, \pi_{n+1}(M_2)) \approx \text{Ext}(T, T \otimes \mathbf{Z}_2) \approx \bigoplus^{s^2} \mathbf{Z}_2$.

We next see that $\mathcal{E}_*(M_2) \approx \mathcal{Z}_*(M_2)$. There is a bijection $\rho : \mathcal{Z}_*(M_2) \rightarrow \mathcal{E}_*(M_2)$ defined by $\rho(g) = g + 1$, for $g \in \mathcal{Z}_*(M_2)$. We show that ρ is a homomorphism : $\rho(g+h) = g+h+1$ and $\rho(g)\rho(h) = (g+1)(h+1) = gh + g + h + 1$. Thus it suffices to show $gh = 0$ for $g, h \in \mathcal{Z}_*(M_2)$. Let $M(\mathfrak{F}', n) \xrightarrow{i'} M_2 \xrightarrow{q'} M(\mathfrak{R}', n+1)$ be the defining cofibre sequence of M_2 . By Proposition 2.5, there exist $a, b : M(\mathfrak{R}', n+1) \rightarrow M(\mathfrak{F}', n)$ such that

$$g = i' a q' \quad \text{and} \quad h = i' b q'.$$

Thus $gh = i' a q' i' b q' = 0$ since $q' i' = 0$. This completes the proof that $\mathcal{E}_*(X) \approx \bigoplus^{(r+s)s} \mathbf{Z}_2$.

Finally, it is straightforward to verify that the elements described in Theorem 3.2 are generators. \square

3.3 Remark. Theorem 3.2 shows that $\mathcal{E}_*(X)$ is always abelian and is the trivial group if and only if G has no 2-torsion. We also note that the decomposition of $\mathcal{E}_*(X)$ into a direct sum of \mathbf{Z}_2 's could have been obtained directly using the isomorphism $\mathcal{E}_*(X) \approx \mathcal{Z}_*(X)$.

We next compare $\mathcal{E}_*(X)$ and $\mathcal{E}^*(X)$. Let $\iota : \mathcal{E}_*(X) \rightarrow \mathcal{E}^*(X)$ be the inclusion and define $\rho : \mathcal{E}^*(X) \rightarrow \text{Hom}(F, T)$ by $\rho \left(\begin{smallmatrix} 1 & f_{12} \\ f_{21} & f_{22} \end{smallmatrix} \right) = (f_{21})_{*n} : F \rightarrow T$.

3.4 Proposition. *The following is a split short exact sequence*

$$0 \rightarrow \mathcal{E}_*(X) \xrightarrow{\iota} \mathcal{E}^*(X) \xrightarrow{\rho} \text{Hom}(F, T) \rightarrow 0,$$

and so $\mathcal{E}^*(X)$ is a semi-direct product of $\mathcal{E}_*(X)$ by $\text{Hom}(F, T)$.

Proof. By Theorem 3.1, ρ is a homomorphism. Furthermore, $\text{Kernel } \rho = \text{Image } \iota$. We define a homomorphism $\sigma : \text{Hom}(F, T) \rightarrow \mathcal{E}^*(X)$ such that $\rho\sigma = 1$. Let $a : F \rightarrow T$ be a homomorphism and $\hat{a} : M_1 \rightarrow M_2$ be the corresponding homotopy class. Then we set

$$\sigma(a) = \left(\begin{smallmatrix} 1 & 0 \\ \hat{a} & 1 \end{smallmatrix} \right). \quad \square$$

3.5 Remarks. (1) The order of the group $\mathcal{E}^*(X)$ is given by $|\mathcal{E}^*(X)| = |T|^r \times 2^{(r+s)s}$.

(2) Proposition 3.4 gives examples of Moore spaces $X = M(G, n)$ such that $\mathcal{E}_*(X) \neq \mathcal{E}^*(X)$. Simply take any $G = F \oplus T$ with $F \neq 0$ and $T \neq 0$.

(3) As a consequence we have the following partial realization result for \mathcal{E}^* : If A is any finite abelian group without 2-torsion, there is a space X such that $\mathcal{E}^*(X) \approx A$. Just set $X = M(\mathbf{Z} \oplus A, n)$.

We next determine the extension in Proposition 3.4. We use the notation of Theorem 3.2 and Proposition 3.4. For $\alpha = 1, \dots, r$ and $\beta = 1, \dots, s$, we define $w_{\beta\alpha} : M_1 \rightarrow M_2$ as the following composition

$$M_1 \xrightarrow{p_\alpha} S^n \xrightarrow{i_\beta} N_\beta \xrightarrow{k_\beta} M_2.$$

If $f_{\beta\alpha} \in \text{Hom}(F, T)$ is $w_{\beta\alpha *n}$, then $\sigma(f_{\beta\alpha}) = \left(\begin{smallmatrix} 1 & 0 \\ w_{\beta\alpha} & 1 \end{smallmatrix} \right) \in \mathcal{E}^*(X)$. Now let $H \subseteq$

$\mathcal{E}^*(X)$ be the subgroup generated by the generators $\tilde{u}_{\alpha'\beta'} = \left(\begin{smallmatrix} 1 & u_{\alpha'\beta'} \\ 0 & 1 \end{smallmatrix} \right)$ and $\tilde{v}_{\gamma\delta} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 + v_{\gamma\delta} \end{smallmatrix} \right)$ of $\mathcal{E}_*(X)$ and $\tilde{w}_{\beta\alpha} = \left(\begin{smallmatrix} 1 & 0 \\ w_{\beta\alpha} & 1 \end{smallmatrix} \right)$, where $\alpha, \alpha' = 1, \dots, r$ and $\beta, \beta', \gamma, \delta = 1, \dots, s$. Then the action of $\text{Hom}(F, T)$ on $\mathcal{E}_*(X)$ in Proposition 3.4 is given by

$$(1) \quad \tilde{w}_{\beta\alpha}^{-1} \tilde{u}_{\alpha'\beta'} \tilde{w}_{\beta\alpha} = \left(\begin{smallmatrix} 1 & u_{\alpha'\beta'} \\ 0 & 1 + w_{\beta\alpha} u_{\alpha'\beta'} \end{smallmatrix} \right) \quad \text{and}$$

$$(2) \quad \tilde{w}_{\beta\alpha}^{-1} \tilde{v}_{\gamma\delta} \tilde{w}_{\beta\alpha} = \tilde{v}_{\gamma\delta}.$$

In (1) if $\alpha \neq \alpha'$, $w_{\beta\alpha} u_{\alpha'\beta'} = 0$; if $\alpha = \alpha'$, $w_{\beta\alpha} u_{\alpha'\beta'} = v_{\beta\beta'}$. Furthermore, $\text{Hom}(F, P)$ clearly acts trivially on $\mathcal{E}_*(X)$.

Recall that a group has nilpotency ≤ 2 if all commutators of length ≥ 3 are trivial.

3.6 Theorem. *The group $\mathcal{E}^*(X)$ is isomorphic to $H \oplus \bigoplus^r P$. Moreover, H is the semi-direct product of $\mathcal{E}_*(X)$ with $\text{Hom}(F, T')$ with action of the latter group on the former group given on generators by (1) and (2). In addition, $\text{nil } \mathcal{E}^*(X) \leq 2$, and $\mathcal{E}^*(X)$ is abelian if and only if $r = 0$ or $s = 0$.*

Proof. Only the last assertion requires proof. It easily follows from (1) and (2) that the commutator

$$[\tilde{u}_{\alpha'\beta'}, \tilde{w}_{\beta\alpha}] = \tilde{u}_{\alpha'\beta'}^{-1} \tilde{w}_{\beta\alpha}^{-1} \tilde{u}_{\alpha'\beta'} \tilde{w}_{\beta\alpha} = \begin{cases} 1 & \text{if } \alpha \neq \alpha' \\ \tilde{v}_{\beta\beta'} & \text{if } \alpha = \alpha', \end{cases}$$

and commutators of all other generators are trivial. The theorem now follows. \square

We next give a simple illustration of Theorem 3.6 in the following corollary.

3.7 Corollary. *$\mathcal{E}^*(M(\mathbf{Z} \oplus \mathbf{Z}_2, n)) \approx D_4$, the dihedral group with 8 elements.*

Proof. Let $A = \mathcal{E}^*(M(\mathbf{Z} \oplus \mathbf{Z}_2, n))$. By Remark 3.5(1), A has 8 elements which are represented by matrices

$$\begin{pmatrix} 1 & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{21} \in \pi_n(M(\mathbf{Z}_2, n))$, $f_{12} \in \pi_n(\mathbf{Z}_2; S^n)$ and $f_{22} \in \mathcal{E}_*(M(\mathbf{Z}_2, n))$. Consider the defining cofibre sequence $S^n \xrightarrow{i'} M(\mathbf{Z}_2, n) \xrightarrow{q'} S^{n+1}$ of $M(\mathbf{Z}_2, n)$ and define $x, y \in A$ by

$$x = \begin{pmatrix} 1 & \eta_n q' \\ i' & 1 + i' \eta_n q' \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 \\ i' & 1 \end{pmatrix}.$$

(In the notation of Theorem 3.6, $x = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$.) Clearly x has order 4, y has order 2 and $xyx = x^{-1}$. Therefore $A \approx D_4$. \square

Now we turn to the group $\mathcal{E}_{\#}^{\dim+r}(X) = \{f \in \mathcal{E}(X) \mid f_{\#i} = 1, \text{ for } i \leq \dim X + r\}$, where $r \geq 0$.

3.8 Theorem. *For the Moore space $X = M(G, n)$,*

- (1) $\mathcal{E}_{\#}^{\dim}(X) = \mathcal{E}_*(X)$ and
- (2) $\mathcal{E}_{\#}^{\dim+1}(X) = 1$, if $n > 3$.

Proof. If G is torsion-free, $\dim X = n$ and the theorem follows easily. Now assume $T \neq 0$ so that $\dim X = n + 1$.

(1) If $f : X \rightarrow X$, then $f_{\#n} = 1 \Leftrightarrow f_{*n} = 1$ by the Hurewicz theorem. Therefore it suffices to show $f \in \mathcal{E}_*(X) \Rightarrow f_{\#n+1} = 1$. Consider the defining cofibre sequence $M(\mathfrak{F}, n) \xrightarrow{i} X \xrightarrow{q} M(\mathfrak{R}, n + 1)$ of X . By Proposition 2.5, $f = 1 + iaq$ for some $a : M(\mathfrak{R}, n + 1) \rightarrow M(\mathfrak{F}, n)$. But $f_{\#n+1} = 1 + i_{\#n+1} a_{\#n+1} q_{\#n+1}$. However,

$$G \otimes \mathbf{Z}_2 = \pi_{n+1}(X) \xrightarrow{q_{\#n+1}} \pi_{n+1}(M(\mathfrak{R}, n + 1)) = \mathfrak{R},$$

and so $q_{\#n+1} = 0$.

(2) We have $\mathcal{E}_{\#}^{\dim+1}(X) \subseteq \mathcal{E}_*(X)$. First note that if G has no 2-torsion, then $\mathcal{E}_{\#}^{\dim+1}(X) = 1$ since $\mathcal{E}_*(X) = 1$. Now suppose that G has s summands of 2-torsion, $s \geq 1$. We write $T = T' \oplus P$, where T' is the 2-torsion subgroup and P is the sum of all other p -torsion subgroups. Then $T' = \mathbf{Z}_{2^{a_1}} \oplus \cdots \oplus \mathbf{Z}_{2^{a_s}}$. If $f \in \mathcal{E}_{\#}^{\dim+1}(X)$, then

$$f = \begin{pmatrix} 1 & f_{12} \\ 0 & f_{22} \end{pmatrix},$$

where $f_{22} \in \mathcal{E}_*(M_2)$. Also $\pi_{n+2}(M_1 \vee M_2) \approx \pi_{n+2}(M_1) \oplus \pi_{n+2}(M_2)$ by Proposition 2.4. Hence by Proposition 2.6,

$$(x, y) = (x + f_{12\#}(y), f_{22\#}(y)),$$

for every $x \in \pi_{n+2}(M_1)$ and $y \in \pi_{n+2}(M_2)$. Thus it suffices to prove

- (i) $(f_{12})_{\#n+2} = 0 \Rightarrow f_{12} = 0$ and
- (ii) $(f_{22})_{\#n+2} = 1$ and $f_{22} \in \mathcal{E}_*(M_2) \Rightarrow f_{22} = 1$.

For (i), let $M_{T'} = M(T', n)$ and $M_P = M(P, n)$ and so $M_2 = M_{T'} \vee M_P$. Then $f_{12}|_{M_P} \in [M_P, M_1] = \pi_n(P; M_1) = \text{Ext}(P, F \otimes \mathbf{Z}_2) = 0$, and so $f_{12}|_{M_P} = 0$. Now let $M_{T'} = N_1 \vee \cdots \vee N_s$, where $N_\alpha = M(\mathbf{Z}_{2^{a_\alpha}}, n)$ and let $k_\alpha : N_\alpha \rightarrow M_2$ be the inclusion. It suffices to show $f_{12}k_\alpha = 0$ for $\alpha = 1, \dots, s$. Let $M(\mathfrak{F}', n) \xrightarrow{i'} M_2 \xrightarrow{q'} M(\mathfrak{X}', n+1)$ be the defining cofibre sequence of $M_2 = M(T, n)$. By Proposition 2.5, there exists $a : M(\mathfrak{X}', n+1) \rightarrow M_1$ such that $f_{12} = aq'$. Let $j_\alpha : S^{n+1} \rightarrow M(\mathfrak{X}', n+1)$ be the inclusion onto the α th sphere, $\alpha = 1, \dots, s$, and let $a_\alpha = aj_\alpha : S^{n+1} \rightarrow M_1$. Then $a_\alpha = (\epsilon_1\eta_n, \dots, \epsilon_r\eta_n)$, where $\epsilon_i = 0$ or 1. Since the diagram

$$\begin{array}{ccc} N_\alpha & \xrightarrow{k_\alpha} & M_2 \\ \downarrow q_\alpha & & \downarrow q' \searrow f_{12} \\ S^{n+1} & \xrightarrow{j_\alpha} & M(\mathfrak{X}', n+1) \xrightarrow{a} M_1 \end{array}$$

is commutative, it suffices to show that $a_{\alpha\#n+2}q_{\alpha\#n+2} = 0$ implies $a_\alpha = 0$, for $\alpha = 1, \dots, s$. Since $n > 3$, the following sequence is exact by Theorem 2.3

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{n+2}(S^n) & \xrightarrow{i_{\alpha\#}} & \pi_{n+2}(N_\alpha) & \xrightarrow{q_{\alpha\#}} & \pi_{n+2}(S^{n+1}) \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \mathbf{Z}_2 & & & & \mathbf{Z}_2 \end{array}$$

Choose $x \in \pi_{n+2}(N_\alpha)$ such that $q_{\alpha\#}(x) = \eta_{n+1}$. Then

$$\begin{aligned} 0 &= a_{\alpha\#}q_{\alpha\#}(x) \\ &= a_{\alpha\#}(\eta_{n+1}) \\ &= (\epsilon_1\eta_n\eta_{n+1}, \dots, \epsilon_r\eta_n\eta_{n+1}). \end{aligned}$$

Since $\eta_n\eta_{n+1} \in \pi_{n+2}(S^n)$ in a generator of \mathbf{Z}_2 , $\epsilon_i = 0$ for all i . Thus $a_\alpha = 0$.

The proof of (ii) is similar and hence omitted. \square

§4 CO-MOORE SPACES

Let G be a finitely-generated abelian group and write $G = F \oplus T$, where F is free-abelian of rank r and T is a finite group. Let $n \geq 3$ and denote by $C(G, n)$ the co-Moore space of type (G, n) defined by

$$\tilde{H}^i(C(G, n)) = \begin{cases} G & i = n \\ 0 & i \neq n. \end{cases}$$

We note that $C(G, n) = M(F, n) \vee M(T, n-1)$. We adopt the following notation in this section: Y denotes $C(G, n)$ with $n \geq 4$, M_1 denotes $M(F, n)$ and M'_2 denotes $M(T, n-1)$. The prime in M'_2 is to distinguish M'_2 from $M_2 = M(T, n)$ of §3.

Given $f \in [Y, Y] = [M_1 \vee M'_2, M_1 \vee M'_2]$ we obtain as in §3, $f_{jk} = p_j f i_k$, for $j, k = 1, 2$, where $f_{11} \in [M_1, M_1]$, $f_{21} \in [M_1, M'_2]$, $f_{12} \in [M'_2, M_1]$ and $f_{22} \in [M'_2, M'_2]$. By Proposition 2.6 the identification of f with the 2×2 matrix

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

is a bijection compatible with multiplication (i.e., composition and matrix multiplication).

In this section we investigate $\mathcal{E}^*(Y)$, $\mathcal{E}_*(Y)$ and $\mathcal{E}_*^{\dim+t}(Y)$. The discussion of $\mathcal{E}^*(Y)$ and $\mathcal{E}_*(Y)$ is completely analogous to that of $\mathcal{E}_*(X)$ and $\mathcal{E}^*(X)$ in §3. Therefore we state most of the results without proof. However, see Remark 4.9 for another approach. Clearly $f \in \mathcal{E}(Y)$ corresponds to the 2×2 matrix

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

where $f_{11} \in \mathcal{E}(M_1)$, $f_{21} \in [M_1, M'_2]$, $f_{12} \in [M'_2, M_1]$ and $f_{22} \in \mathcal{E}(M'_2)$.

4.1 Theorem. $f \in \mathcal{E}_*(Y) \iff f_{11} = 1$ and $f_{22} \in \mathcal{E}_*(M'_2) = \mathcal{E}^*(M'_2)$. Furthermore, if $f, g \in \mathcal{E}_*(Y)$,

$$fg = \begin{pmatrix} 1 & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} 1 & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & f_{12} + g_{12} \\ f_{21} + g_{21} & f_{21}g_{12} + f_{22}g_{22} \end{pmatrix}.$$

4.2 Theorem. $f \in \mathcal{E}^*(Y) \iff f_{11} = 1, f_{12} = 0$ and $f_{22} \in \mathcal{E}_*(M'_2)$. Furthermore, if $f, g \in \mathcal{E}^*(Y)$,

$$fg = \begin{pmatrix} 1 & 0 \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f_{21} + g_{21} & f_{22}g_{22} \end{pmatrix}.$$

In addition,

$$\mathcal{E}^*(Y) \approx \bigoplus^{(r+s)s} \mathbf{Z}_2,$$

where r is the rank of G and s is the number of summands of the form \mathbf{Z}_{2^a} in G .

Proof. We briefly comment on the last assertion. We have $\mathcal{E}^*(Y) \approx [M_1, M'_2] \oplus \mathcal{E}_*(M'_2)$. By Theorem 3.2, $\mathcal{E}_*(M'_2) \approx \bigoplus^{rs} \mathbf{Z}_2$. Now $[M_1, M'_2] = \pi_n(F; M'_2) = \text{Hom}(F, \pi_n(M'_2)) = \text{Hom}(F, T \otimes \mathbf{Z}_2) = \bigoplus^{rs} \mathbf{Z}_2$. The result follows. \square

In the discussion preceding Theorem 4.8 we indicate the generators of $\mathcal{E}^*(Y)$ in analogy to Theorem 3.2. Also we discuss $\mathcal{E}_*(C(G, 3))$ and $\mathcal{E}^*(C(G, 3))$ in Remark 4.10.

Next we consider $\mathcal{E}_{\#}^{\dim}(Y)$. Note that $Y = M_1 \vee M'_2$ has dimension n and so $\mathcal{E}_{\#}^{\dim+t}(Y)$ is the set of all $f \in \mathcal{E}(Y)$ such that $f_{\#i} = 1$ for $i \leq n + t$.

4.3 Lemma. $\mathcal{E}_{\#}^{\dim}(Y) \subseteq \mathcal{E}_*(Y)$.

Proof. Let $f : Y \rightarrow Y$ with $f_{\#n-1} = 1$ and $f_{\#n} = 1$. We have the commutative diagram

$$\begin{array}{ccc} \pi_i(Y) & \xrightarrow{f_{\#i}} & \pi_i(Y) \\ \downarrow h_i & & \downarrow h_i \\ H_i(Y) & \xrightarrow{f_{*i}} & H_i(Y), \end{array}$$

where h_i is the Hurewicz homomorphism. If $i = n - 1$, h_i is an isomorphism, and so $f_{*n-1} = 1$. If $i = n$, h_i is an epimorphism and so $f_{*n} = 1$. \square

4.4 Proposition. Consider the element

$$f = \begin{pmatrix} 1 & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathcal{E}_*(Y),$$

where $f_{22} \in \mathcal{E}_*(M'_2)$. Then $f \in \mathcal{E}_{\#}^{\dim}(Y) \Leftrightarrow f_{21} = 0$. For $f, g \in \mathcal{E}_{\#}^{\dim}(Y)$, the product is given by

$$fg = \begin{pmatrix} 1 & f_{12} \\ 0 & f_{22} \end{pmatrix} \begin{pmatrix} 1 & g_{12} \\ 0 & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & f_{12} + g_{12} \\ 0 & f_{22}g_{22} \end{pmatrix}.$$

Proof. If $f \in \mathcal{E}_*(Y)$, then

$$f_{\#n}(x, y) = (x + f_{12\#}(y), f_{21\#}(x) + f_{22\#}(y)),$$

for $x \in \pi_n(M_1)$ and $y \in \pi_n(M'_2)$ by Proposition 2.6. But $f_{12\#} : \pi_n(M'_2) = T \otimes \mathbf{Z}_2 \rightarrow \pi_n(M_1) = F$ is zero. Moreover, $f_{22} \in \mathcal{E}_*(M'_2) = \mathcal{E}_{\#}^{\dim}(M'_2)$ by Theorem 3.8. Therefore $f_{22\#} = 1$ and so

$$f_{\#n}(x, y) = (x, f_{21\#}(x) + y).$$

Hence $f_{\#n} = 1 \Leftrightarrow (f_{21})_{\#n} = 0$. Now we show $(f_{21})_{\#n} = 0 \Leftrightarrow f_{21} = 0$. Consider the homomorphism of the universal coefficient theorem (2.1)

$$[M_1, M'_2] = \pi_n(F; M'_2) \xrightarrow{\lambda} \text{Hom}(\pi_n(M_1), \pi_n(M'_2)) = \text{Hom}(F, \pi_n(M'_2)),$$

where $\lambda(g) = g_{\#n}$. Since $\text{Ext}(F, \pi_{n+1}(M'_2)) = 0$, λ is an isomorphism. Thus $f_{21\#n} = 0 \Leftrightarrow f_{21} = 0$. This completes the proof. \square

4.5 Corollary. $\mathcal{E}_{\#}^{\dim}(Y) \approx \left(\bigoplus^r T \right) \oplus \left(\bigoplus^{s^2} \mathbf{Z}_2 \right)$.

Proof. By Proposition 4.4, $\mathcal{E}_{\#}^{\dim}(Y) \approx [M'_2, M_1] \oplus \mathcal{E}_*(M'_2)$. However $\mathcal{E}_*(M'_2) \approx \bigoplus^{s^2} \mathbf{Z}_2$ by Theorem 3.2. Furthermore, $[M'_2, M_1] = \pi_{n-1}(T; M_1) \approx \text{Ext}(T, \pi_n(M_1)) \approx \text{Ext}(T, F) \approx \bigoplus^r T$. \square

It follows that $\mathcal{E}_{\#}^{\dim}(Y)$ and $\mathcal{E}^*(Y)$ are distinct groups in general. This differs from Moore spaces X since $\mathcal{E}_{\#}^{\dim}(X) = \mathcal{E}_*(X)$.

By Theorem 4.2, $\mathcal{E}^*(Y) \subseteq \mathcal{E}_*(Y)$ and by Lemma 4.3, $\mathcal{E}_{\#}^{\dim}(Y) \subseteq \mathcal{E}_*(Y)$. We denote these inclusion maps by $\iota : \mathcal{E}^*(Y) \rightarrow \mathcal{E}_*(Y)$ and $\kappa : \mathcal{E}_{\#}^{\dim}(Y) \rightarrow \mathcal{E}_*(Y)$. There are homomorphisms $\rho : \mathcal{E}_*(Y) \rightarrow \text{Hom}(F, T)$ and $\tau : \mathcal{E}_*(Y) \rightarrow \text{Hom}(F, T \otimes \mathbf{Z}_2)$ defined by $\rho(f) = f_{12}^{*n} : F \rightarrow T$ and $\tau(f) = f_{21\#n} : F \rightarrow \pi_n(M'_T) = T \otimes \mathbf{Z}_2$.

4.6 Proposition. *The following are split short exact sequences*

$$\begin{aligned} 0 \rightarrow \mathcal{E}^*(Y) \xrightarrow{\iota} \mathcal{E}_*(Y) \xrightarrow{\rho} \text{Hom}(F, T) \rightarrow 0 \quad \text{and} \\ 0 \rightarrow \mathcal{E}_{\#}^{\dim}(Y) \xrightarrow{\kappa} \mathcal{E}_*(Y) \xrightarrow{\tau} \text{Hom}(F, T \otimes \mathbf{Z}_2) \rightarrow 0. \end{aligned}$$

We make some remarks on the preceding results.

4.7 Remarks. (1) By Theorem 4.2, $\mathcal{E}^*(Y)$ is abelian, and $\mathcal{E}^*(Y) = 1$ if and only if G has no 2-torsion.

- (2) The order of the group $\mathcal{E}_*(Y)$ is given by $|\mathcal{E}_*(Y)| = |T| \times 2^{(r+s)s}$. This follows from either exact sequence of Proposition 4.6.
- (3) We easily obtain examples from Proposition 4.6 of co-Moore spaces Y with $\mathcal{E}^*(Y) \neq \mathcal{E}_*(Y)$ and $\mathcal{E}_{\#}^{\dim}(Y) \neq \mathcal{E}_*(Y)$.
- (4) We obtain a partial realization result from Proposition 4.6: Given any finite abelian group A without 2-torsion, then $\mathcal{E}_*(C(\mathbf{Z} \oplus A, n)) \approx \mathcal{E}_{\#}^{\dim}(C(\mathbf{Z} \oplus A, n)) \approx A$.

Next we determine the extension of the first exact sequence of Proposition 4.6 (the second can be obtained similarly) in analogy with Theorem 3.6. We first define certain generators of $\mathcal{E}_*(Y)$. As in §3 we write $T = T' \oplus P$ and set $M'_{T'} = M(T', n-1)$ and $M'_P = M(P, n-1)$. Furthermore, $T' = \mathbf{Z}_{2^{a_1}} \oplus \cdots \oplus \mathbf{Z}_{2^{a_s}}$ and we set $N'_{\beta} = M(\mathbf{Z}_{2^{a_{\beta}}}, n-1)$. Now let $x_{\beta\alpha} : M_1 \rightarrow M'_2$ and $y_{\gamma\delta} : M'_2 \rightarrow M'_2$ be defined as the respective compositions

$$\begin{aligned} M_1 \xrightarrow{p_{\alpha}} S^n \xrightarrow{\eta_{n-1}} S^{n-1} \xrightarrow{i_{\beta}} N'_{\beta} \xrightarrow{k_{\beta}} M'_2 \quad \text{and} \\ M'_2 \xrightarrow{r_{\delta}} N'_{\delta} \xrightarrow{q_{\delta}} S^n \xrightarrow{\eta_{n-1}} S^{n-1} \xrightarrow{i_{\gamma}} N'_{\gamma} \xrightarrow{k_{\gamma}} M'_2 \end{aligned}$$

for $\alpha = 1, \dots, r$ and $\beta, \gamma, \delta = 1, \dots, s$. Note that the suspension of $y_{\gamma\delta}$ is the map $v_{\gamma\delta}$ of §3. Then $\mathcal{E}^*(Y)$ is generated by the elements

$$\tilde{x}_{\beta\alpha} = \begin{pmatrix} 1 & 0 \\ x_{\beta\alpha} & 1 \end{pmatrix} \quad \text{and} \quad \tilde{y}_{\gamma\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + y_{\gamma\delta} \end{pmatrix}.$$

Now define $z_{\alpha\beta} : M'_2 \rightarrow M_1$ as the composition

$$M'_2 \xrightarrow{r_\beta} N'_\beta \xrightarrow{q_\beta} S^n \xrightarrow{j_\alpha} M_1,$$

for $\alpha = 1, \dots, r$ and $\beta = 1, \dots, s$. In the split exact sequence

$$0 \rightarrow \mathcal{E}^*(Y) \xrightarrow{\iota} \mathcal{E}_*(Y) \xrightleftharpoons[\sigma]{\rho} \text{Hom}(F, T) \rightarrow 0,$$

the 2-torsion part of Image σ is generated by all

$$\tilde{z}_{\alpha\beta} = \begin{pmatrix} 1 & z_{\alpha\beta} \\ 0 & 1 \end{pmatrix}.$$

Let $H \subseteq \mathcal{E}_*(Y)$ be the subgroup generated by all $\tilde{x}_{\beta'\alpha'}$, $\tilde{y}_{\gamma\delta}$ and $\tilde{z}_{\alpha\beta}$. Then the action of $\text{Hom}(F, T)$ on $\mathcal{E}^*(Y)$ is given by

$$(1) \quad \tilde{z}_{\alpha\beta}^{-1} \tilde{x}_{\beta'\alpha'} \tilde{z}_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ x_{\beta'\alpha'} & 1 + x_{\beta'\alpha'} z_{\alpha\beta} \end{pmatrix} \quad \text{and}$$

$$(2) \quad \tilde{z}_{\alpha\beta}^{-1} \tilde{y}_{\gamma\delta} \tilde{z}_{\alpha\beta} = \tilde{y}_{\gamma\delta}.$$

In (1), if $\alpha \neq \alpha'$, $x_{\beta'\alpha'} z_{\alpha\beta} = 0$; if $\alpha = \alpha'$, $x_{\beta'\alpha'} z_{\alpha\beta} = y_{\beta'\beta}$. Furthermore, $\text{Hom}(F, P)$ acts trivially on $\mathcal{E}^*(Y)$.

4.8 Theorem. *The group $\mathcal{E}_*(Y)$ is isomorphic to $H \oplus \bigoplus^r P$. Moreover, H is the semi-direct product of $\mathcal{E}^*(Y)$ with $\text{Hom}(F, T)$ with the action of the latter group on the former group given by (1) and (2). In addition, $\text{nil } \mathcal{E}_*(Y) \leq 2$ and $\mathcal{E}_*(Y)$ is abelian if and only if $r = 0$ or $s = 0$.*

4.9 Remark. It follows from §§3 and 4 that for $n > 3$,

$$\begin{aligned} \mathcal{E}_*(M(G, n)) &\approx \mathcal{E}^*(C(G, n)) \quad \text{and} \\ \mathcal{E}^*(M(G, n)) &\approx \mathcal{E}_*(C(G, n)). \end{aligned}$$

We explain this by appealing to S -duality theory (for more details, see [Sp₁] and [Sp₂, pp. 462-63]). First note that under suspension

$$\begin{aligned} \mathcal{E}_*(M(G, 3)) &\approx \mathcal{E}_*(M(G, 4)) \approx \dots \quad \text{and} \\ \mathcal{E}^*(M(G, 3)) &\approx \mathcal{E}^*(M(G, 4)) \approx \dots \end{aligned}$$

Thus we define *stable groups of equivalences* by $\mathcal{E}_{*M}^{st}(G) = \mathcal{E}_*(M(G, n))$ and $\mathcal{E}_{st}^{*M}(G) = \mathcal{E}^*(M(G, n))$, for all $n \geq 3$. Similarly for co-Moore spaces $C(G, n)$ with $n \geq 4$, we obtain stable groups $\mathcal{E}_{st}^{*C}(G)$ and $\mathcal{E}_{*C}^{st}(G)$. Now let $X = M(G, n)$ be a Moore space and X^* be the N -dual of X . Then X^* is a co-Moore space $C(G, N - n)$. Furthermore, for any spaces A and B , the group of stable homotopy classes $\{A, B\}$ is isomorphic to the group $\{B^*, A^*\}$ under an isomorphism that reverses composition. From this we easily obtain that

$\mathcal{E}_{*M}^{st}(G)$ is anti-isomorphic to $\mathcal{E}_{st}^{*C}(G)$. Thus we conclude that $\mathcal{E}_*(M(G, n)) \approx \mathcal{E}^*(C(G, n))$. By starting with a co-Moore space and taking its N -dual, we similarly conclude that $\mathcal{E}_*(C(G, n)) \approx \mathcal{E}^*(M(G, n))$.

4.10 Remark. We comment on $\mathcal{E}_*(C(G, n))$ and $\mathcal{E}^*(C(G, n))$ when $n = 3$. Then $Y = C(G, 3) = M_1 \vee M'_2$, where $M_1 = M(F, 3)$ and $M'_2 = M(T, 2)$. Theorems 4.1 and 4.2 and Proposition 4.6 hold for $Y = C(G, 3)$ with the exception of the direct sum decomposition of $\mathcal{E}^*(Y)$ in Theorem 4.2. To obtain a decomposition for $\mathcal{E}^*(Y)$ we proceed as in the proof of Theorem 4.2 and conclude that $\mathcal{E}^*(Y) \approx \text{Hom}(F, \pi_3(M'_2)) \oplus \text{Ext}(T, \pi_3(M'_2))$. But $\pi_3(M'_2) = \Gamma(T)$, where Γ is Whitehead's quadratic functor (see [Wh] and [Ba, p.268]). Thus we have

$$\mathcal{E}^*(Y) \approx \left(\bigoplus^r \Gamma(T) \right) \oplus \text{Ext}(T, \Gamma(T)).$$

Next we consider $\mathcal{E}_{\#}^{\dim+1}(Y)$. Note that $[N'_{\alpha}, S^n] = \pi_{n-1}(\mathbf{Z}_{2^{a_{\alpha}}}; S^n) \approx \text{Ext}(\mathbf{Z}_{2^{a_{\alpha}}}, \mathbf{Z}) \approx \mathbf{Z}_{2^{a_{\alpha}}}$, where $N'_{\alpha} = M(\mathbf{Z}_{2^{a_{\alpha}}}, n-1)$, and a generator of this group is the projection $q_{\alpha} : N'_{\alpha} \rightarrow S^n$. Thus elements of $[N'_{\alpha}, S^n]$ can be represented as $m_{\alpha}q_{\alpha}$, $m_{\alpha} = 0, 1, \dots, 2^{a_{\alpha}} - 1$.

4.11 Proposition. *If $n > 4$, then the group $\mathcal{E}_{\#}^{\dim+1}(Y)$ consists of all $f = \begin{pmatrix} 1 & f_{12} \\ 0 & 1 \end{pmatrix}$, where $f_{12} : M'_2 \rightarrow M_1$ is such that $p_{\beta}f_{12}k_{\alpha} : N'_{\alpha} \rightarrow S^n$ is $m_{\alpha\beta}q_{\alpha}$, for $m_{\alpha\beta} = 0, 2, \dots, 2^{a_{\alpha}} - 2$, and $\alpha = 1, \dots, s$ and $\beta = 1, \dots, r$.*

Proof. Let $f = \begin{pmatrix} 1 & f_{12} \\ 0 & f_{22} \end{pmatrix} \in \mathcal{E}_{\#}^{\dim}(Y)$, where $f_{22} \in \mathcal{E}_*(M'_2)$. Then for $x \in \pi_{n+1}(M_1)$ and $y \in \pi_{n+1}(M'_2)$,

$$f_{\#}(x, y) = (x + f_{12\#}(y), f_{22\#}(y)).$$

Thus $f \in \mathcal{E}_{\#}^{\dim+1}(Y) \Leftrightarrow f_{12\#n+1} = 0$ and $f_{22\#n+1} = 1$. But the latter is equivalent to $f_{22} \in \mathcal{E}_{\#}^{\dim+1}(M'_2)$. Therefore this is equivalent to $f_{22} = 1$ by Theorem 3.8 since $n > 4$. Now consider $f_{12} : M'_2 \rightarrow M_1$ and set $h = f_{12}$. Then $M'_2 = M'_T \vee M'_P = N'_1 \vee \dots \vee N'_s \vee M'_P$ and we consider $h|_{M'_P} : M'_P \rightarrow M_1$. We have that $\pi_{n+1}(M'_P) = 0$ since it is an extension of $\text{Tor}(P, \mathbf{Z}_2)$ by $P \otimes \mathbf{Z}_2$ according to §1. Therefore $(h|_{M'_P})_{\#n+1} = 0$. Hence $h_{\#n+1} = 0$ if and only if $(p_{\beta}hk_{\alpha})_{\#n+1} = 0$,

$$N'_{\alpha} \xrightarrow{k_{\alpha}} M'_2 \xrightarrow{h} M_1 \xrightarrow{p_{\beta}} S^n,$$

$\alpha = 1, \dots, s$, $\beta = 1, \dots, r$. Now consider the defining cofibre sequence $S^{n-1} \xrightarrow{i_{\alpha}} N'_{\alpha} \xrightarrow{q_{\alpha}} S^n$ and the corresponding exact sequence (2.3)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{n+1}(S^{n-1}) & \xrightarrow{i_{\alpha\#}} & \pi_{n+1}(N'_{\alpha}) & \xrightarrow{q_{\alpha\#}} & \pi_{n+1}(S^n) & \longrightarrow & 0. \\ & & \parallel & & \downarrow (p_{\beta}hk_{\alpha})_{\#} & & \parallel & & \\ & & \mathbf{Z}_2 & & \pi_{n+1}(S^n) & & \mathbf{Z}_2 & & \end{array}$$

We have $p_\beta h k_\alpha = m_{\alpha\beta} q_\alpha$ for some $m_{\alpha\beta} = 0, 1, \dots, 2^{a_\alpha} - 1$. It follows that $(p_\beta h k_\alpha)_\# i_{\alpha\#} = m_{\alpha\beta} q_\alpha i_\alpha = 0$. Choose $z \in \pi_{n+1}(N'_\alpha)$ such that $q_{\alpha\#}(z) = \eta_n \in \pi_{n+1}(S^n)$. Then

$$\begin{aligned} (p_\beta h k_\alpha)_\#(z) &= m_{\alpha\beta} q_{\alpha\#}(z) \\ &= m_{\alpha\beta} \eta_n. \end{aligned}$$

Thus $(p_\beta h k_\alpha)_\# = 0$ if and only if $m_{\alpha\beta}$ is even. Therefore those maps $f_{12} : M'_2 \rightarrow M_1$ such that $(f_{12})_{\#n+1} = 0$ are precisely those such that $p_\beta f_{12} k_\alpha = m_{\alpha\beta} q_\alpha$ for $m_{\alpha\beta} = 0, 2, \dots, 2^{a_\alpha} - 2$, where $\alpha = 1, \dots, s$ and $\beta = 1, \dots, r$. \square

We have $T = T' \oplus P$, where $T' = \mathbf{Z}_{2^{a_1}} \oplus \dots \oplus \mathbf{Z}_{2^{a_s}}$, and we denote by $2T'$ the subgroup of T' which is represented in each component by an even integer.

4.12 Corollary. For $n > 4$,

$$\mathcal{E}_\#^{\dim+1}(Y) \approx \bigoplus_{\#}^r (2T' \oplus P) \approx \bigoplus_{\#}^r (\mathbf{Z}_{2^{a_1-1}} \oplus \dots \oplus \mathbf{Z}_{2^{a_s-1}} \oplus P).$$

For any space X , let $\mathcal{E}_\#^\infty(X) = \bigcap_{t \geq 0} \mathcal{E}_\#^{\dim+t}(X)$. Then the following question is suggested by Corollary 4.12: Is $\mathcal{E}_\#^\infty(Y) = 1$ for every co-Moore space Y ?

Finally, we consider the co-Moore space $C(\mathbf{Z} \oplus \mathbf{Z}_2, n)$ to illustrate many of the results of this section.

4.13 Corollary. If $Y = C(\mathbf{Z} \oplus \mathbf{Z}_2, n)$, then $\mathcal{E}_*(Y) \approx D_4$, the dihedral group of 8 elements, $\mathcal{E}^*(Y) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$, $\mathcal{E}_\#^{\dim}(Y) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $\mathcal{E}_\#^{\dim+1}(Y) = 1$. Furthermore, $\mathcal{E}^*(Y)$ and $\mathcal{E}_\#^{\dim}(Y)$ are distinct subgroups of $\mathcal{E}_*(Y)$.

§5 SPACES WITH $\mathcal{E}_* = \mathcal{E}^*$

Recall that a finite CW-complex X is called a *Poincaré complex of dimension n* if

- (i) $H_i(X) = 0$ for $i > n$.
- (ii) $H_n(X) \approx \mathbf{Z}$ generated by μ .
- (iii) The homomorphism $P : H^q(X) \rightarrow H_{n-q}(X)$ defined by $P(x) = x \cap \mu$ is an isomorphism for all $0 \leq q \leq n$.

5.1 Proposition. If X is a Poincaré complex, then $\mathcal{E}_*(X) = \mathcal{E}^*(X)$.

Proof. If $f : X \rightarrow X$, then $f_*(f^*(x) \cap \mu) = x \cap f_*(\mu)$. Now let $f \in \mathcal{E}_*(X)$. Then

$$\begin{aligned} P(f^*(x)) &= f^*(x) \cap \mu \\ &= x \cap f_*(\mu) \\ &= P(x) \end{aligned}$$

and so $f^*(x) = x$. Therefore $f \in \mathcal{E}^*(X)$ and so, $\mathcal{E}_*(X) \subseteq \mathcal{E}^*(X)$. For the opposite inclusion, we conclude from the universal coefficient theorem [Sp₂, p. 248],

$$0 \rightarrow \text{Ext}(H^{n+1}(X), \mathbf{Z}) \rightarrow H_n(X) \rightarrow \text{Hom}(H^n(X), \mathbf{Z}) \rightarrow 0,$$

that $H_n(X) \approx \text{Hom}(H^n(X), \mathbf{Z})$. Thus if $f : X \rightarrow X$ and $f^{*n} = 1$, then $f_*(\mu) = \mu$. Now let $f \in \mathcal{E}^*(X)$. Then

$$\begin{aligned} f_*(P(x)) &= f_*(x \cap \mu) \\ &= f_*(f^*(x) \cap \mu) \\ &= x \cap f_*(\mu) \\ &= P(x). \end{aligned}$$

Thus $f_* = 1$, and so $\mathcal{E}^*(X) \subseteq \mathcal{E}_*(X)$. \square

5.2 Corollary. (1) If X is an H -space, then $\mathcal{E}_*(X) = \mathcal{E}^*(X)$.

(2) If X is a compact, oriented manifold, then $\mathcal{E}_*(X) = \mathcal{E}^*(X)$.

Proof. (1) Browder proved that X is a Poincaré complex [Ka, p. 31].

(2) By the Poincaré duality theorem [Mas, p. 365], X is a Poincaré complex. \square

Now let X be a space and for every $n > 0$ write

$$H_n(X) = F_n \oplus T_n,$$

where F_n is a free-abelian group and T_n is a finite group.

5.3 Proposition. (1) If for every $n > 0$, $\text{Hom}(F_n, T_{n-1}) = 0$, then $\mathcal{E}_*(X) \subseteq \mathcal{E}^*(X)$.

(2) If for every $n > 0$, $\text{Hom}(F_n, T_n) = 0$, then $\mathcal{E}^*(X) \subseteq \mathcal{E}_*(X)$.

Proof. (1) We have $\text{Hom}(H_n(X), \mathbf{Z}) \approx F_n$ and $\text{Ext}(H_{n-1}(X), \mathbf{Z}) \approx T_{n-1}$. Thus the universal coefficient theorem for cohomology yields the short exact sequence

$$0 \rightarrow T_{n-1} \xrightarrow{\iota} H^n(X) \xrightarrow{\pi} F_n \rightarrow 0.$$

If $f \in \mathcal{E}_*(X)$, we have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{n-1} & \xrightarrow{\iota} & H^n(X) & \xrightarrow{\pi} & F_n \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow f^* & & \downarrow 1 \\ 0 & \longrightarrow & T_{n-1} & \xrightarrow{\iota} & H^n(X) & \xrightarrow{\pi} & F_n \longrightarrow 0. \end{array}$$

Then the difference $f^* - 1 = \iota\theta\pi$, for some homomorphism $\theta : F_n \rightarrow T_{n-1}$. Thus if $\text{Hom}(F_n, T_{n-1}) = 0$, $f^* = 1$, and so $\mathcal{E}_*(X) \subseteq \mathcal{E}^*(X)$.

(2) This is proved as in (1) using the universal coefficient theorem which expresses homology in terms of cohomology (see the proof of Proposition 5.1). \square

The following corollary gives two extreme cases of Proposition 5.3.

5.4 Corollary. (1) If $H_i(X)$ has no torsion for all $i > 0$, then $\mathcal{E}_*(X) = \mathcal{E}^*(X)$.

(2) If $H_i(X)$ is a finite group for all $i > 0$, then $\mathcal{E}_*(X) = \mathcal{E}^*(X)$.

§6 HOMOTOPY EQUIVALENCES WHICH INDUCE THE IDENTITY ON HOMOTOPY GROUPS

From earlier sections it is not clear if there exist spaces X with $\mathcal{E}_{\#}^{\infty}(X) \neq 1$, where $\mathcal{E}_{\#}^{\infty}(X) = \bigcap_{t \geq 0} \mathcal{E}_{\#}^{\dim+t}(X)$. We show that such examples exist.

6.1 Proposition. *If $X = S^m \times S^n$, with $n > m \geq 2$, then the group $\mathcal{E}_{\#}^{\dim}(X)$ is abelian and equals $\mathcal{E}_{\#}^{\dim+t}(X)$ for all $t > 0$.*

Proof. We use the exact sequence of [Sa, p. 71]

$$\pi_{m+n}(X) \xrightarrow{\lambda} \mathcal{E}(X) \xrightarrow{\rho} \mathcal{E}(S^m \vee S^n),$$

where ρ is given by restriction to the n -skeleton and λ is defined as follows: Consider the cofibre sequence

$$S^m \vee S^n \xrightarrow{i} X \xrightarrow{q} S^m \wedge S^n = S^{m+n},$$

and the corresponding pinching map $\ell : X \rightarrow X \vee S^{m+n}$. Then if $z \in \pi_{m+n}(X)$, $\lambda(z)$ is the composition

$$X \xrightarrow{\ell} X \vee S^{m+n} \xrightarrow{1 \vee z} X \vee X \xrightarrow{\nabla} X,$$

where ∇ is the folding map. We first show $\rho(\mathcal{E}_{\#}^{\dim}(X)) = 1$. Let $f \in \mathcal{E}_{\#}^{\dim}(X)$ and let $i_1 : S^m \rightarrow S^m \vee S^n$ and $i_2 : S^n \rightarrow S^m \vee S^n$ be the inclusions. Then it suffices to show $\rho(f)i_1 = i_1$ and $\rho(f)i_2 = i_2$. But

$$\begin{aligned} i_{\#}(\rho(f)i_1) &= f_{\#}(ii_1) \\ &= ii_1 \\ &= i_{\#}(i_1). \end{aligned}$$

Since $i_{\#} : \pi_n(S^m \vee S^n) \rightarrow \pi_n(X)$ is an isomorphism, $\rho(f)i_1 = i_1$. Similarly $\rho(f)i_2 = i_2$. Thus

$$\mathcal{E}_{\#}^{\dim}(X) \subseteq \text{Kernel } \rho = \text{Image } \lambda.$$

Therefore any element of $\mathcal{E}_{\#}^{\dim}(X)$ is of the form $\lambda(z) = \nabla(1 \vee z)\ell$, for some $z \in \pi_{m+n}(X)$. To complete the proof it suffices to show $\lambda(z)_{\#k} = 1 : \pi_k(X) \rightarrow \pi_k(X)$ for all k . It is known that the following diagram commutes

$$\begin{array}{ccc} S^m \vee S^n & \xrightarrow{i} & X \\ \downarrow i & & \downarrow j_1 \\ X & \xrightarrow{\ell} & X \vee S^{m+n}, \end{array}$$

where j_1 is the inclusion. Furthermore $i_{\#} : \pi_k(S^m \vee S^n) \rightarrow \pi_k(X)$ is onto for all k . Thus if $y \in \pi_k(X)$, $y = i_{\#}(x)$ for some $x \in \pi_k(S^m \vee S^n)$, and so

$$\begin{aligned} \lambda(z)_{\#}(y) &= \nabla_{\#}(1 \vee z)_{\#}\ell_{\#}i_{\#}(x) \\ &= \nabla_{\#}(1 \vee z)_{\#}j_{1\#}i_{\#}(x) \\ &= i_{\#}(x) \\ &= y. \quad \square \end{aligned}$$

The proof shows that $\text{Image } \lambda = \mathcal{E}_{\#}^{\dim+t}(S^m \times S^n)$ for all $t \geq 0$. From [Sa, Thm. 2.6, Lem. 4.1 and (5.1)] we obtain

6.2 Corollary. *If $n > m \geq 2$, then*

$$\begin{aligned} \mathcal{E}_{\#}^{\infty}(S^m \times S^n) &\approx \mathcal{E}_{\#}^{\dim}(S^m \times S^n) \\ &\approx (\pi_{m+n}(S^m)/[\iota_m, \pi_{n+1}(S^m)]) \oplus (\pi_{m+n}(S^n)/[\iota_n, \pi_{m+1}(S^n)]), \end{aligned}$$

where $\iota_k \in \pi_k(S^k)$ is the identity map and the brackets denote Whitehead products. In particular,

$$\begin{aligned} \mathcal{E}_{\#}^{\infty}(S^2 \times S^n) &\approx \pi_{n+2}(S^2) \oplus \mathbf{Z}_2 \quad \text{and} \\ \mathcal{E}_{\#}^{\infty}(S^3 \times S^n) &\approx \pi_{n+3}(S^3) \oplus \pi_{n+3}(S^n). \end{aligned}$$

Next we show that $\mathcal{E}_{\#}^{\dim}(X)$ can be infinite while $\mathcal{E}_{\#}^{\dim+1}(X)$ is trivial.

6.3 Proposition. *If $X = CP^n \vee S^{2n}$, then $\mathcal{E}_{\#}^{\dim}(X) \approx \mathbf{Z}$ and $\mathcal{E}_{\#}^{\dim+1}(X) = 1$.*

Proof. Let $i_1 : CP^n \rightarrow X$ and $i_2 : S^{2n} \rightarrow X$ be the inclusions and $p_1 : X \rightarrow CP^n$ and $p_2 : X \rightarrow S^{2n}$ the projections. If $f \in [X, X]$, we write $f_{rt} = p_r f i_t$, for $r, t = 1, 2$. Let $u \in H^2(CP^n) = \mathbf{Z}$ and $v \in H^{2n}(S^{2n}) = \mathbf{Z}$ be generators and let $x = p_1^*(u) \in H^2(X)$ and $s = p_2^*(v) \in H^{2n}(X)$. Then $H^{2n}(X) = \mathbf{Z} \oplus \mathbf{Z}$ and is generated by x^n and s . If $f \in \mathcal{E}_{\#}^{\dim}(X)$, then $f^*(s) = kx^n + \ell s$ for $k, \ell \in \mathbf{Z}$. But $f_{\#2n} = 1$ and so $f_{22} = 1$. Thus $\ell = 1$ and hence

$$f^*(s) = kx^n + s.$$

Define $\theta : \mathcal{E}_{\#}^{\dim}(X) \rightarrow \mathbf{Z}$ by $\theta(f) = k$. Note that if $f \in \mathcal{E}_{\#}^{\dim}(X)$, $f^*(x) = x$ since $f_{\#2} = 1$. We now show θ is a homomorphism: Given $f, f' \in \mathcal{E}_{\#}^{\dim}(X)$, where $f'^*(s) = k'x^n + s$. Then

$$\begin{aligned} (ff')^*(s) &= f'^*(kx^n + s) \\ &= kx^n + (k'x^n + s) \\ &= (k + k')x^n + s. \end{aligned}$$

Thus $\theta(ff') = \theta(f) + \theta(f')$. Now we show that θ is a monomorphism: By Proposition 2.4 and the fact that $[S^{2n}, CP^n] = 0$, we have that $f, f' \in [X, X]$ are equal $\Leftrightarrow f_{rt} = f'_{rt}$ for $(r, t) = (1, 1), (2, 1), (2, 2)$. But if $f, f' \in \mathcal{E}_{\#}^{\dim}(X)$, then $f_{22} = 1 = f'_{22}$. Furthermore, $f^*(x) = x = f'^*(x)$ and so $f_{11} = f'_{11}$. Thus $f, f' \in \mathcal{E}_{\#}^{\dim}(X)$ are equal $\Leftrightarrow f_{21} = f'_{21} \in [CP^n, S^{2n}]$. But the latter holds if and only if $f_{21}^*(v) = f'_{21}^*(v)$. Now $f_{21}^*(v) = (p_2 f i_1)^*(v) = i_1^* f^*(s) = i_1^*(kx^n + s) = ku^n$ and similarly $f'_{21}^*(v) = k'u^n$. Therefore $\theta(f) = k = k' = \theta(f')$ implies $f = f'$. Finally we show that θ is an epimorphism: Given $k \in \mathbf{Z}$, let $g \in [CP^n, S^{2n}]$ be such that $g^*(v) = ku^n$. Then define $f \in \mathcal{E}_{\#}^{\dim}(X)$ by $f_{11} = 1, f_{12} = 0, f_{21} = g$ and $f_{22} = 1$. Clearly $\theta(f) = k$. Thus $\mathcal{E}_{\#}^{\dim}(X) \approx \mathbf{Z}$.

To see that $\mathcal{E}_{\#}^{\dim+1}(X) = 1$, we use the Sullivan minimal model \mathcal{M} of X [G-M]. Let $\mathcal{M}(k) \subseteq \mathcal{M}$ be the subminimal algebra of \mathcal{M} generated by all free algebra generators of degree $\leq k$. Then $\mathcal{E}_{\#}^k(\mathcal{M})$ is the group of homotopy classes of differential graded algebra automorphisms $\phi : \mathcal{M}(k) \rightarrow \mathcal{M}(k)$ such that for every free algebra generator x of $\mathcal{M}(k)$, $\phi(x) = x + \chi$, where χ is a decomposable element of $\mathcal{M}(k)$. With subscripts denoting degree, the free algebra generators of $\mathcal{M}(2n+1)$ are u_2, v_{2n+1}, x_{2n} and y_{2n+1} with differential d given by $du = 0, dx = 0, dv = u^{n+1}$ and $dy = xu$. Then each element of $\mathcal{E}_{\#}^{2n+1}(\mathcal{M})$ is represented by a differential graded algebra automorphism $\phi_q : \mathcal{M}(2n+1) \rightarrow \mathcal{M}(2n+1)$ which is the identity on u, v and y and such that $\phi_q(x) = x + qu^n$ for some rational $q \in \mathbf{Q}$. Hence

$$\begin{aligned} xu &= dy \\ &= d(\phi_q(y)) \\ &= \phi_q d(y) \\ &= (x + qu^n)u \\ &= xu + qu^{n+1} \end{aligned}$$

Thus $q = 0$, and so $\mathcal{E}_{\#}^{2n+1}(\mathcal{M}) = 1$. But by [Mar], there is a rationalization homomorphism $\mathcal{E}_{\#}^{\dim+1}(X) \rightarrow \mathcal{E}_{\#}^{2n+1}(\mathcal{M})$. It follows that $\mathcal{E}_{\#}^{\dim+1}(X)$ is a finite group. Since $\mathcal{E}_{\#}^{\dim+1}(X) \subseteq \mathcal{E}_{\#}^{\dim}(X) = \mathbf{Z}$, we have that $\mathcal{E}_{\#}^{\dim+1}(X) = 1$. \square

We mention another example, taken directly from [A-L, Prop. 6.3].

6.4 Example. If $U(\ell)$ denotes the unitary group and X is the homogeneous space

$$U(n)/(U(n_1) \times \cdots \times U(n_k)),$$

where $n_1 \leq \cdots \leq n_k$ and $n - (n_1 + \cdots + n_k) \geq 2$, then $\mathcal{E}_{\#}^{\dim+t}(X)$ is infinite for every $t \geq 0$.

We close with a conjecture regarding the stability of the groups $\mathcal{E}_{\#}^{\dim+t}(X)$.

6.5 Conjecture. *If X is a finite CW-complex, then there exists an integer N such that*

$$\mathcal{E}_{\#}^{\dim+t}(X) = \mathcal{E}_{\#}^{\dim+N}(X)$$

for all $t \geq N$.

Clearly this is true if $\mathcal{E}_{\#}^{\dim+s}(X)$ is finite for some s .

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