THE CONE LENGTH OF A PRODUCT OF CO-H-SPACES AND A PROBLEM OF GANEA

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Abstract. It is proved that the cone length or strong category of a product of two co-H-spaces is less than or equal to two. This yields the following positive solution to a problem of Ganea: Let $\alpha \in \pi_{2p}(S^3)$ be an element of order p, p a prime ≥ 3, and let $X(p) = S^3 \cup_\alpha e^{2p+1}$. Then $X(p) \times X(p)$ is the mapping cone of some map $\varphi : Y \to Z$, where Z is a suspension.

§1. Introduction

The (Lusternik-Schnirelmann) *category* of a topological space X, denoted $\operatorname{cat} X$, is a numerical invariant of homotopy type which has been extensively studied (see [Ja$_1$] and [Ja$_2$] for surveys). It follows easily from the definitions that $\operatorname{cat} X = 0$ if and only if X is contractible and $\operatorname{cat} X \leq 1$ if and only if X is a co-H-space. A related numerical invariant is the strong category or *cone length* of a space X, denoted $\operatorname{cl} X$ [Ga$_1$]. This is defined as follows: $\operatorname{cl} X$ is the least integer $n \geq 0$ such that there exists n cofibration sequences

$$L_i \longrightarrow X_i \longrightarrow X_{i+1}, \quad 0 \leq i < n,$$

with X_0 contractible and X_n having the homotopy type of X. Then $\operatorname{cl} X \leq 1$ if and only if X has the homotopy type of a suspension. Also $\operatorname{cl} X \leq 2$ if and only if there exists a cofibration sequence $L_1 \xrightarrow{\varphi} X_1 \longrightarrow C_\varphi$, where X_1 is a suspension and the mapping cone C_φ of φ has the homotopy type of X. It can happen that $\operatorname{cat} X \neq \operatorname{cl} X$ [St$_1$], but it has been shown [Ga$_1$] that

$$\operatorname{cat} X \leq \operatorname{cl} X \leq \operatorname{cat} X + 1.$$

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In addition, there are inequalities for the category and the cone length of a cartesian product, namely,

$$\mathrm{cat}\,(X \times Y) \leq \mathrm{cat}\, X + \mathrm{cat}\, Y \quad \text{and}$$

$$\mathrm{cl}\,(X \times Y) \leq \mathrm{cat}\, X + \max\{\mathrm{cl}\, Y, 1\}.$$

The first is due to Bassi [Fo] and the second to Takens [Ta]. Thus if X and Y are co-H-spaces, $\mathrm{cat}\,(X \times Y) \leq 2$ and $\mathrm{cl}\,(X \times Y) \leq 3$. The following theorem, which is our main result, shows that the cone length is in fact less than or equal to 2.

Theorem. *If A and A' are co-H-spaces of the homotopy type of 1-connected CW complexes, then $cl\,(A \times A') \leq 2$.*

The theorem is proved by constructing a map from a mapping cone C_φ to $A \times A'$, where φ maps into a suspension, and then showing that this map induces homology isomorphisms (Corollary 3.3).

¿From the theorem we obtain a positive solution to the following problem of Ganea. Let p be a prime ≥ 3 and let $\alpha \in \pi_{2p}(S^3)$ be an element of order p. We attach a cell to S^3 by α and form the cell complex $X(p) = S^3 \cup e^{2p+1}$. It is known that $X(p)$ is a co-H-space but not a suspension [B-H], and so $\mathrm{cat}\, X(p) = 1$ and $\mathrm{cl}\, X(p) = 2$. Ganea asked [Ga_2, Problem 8] if $X(p) \times X(p)$ can be the mapping cone C_φ of some map $\varphi : Y \to Z$, where Z is a suspension, i.e., if $\mathrm{cl}\,(X(p) \times X(p))$ is less than or equal to 2. This is answered affirmatively by the above theorem. We note that in [F-S] Fernández-Suárez proved by a different method the following related result: If Y is the 3-localization of $X(3)$, then $\mathrm{cl}\,(Y \times Y) \leq 2$.

Ganea does not give any reasons for stating Problem 8, but it seems clear that it is related to Takens's inequality. More precisely, if X is a space such that $\mathrm{cat}\, X = 1$, $\mathrm{cl}\, X = 2$ and $\mathrm{cl}\,(X \times X) \leq 2$, then Takens's inequality is strict for $X \times X$. Until recently the only known examples where Takens's inequality is strict occurred when both of the spaces had torsion in their homology. Thus an affirmative answer to Problem 8 shows that there are spaces X without homological torsion such that $\mathrm{cl}\,(X \times X) < \mathrm{cat}\, X + \mathrm{cl}\, X$. Hence by the theorem the $X(p)$ are such spaces and by [F-S] so is the 3-localization of $X(3)$. Of course the theorem provides many more examples of this phenomenon. Incidentally, there are torsion-free examples where $\mathrm{cl}\,(X \times X)$ and $\mathrm{cat}\, X + \mathrm{cl}\, X$ differ by 2 [St_2].

For the remainder of this section we present our notation and conventions. All topological spaces will be based and have the based homotopy type of connected CW-complexes. All maps and homotopies will preserve base points. The standard notation of homotopy theory will be used: 'Σ' for reduced suspension, '$\vee$' for the wedge of spaces and '$\wedge$' for the smashed product of spaces. We write '$\approx$' to denote natural homeomorphism of spaces or isomorphism of groups and '$\equiv$' to denote same homotopy type of spaces. For spaces R and S, the following natural homeomorphisms will be frequently used:

$$(\Sigma R) \wedge S \approx \Sigma(R \wedge S) \approx R \wedge (\Sigma S).$$

The identity map of a space R is written $\mathrm{id}\colon R \to R$ and the constant map is written $0 : R \to S$. We denote the natural inclusion by $\iota : R \vee S \to R \times S$ and the natural projection by $\chi : R \times S \to R \wedge S$, so that $R \vee S \xrightarrow{\iota} R \times S \xrightarrow{\chi} R \wedge S$ is a cofibration sequence. By a commutative diagram of spaces and maps we will mean one that is commutative up to homotopy. For maps f and g, we write $f = g$ to denote equality or homotopy of maps.

§2. Preliminaries

Our standing hypothesis in this section is that there are spaces X, X', A, A', W and W' such that $\Sigma X \equiv A \vee \Sigma W$ and $\Sigma X' \equiv A' \vee \Sigma W'$. We shall use the following notation: $i : A \to \Sigma X$ and $j : \Sigma W \to \Sigma X$ are the inclusions and $p : \Sigma X \to A$ and $q : \Sigma X \to \Sigma W$ are the projections. Using X', A' and W' in place of X, A and W, we obtain analogous maps i', j', p' and q'.

2.1. Definition. We respectively define maps $h' : A \wedge X' \to A \wedge W'$ and $k' : A \wedge X' \to \Sigma X \vee \Sigma X'$ as the following compositions:

$$A \wedge X' \xrightarrow{i \wedge \mathrm{id}} (\Sigma X) \wedge X' \approx X \wedge (\Sigma X') \xrightarrow{\mathrm{id} \wedge q'} X \wedge (\Sigma W') \approx (\Sigma X) \wedge W' \xrightarrow{p \wedge \mathrm{id}} A \wedge W'$$

and

$$A \wedge X' \xrightarrow{i \wedge \mathrm{id}} (\Sigma X) \wedge X' \approx \Sigma(X \wedge X') \xrightarrow{w} \Sigma X \vee \Sigma X',$$

where w is the generalized Whitehead product map [Ar].

2.2. Lemma. *There is a commutative diagram of homology groups*

$$\begin{array}{ccc} H_*\big(\Sigma(A \wedge X')\big) & \approx & H_*\big(A \wedge (\Sigma X')\big) \\ \downarrow{\scriptstyle (\Sigma h')_*} & & \downarrow{\scriptstyle (\mathrm{id} \wedge q')_*} \\ H_*\big(\Sigma(A \wedge W')\big) & \approx & H_*\big(A \wedge (\Sigma W')\big), \end{array}$$

where the horizontal isomorphisms are induced by the natural homeomorphisms. Consequently, $(\Sigma h')_*$ is onto and, if e is the composition

$$A \wedge A' \xrightarrow{\mathrm{id} \wedge i'} A \wedge (\Sigma X') \approx \Sigma(A \wedge X'),$$

then Kernel $(\Sigma h')_* =$ Image e_*.

Proof. The proof consists of showing that the maps $\Sigma(A\wedge X') \xrightarrow{\Sigma h'} \Sigma(A\wedge W') \approx A\wedge(\Sigma W')$ and $\Sigma(A\wedge X') \approx A\wedge(\Sigma X') \xrightarrow{\mathrm{id}\wedge q'} A\wedge(\Sigma W')$ are homotopic. This is a long, but straightforward calculation, and hence omitted. □

2.3. Lemma. $k'_* = 0 : H_*(A\wedge X') \longrightarrow H_*(\Sigma X \vee \Sigma X')$.

Proof. This is a consequence of $w_* = 0$. The latter is seen as follows: If $\chi : X \times X' \longrightarrow X\wedge X'$ is the projection, then $w\,\Sigma\chi : \Sigma(X\times X') \longrightarrow \Sigma X \vee \Sigma X'$ is defined as a commutator using the suspension structure of $\Sigma(X\times X')$ (see [Ar] for details). Therefore $(w\,\Sigma\chi)_* = 0$. Since $(\Sigma\chi)_*$ is onto, we obtain $w_* = 0$. □

We note that A is a co-H-space since it is a retract of ΣX. Thus $A\wedge X'$ is a co-H-space. Therefore one can add two maps defined on $A\wedge X'$.

2.4. Definition. We define $h, k : A\wedge X' \longrightarrow (\Sigma X\vee\Sigma X')\vee(A\wedge W')$ as the respective compositions

$$A\wedge X' \xrightarrow{h'} A\wedge W' \xrightarrow{i_2} (\Sigma X\vee\Sigma X')\vee(A\wedge W') \text{ and}$$
$$A\wedge X' \xrightarrow{k'} \Sigma X\vee\Sigma X' \xrightarrow{i_1} (\Sigma X\vee\Sigma X')\vee(A\wedge W'),$$

where i_1 and i_2 are the inclusions. Then define $g : A\wedge X' \longrightarrow (\Sigma X\vee\Sigma X')\vee(A\wedge W')$ by $g = h + k$.

2.5. Lemma. $g_* = h_* : H_*(A\wedge X') \to H_*\big((\Sigma X\vee\Sigma X')\vee(A\wedge W')\big)$.

Proof. This follows from Lemma 2.3 since $g_* = h_* + k_*$. □

§3. The Main Result

The standing hypothesis of §2 also holds in this section. We consider the commutative diagram

$$\begin{array}{ccccccc}
A \wedge X' & \xrightarrow{g} & \Sigma X \vee \Sigma X' \vee (A \wedge W') & \xrightarrow{a} & C_g & \xrightarrow{b} & \Sigma(A \wedge X') \\
\downarrow^{\mathrm{id}} & & \downarrow^{r} & & \downarrow^{\tilde{r}} & & \\
A \wedge X' & \xrightarrow{k'} & \Sigma X \vee \Sigma X' & \longrightarrow & C_{k'} & & \\
\downarrow^{i \wedge \mathrm{id}} & & & & & & \\
(\Sigma X) \wedge X' & & \downarrow^{\mathrm{id}} & & \downarrow^{s} & & \\
\downarrow^{\approx} & & & & & & \\
\Sigma(X \wedge X') & \xrightarrow{w} & \Sigma X \vee \Sigma X' & \xrightarrow{\iota} & \Sigma X \times \Sigma X' & & \\
 & & \downarrow^{p \vee p'} & & \downarrow^{p \times p'} & & \\
 & & A \vee A' & \xrightarrow{\iota} & A \times A' & \xrightarrow{\chi} & A \wedge A',
\end{array}$$

where the first three horizontal lines are mapping cone sequences, r is the projection, $\widetilde{r}$ is induced by the upper left hand square and s is induced by the lower left hand square. Note that by [Ar] the mapping cone C_w of the generalized Whitehead product map w is homotopically equivalent to $\Sigma X \times \Sigma X'$. For notational convenience set

$$\pi = (p \vee p')r \quad \text{and} \quad \lambda = (p \times p')s\widetilde{r}$$

so we have a commutative diagram

$$\begin{array}{ccc}
\Sigma X \vee \Sigma X' \vee (A \wedge W') & \xrightarrow{\quad a \quad} & C_g \\
\Big\downarrow \pi & & \Big\downarrow \lambda \\
A \vee A' & \xrightarrow{\quad \iota \quad} & A \times A'.
\end{array}$$

Now let ℓ be the composition

$$\Sigma W \vee \Sigma W' \xrightarrow{j \vee j'} \Sigma X \vee \Sigma X' \xrightarrow{i_1} (\Sigma X \vee \Sigma X') \vee (A \wedge W')$$

and form the mapping cone sequence

$$\Sigma W \vee \Sigma W' \xrightarrow{a\ell} C_g \xrightarrow{c} C_{a\ell},$$

that is, $C_{a\ell}$ is the mapping cone of $a\ell$ with inclusion $c : C_g \to C_{a\ell}$. Then $\lambda a \ell = \iota \pi \ell = \iota(p \vee p')ri_1(j \vee j') = 0$ since $ri_1 = \mathrm{id}$, $pj = 0$ and $p'j' = 0$. Thus there exists an $f : C_{a\ell} \to A \times A'$ such that the following diagram commutes

$$\begin{array}{ccc}
C_g & & \\
\downarrow^{c} & \overset{\lambda}{\searrow} & \\
C_{a\ell} & \xrightarrow{f} & A \times A'.
\end{array}$$

3.1. Proposition. *The map f induces a homology isomorphism $f_* : H_*(C_{a\ell}) \to H_*(A \times A')$. Consequently, f is a homotopy equivalence.*

Proof. The proof proceeds by a sequence of numbered steps.

(1) There exists a map $\widetilde{\lambda} : \Sigma(A \wedge X') \to A \wedge A'$ such that the following diagram commutes

$$\begin{array}{ccc} C_g & \xrightarrow{b} & \Sigma(A \wedge X') \\ \big\downarrow \lambda & & \big\downarrow \tilde{\lambda} \\ A \times A' & \xrightarrow{\chi} & A \wedge A'. \end{array}$$

By Lemmas 2.2 and 2.5,

$$\operatorname{Image} b_* = \operatorname{Kernel}(\Sigma g)_* = \operatorname{Kernel}(\Sigma h)_* = \operatorname{Kernel}(\Sigma h')_* = \operatorname{Image} e_*,$$

where e is $A \wedge A' \xrightarrow{\mathrm{id} \wedge i'} A \wedge \Sigma X' \approx \Sigma(A \wedge X')$.

(2) We show that $\widetilde{\lambda}_* e_* = \mathrm{id} : H_*(A \wedge A') \longrightarrow H_*(A \wedge A')$. By [Ru, Theorem 1], the following square commutes

$$\begin{array}{ccc} C_{k'} & \longrightarrow & \Sigma(A \wedge X') \\ & & \big\downarrow \Sigma(i \wedge \mathrm{id}) \\ \big\downarrow s & & \Sigma((\Sigma X) \wedge X') \\ & & \big\downarrow \approx \\ \Sigma X \times \Sigma X' & \xrightarrow{\chi} & \Sigma X \wedge \Sigma X'. \end{array}$$

Since $\lambda = (p \times p')s\widetilde{r}$, it follows that $\widetilde{\lambda}$ can be taken to be the composition

$$\Sigma(A \wedge X') \xrightarrow{\Sigma(i \wedge \mathrm{id})} \Sigma\big((\Sigma X) \wedge X'\big) \approx \Sigma X \wedge \Sigma X' \xrightarrow{p \wedge p'} A \wedge A'.$$

Therefore $\widetilde{\lambda} e$ is

$$A \wedge A' \xrightarrow{\mathrm{id} \wedge i'} A \wedge \Sigma X' \approx \Sigma(A \wedge X') \xrightarrow{\Sigma(i \wedge \mathrm{id})} \Sigma(\Sigma X \wedge X') \approx \Sigma X \wedge \Sigma X' \xrightarrow{p \wedge p'} A \wedge A'.$$

But this latter composition is just $(p \wedge p')(i \wedge i') = \mathrm{id}$, and so $\widetilde{\lambda}_* e_* = \mathrm{id}$.

(3) Now we show that f_* is onto. Let $z \in H_*(A \times A')$. Then

$$\begin{aligned} \chi_*(z) &= \widetilde{\lambda}_* b_*(u) \quad \text{for some} \quad u \in H_*(C_g) \quad \text{by (1) and (2)} \\ &= \chi_* \lambda_*(u). \end{aligned}$$

Therefore

$$\begin{aligned} z - \lambda_*(u) &= \iota_*(v) \qquad \text{for some} \quad v \in H_*(A \vee A) \\ &= \iota_* \pi_*(w) \quad \text{for some} \quad w \in H_*(\Sigma X \vee \Sigma X' \vee (A \wedge W')) \\ &= \lambda_* a_*(w). \end{aligned}$$

Therefore

$$z = \lambda_*\big(u + a_*(w)\big) = f_* c_*\big(u + a_*(w)\big).$$

Thus $z \in$ Image f_*, and so f_* is onto.

(4) Next we consider the exact homology mapping cone sequence

$$H_*(A \wedge X') \xrightarrow{g_*} H_*\big(\Sigma X \vee \Sigma X' \vee (A \wedge W')\big) \xrightarrow{a_*} H_*(C_g) \xrightarrow{b_*} H_*\big(\Sigma(A \wedge X')\big).$$

Now $g_* = i_{2*} h'_*$ by Lemma 2.5 and $h'_* : H_*(A \wedge X') \longrightarrow H_*(A \wedge W')$ is onto by Lemma 2.2. Thus we have a short exact sequence

$$0 \longrightarrow H_*(\Sigma X \vee \Sigma X') \xrightarrow{(ai_1)_*} H_*(C_g) \xrightarrow{b_*} \text{Image}\, b_* = \text{Image}\, e_* \longrightarrow 0.$$

(5) Now we show Kernel $\lambda_* \subseteq$ Kernel c_*. Suppose $x \in H_*(C_g)$ and $\lambda_*(x) = 0$. Therefore

$$\widetilde{\lambda}_* b_*(x) = \chi_* \lambda_*(x) = 0.$$

But $b_*(x) = e_*(y)$ for some $y \in H_*(A \wedge A')$ by (1). Thus by (2)

$$0 = \widetilde{\lambda}_* e_*(y) = y$$

and so $b_*(x) = 0$. By (4), there exists $z \in H_*(\Sigma X \vee \Sigma X')$ such that $a_* i_{1*}(z) = x$. Consequently,

$$\lambda_* a_* i_{1*}(z) = \lambda_*(x) = 0.$$

Therefore, $\iota_* \pi_* i_{1*}(z) = 0$, and so $\pi_* i_{1*}(z) = 0$. ¿From the definition of π, we have $(p \vee p')_*(z) = 0$. Thus $z = (j \vee j')_*(w)$ for some $w \in H_*(\Sigma W \vee \Sigma W')$. Hence

$$x = a_* i_{1*}(z) = a_* i_{1*}(j \vee j')_*(w) = (al)_*(w).$$

By exactness, $c_*(x) = 0$, so $x \in$ Kernel c_*. This shows Kernel $\lambda_* \subseteq$ Kernel c_*.

(6) Next we see that $c_* : H_*(C_g) \longrightarrow H_*(C_{a\ell})$ is onto. Consider the mapping cone sequence of $a\ell$

$$\Sigma W \vee \Sigma W' \xrightarrow{a\ell} C_g \xrightarrow{c} C_{a\ell} \xrightarrow{t} \Sigma(\Sigma W \vee \Sigma W') \xrightarrow{\Sigma(a\ell)} \Sigma C_g.$$

Since $(a\ell)_*$ is a monomorphism by (4), $\Sigma(a\ell)_*$ is a monomorphism. Therefore $t_* = 0 : H_*(C_{a\ell}) \to H_*\big(\Sigma(\Sigma W \vee \Sigma W')\big)$, and so c_* is onto.

(7) Finally, we show that $f_* : H_*(C_{a\ell}) \to H_*(A \times A')$ is one-one. Suppose $f_*(x) = 0$ for $x \in H_*(C_{a\ell})$. By (6), $x = c_*(y)$ for some $y \in H_*(C_g)$. Thus

$$0 = f_*(x) = f_* c_*(y) = \lambda_*(y).$$

Consequently $y \in$ Kernel λ_* and so $y \in$ Kernel c_* by (5). Hence $x = c_*(y) = 0$. Therefore f_* is one-one.

This proves that f_* is an isomorphism. To complete the proof we first note that A and A' are 1-connected since they are retracts of ΣX and $\Sigma X'$ respectively. Thus f is a homotopy equivalence since $C_{a\ell}$ is also 1-connected. □

3.2. Theorem. *Let A, W and X be spaces such that $\Sigma X \equiv A \vee \Sigma W$ and A', W' and X' be spaces such that $\Sigma X' \equiv A' \vee \Sigma W'$ and set*

$$Y = (A \wedge X') \vee \Sigma W \vee \Sigma W' \vee \Sigma(W \wedge W') \quad \textit{and}$$

$$Z = \Sigma X \vee \Sigma X' \vee (A \wedge W') \vee \Sigma(W \wedge W').$$

Define $\varphi : Y \to Z$ by

$$\varphi\big|_{A\wedge X'} = i_0 g, \quad \varphi\big|_{\Sigma W} = i_1 j, \quad \varphi\big|_{\Sigma W'} = i_2 j', \textit{ and } \quad \varphi\big|_{\Sigma(W\wedge W')} = i_4,$$

where $g : A \wedge X' \to \Sigma X \vee \Sigma X' \vee (A \wedge W')$ is given by Definition 2.4 and $i_0 : \Sigma X \vee \Sigma X' \vee (A \wedge W') \to Z$, $i_1 : \Sigma X \to Z$, $i_2 : \Sigma X' \to Z$ and $i_4 : \Sigma(W \wedge W') \to Z$ are all inclusions. If C_φ is the mapping cone of φ, then $A \times A' \equiv C_\varphi$.

Proof. Let $\varphi' = \varphi\big|_{(A\wedge X')\vee\Sigma W\vee\Sigma W'} : (A \wedge X') \vee \Sigma W \vee \Sigma W' \to \Sigma X \vee \Sigma X' \vee (A \wedge W')$. Then $\varphi = \varphi' \vee$ id. Therefore $C_\varphi \equiv C_{\varphi'}$. But by Proposition 3.1, we have $C_{\varphi'} \equiv C_{a\ell} \equiv A \times A'$. □

The following corollary immediately implies the theorem stated in the introduction.

3.3. Corollary. *Given 1-connected co-H-spaces A and A'. Then there exists a space Y, a suspension Z and a map $\varphi : Y \to Z$ such that $A \times A' \equiv C_\varphi$, the mapping cone of φ.*

Proof. By [Ta], there exist spaces X, X', W and W' such that $\Sigma X \equiv A \vee \Sigma W$ and $\Sigma X' \equiv A' \vee \Sigma W'$. If $\varphi : Y \to Z$ is as defined in Theorem 3.2, then $A \times A' \equiv C_\varphi$. To complete the proof we show that Z is a suspension:

$$\begin{aligned} Z &= \Sigma X \vee \Sigma X' \vee (A \wedge W') \vee \Sigma(W \wedge W') \\ &\approx \Sigma X \vee \Sigma X' \vee \big((A \vee \Sigma W) \wedge W'\big) \\ &\equiv \Sigma X \vee \Sigma X' \vee (\Sigma X \wedge W') \\ &\approx \Sigma\big(X \vee X' \vee (X \wedge W')\big). \end{aligned}$$

□

3.4. Remarks. (1) Corollary 3.3 implies that if A and A' are co-H-spaces that are not suspensions, then Takens's inequality

$$\mathrm{cl}\,(A \times A') \leq \ \mathrm{cat}\ A + \ \max\{\mathrm{cl}\, A', 1\}$$

is strict since the right hand side is 3. In particular, we could take $A = A' = X(p)$, for p an odd prime.

(2) As observed earlier in this section, the product $\Sigma R \times \Sigma S$ of two suspensions has the homotopy type of the mapping cone of a map $Y \to \Sigma R \vee \Sigma S$. In fact, $Y = \Sigma(R \wedge S) \approx (\Sigma R) \wedge S$ and the map is the generalized Whitehead product map. Rutter [Ru] has shown how to extend this result to the case of the product of a suspension and a co-H-space. He proves that if A' is a co-H-space, then $(\Sigma R) \times A'$ has the homotopy type of the mapping cone of a map $Y \to \Sigma R \vee A'$, where $Y = R \wedge A'$. This raises the question (see [F-S, §1]) of whether the product $A \times A'$ of two co-H-spaces can be represented as the mapping cone of some map $Y \to A \vee A'$. The theorem in §1 shows that $A \times A'$ can be represented as a mapping cone.

(3) The theorem in Section 1 shows that

$$\mathrm{cl}\,(A \times A') \leq \mathrm{cat}\, A + \mathrm{cat}\, A'$$

in the case when $\mathrm{cat}\, A = \mathrm{cat}\, A' = 1$. We ask: Is this formula true for other values of $\mathrm{cat}\, A$ and $\mathrm{cat}\, A'$?

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