

The Sectional Category of a Map

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Abstract

We study a generalization of the Svarc genus of a fiber map. For an arbitrary collection \mathcal{E} of spaces and a map $f : X \rightarrow Y$, we define a numerical invariant, the \mathcal{E} -sectional category of f , in terms of open covers of Y . We obtain several basic properties of \mathcal{E} -sectional category, including those dealing with homotopy domination and homotopy pushouts. We then give three simple axioms which characterize the \mathcal{E} -sectional category. In the final section we obtain inequalities for the \mathcal{E} -sectional category of a composition and inequalities relating the \mathcal{E} -sectional category to the Fadell-Hussein category of a map and the Clapp-Puppe category of a map.

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1 Introduction

The sectional category of a fiber map $f : X \rightarrow Y$, denoted $\text{secat}(f)$, is one less than the number of sets in the smallest open cover of Y such that f admits a cross-section over each member of the cover. This simple and natural numerical invariant of fiber maps was developed and studied extensively by Svarc [21] who called it the genus of f . Subsequently Fet [11] and Berstein-Ganea [2] extended the definition to arbitrary maps and related it to the Lusternik-Schnirelmann category. There have been many applications of sectional category to questions of classification of bundles, embeddings of spaces and existence of regular maps [21] as well as applications outside of algebraic and differential topology [7, 10]. However, since Svarc's papers, the

actual study of sectional category has been sporadic and has often appeared as subsidiary results within a larger work [2, §2], [15, §8], [3, §4], [6, 9.3]. (An exception to this is the paper of Stanley [20] which deals with sectional category in the context of rational homotopy theory.)

Recently there has been considerable interest in the study of numerical invariants of the homotopy type of spaces and of the homotopy class of maps. Classically the Lusternik-Schnirelmann category and cone length of a space have been studied in [16, 12, 13, 4] and, more recently, the cone length [17], Clapp-Puppe category [3] and Fadell-Husseini category of a map [9, 5] have been investigated.

In the paper [1], we gave a unified axiomatic development of many of these invariants. For the category of spaces and maps and a fixed collection \mathcal{E} of objects, these axioms were of two basic types, namely, those dealing with numerical functions relative to \mathcal{E} defined on the objects of the category and those dealing with numerical functions relative to \mathcal{E} defined on the morphisms of the category. By specializing the collection \mathcal{E} , we obtained some of the previously studied invariants as well as several new invariants. For instance, by setting $\mathcal{E} = \{\text{all spaces}\}$, our axioms for morphisms yield the Fadell-Husseini category of a map. One invariant that was not dealt with in [1] was the sectional category. Whereas the various versions of category and cone length have been defined in numerous homotopy-invariant ways, this is not the case for sectional category. The fact that sectional category does not fit nicely into the general framework that so neatly encapsulates the category and cone length of spaces and maps may account for its marginal status in the decades since its introduction.

We begin in Section 3 by defining a generalization of the sectional category of a map with respect to a collection \mathcal{E} of spaces using open covers of the target of the map. This is a straightforward extension of the classical definition. We derive several simple basic properties of this invariant. In particular, when $\mathcal{E} = \{\text{all spaces}\}$, we see that $\mathcal{E}\text{-secat}(f) = \text{secat}(f)$. This treatment of sectional category leads to new invariants obtained by varying the collection \mathcal{E} of spaces.

In Section 4 we bring sectional category of maps in line with the other invariants studied in [1]. This is done by considering maps as *objects* in the category whose objects are maps of spaces and whose morphisms are given by commutative squares. In this category, we apply the axiomatic approach for invariants of objects, relative to the collection of all maps with sections which factor through a space in

\mathcal{E} . The unique numerical invariant obtained from the axioms is then proved to be \mathcal{E} -secat. This is the content of Theorem 4.7.

Another case of this axiomatic approach is obtained by working in the category of maps, relative to the collection of all maps which factor through some space in \mathcal{E} . The unique invariant obtained from the axioms can then be shown to be the \mathcal{E} -Clapp-Puppe category of a map (see Remark 4.8). We hope to return to this in the future.

All invariants that fall into our axiomatic scheme will of course share certain basic properties that follow formally from the axioms. A major interest, however, is in the new questions which arise regarding \mathcal{E} -sectional category. Many of these are considered in Section 5 and take the form of inequalities. There we concentrate on the following: (1) How is \mathcal{E} -secat(f) related to the domain and the target of f ? (2) How does \mathcal{E} -sectional category behave with respect to composition of maps? (3) What is the relation between the \mathcal{E} -sectional category, the Clapp-Puppe category and the Fadell-Husseini category?

2 Preliminaries

In this section we establish our notation and recall some basic definitions and results which we shall use. All spaces are to have the homotopy type of connected CW-complexes. We do not assume that spaces have base points and hence maps are not base point preserving. For spaces X and Y , $X \equiv Y$ denotes that X and Y have the same homotopy type. We let $*$ denote a space with one point and we also write $*$: $X \rightarrow Y$ for any constant map from X to Y . We denote by id_X or id the identity map of X . If $f, g : X \rightarrow Y$ are two maps, then $f \simeq g$ signifies that f and g are homotopic. Given maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$, if $fg \simeq \text{id}_Y$, we say that g is a **section** of f or that Y is a **retract** of X . For maps $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$, if there is a homotopy commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & X' & \xrightarrow{r} & X \\ \downarrow f & & \downarrow f' & & \downarrow f \\ Y & \xrightarrow{j} & Y' & \xrightarrow{s} & Y \end{array}$$

such that $ri \simeq \text{id}$ and $sj \simeq \text{id}$, then we say that f' **dominates** f or that f is a **retract** of f' . If, in addition, $ir \simeq \text{id}$ and $js \simeq \text{id}$, then f and f' are called **equivalent**, and we write $f \equiv f'$.

We will also use certain basic constructions in homotopy theory. The **pushout** of a diagram of the form

$$C \xleftarrow{g} A \xrightarrow{f} B$$

is the quotient space of $B \cup C$ by the equivalence relation which sets $f(a)$ equivalent to $g(a)$, for every $a \in A$. The **homotopy pushout** H of the diagram is the quotient space of $B \cup (A \times [0, 1]) \cup C$ by the equivalence relation which sets $(a, 0)$ equivalent to $f(a)$ and $(a, 1)$ equivalent to $g(a)$, for every $a \in A$. The pushout and homotopy pushout constructions are clearly functors from the category of given diagrams to the category of spaces.

There are two homotopy pushouts of special interest. The first is the homotopy pushout of

$$* \xleftarrow{\quad} A \xrightarrow{f} B$$

which is the **mapping cone** of f and is denoted $B \cup CA$. The second is the homotopy pushout of

$$A \xleftarrow{\text{id}_A} A \xrightarrow{f} B$$

which is the **mapping cylinder** of f and is denoted M_f . We note for later use that the homotopy pushout of

$$C \xleftarrow{g} A \xrightarrow{f} B$$

is homeomorphic to the pushout of the associated mapping cylinder diagram

$$M_g \xleftarrow{\quad} A \xrightarrow{\quad} M_f.$$

We next consider two versions of the category of a map. Given $f : X \rightarrow Y$, we first define the (reduced) **Fadell-Husseini category** of f , denoted $\text{cat}_{\text{FH}}(f)$ [9]. We note that f is equivalent to the inclusion of X into M_f , and so we regard f as an inclusion. Then $\text{cat}_{\text{FH}}(f)$ is the smallest n such that there is an open cover $\{U_0, U_1, \dots, U_n\}$ of Y with the following properties: (1) if $j_i : U_i \rightarrow Y$ is the inclusion, then $j_i \simeq *$ for $i = 0, 1, \dots, n$, (2) $X \subseteq U_0$ and (3) there is a map $r : U_0 \rightarrow X$ and a homotopy of pairs $j_0 \simeq r : (U_0, X) \rightarrow (Y, X)$. It follows that $\text{cat}_{\text{FH}}(* \rightarrow Y)$ is just the Lusternik-Schnirelmann category $\text{cat}(Y)$ of Y .

By a **collection** \mathcal{E} we mean any collection of spaces which contains the one point space $*$. The second notion of category of a map $f : X \rightarrow Y$ that we consider is the (reduced) **\mathcal{E} -Clapp-Puppe category** of f , denoted $\mathcal{E}\text{-cat}_{\text{CP}}(f)$ [3]. This is the smallest non-negative integer n with the following properties: (1) there exists an open cover $\{U_0, U_1, \dots, U_n\}$ of X , (2) there exists spaces $E_i \in \mathcal{E}$ and maps $u_i : U_i \rightarrow E_i$ and $v_i : E_i \rightarrow Y$, for $i = 0, 1, \dots, n$ and (3) $f|_{U_i} \simeq v_i u_i$ for each i . For a space X , the \mathcal{E} -Clapp-Puppe category of X is defined by $\mathcal{E}\text{-cat}_{\text{CP}}(X) = \mathcal{E}\text{-cat}_{\text{CP}}(\text{id}_X)$. If $\mathcal{E} = \{*\}$, the collection consisting of a one point space, then $\mathcal{E}\text{-cat}_{\text{CP}}(f)$ is the category of the map f , as discussed by Bernstein-Ganea [2] and others, which we will denote by $\text{cat}_{\text{BG}}(f)$. We note that $\text{cat}_{\text{BG}}(\text{id}_X) = \text{cat}(X)$.

3 Definition and Basic Properties

In this section we define the sectional category of a map relative to a collection \mathcal{E} of spaces. We then establish basic properties of this invariant. Of particular importance are the Domination Proposition (Prop. 3.6) and the Homotopy Pushout Theorem (Thm. 3.9).

Definition 3.1 1. If \mathcal{E} is a collection, $f : X \rightarrow Y$ a map and $i : U \hookrightarrow Y$ the inclusion map, then U is **\mathcal{E} -section-categorical** (with respect to f) if there is space $E \in \mathcal{E}$ and maps $u : U \rightarrow E$ and $v : E \rightarrow X$ such that the following diagram is homotopy-commutative:

$$\begin{array}{ccc} E & \xrightarrow{v} & X \\ u \uparrow & & \downarrow f \\ U & \xrightarrow{i} & Y. \end{array}$$

The map $vu : U \rightarrow X$ is called an **\mathcal{E} -section of f over U** .

2. For a map $f : X \rightarrow Y$, the **(reduced) \mathcal{E} -sectional category** of f , written $\mathcal{E}\text{-secat}(f)$, is the smallest integer n such that there exists an open cover of Y by $n + 1$ subsets, each of which is \mathcal{E} -section-categorical. If no such integer exists, then $\mathcal{E}\text{-secat}(f) = \infty$.

If \mathcal{E} is the collection of all spaces, then we write $\text{secat}(f)$ for $\mathcal{E}\text{-secat}(f)$ and note that $\text{secat}(f)$ is just the sectional category of f as defined in [2, Def. 2.1].

The following result lists a number of useful properties of \mathcal{E} -secat. The proofs are all straightforward, and we omit them.

Proposition 3.2 *Let \mathcal{E} be a collection.*

1. *If $f \simeq f'$, then $\mathcal{E}\text{-secat}(f) = \mathcal{E}\text{-secat}(f')$.*
2. *If $\mathcal{E} = \{*\}$, then $\mathcal{E}\text{-secat}(f) = \text{cat}(Y)$, the category of Y .*
3. *If \mathcal{E} and \mathcal{F} are two collections and $\mathcal{F} \subseteq \mathcal{E}$, then $\mathcal{E}\text{-secat}(f) \leq \mathcal{F}\text{-secat}(f)$.*
4. *If $* : X \rightarrow Y$ is a constant map, then $\mathcal{E}\text{-secat}(*) = \text{cat}(Y)$.*

Assertions (2) and (4) show that the Lusternik-Schnirelmann category of Y can be obtained from the \mathcal{E} -sectional category of f in two ways: either by making \mathcal{E} trivial or by making f trivial. Since $\{*\} \subseteq \mathcal{E} \subseteq \{\text{all spaces}\}$, we obtain from (3) the following.

Corollary 3.3 *For any map $f : X \rightarrow Y$ and collection \mathcal{E} ,*

$$\text{secat}(f) \leq \mathcal{E}\text{-secat}(f) \leq \text{cat}(Y).$$

Next we consider some basic properties of the identity map.

Proposition 3.4 *Let \mathcal{E} be a collection and X be a space.*

1. $\mathcal{E}\text{-secat}(\text{id}_X) = \mathcal{E}\text{-cat}_{\text{CP}}(X)$.
2. *X is a retract of a space in \mathcal{E} if and only if $\mathcal{E}\text{-secat}(\text{id}_X) = 0$. In particular, $\text{secat}(\text{id}_X) = 0$ for every space X .*

The following examples show that the inequalities in Proposition 3.2 and Corollary 3.3 can be strict.

Example 3.5 1. For the inequality in 3.2, let \mathcal{E} be the collection of all spaces and \mathcal{F} a collection which does not contain a space having X as a retract. Then

$$\mathcal{E}\text{-secat}(\text{id}_X) = 0 < \mathcal{F}\text{-secat}(\text{id}_X).$$

For a specific example, let $\mathcal{F} = \{*\}$ and take any $X \neq *$.

2. For the inequality in 3.3, consider $f : X \rightarrow Y$ and the inclusion $j : Y \rightarrow Y \cup CX$. Then $\text{secat}(j) \leq 1$ by Lemma 5.3 (below). If $\text{cat}(Y \cup CX) > 1$, then

$$\text{secat}(j) \leq 1 < \text{cat}(Y \cup CX).$$

This is the case, for example, when $f : S^{2n+1} \rightarrow \mathbb{C}P^n$ is the Hopf map.

We next establish a basic property of \mathcal{E} -sectional category. In the next section this will be shown to be one of three properties which characterize \mathcal{E} -secat.

Proposition 3.6 (Domination) *If $f : X \rightarrow Y$ is dominated by $f' : X' \rightarrow Y'$, then*

$$\mathcal{E}\text{-secat}(f) \leq \mathcal{E}\text{-secat}(f').$$

Proof We are given $j : Y \rightarrow Y'$ and $r : X' \rightarrow X$ as in the definition of domination in §2. Let $\mathcal{E}\text{-secat}(f') = n$ with \mathcal{E} -section-categorical cover $\{U'_0, U'_1, \dots, U'_n\}$ of Y' and maps

$$U'_i \xrightarrow{u_i} E_i \xrightarrow{v_i} X'$$

such that $f'v_iu_i \simeq j_i : U'_i \rightarrow Y'$, where $E_i \in \mathcal{E}$ and j_i is the inclusion. If $U_i = j^{-1}(U'_i)$, then $\{U_0, U_1, \dots, U_n\}$ is an open cover of Y and

$$U_i \xrightarrow{j|_{U_i}} U'_i \xrightarrow{u_i} E_i \xrightarrow{v_i} X' \xrightarrow{r} X$$

is the desired \mathcal{E} -section of f over U_i . \square

An immediate corollary of Proposition 3.6 is that \mathcal{E} -secat is an invariant of homotopy equivalence of maps.

Corollary 3.7 *If $f \equiv f'$, then $\mathcal{E}\text{-secat}(f) = \mathcal{E}\text{-secat}(f')$.*

It is well-known that every map is homotopy equivalent to a fiber map [18, p. 48]. Corollary 3.7 implies that the \mathcal{E} -sectional category of an arbitrary map is equal to the \mathcal{E} -sectional category of the equivalent fiber map. Therefore Svarc's definition of sectional category [21], which applies only to fiber maps, is equivalent to Definition 3.1 in the special case $\mathcal{E} = \{\text{all spaces}\}$.

We next prove a result about the \mathcal{E} -sectional category of the maps of one homotopy pushout into another. To establish notation, let

$$\begin{array}{ccccc} C & \xleftarrow{g} & A & \xrightarrow{f} & B \\ \downarrow c & & \downarrow a & & \downarrow b \\ C' & \xleftarrow{g'} & A' & \xrightarrow{f'} & B' \end{array}$$

be a commutative diagram, let D and D' be the homotopy pushouts of the top and bottom rows, respectively, and let $d : D \rightarrow D'$ be the induced map. We begin with a lemma.

Lemma 3.8 *If $\mathcal{E}\text{-secat}(b) = n$, then there exists \mathcal{E} -section-categorical open sets N_0, N_1, \dots, N_n of D' with respect to d such that $M_{f'} \subseteq \bigcup N_i$.*

Proof Let $\tilde{d} : M_f \rightarrow M_{f'}$ be the map d with restricted domain and target. Then $\tilde{d} \equiv b$, so $\mathcal{E}\text{-secat}(\tilde{d}) = \mathcal{E}\text{-secat}(b) = n$ by Corollary 3.7. To complete the proof, observe that if $U \subseteq M_{f'}$ is \mathcal{E} -section-categorical with respect to \tilde{d} , then it is also \mathcal{E} -section-categorical with respect to d . \square

Now we can prove the Homotopy Pushout Theorem, which is another basic property which we use in §4 to characterize \mathcal{E} -sectional category.

Theorem 3.9 (Homotopy Pushout) *With the notation above,*

$$\mathcal{E}\text{-secat}(d) \leq \mathcal{E}\text{-secat}(b) + \mathcal{E}\text{-secat}(c) + 1.$$

Proof Let $\mathcal{E}\text{-secat}(b) = n$ and $\mathcal{E}\text{-secat}(c) = m$. Let $\{U_0, U_1, \dots, U_n\}$ be a minimal open \mathcal{E} -section-categorical cover of B' with respect to b . Then there are \mathcal{E} -section-categorical open sets N_0, N_1, \dots, N_n of D' with respect to d which cover $M_{f'}$ by Lemma 3.8. Similarly, if $\{V_0, V_1, \dots, V_m\}$ is a minimal open \mathcal{E} -section-categorical cover of C' with respect to c , then there are \mathcal{E} -section-categorical open sets M_0, M_1, \dots, M_m of D' with respect to d which cover $M_{g'}$. It follows that $\{N_0, \dots, N_n, M_0, \dots, M_m\}$ is an \mathcal{E} -section-categorical cover of D' with respect to d . Thus $\mathcal{E}\text{-secat}(d) \leq \mathcal{E}\text{-secat}(b) + \mathcal{E}\text{-secat}(c) + 1$. \square

We conclude this section with a simple application of Theorem 3.9 to the maps in a homotopy pushout square.

Corollary 3.10 *If*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow r \\ C & \xrightarrow{s} & D \end{array}$$

is a homotopy pushout square, then

$$\mathcal{E}\text{-secat}(r) \leq \mathcal{E}\text{-secat}(g) + \mathcal{E}\text{-cat}_{\text{CP}}(B) + 1.$$

Proof The map of homotopy pushouts obtained from the commutative diagram

$$\begin{array}{ccccc}
 A & \xlongequal{\quad} & A & \xrightarrow{f} & B \\
 \downarrow g & & \parallel & & \parallel \\
 C & \xleftarrow{g} & A & \xrightarrow{f} & B
 \end{array}$$

is equivalent to $r : B \rightarrow D$. Now apply Theorem 3.9, using Proposition 3.4(1). \square

4 Axioms

In this section we characterize the \mathcal{E} -sectional category by simple axioms. For reasons given in Remark 4.8 we state our axioms in greater generality than is needed for sectional category.

Definition 4.1 We denote by \mathcal{S} a non-empty collection of maps. An **\mathcal{S} -length function** is a function $\gamma = \gamma_{\mathcal{S}}$ which assigns to every map f an integer $0 \leq \gamma(f) \leq \infty$ such that

1. If $f \in \mathcal{S}$, then $\gamma(f) = 0$.
2. Let

$$\begin{array}{ccccc}
 C & \xleftarrow{g} & A & \xrightarrow{f} & B \\
 \downarrow c & & \downarrow a & & \downarrow b \\
 C' & \xleftarrow{g'} & A' & \xrightarrow{f'} & B'
 \end{array}$$

be a commutative diagram with induced map $d : D \rightarrow D'$ of the homotopy pushout of the first row into the homotopy pushout of the second row. If $c \in \mathcal{S}$, then $\gamma(d) \leq \gamma(b) + 1$.

3. If f is dominated by f' , then $\gamma(f) \leq \gamma(f')$.

We call (1)–(3) the **axioms for \mathcal{S} -length functions**.

It is an immediate consequence of Axiom (3) that if $f \equiv f'$, then $\gamma(f) = \gamma(f')$. Then, since homotopic maps are equivalent (§2), it follows that $f \simeq f'$ implies that $\gamma(f) = \gamma(f')$.

Remark 4.2 These axioms are analogous to the axioms for the \mathcal{A} -category of a *space* given in [1, Prop. 5.6(2)]. The following comments are made to elucidate the analogy. In Definition 4.1 we define the \mathcal{S} -length function on the objects in the category of maps of spaces. In [1,

p. 26] an \mathcal{S} -length function was defined on the objects in the category of spaces in the case where \mathcal{S} is the collection of contractible spaces, though we could have taken an arbitrary collection \mathcal{S} in the definition. This is why the analog of Axiom (3) in [1] deals with mapping cone sequences instead of homotopy pushouts. In addition, the axioms in [1, p. 26] were made with respect to an arbitrary collection \mathcal{A} of spaces. It is possible to incorporate an arbitrary collection \mathcal{A} of maps into Definition 4.1, but we have not given the definition in this generality.

We are led to the following definition.

Definition 4.3 The **maximal \mathcal{S} -length function** $\Gamma_{\mathcal{S}}$ is defined by

$$\Gamma_{\mathcal{S}}(f) = \max\{\gamma_{\mathcal{S}}(f) \mid \text{for every } \mathcal{S}\text{-length function } \gamma_{\mathcal{S}}\}.$$

Note that $\Gamma_{\mathcal{S}}$ satisfies the axioms for \mathcal{S} -length functions.

We wish to show that \mathcal{E} -secat equals $\Gamma_{\mathcal{S}}$ for some \mathcal{S} . For this we need to choose a collection \mathcal{S} of maps that is closely related to \mathcal{E} .

Definition 4.4 For a collection of spaces \mathcal{E} , let

$$\mathcal{S}(\mathcal{E}) = \{f : X \rightarrow Y \mid \text{there is an } \mathcal{E}\text{-section of } f \text{ over } Y\}.$$

The main result of this section is that $\Gamma_{\mathcal{S}(\mathcal{E})} = \mathcal{E}$ -secat. In order to prove this, we need to show that we can replace open covers with covers by simplicial subcomplexes in the definition of \mathcal{E} -sectional category. This is crucial in dealing with maps that are defined by homotopy pushouts. We require the following lemma.

Lemma 4.5 *Let $f : X \rightarrow Y$ be a map and $B \subseteq Y$ an \mathcal{E} -section-categorical subset. If $A \subseteq B$, then A is \mathcal{E} -section-categorical. If $B \subseteq C$ is a deformation retract, then C is \mathcal{E} -section-categorical.*

The proof is elementary, and we omit it.

Proposition 4.6 *Let Y be a simplicial complex and $f : X \rightarrow Y$ a map. Then \mathcal{E} -secat(f) $\leq n \iff$ there is a simplicial structure on Y such that Y can be covered by $n+1$ \mathcal{E} -section-categorical subcomplexes L_0, L_1, \dots, L_n .*

Proof (\Leftarrow) If N_i is the second regular neighborhood of L_i in Y , then L_i is a deformation retract of the closure \overline{N}_i [8, p. 72]. Thus \overline{N}_i is \mathcal{E} -section-categorical, and hence so is the open set N_i by Lemma 4.5.

(\Rightarrow) Consider an \mathcal{E} -section-categorical cover $\{U_0, U_1, \dots, U_n\}$ of Y . It is known that there is a simplicial structure on Y with subcomplexes L_0, \dots, L_n which cover Y such that $L_i \subseteq U_i$ for each i [14, Lem. 2.3]. The result follows from Lemma 4.5. \square

Theorem 4.7 *If \mathcal{E} is a collection of spaces and $f : X \rightarrow Y$ is a map, then*

$$\Gamma_{\mathcal{S}(\mathcal{E})}(f) = \mathcal{E}\text{-secat}(f).$$

Proof In the proof we write Γ for $\Gamma_{\mathcal{S}(\mathcal{E})}$. To show $\mathcal{E}\text{-secat}(f) \leq \Gamma(f)$, it suffices to show that $\mathcal{E}\text{-secat}$ satisfies the axioms of Definition 4.1 for an \mathcal{S} -length function. But this follows from §3.

Next we show $\Gamma(f) \leq \mathcal{E}\text{-secat}(f)$. We suppose that $\mathcal{E}\text{-secat}(f) < \infty$ and prove the result by induction on $\mathcal{E}\text{-secat}(f)$. If $\mathcal{E}\text{-secat}(f) = 0$, then $f \in \mathcal{S}$, and so $\Gamma(f) = 0$. Suppose next that the inequality holds for all maps g with $\mathcal{E}\text{-secat}(g) < n$ and let $f : X \rightarrow Y$ with $\mathcal{E}\text{-secat}(f) = n$. Now X and Y have the homotopy type of simplicial complexes [19, Thm. 2], so we may assume that X and Y are simplicial complexes. By the Simplicial Approximation Theorem, we can take f to be a simplicial map. By Proposition 4.6, there is an \mathcal{E} -section-categorical cover $\{L_0, L_1, \dots, L_n\}$ of Y by subcomplexes with respect to f . We set $K_i = f^{-1}(L_i)$ so that $\{K_0, K_1, \dots, K_n\}$ is a cover of X by subcomplexes.

Thus the following diagram is commutative

$$\begin{array}{ccccc} K_n & \longleftarrow & K_n \cap (K_0 \cup \dots \cup K_{n-1}) & \longrightarrow & K_0 \cup \dots \cup K_{n-1} \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ L_n & \longleftarrow & L_n \cap (L_0 \cup \dots \cup L_{n-1}) & \longrightarrow & L_0 \cup \dots \cup L_{n-1}, \end{array}$$

where each f_i is induced by f . Clearly $f_1 \in \mathcal{S}(\mathcal{E})$, and so $\Gamma(f_1) = 0$. Furthermore, $\mathcal{E}\text{-secat}(f_3) \leq n - 1$, and hence $\Gamma(f_3) \leq n - 1$ by our inductive hypothesis. But the pushout of the rows of the diagram yields the map $f : X \rightarrow Y$. However this is homotopy equivalent to the homotopy pushout since all horizontal maps are simplicial inclusions and hence cofibrations [18, p. 78]. By Definition 4.1, $\Gamma(f) \leq \Gamma(f_3) + 1 \leq n$. This completes the proof. \square

Remark 4.8 1. For a collection \mathcal{E} of spaces, the \mathcal{E} -Clapp-Puppe category of f fits nicely into our axiomatic framework. Precisely, let $\mathcal{T}(\mathcal{E})$ be the collection of all maps $g : X \rightarrow Y$ such that $g \simeq vu$, where $u : X \rightarrow E$ and $v : E \rightarrow Y$, for some $E \in \mathcal{E}$. It can be shown that $\mathcal{E}\text{-cat}_{\text{CP}}(f) = \Gamma_{\mathcal{T}(\mathcal{E})}(f)$ for all maps f .

2. The axioms of Definition 4.1 can be dualized in the sense of Eckmann-Hilton. This just consists of replacing Axiom 3 with an appropriate homotopy pull-back axiom. As in Definition 4.3, we set $\Lambda_{\mathcal{S}}$ equal to the maximum all functions which satisfy the dual axioms. If \mathcal{E} is any collection of spaces, we define

$$\mathcal{S}^*(\mathcal{E}) = \{g : X \rightarrow Y \mid \text{there exists } h : Y \rightarrow X \text{ with } hg \simeq \text{id}_X\}.$$

Then $\Lambda_{\mathcal{S}^*(\mathcal{E})}(f)$, which we can denote by $\mathcal{E}\text{-cosecat}(f)$, is the dual of the \mathcal{E} -sectional category of f . It would be interesting to investigate this invariant.

5 Inequalities

In this section we consider the questions raised in the introduction dealing with the \mathcal{E} -sectional category of a composition and the relationship between the various invariants. The main theorem of this section is Theorem 5.4.

5.1 Composition of Maps

The composition results in this subsection are proved by elementary covering arguments.

Proposition 5.1 *For any maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$,*

$$\mathcal{E}\text{-secat}(g) \leq \mathcal{E}\text{-secat}(gf).$$

The proof is obvious, and hence omitted.

The next result deals with maps that have sections.

Proposition 5.2 *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then*

1. *If f has a section, then $\mathcal{E}\text{-secat}(gf) = \mathcal{E}\text{-secat}(g)$.*
2. *If g has a section, then $\mathcal{E}\text{-secat}(gf) \leq \mathcal{E}\text{-secat}(f)$.*

Proof First assume that f has a section s . Let $U \subseteq Z$ be \mathcal{E} -section-categorical with respect to g . We claim that U is also \mathcal{E} -section-categorical with respect to gf . This follows easily from the following homotopy-commutative diagram

$$\begin{array}{ccc}
 & & X \\
 & \nearrow^{sv} & \downarrow f \\
 E & \xrightarrow{v} & Y \\
 \uparrow u & & \downarrow g \\
 U & \longrightarrow & Z.
 \end{array}$$

This proves $\mathcal{E}\text{-secat}(gf) \leq \mathcal{E}\text{-secat}(g)$, and Proposition 5.1 completes the proof of (1). When g has a section t , one proves (2) by showing that if $U \subseteq Y$ is \mathcal{E} -section-categorical with respect to f , then $t^{-1}(U)$ is \mathcal{E} -section-categorical with respect to gf . \square

Next we derive additional results on the sectional category of a composition. We begin with a lemma.

Lemma 5.3 *If $A \rightarrow Y \xrightarrow{j} Y \cup CA$ is a mapping cone sequence and $f : X \rightarrow Y$ is a map, then*

$$\mathcal{E}\text{-secat}(jf) \leq \mathcal{E}\text{-secat}(f) + 1.$$

Proof Consider the diagram

$$\begin{array}{ccccc}
 * & \longleftarrow & * & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow f \\
 * & \longleftarrow & A & \longrightarrow & Y
 \end{array}$$

which induces the map $jf : X \rightarrow Y \cup CA$ of homotopy pushouts. The result now follows from Theorem 3.9. \square

Our main result gives an upper bound for the \mathcal{E} -sectional category of a composition.

Theorem 5.4 *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then*

$$\mathcal{E}\text{-secat}(gf) \leq \text{cat}_{\text{FH}}(g) + \mathcal{E}\text{-secat}(f).$$

Proof Suppose $\text{cat}_{\text{FH}}(g) = n$. Then we can choose a decomposition of g [1, §3]

$$\begin{array}{ccccccc}
 Y_0 & \xrightarrow{j_0} & Y_1 & \xrightarrow{j_1} & \cdots & \xrightarrow{j_{n-2}} & Y_{n-1} & \xrightarrow{j_{n-1}} & Y_n \\
 \parallel & & & & & & & & \parallel \\
 Y & \xrightarrow{g} & & & & & & & Z
 \end{array}$$

$g_n \updownarrow s_n$

where (1) $g \simeq g_n j_{n-1} \cdots j_1 j_0$, (2) $s_n g \simeq j_{n-1} \cdots j_1 j_0$, (3) $g_n s_n \simeq \text{id}$ and (4) there is a mapping cone sequence $A_i \longrightarrow Y_i \xrightarrow{j_i} Y_{i+1}$. Since g is dominated by $j_{n-1} \cdots j_1 j_0$, it follows that gf is dominated by $j_{n-1} \cdots j_1 j_0 f$. By Lemma 5.3,

$$\begin{aligned}
 \mathcal{E}\text{-seccat}(gf) &\leq \mathcal{E}\text{-secat}(j_{n-1} \cdots j_1 j_0 f) \\
 &\leq \mathcal{E}\text{-secat}(j_{n-2} \cdots j_1 j_0 f) + 1 \\
 &\vdots \\
 &\leq \mathcal{E}\text{-secat}(f) + n.
 \end{aligned}$$

□

We conclude this subsection by showing that the inequality in Theorem 5.4 can be strict.

Example 5.5 Let $f : X \rightarrow Y$ be any map with $\mathcal{E}\text{-secat}(f) > 0$, let $Z = *$ and let $g : Y \rightarrow Z$ be the constant map. Then

$$\mathcal{E}\text{-secat}(gf) = \text{cat}(*) = 0 < \text{cat}_{\text{FH}}(g) + \mathcal{E}\text{-secat}(f).$$

5.2 Relations Between Invariants

We begin this subsection with an inequality which provides a lower bound for the Clapp-Puppe category of a composition.

Proposition 5.6 *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then*

$$\mathcal{E}\text{-cat}_{\text{CP}}(g) + 1 \leq (\text{secat}(f) + 1)(\mathcal{E}\text{-cat}_{\text{CP}}(gf) + 1).$$

Proof Write $\text{secat}(f) = n$ and $\mathcal{E}\text{-cat}_{\text{CP}}(gf) = m$. Then we have a cover $\{U_0, U_1, \dots, U_n\}$ of Y and sections $s_i : U_i \rightarrow X$ for each i . Now

$$\mathcal{E}\text{-cat}_{\text{CP}}(g|_{U_i}) = \mathcal{E}\text{-cat}_{\text{CP}}(gf s_i) \leq \mathcal{E}\text{-cat}_{\text{CP}}(gf) = m,$$

so $U_i = V_{i0} \cup \dots \cup V_{im}$ with each $g|_{V_{ij}}$ factoring through some member of \mathcal{E} . Then $Y = \{V_{ij} \mid i = 0, 1, 2, \dots, n; j = 0, 1, 2, \dots, m\}$ is desired cover of Y by $(n+1)(m+1)$ \mathcal{E} -categorical subsets. \square

If $*$ is a constant map, then clearly $\mathcal{E}\text{-cat}_{\text{CP}}(*) = 0$. Thus if $\mathcal{E} = \{*\}$ and $gf \simeq *$, then Proposition 5.6 yields the inequality

$$\text{cat}_{\text{BG}}(g) \leq \text{secat}(f)$$

of Berstein-Ganea [2, Prop 2.6]. More generally, if $gf \simeq *$, then

$$\mathcal{E}\text{-cat}_{\text{CP}}(g) \leq \text{secat}(f)$$

for any collection \mathcal{E} by Proposition 5.6. This gives $\mathcal{E}\text{-cat}_{\text{CP}}(g) \leq \mathcal{E}\text{-secat}(f)$ if $gf \simeq *$. However this inequality holds without the latter assumption.

Proposition 5.7 *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Then*

$$\mathcal{E}\text{-cat}_{\text{CP}}(g) \leq \mathcal{E}\text{-cat}_{\text{CP}}(Y) \leq \mathcal{E}\text{-secat}(f).$$

Proof The left inequality follows immediately. The right inequality follows from the observation that if $\{U_0, U_1, \dots, U_n\}$ is an open cover of Y which is \mathcal{E} -section-categorical for f , then it is an open cover for the \mathcal{E} -Clapp-Puppe category of id_Y . \square

Next we turn to a corollary to Theorem 5.4

Corollary 5.8 *For any map $f : X \rightarrow Y$,*

$$\mathcal{E}\text{-secat}(f) \leq \text{cat}_{\text{FH}}(f) + \mathcal{E}\text{-cat}_{\text{CP}}(X).$$

In particular, $\text{secat}(f) \leq \text{cat}_{\text{FH}}(f)$.

Proof Apply Theorem 5.4 to the composition $X \xrightarrow{\text{id}} X \xrightarrow{f} Y$ and use Proposition 3.4(1). \square

This corollary is used to prove the following relationships between the invariants $\mathcal{E}\text{-secat}$, $\mathcal{E}\text{-cat}_{\text{CP}}$ and $\mathcal{E}\text{-cat}_{\text{FH}}$.

Proposition 5.9 *Let \mathcal{E} be a collection and $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be maps. Then*

1. $\mathcal{E}\text{-cat}_{\text{CP}}(g) \leq \text{cat}_{\text{FH}}(f) + \mathcal{E}\text{-cat}_{\text{CP}}(X)$.

2. If $X \in \mathcal{E}$, then $\mathcal{E}\text{-cat}_{\text{CP}}(g) \leq \text{cat}_{\text{FH}}(f)$.
3. If $gf \simeq *$, then $\text{cat}_{\text{BG}}(g) \leq \text{cat}_{\text{FH}}(f)$.

Proof By Proposition 5.7 and Corollary 5.8, we have

$$\begin{aligned} \mathcal{E}\text{-cat}_{\text{CP}}(g) &\leq \mathcal{E}\text{-secat}(f) \\ &\leq \text{cat}_{\text{FH}}(f) + \mathcal{E}\text{-cat}_{\text{CP}}(X). \end{aligned}$$

Also (2) follows from (1) since $\mathcal{E}\text{-cat}_{\text{CP}}(X) = 0$.

For (3) we specialize to $\mathcal{E} = \{\text{all spaces}\}$ in Corollary 5.8 and use the Berstein-Ganea inequality $\text{cat}_{\text{BG}}(g) \leq \text{secat}(f)$ mentioned above. \square

In the special case $\mathcal{E} = \{*\}$, Proposition 5.9(1) reduces to

$$\text{cat}_{\text{BG}}(g) \leq \text{cat}_{\text{FH}}(f) + \text{cat}(X).$$

Next we relate sectional category and the Clapp-Puppe category.

Proposition 5.10 *For any collection \mathcal{E} and any map $f : X \rightarrow Y$,*

$$\mathcal{E}\text{-cat}_{\text{CP}}(f) \leq \mathcal{E}\text{-secat}(f).$$

Proof Let $\mathcal{E}\text{-secat}(f) = n$ and let $\{U_0, U_1, \dots, U_n\}$ be a \mathcal{E} -section-categorical cover of Y . Then there are spaces $E_i \in \mathcal{E}$ and maps $u_i : U_i \rightarrow E_i$ and $v_i : E_i \rightarrow X$ such that $f v_i u_i \simeq j_i$, where $j_i : U_i \hookrightarrow Y$ is the inclusion map. We set $V_i = f^{-1}(U_i)$. Then $\{V_0, V_1, \dots, V_n\}$ is an open cover of X and the maps

$$V_i \xrightarrow{f|_{V_i}} U_i \xrightarrow{u_i} E_i \xrightarrow{v_i} X \xrightarrow{f} Y$$

show that $\mathcal{E}\text{-cat}_{\text{CP}}(f) \leq n$. \square

Remark 5.11 This proposition does *not* show that $\text{cat}_{\text{BG}}(f) \leq \text{secat}(f)$. This is because $\mathcal{E}\text{-cat}_{\text{CP}}(f) = \text{cat}_{\text{BG}}(f)$ when $\mathcal{E} = \{*\}$ and $\mathcal{E}\text{-secat}(f) = \text{secat}(f)$ when $\mathcal{E} = \{\text{all spaces}\}$. In a sense, $\mathcal{E}\text{-cat}_{\text{CP}}$ and $\mathcal{E}\text{-secat}$ are dual to each other.

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