The Inverses of an H-Space

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Abstract A multiplication on an H-space X has a left inverse λ and a right inverse ρ. They are mutual inverses and $\lambda = \rho$ if and only if $\lambda^2 = id$. In this paper we investigate the order $|\lambda|$ of λ. We give an example of a multiplication with $|\lambda| = 6$, and prove that for any finite H-complex X there are finitely many left inverses of finite order. Conditions are given for there to be infinitely many multiplications on X with the same left inverse. We then give conditions for a left inverse to have infinite order. We apply these results to specific Lie groups.

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1 Introduction

An *H-space* is a pair (X, μ) consisting of a based topological space X and a homotopy class $\mu \in [X \times X, X]$, called the *multiplication*, whose restriction to each factor of $X \times X$ is id, the homotopy class of the identity map of X. If, in addition, X is a (finite) CW-complex, we call X or (X, μ) a *(finite) H-complex.* An H-space (X, μ) is *group-like* if μ is *homotopy associative*, i.e., $\mu(\mu \times id) = \mu(id \times \mu) \in [X \times X \times X, X]$ and if μ has a *homotopy inverse.* The latter condition is that there exists $\iota \in [X, X]$ such that $\mu(\iota \times id)\Delta = 0 = \mu(id \times \iota)\Delta$, where Δ is the homotopy class of the diagonal map of X and 0 is the constant homotopy class.

For any based space A, a multiplication μ of X induces a binary operation on $[A, X]$ defined by $\alpha + \beta = \mu(\alpha \times \beta)\Delta$ such that $\alpha + 0 = 0 + \alpha = \alpha$. If μ is homotopy associative, then $[A, X]$ is associative. If (X, μ) is group-like, then $[A, X]$ is a group.

A great deal of work has been done on the homotopy associativity condition of an H-space [St1, St2, Za]. On the other hand, the inverse condition has been studied very little. The reasons for this may be the following: (1) the inverse condition rarely appears as a hypothesis in theorems about H-spaces (2) the result of James which we discuss next.

Recall that an (algebraic) *loop* L is a set with an additive binary operation such that for every $a, b \in L$ the equations $a + x = b$ and $y + a = b$ have unique solutions $x, y \in L$. Then James has proved [Ja3, Thm. 1.1] that $[A, X]$ is a loop if A is a CW-complex and (X, μ) is an H-space. Thus every $\alpha \in [A, X]$ has a unique left inverse α_L and a unique right inverse α_R defined by $\alpha_L + \alpha = 0$ and $\alpha + \alpha_R = 0$. Hence if (X, μ) is an H-complex, there are unique elements $\lambda, \rho \in [X, X]$, called the *left and right inverse of* μ, which are the left and right inverse of id, respectively, i.e., $\mu(\lambda \times id)\Delta = 0 = \mu(id \times \rho)\Delta$. It follows as in group theory that if μ is homotopy associative, then $\lambda = \rho$, and so (X, μ) is group-like. Thus the inverse property automatically holds for homotopy associative H-complexes, and this may be a reason why this condition has not received much attention. However, many familiar homotopy associative H-complexes such as compact, connected Lie groups, admit infinitely many multiplications which are not homotopy associative [Cu, Thm. II] and thus may have left and right inverses which are not equal [AL, Cor. 4.4]. It is therefore reasonable to study the inverse condition for arbitrary multiplications. We do this in this paper.

We next give a brief outline of the paper. We let (X, μ) be an H-complex with left inverse λ and right inverse ρ. In §2 it is shown that λ and ρ are homotopy equivalences and mutual inverses and that $\lambda = \rho$ if and only if $\lambda \circ \lambda = \lambda^2 = id$. We construct a class of multiplications on any H-space all of which have the same left inverse and the same right inverse. In §3 we consider the order $|\lambda|$ of a left inverse λ of a multiplication on X, i.e., the smallest positive integer n such that $\lambda^n = id$; or ∞, if there is no such integer. Since $|\lambda| = 2$ corresponds to $\lambda = \rho$, we regard the order of λ as a measure of how much λ differs from ρ. Since $\lambda = \rho$ for homotopy associative multiplications, we could also regard a large order $|\lambda|$ as indicating that the multiplication is highly nonhomotopy associative. In §4 we use methods of rational homotopy theory to study inverses on an H-space X by investigating multiplications and inverses on the Sullivan minimal model $\mathcal{M}$ of X. We prove that for any finite H-complex, the set of all left inverses of finite order is a finite set. We obtain an easily verifiable condition on a homotopy associative finite H-complex X (in terms of the rational cohomology of X) for there to be infinitely many multiplications with the same left inverse. This leads to a determination of which 1–connected simple Lie groups have this property. Finally, we give necessary and sufficient conditions for X to admit a multiplication with $|\lambda| = \infty$, and determine which 1–connected simple Lie groups have this property.

We conclude this section by giving our notation and terminology. The usual conventions of homotopy theory will hold. All spaces will be based and will have the homotopy type of CW-complexes. They will be assumed to be nilpotent, and will often be 1–connected. All maps and homotopies will preserve base points. We will not distinguish notationally between a map and its homotopy class, but will refer to an actual map as a function. For spaces A and X, we let $[A, X]$ denote the set of homotopy classes from A to X. A map (or homotopy class) $f : A \to A'$ determines a function $f^* : [A', X] \to [A, X]$ in the obvious way. Furthermore, f induces a homomorphism of homotopy groups, denoted $f_\# : \pi_s(A) \to \pi_s(A')$.

2 Basic Properties

Let X be an H-complex with multiplication μ, left inverse λ and right inverse ρ and let A be a based space. If $\alpha \in [A, X]$, then it easily follows that $\alpha_L = \lambda\alpha$ and $\alpha_R = \rho\alpha$. From $\lambda + id = 0 = id + \rho$, we obtain $id = \lambda_R = \rho\lambda$ and $id = \rho_L = \lambda\rho$.

Lemma 2.1 *$\lambda\rho = id$ and $\rho\lambda = id$. Thus λ and ρ are homotopy equivalences and mutual inverses.* □

For an element $\alpha \in [X, X]$ and positive integer n, let $\alpha^n = \alpha\alpha\cdots\alpha$ (n times). The following lemma is then obvious.

Lemma 2.2 $\lambda^2 = id \iff \lambda = \rho \iff \rho^2 = id$. □

Lemma 2.3 *For every $i > 0$ and for every $x \in \pi_i(X)$, we have $\lambda_\#(x) = -x = \rho_\#(x)$.*

Proof. This follows since $\lambda_\# + id_\# = (\lambda + id)_\# = 0_\# = 0 : \pi_i(X) \to \pi_i(X)$, and similarly for ρ. □

Now we construct a class of multiplications related to a given one by commutators. Let (X, μ_0) be an H-space with left and right inverses λ_0 and ρ_0, respectively. We define a commutator $\phi_0 \in [X \times X, X]$ as follows. Let $p_1, p_2 \in [X \times X, X]$ be the two projections and set

$$\phi_0 = (p_1 + p_2) + (\rho p_1 + \rho p_2).$$

Note that if μ_0 is homotopy-associative, $\phi_0 = p_1 + p_2 - p_1 - p_2 = [p_1, p_2]$, the commutator in the group $[X \times X, X]$.

For a fixed multiplication μ_0 on X, a positive integer n and an element $\alpha \in [A, X]$, define $n\alpha$ inductively by $1\alpha = \alpha$, $2\alpha = \alpha + \alpha$, $\ldots, n\alpha = (n-1)\alpha + \alpha$.

Definition 2.4 Given μ_0, λ_0 and ρ_0 as above, define a multiplication μ_s on X by

$$\mu_s = \mu_0 + (s\phi_0),$$

where $s \geq 0$ is an integer.

Remark 2.5 (1) If μ_0 is the standard multiplication on a compact, connected Lie group X, then James has proved that ϕ_0 has finite order [Ja2, Lem. 4.1]. If k is the order of ϕ_0, there are k multiplications of the form μ_s. Furthermore if X is not a product of circles, then $k \geq 2$ [Hu, Thm. 1.1].

(2) Another definition of a commutator element is $\psi_0 = (\lambda_0 p_1 + \lambda_0 p_2) + (p_1 + p_2) \in [X \times X, X]$. When μ_0 is homotopy-associative, $\psi_0 = -p_1 - p_2 + p_1 + p_2$. One could use ψ_0 in place of ϕ_0 to define a class of multiplications analogous to μ_s.

Proposition 2.6 *If λ_s and ρ_s are the left and right inverses of μ_s, then $\lambda_s = \lambda_0$ and $\rho_s = \rho_0$.*

Proof. Let $+_s$ denote the binary operation in $[X, X]$ induced by μ_s. Consider

$$\begin{aligned} \lambda_0 +_s id &= (\mu_0 + s\phi_0)(\lambda_0 \times id)\Delta \\ &= \mu_0(\lambda_0 \times id)\Delta + s\phi_0(\lambda_0 \times id)\Delta. \end{aligned}$$

But $\mu_0(\lambda_0 \times id)\Delta = \lambda_0 + id = 0$ and

$$\begin{aligned} \phi_0(\lambda_0 \times id)\Delta &= ((p_1 + p_2) + (\rho_0 p_1 + \rho_0 p_2))(\lambda_0 \times id)\Delta \\ &= (\lambda_0 + id) + (\rho_0\lambda_0 + \rho_0) \\ &= 0 + (id + \rho_0) \\ &= 0. \end{aligned}$$

Thus $\lambda_0 +_s id = 0 + s0 = 0$, and so $\lambda_0 = \lambda_s$. A similar argument shows

$$id +_s \rho_0 = s(\rho_0 + \rho_0^2).$$

But $\rho_0 + \rho_0^2 = (id + \rho_0)\rho_0 = 0$. Therefore $\rho_0 = \rho_s$. □

Remark 2.7 We have noted that if μ is homotopy-associative, then $\lambda = \rho$. We now observe that the converse is false. For if μ_0 is a homotopy-associative multiplication, then $\lambda_0 = \rho_0$. Thus for any $s <$ the order of ϕ_0, we have $\lambda_s = \rho_s$. But μ_s need not be homotopy-associative. One concrete example of this occurs with $X = S^3$ and $s = 1, 4, 7$ or 10 [AC3, Rem. 1]. Another example occurs with the exceptional Lie group $(\mathrm{G}_2, \mu_\mathrm{s})$. Since $\lambda_s = \lambda_0 = \rho_0 = \rho_s$, we show that μ_s is not homotopy-associative for $s = 1, 4, 7, 10, \ldots$. If $i : S^3 \hookrightarrow \mathrm{G}_2$ is the inclusion, then $i : (S^3, \mu_s) \to (\mathrm{G}_2, \mu_\mathrm{s})$ is an H-map. Hence $i_*\langle\iota_3, \iota_3\rangle_s = \langle i, i\rangle_s$ where $\langle -, -\rangle_s$ is the Samelson product with respect to μ_s and ι_3 is the identity map of S^3. By [Sc, Thm. 0.1], if μ_s is homotopy-associative, then $\langle i, i\rangle_s$ is a generator of $\pi_6(\mathrm{G}_2) = \mathbb{Z}/3\mathbb{Z}$ [Mi]. Hence $\langle i, i\rangle_0$ is a generator of $\pi_6(\mathrm{G}_2)$. Since $\langle\iota_3, \iota_3\rangle_s = (2s+1)\langle\iota_3, \iota_3\rangle_0$ by [AC3, Lem. 4], we have that $\langle i, i\rangle_s = i_*\langle\iota_3, \iota_3\rangle_s = (2s+1)i_*\langle\iota_3, \iota_3\rangle_0 = (2s+1)\langle i, i\rangle_0$. Therefore if $s \equiv 1 \pmod 3$, then $\langle i, i\rangle_s = 0$ so that μ_s is not homotopy-associative.

3 The Order of Inverses

Let (X, μ) be an H-space with left inverse λ and right inverse ρ. The *(multiplicative* or *composition) order* of λ is the smallest positive integer n such that $\lambda^n = id$, and we write $|\lambda| = n$. If there is no such positive integer, let $|\lambda| = \infty$. Then $|\lambda| = |\rho|$ by Lemma 2.1. If $2\pi_i(X) \neq 0$, for some i, then $\lambda \neq id$ since $\lambda_\# = -1$. The condition $2\pi_i(X) \neq 0$, for some i, holds for any non-contractible, finite H-complex [Cl, Thm. 1]. Thus if X is a non-contractible finite H-complex, then $|\lambda| = |\rho| = 2$ if μ is homotopy-associative. More generally, if $[X, X]$ is a group, then $|\lambda| = |\rho| = 2$. In particular, this is the case for every multiplication on S^1, S^3, S^7, $S^m \times S^n$ $(m, n \in \{1, 3, 7\})$, SU(3) and Sp(2).

Now assume that (X, μ) is a non-contractible, finite H-complex, but μ is not necessarily homotopy-associative. What is $|\lambda|$? First note that if $|\lambda| < \infty$, then $|\lambda|$ is even. For if $|\lambda| = n$ with n odd, then for all $x \in \pi_i(X)$,

$$x = \lambda_\#^n(x) = (-1)^n x = -x.$$

But since X is a non-contractible, finite H-space, $2\pi_i(X) \neq 0$, for some i. This contradicts $x = -x$.

We next give an example of a multiplication whose left inverse has order 6. Let S^3 have the standard multiplication μ_0 induced from the quaternion structure of $\mathbb{R}^4$ and let $\phi_0 \in [S^3 \times S^3, S^3]$ be the commutator. Let $p_i : S^3 \times S^3 \to S^3$, $i = 1, 2$, be the two projections and let $\pi_j : S^3 \times S^3 \times S^3 \to S^3$, $j = 1, 2, 3$, be the three projections.

Lemma 3.1 *(1)* $[-\pi_2, \pi_1] = [\pi_2, -\pi_1] = [\pi_1, \pi_2]$ *and*
(2) $[[\pi_1, \pi_2], \pi_3] \in [S^3 \times S^3 \times S^3, S^3]$ *has order 3.*

Proof. (1) Let $p : S^3 \times S^3 \times S^3 \to S^3 \times S^3$ project onto the first two factors. Then for $i = 1, 2$, we have $\pi_i = p_i p$ so

$$[-\pi_2, \pi_1] = [-p_2, p_1]p.$$

But $[S^3 \times S^3, S^3]$ is a group of nilpotency ≤ 2 [Wh, p. 464], and for such groups $[na, b] = n[a, b]$ for any integer n [Ro, Lem. 5.42]. Therefore

$$[-\pi_2, \pi_1] = -[p_2, p_1]p = [p_1, p_2]p = [\pi_1, \pi_2].$$

(2) We have $[[\pi_1, \pi_2], \pi_3] = \gamma q = q^*(\gamma)$, where $q : S^3 \times S^3 \times S^3 \to S^9$ is the quotient map onto the smash product and $\gamma \in \pi_9(S^3)$ is the triple Samelson product $\langle\langle\iota_3, \iota_3\rangle, \iota_3\rangle$ of the identity class $\iota_3 \in \pi_3(S^3)$. But γ has order 3 [Ja1, §3] and $q^* : [S^9, S^3] \to [S^3 \times S^3 \times S^3, S^3]$ is a monomorphism. Therefore $[[\pi_1, \pi_2], \pi_3]$ has order 3. □

Let S^3 have the standard multiplication μ_0 and let $\phi_0 : S^3 \times S^3 \to S^3$ be the commutator map. If $X = S^3 \times S^3 \times S^3$ and $\theta : A \to X$ is the map such that $\pi_1\theta = \theta_1$, $\pi_2\theta = \theta_2$ and $\pi_3\theta = \theta_3$, then we write $\theta = (\theta_1, \theta_2, \theta_3)$.

Example 3.2 Define a multiplication μ on X by

$$\mu = (\mu_0(\pi_1 \times \pi_1), \mu_0(\pi_2 \times \pi_2), \mu_0(\pi_3 \times \pi_3) + \phi_0(\pi_1 \times \pi_2)),$$

(Cf. [AL, Prop. 6.2]). This homotopy class can be represented by the function $m : X \times X \to X$ given by

$$m((a, b, c), (a', b', c')) = (a + a', b + b', c + c' + [a, b']).$$

Then

$$\lambda = (-\pi_1, -\pi_2, [\pi_2, -\pi_1] - \pi_3)$$

and can be represented by the function $l : X \to X$ defined by

$$l(a, b, c) = (-a, -b, [b, -a] - c).$$

Then a simple calculation gives

$$\begin{aligned} \lambda^2 &= (\pi_1, \pi_2, [-\pi_2, \pi_1] + \pi_3 + [-\pi_1, \pi_2]) \\ &= (\pi_1, \pi_2, [\pi_1, \pi_2] + \pi_3 - [\pi_1, \pi_2]) \end{aligned}$$

by Lemma 3.1. But $[\pi_1, \pi_2] + \pi_3 - [\pi_1, \pi_2] = [[\pi_1, \pi_2], \pi_3] + \pi_3$. Therefore

$$\lambda^2 = (\pi_1, \pi_2, [[\pi_1, \pi_2], \pi_3] + \pi_3).$$

Since the group $[X, S^3]$ has nilpotency class ≤ 3, it follows that

$$\lambda^4 = (\pi_1, \pi_2, 2[[\pi_1, \pi_2], \pi_3] + \pi_3)$$

and

$$\lambda^6 = (\pi_1, \pi_2, 3[[\pi_1, \pi_2], \pi_3] + \pi_3).$$

Thus $\lambda^2 \neq id$, $\lambda^4 \neq id$ and $\lambda^6 = id$ by Lemma 3.1. Therefore $|\lambda| = 6$.

4 Rational Methods

In this section we use methods of rational homotopy theory to investigate the inverses of an H-space. This section is a continuation and further development of some of the ideas in [AL]. We begin by summarizing several basic facts about rational homotopy theory which we shall need. Some references for this material are [GM], [Ha], [HMR] and [Su]. If X is a nilpotent, finite complex, then $X_{\mathbb{Q}}$ denotes the rationalization or $\mathbb{Q}$-localization of X and $\alpha_{\mathbb{Q}} \in [X_{\mathbb{Q}}, Y_{\mathbb{Q}}]$ denotes the rationalization of $\alpha \in [X, Y]$. Furthermore, $\mathcal{M} = \mathcal{M}_X$ denotes the minimal algebra of X (also called the Sullivan minimal model of X) which is a free-commutative, differential graded algebra over the rationals $\mathbb{Q}$ with decomposable differential whose cohomology is $H^*(X;\mathbb{Q})$. If X is a finite H-complex, the differential of $\mathcal{M}$ is zero and so $\mathcal{M} = H^*(X;\mathbb{Q}) = H^*(X_{\mathbb{Q}};\mathbb{Q}) = \Lambda(x_1, \ldots, x_k)$, the exterior algebra on generators $x_1, \ldots, x_k$ of odd degree. The homotopy category of rational spaces of finite type is equivalent to the homotopy category of minimal algebras of finite type. In particular, $[X, Y]$ corresponds bijectively to $[\mathcal{M}_Y, \mathcal{M}_X]$. If X and Y are H-spaces, $[X, Y] \approx \mathrm{Hom}(\mathcal{M}_{\mathrm{Y}}, \mathcal{M}_{\mathrm{X}})$.

If $\mathcal{M} = \Lambda(x_1, \ldots, x_k)$ is the minimal algebra of a finite H-complex X, where $|x_i| = n_i$ is odd, then a multiplication on $\mathcal{M}$ is a homomorphism $\phi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ whose composition with the two projections $\mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ is the identity homomorphism. (We have called ϕ a multiplication instead of a comultiplication because it corresponds to a multiplication on X.) In $\mathcal{M} \otimes \mathcal{M}$ we denote $x \otimes 1$ by x' and $1 \otimes x$ by x''. Then a multiplication ϕ on $\mathcal{M}$ can be written

$$\phi(x_i) = x_i' + x_i'' + P(x_i),$$

where $P(x_i)$ is a polynomial in $x_1', \ldots, x_k', x_1'', \ldots, x_k''$ each monomial of which contains as a factor at least one of $x_1', \ldots, x_k'$ and at least one of $x_1'', \ldots, x_k''$. Then P is called the perturbation of ϕ [AL, Def. 2.1]. Clearly the perturbation determines the multiplication. A left inverse for a multiplication ϕ on $\mathcal{M}$ is a homomorphism $\gamma : \mathcal{M} \to \mathcal{M}$ such that $\nabla(\gamma \otimes id)\phi = 0 : \mathcal{M} \to \mathcal{M}$, where $\nabla : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ is the product homomorphism. The right inverse of ϕ is similarly defined.

Definition 4.1 Let X be a finite H-complex and $\chi \in [X, X]$. Then χ is called a *quasi-inverse of* X if χ induces multiplication by -1 on all homotopy groups of X. Let $\mathcal{M} = \Lambda(x_1, \ldots, x_k)$ with $|x_i| = n_i$ odd. A homomorphism $\gamma : \mathcal{M} \to \mathcal{M}$ is called a *quasi-inverse of* $\mathcal{M}$ if γ induces multiplication by -1 on the vector space of indecomposables. Thus a left or right inverse for any multiplication of X is a quasi-inverse of X and a left or right inverse for any multiplication of $\mathcal{M}$ is a quasi-inverse of $\mathcal{M}$. The function defined by $\gamma_0(x_i) = -x_i$, for all i is called the *trivial* quasi-inverse of $\mathcal{M}$.

Remark 4.2 *If γ is a quasi-inverse of $\mathcal{M}$, note that*

$$\gamma(x_i) = -x_i + Q(x_i),$$

where $Q(x_i)$ is a rational polynomial in $x_1, \ldots, x_k$ of degree ≥ 3, i.e., $Q(x_i)$ is a decomposable element of $\mathcal{M}$.

Proposition 4.3 *If γ is a non-trivial quasi-inverse of $\mathcal{M}$, then $|\gamma| = \infty$.*

Proof. Let j be the smallest positive integer such that $Q(x_j) \neq 0$. Then $\gamma(x_i) = -x_i$ for all $i < j$. Thus

$$\gamma^2(x_j) = -\gamma(x_j) + \gamma Q(x_j) = x_j - 2Q(x_j).$$

Then by induction,

$$\gamma^n(x_j) = (-1)^n(x_j - nQ(x_j)).$$

Therefore $|\gamma| = \infty$. □

Definition 4.4 Let X be a finite H-complex and $\mathcal{M}$ its minimal algebra. Then define a function $e_1 : [X, X] \to \mathrm{Hom}(\mathcal{M}, \mathcal{M})$ to be the composition

$$[X, X] \xrightarrow{l} [X_{\mathbb{Q}}, X_{\mathbb{Q}}] \xrightarrow{\approx} \mathrm{Hom}(\mathcal{M}, \mathcal{M})$$

and a function $e_2 : [X \times X, X] \to \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \mathcal{M})$ to be the composition

$$[X \times X, X] \xrightarrow{l} [X_{\mathbb{Q}} \times X_{\mathbb{Q}}, X_{\mathbb{Q}}] \xrightarrow{\approx} \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \mathcal{M}),$$

where l is the $\mathbb{Q}$-localization functor. Thus

$$e_1(\chi) = \chi_{\mathbb{Q}}^* : \mathcal{M} = H^*(X_{\mathbb{Q}}; \mathbb{Q}) \to \mathcal{M} = H^*(X_{\mathbb{Q}}; \mathbb{Q})$$

and $e_2(\mu) = \mu_{\mathbb{Q}}^* : \mathcal{M} = H^*(X_{\mathbb{Q}}; \mathbb{Q}) \to \mathcal{M} \otimes \mathcal{M} = H^*(X_{\mathbb{Q}}; \mathbb{Q}) \otimes H^*(X_{\mathbb{Q}}; \mathbb{Q})$.

Proposition 4.5 *Let X be a finite H-complex. Then the set of quasi-inverses of X of finite order is a finite set. In particular, the set of left inverses of multiplications of X of finite order is a finite set.*

Proof. Let $\mathcal{E}_{-1}(X)$ be the set of quasi-inverses of X and let $\mathcal{E}_{-1,f}(X) \subseteq \mathcal{E}_{-1}(X)$ be those quasi-inverses of finite order. Let $Q(\mathcal{M})$ be the set of quasi-inverses of $\mathcal{M}$. Clearly the function e_1 of Definition 4.4 induces a function $\epsilon_1 : \mathcal{E}_{-1}(X) \to Q(\mathcal{M})$ defined by $\epsilon_1(\chi) = \chi_{\mathbb{Q}}^*$. Then ϵ_1 carries elements of finite order into elements of finite order. Thus by Proposition 4.3, $\epsilon_1(\mathcal{E}_{-1,f}(X)) = \gamma_0$, the trivial quasi-inverse. But $e_1 : [X, X] \to \mathrm{Hom}(\mathcal{M}, \mathcal{M})$ is a finite-to-one function by [HMR, Cor. 5.4]. Thus $\mathcal{E}_{-1,f}(X)$ is a finite set. □

We have seen that several different multiplications may have the same left inverse, e.g., the multiplications μ_s of Proposition 2.6. But there are finitely many of these multiplications. We next consider if there exist infinitely many multiplications with the same left inverse. We first discuss some preliminaries.

Let X be a finite H-complex and $M(X) \subseteq [X \times X, X]$ the set of homotopy classes of multiplications of X. Let $\mathcal{M} = \Lambda(x_1, \ldots, x_k)$, with $|x_i| = n_i$ odd, be the minimal algebra of X and $M(\mathcal{M}) \subseteq \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \mathcal{M})$ the set of multiplications of $\mathcal{M}$. Then the function $e_2 : [X \times X, X] \to \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \mathcal{M})$ of Definition 4.4 induces a function $\epsilon_2 : M(X) \to M(\mathcal{M})$. If N is a positive integer and $\phi \in M(\mathcal{M})$ has perturbation P, then we denote by $\phi^{(N)} \in M(\mathcal{M})$ the multiplication having perturbation NP. It has been proved in [AL, Lem. 4.2] that if X is homotopy associative and $\phi \in M(\mathcal{M})$, then there is a positive integer N and a $\mu \in M(X)$ such that $\epsilon_2(\mu) = \mu^*_{\mathbb{Q}} = \phi^{(N)}$.

Proposition 4.6 *Let X be a finite, homotopy associative H-complex and suppose that $H^*(X; \mathbb{Q}) = \Lambda(x_1, \ldots, x_k)$ with $|x_i| = n_i$ odd. If some $n_i = \varepsilon_1 n_{j_1} + \varepsilon_2 n_{j_2} + \cdots + \varepsilon_t n_{j_t}$, where $j_1 < j_2 < \cdots < j_t$, each $\varepsilon_s = 1$ or 2 and some $\varepsilon_s = 2$, then there are infinitely many multiplications on X which have the same left inverse.*

Proof. Define a multiplication ψ on $\mathcal{M}$ with perturbation P as follows: If $j \neq i$, set $P(x_j) = 0$; otherwise set

$$P(x_i) = x'_{j_1} \cdots x'_{j_t} (x''_{j_1})^{a_1} \cdots (x''_{j_t})^{a_t},$$

where $a_s = \varepsilon_s - 1$. Then ψ is a multiplication on $\mathcal{M}$ with left inverse γ_0, in fact, for any positive integer N, $\psi^{(N)}$ is a multiplication with left inverse γ_0. Then there exists a positive integer N_1 and a multiplication μ_1 on X such that $\epsilon_2(\mu_1) = \psi^{(N_1)}$. Now repeat this process with $\psi^{(N_1)}$ replacing ψ. We obtain a positive integer N_2 and a multiplication μ_2 on X such that $\epsilon_2(\mu_2) = (\psi^{(N_1)})^{(N_2)} = \psi^{(N_1 N_2)}$. We continue in this way and obtain infinitely many multiplications $\mu_1, \ldots, \mu_s, \ldots$ on X. Let λ_s be the left inverse of μ_s. Hence $\epsilon_1(\lambda_s) = (\lambda_s)^*_{\mathbb{Q}}$ is a left inverse for $\psi^{(N_1 \cdots N_s)}$. But $\mathrm{Hom}(\mathcal{M}, \mathcal{M})$, with binary operation induced by $\psi^{(N_1 \cdots N_s)}$, is a loop [AL, Lem. 3.1], and so $\epsilon_1(\lambda_s) = (\lambda_s)^*_{\mathbb{Q}} = \gamma_0$. But $\epsilon_1^{-1}(\gamma_0)$ is finite (see the proof of Proposition 4.5). Thus there exists infinitely many positive integers $s_1, \ldots, s_n, \ldots$ such that $\lambda_{s_1} = \cdots = \lambda_{s_n} = \cdots$. Therefore all the multiplications μ_{s_n} have the same left inverse. □

We note that the condition in Proposition 4.6 is easily checked for any H-space whose rational cohomology is known. In particular, this is so for the simple Lie groups. Thus we have

Example 4.7 The following simple Lie groups have infinitely many multiplications with the same left inverse: $\mathrm{SU}(n)$, $n \geq 6$; $\mathrm{Sp}(n)$, $n \geq 8$; $\mathrm{Spin}(2n)$, $n = 5$, 7 and $n \geq 9$; $\mathrm{Sp}(2n+1)$, $n \geq 8$; E_6 and E_8.

Finally we give necessary and sufficient conditions for an H-space to admit a multiplication whose left inverse has infinite order. We let $\mathcal{E}_{\#}(X) \subseteq [X, X]$ denote the group of homotopy classes which induce the identity homomorphism on all homotopy groups.

Proposition 4.8 *If X is a finite, homotopy-associative H-space with $H^*(X;\mathbb{Q}) = \Lambda(x_1,\ldots,x_k)$, $|x_i| = n_i$, then the following are equivalent:*

(1) X admits a multiplication whose left inverse has infinite order.

(2) $\mathcal{E}_\#(X)$ is an infinite group.

(3) Some $n_i = n_{i_1} + \ldots + n_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ and $r \geq 3$.

Proof. The hypothesis of homotopy-associativity is only used for (3) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2): Let λ be a left inverse of infinite order. Then $\lambda^2 \in \mathcal{E}_\#(X)$ and has infinite order.
(2) $\Rightarrow$ (3): This is proved in [AC2, pp. 31–33].
(3) $\Rightarrow$ (1): Consider the minimal algebra of X, $\mathcal{M} = \Lambda(x_1,\ldots,x_k)$. Let $n_i = n_{i_1} + \cdots + n_{i_r}$ and let N be any positive integer. Let $\phi^{(N)} : \mathcal{M} \to \mathcal{M}\otimes\mathcal{M}$ be a multiplication given by

$$\phi^{(N)}(x_j) = \begin{cases} x'_j + x''_j & \text{if } j \neq i \\ x'_i + x''_i - N x'_{i_1} x''_{i_2} \cdots x''_{i_r} & \text{if } j = i. \end{cases}$$

Then a left inverse $\gamma^{(N)}$ to $\phi^{(N)}$ is given by

$$\gamma^{(N)}(x_j) = \begin{cases} -x_j & \text{if } j \neq i \\ -x_i - N x_{i_1} x_{i_2} \cdots x_{i_r} & \text{if } j = i\ . \end{cases}$$

Since

$$(\gamma^{(N)})^n(x_i) = (-1)^n(x_i - nN x_{i_1} x_{i_2} \cdots x_{i_r}),$$

$\gamma^{(N)}$ has infinite order.

Now let $\phi = \phi^{(1)} \in M(\mathcal{M})$ and consider the function $\epsilon_2 : M(X) \to M(\mathcal{M})$. There is a positive integer N such that $\phi^{(N)} = \epsilon_2(\mu) = \mu^*_{\mathbb{Q}}$ for some multiplication μ on X. Thus if λ is the left inverse of μ, then $\lambda^*_{\mathbb{Q}}$ is the left inverse of $\phi^{(N)}$. But $\mathrm{Hom}(\mathcal{M},\mathcal{M})$ is a loop, and so $\lambda^*_{\mathbb{Q}} = \gamma^{(N)}$. Therefore λ has infinite order. □

For completeness, we apply Proposition 4.8 to the simple Lie groups.

Example 4.9 Each of the following simple Lie groups admits a multiplication whose left inverse has infinite order: $\mathrm{SU}(n)$, $n \geq 8$; $\mathrm{Sp}(n)$, $n \geq 14$; $\mathrm{Spin}(2n)$, $n = 7, 9, 11, 13$ and $n \geq 15$; $\mathrm{Sp}(2n+1)$, $n \geq 14$; E_6.

References

[AC1] Arkowitz, M. and Curjel, C.: On the number of multiplications of an H-space. Topology **2**, 205–208 (1963).

[AC2] Arkowitz, M. and Curjel, C.: Groups of Homotopy Classes, Lecture Notes in Math. **4**, Berlin: Springer-Verlag 1964.

[AC3] Arkowitz, M. and Curjel, C.: Some properties of the exotic multiplications on the three-sphere. Quart. J. Math. **78**, 171–176 (1969).

[AL] Arkowitz, M. and Lupton, G.: Loop-theoretic properties of H-spaces. Math. Proc. Camb. Phil. Soc. **110**, 121–136 (1991).

[Cl] Clark, A.: On π_3 of finite dimensional H-spaces. Ann. Math. **78**, 193–196 (1963).

[Cu] Curjel, C.: On the H-space structures of finite complexes. Comm. Math. Helv. **43**, 1–17 (1968).

[GM] Griffiths, P. and Morgan, J.: Rational Homotopy Theory and Differential Forms. Progress in Math. **16**, Boston: Birkhäuser 1981.

[Ha] Halperin, S.: Lectures on Minimal Models, Memoire S. M. F. 9–10 1983.

[HMR] Hilton, P., Mislin, G. and Roitberg, J.: Localization of Nilpotent Groups and Spaces. Math. Studies **15**, New York: North Holland 1975.

[Hu] Hubbuck, J.: On homotopy commutative H-spaces. Topology **8**, 119–126 (1969).

[Ja1] James, I.: Multiplications on spheres II. Trans. Amer. Math. Soc. **84**, 545–558 (1957).

[Ja2] James, I.: On Lie groups and their homotopy groups. Proc. Camb. Phil. Soc. **55**, 244–247 (1959).

[Ja3] James, I.: On H-spaces and their homotopy groups. Quart. J. Math. **11**, 161–179 (1960).

[Mi] Mimura, M.: The homotopy groups of Lie groups of low rank. J. Math. Kyoto Univ. **6**, 131–176 (1967).

[Ro] Rotman, J.: An Introduction to the Theory of Groups. Graduate Texts in Math. **148**, New York: Springer-Verlag 1995.

[Sc] Schiffman, S.: A Samelson product and homotopy-associativity. Proc. Amer. Math. Soc. **7**, 189–195 (1978).

[St1] Stasheff, J.: Homotopy associativity of H-spaces I, Trans. Amer. Math. Soc. **108**, 275–292 (1963).

[St2] Stasheff, J.: Homotopy associativity of H-spaces II. Trans. Amer. Math. Soc. **108**, 293–312 (1963).

[Su] Sullivan, D.: Infinitesimal Computations in Topology. Publ. Math. I. H. E. S. **47**, 269–331 (1977).

[Wh] Whitehead, G.: Elements of Homotopy Theory. Graduate Texts in Math. **61**, New York: Springer-Verlag 1978.

[Za] Zabrodsky, A.: Homotopy associativity and finite CW complexes. Topology **9**, 121–128 (1970).

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