PARTITION COMPLEXES, TITS BUILDINGS AND SYMMETRIC PRODUCTS

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ABSTRACT. We construct a homological approximation to the partition complex, and identify it as the Tits building. This gives a homological relationship between the symmetric group and the affine group, leads to a geometric tie between symmetric powers of spheres and the Steinberg idempotent, and allows us to use the self-duality of the Steinberg module to study layers in the Goodwillie tower of the identity functor.

1. INTRODUCTION

A partition complex is a geometric object associated to the poset of equivalence relations on a finite set; a Tits building is a geometric object associated to the poset of subspaces of a vector space. The two objects are formally alike, but aside from that they don't seem to have much in common. For instance, they have very different symmetries; permutation groups act on partition complexes, while affine groups or general linear groups act on Tits buildings. Nevertheless, in this paper we describe a close relationship between the two, based upon the observation that when we feed the partition complex to a particular homological approximation machine, it is essentially a Tits building that comes back out (§4-5) . This gives some surprising connections between the homology of symmetric groups and the homology of affine groups or general linear groups (1.2, 1.5). In view of a tie we establish between the partition complex and symmetric powers of spheres (1.11), it also explains, in a geometric way, why the homology of these symmetric powers is related to the Steinberg idempotent. Finally, although the partition complex is not stably self-dual, the Tits building is. The relationship between the two leads to a kind of shifted self-duality for the partition complex, and this allows us to prove some conjectures of Arone and Mahowald about the layers in the Goodwillie tower of the identity functor (1.16).

Date: January 28, 1999.

Both authors were partially supported by the NSF.

In the course of the paper, we clarify some aspects of [1] and weave together many threads from earlier work of Kuhn [11], Kuhn and Priddy [12], Mitchell [16], Mitchell and Priddy [17], and others. Part of the appeal of our approach is that it amounts to setting up a general approximation technique and then just working out its consequences in one particular case.

We will now describe the results of the paper in more detail.

Partitions and subspaces. Let $\mathbf{n}$ denote the finite set $\{1, \ldots, n\}$. An equivalence relation λ on $\mathbf{n}$ is said to be *nontrivial* if there is more than one λ-equivalence class and at least two distinct elements of $\mathbf{n}$ are λ-equivalent to one another. The collection of nontrivial equivalence relations on $\mathbf{n}$ is a poset (with the convention that $\lambda_1 \le \lambda_2$ if λ_1 is a refinement of λ_2) and the associated space $\mathbf{P}_n$ is called the *partition complex* of $\mathbf{n}$ (see 1.18). The group Σ_n of permutations of $\mathbf{n}$ acts on $\mathbf{P}_n$ in a natural way.

Suppose that $n = p^k$ (p a prime) and identify $\mathbf{n}$ in some way with the vector space $\Delta = (\mathbb{F}_p)^k$ over the field $\mathbb{F}_p$. Say that an equivalence relation λ on Δ is *linear* if it is stable under translations (i.e., $v \sim_\lambda w \Rightarrow v + x \sim_\lambda w + x$). Giving a linear equivalence relation amounts to specifying the subspace of all elements equivalent to 0, and so the poset of nontrivial linear equivalence relations is isomorphic to the poset (under inclusion) of nontrivial proper subspaces of Δ. The associated space $\mathbf{T}_k$ is called the *Tits building* of Δ. The affine group $\mathrm{Aff}_{k,p} = \mathrm{GL}_k(\mathbb{F}_p) \ltimes \Delta$ acts on Δ and on $\mathbf{T}_k$. The identification of Δ with $\mathbf{n}$ gives a subgroup inclusion $\mathrm{Aff}_{k,p} \to \Sigma_n$, and treating linear equivalence relations as equivalence relations gives a map $\mathbf{T}_k \to \mathbf{P}_n$ which is equivariant with respect to this inclusion.

An approximation theorem. Our first result states that the map $\mathbf{T}_k \to \mathbf{P}_n$ is in a certain sense an equivariant homology approximation. We need some notation to describe this. If G is a subgroup of a symmetric group and M is a G-module, let $M^{\pm}$ denote the twist of M by the sign representation of G; in particular, $\mathbb{F}_p^{\pm}$ denotes the sign representation of G on $\mathbb{F}_p$. For a pair (X, Y) of G-spaces, let $\mathrm{H}_*^G(X, Y; M)$ denote the relative Borel construction homology

$$\mathrm{H}_*(\,(\mathrm{E}G \times X)/G,\ (\mathrm{E}G \times Y)/G;\ M\,)\ .$$

This is relative homology with local coefficients. If X has a G-basepoint x_0, the reduced homology $\tilde{\mathrm{H}}_*^G(X; M)$ is defined to be $\mathrm{H}_*^G(X, x_0; M)$. For any G-space X, the unreduced suspension $X^\diamondsuit$ of X is the pointed G-space obtained by collapsing the base of the cone on X to a point and using this as the basepoint.

1.1. Theorem. *Suppose that $n > 1$ is an integer. If n is not a power of p, then $\tilde{\mathrm{H}}_*^{\Sigma_n}(\mathbf{P}_n^\diamondsuit;\mathbb{F}_p^\pm)$ vanishes. If $n = p^k$, then the map $\mathbf{T}_k \to \mathbf{P}_n$ induces an isomorphism*

$$\tilde{\mathrm{H}}_*^{\mathrm{Aff}_{k,p}}(\,\mathbf{T}_k^\diamondsuit;\,\mathbb{F}_p^\pm\,) \cong \tilde{\mathrm{H}}_*^{\Sigma_n}(\,\mathbf{P}_n^\diamondsuit;\,\mathbb{F}_p^\pm\,)\ .$$

Group homology interpretations. Theorem 1.1 leads to a surprising group homology calculation. The spaces $\mathbf{T}_k$ and $\mathbf{P}_n$ above are *spherical*, in the sense that each is homotopy equivalent to a wedge of spheres which all have the same dimension. For $\mathbf{T}_k$ this dimension is $k-2$, while for $\mathbf{P}_n$ it is $n-3$. (See [20] [2] or [19, Th. 2] for the Tits building, and [18, 4.109] for the partition complex.) The suspended spaces $\mathbf{T}_k^\diamondsuit$ and $\mathbf{P}_n^\diamondsuit$ are also spherical, with the dimensions of the spheres involved increased by one. The top group $\tilde{\mathrm{H}}_{k-1}(\mathbf{T}_k^\diamondsuit;\mathbb{Z})$ is a module over $\mathrm{Aff}_{k,p}$ which is called the *Steinberg module* St_k. The top group $L = \tilde{\mathrm{H}}_{n-2}(\mathbf{P}_n^\diamondsuit;\mathbb{Z})$ is a module over Σ_n without a standard name, but since the $\mathbb{Z}$-dual of the twisted module $L^\pm$ is sometimes denoted Lie_n, we will denote $L^\pm$ itself by Lie_n^*.

1.2. Theorem. *Suppose that $n > 1$ is an integer. If n is not a power of p, then $\mathrm{H}_*(\Sigma_n;\mathbb{F}_p\otimes\mathrm{Lie}_n^*)$ vanishes. If $n = p^k$, then for $i \geq 0$ there are isomorphisms*

$$\mathrm{H}_i(\mathrm{Aff}_{k,p};\mathbb{F}_p\otimes\mathrm{St}_k^\pm) \cong \mathrm{H}_{i-n+k+1}(\Sigma_n;\mathbb{F}_p\otimes\mathrm{Lie}_n^*)\ .$$

1.3. *Remark.* In the above statement and in similar situations involving group homology later on in the paper, $\mathrm{H}_i = 0$ for $i < 0$. There is also a cohomological version of this theorem, which appears in 6.2. The calculations of 6.2 show that Lie_n and Lie_n^* are not in general isomorphic as Σ_n-modules.

Since the subgroup $(\mathbb{F}_p)^k$ of translations in $\mathrm{Aff}_{k,p} = \mathrm{GL}_k(\mathbb{F}_p)\ltimes(\mathbb{F}_p)^k$ acts trivially on the complex $\mathbf{T}_k$ of linear equivalence relations, St_k is actually a module over $\mathrm{GL}_k(\mathbb{F}_p) = \mathrm{GL}_{k,p}$. It turns out that $\mathbb{F}_p\otimes\mathrm{St}_k$ is projective as a module over $\mathbb{F}_p[\mathrm{GL}_{k,p}]$, and that there is an idempotent $\epsilon_k^{\mathrm{St}} \in \mathbb{F}_p[\mathrm{GL}_{k,p}]$, called the *Steinberg idempotent*, such that for any $\mathbb{F}_p[\mathrm{GL}_{k,p}]$-module M there is an isomorphism

$$\epsilon_k^{\mathrm{St}}\cdot M \cong \mathrm{H}_0(\mathrm{GL}_{k,p};M\otimes\mathrm{St}_k)\ . \tag{1.4}$$

See [21], [17], or [10]. A little bit of calculation with a collapsing spectral sequence leads to the following consequence of 1.2.

1.5. Theorem. *Suppose that $n = p^k > 1$. Let $\Delta = (\mathbb{F}_p)^k$ and let $\mathrm{GL}_{k,p} = \mathrm{Aut}(\Delta) \subset \Sigma_n$ act on the group homology $\mathrm{H}_*(\Delta;\mathbb{F}_p^\pm)$ in the natural way. Then for $i \geq 0$ there are isomorphisms*

$$\epsilon_k^{\mathrm{St}}\cdot\mathrm{H}_i(\Delta;\mathbb{F}_p^\pm) \cong \mathrm{H}_{i-n+k+1}(\Sigma_n;\mathbb{F}_p\otimes\mathrm{Lie}_n^*)\ .$$

1.6. *Remark.* In the above statement, the twist in $\mathbb{F}_p^{\pm}$ affects the action of $\mathrm{GL}_{k,p}$ on $\mathrm{H}_*(\Delta;\mathbb{F}_p^{\pm})$ but does not affect the group homology itself.

A geometric interpretation. Theorem 1.1 also has a more geometric interpretation. If X is a G-space, let $X_{\mathrm{h}G}$ denote the Borel construction $(\mathrm{E}G\times X)/G$; if X has basepoint x_0, let $X_{\tilde{\mathrm{h}}G}$ denote the reduced Borel construction $X_{\mathrm{h}G}/(x_0)_{\mathrm{h}G} = X_{\mathrm{h}G}/\,\mathrm{B}G$.

1.7. Theorem. *Suppose that $n>1$, that ℓ is odd, and that Σ_n acts on the sphere $S^{\ell n}$ by permuting the factors of $S^{\ell n}=(S^\ell)^{\wedge n}$. If n is not a power of p, then the reduced mod p homology of $(S^{\ell n}\wedge \mathbf{P}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n}$ vanishes. If $n=p^k$, then the inclusion $\mathbf{T}_k\to\mathbf{P}_n$ induces a mod p homology isomorphism*

$$(S^{\ell n}\wedge \mathbf{T}_k^\diamondsuit)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}} \to (S^{\ell n}\wedge \mathbf{P}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n}\ .$$

In the course of proving 1.7 we also prove a dual result, which it turns out is needed in studying the Goodwillie tower. If X is a space or spectrum, let $X^{\#}$ denote the Spanier-Whitehead dual of X (1.18).

1.8. Theorem. *Suppose that $n>1$, that ℓ is odd, and that Σ_n acts on the sphere $S^{\ell n}$ by permuting the factors of $S^{\ell n}=(S^\ell)^{\wedge n}$. If n is not a power of p, then the reduced mod p homology of $(S^{\ell n}\wedge (\mathbf{P}_n^\diamondsuit)^{\#})_{\tilde{\mathrm{h}}\Sigma_n}$ vanishes. If $n=p^k$, then the inclusion $\mathbf{T}_k\to\mathbf{P}_n$ gives rise to a mod p homology isomorphism*

$$(S^{\ell n}\wedge (\mathbf{P}_n^\diamondsuit)^{\#})_{\tilde{\mathrm{h}}\Sigma_n} \to (S^{\ell n}\wedge (\mathbf{T}_k^\diamondsuit)^{\#})_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}}\ .$$

1.9. *Remark.* Say that an equivalence relation on $\mathbf{n}$ is *completely regular* if all of the equivalence classes have the same size, and let $\mathbf{R}_n$ denote the space associated to the poset of nontrivial completely regular equivalence relations on $\mathbf{n}$. If $n=p^k$, the inclusion $\mathbf{T}_k\to\mathbf{P}_n$ factors as $\mathbf{T}_k\to\mathbf{R}_n\to\mathbf{P}_n$, and for odd ℓ Theorem 1.7 guarantees that the induced composite

$$(S^{\ell n}\wedge \mathbf{T}_k^\diamondsuit)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}} \to (S^{\ell n}\wedge \mathbf{R}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n} \to (S^{\ell n}\wedge \mathbf{P}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n}$$

gives an isomorphism on mod p homology. It follows that the p-completion of $(S^{\ell n}\wedge \mathbf{P}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n}$ is a retract of the p-completion of $(S^{\ell n}\wedge \mathbf{R}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n}$. This answers a question of Kuhn [9]; the space $(S^{n}\wedge \mathbf{R}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n}$ is related to the space Y_k of [8, §5].

1.10. *Remark.* Theorems 1.7 and 1.8 also hold (in the appropriate stable sense) for negative odd values of ℓ. If $p=2$, the assumption that ℓ is odd can be removed from both theorems.

Relationship to symmetric powers. If X is a space, let $\mathrm{SP}^n(X)$ denote the n'th symmetric power of X, i.e., the quotient space X^n/Σ_n. Choice of a basepoint gives an inclusion $\mathrm{SP}^{n-1}(X) \to \mathrm{SP}^n(X)$.

1.11. Theorem. *Suppose that $n \geq 1$ and that S^ℓ is the ℓ-sphere. Then in the stable range with respect to ℓ there is an equivalence*

$$\mathrm{SP}^n(S^\ell)/\,\mathrm{SP}^{n-1}(S^\ell) \simeq S^\ell \wedge (S^n \wedge \mathbf{P}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n} \ .$$

1.12. *Remark.* A map is an equivalence "in the stable range with respect to ℓ" if it is an m-equivalence for $m = 2\ell - \epsilon$, where ϵ is a small constant. We leave it to the reader to determine whether in any particular case that comes up in this paper ϵ should be 1, 2, or 3.

Passing to the limit and making a homology calculation leads to the following result.

1.13. Theorem. *Suppose that $n > 1$ and that $\mathcal{S}^0$ is the stable zero sphere. Then there is an equivalence*

$$\mathrm{SP}^n(\mathcal{S}^0)/\,\mathrm{SP}^{n-1}(\mathcal{S}^0) \simeq \Sigma^\infty (S^n \wedge \mathbf{P}_n^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma_n} \ .$$

The i'th mod p homology group of this spectrum is isomorphic to

$$\mathrm{H}_{i-2n+2}(\Sigma_n; \mathbb{F}_p \otimes \mathrm{Lie}_n^*) \ .$$

1.14. *Remark.* The statement that $\mathrm{SP}^n(\mathcal{S}^0)/\,\mathrm{SP}^{n-1}(\mathcal{S}^0)$ is the suspension spectrum of a space appears in a paper of Lesh [13]; in the course of our arguments we recover her identification of the space involved as the classifying space of a particular family of subgroups of Σ_n (7.4). Combining the above result with Theorem 1.5 gives a connection between the homology of symmetric power spectra and certain images of the Steinberg idempotent. Relationships like this first appeared in a paper of Mitchell and Priddy [17]. Theorems 1.7 and 1.13 give a direct geometric explanation for such a relationship.

1.15. Goodwillie layers. As described in [1], evaluating the n'th layer in the Goodwillie tower for the identity functor at a pointed space X yields the spectrum

$$D_n(X) = \mathrm{Map}_*(S^1 \wedge \mathbf{P}_n^\diamondsuit,\, \Sigma^\infty X^{\wedge n})_{\tilde{\mathrm{h}}\Sigma_n} \ .$$

The following theorem was conjectured by Arone and Mahowald; it is a type of duality statement. The proof depends upon combining the relationship described above between the Tits building and the partition complex with the fact that the Tits building is stably self-dual.

1.16. Theorem. *Suppose that X is an odd sphere and that $n = p^k$. Then after p-completion there is an equivalence*

$$S^{2(k-1)+1} \wedge D_n(X) \underset{p}{\sim} (\Sigma^\infty \mathbf{P}_n^\diamondsuit \wedge X^{\wedge n})_{\tilde{\mathrm{h}}\Sigma_n} .$$

Comparing this with 1.13 gives the following result, which was first proved by N. Kuhn with another technique.

1.17. Theorem. *Suppose that $\mathcal{S}^0$ is the stable zero sphere and that $n = p^k$. Then after p-completion there is an equivalence*

$$S^{2(k-1)+1} \wedge D_n(S^1) \underset{p}{\sim} \mathrm{SP}^n(\mathcal{S}^0)/\mathrm{SP}^{n-1}(\mathcal{S}^0) .$$

We also obtain a retraction (9.6) involving the p-completion of $D_{p^k}S^{2\ell+1}$. The existence of a retraction like this was conjectured by Mahowald.

Organization of the paper. In §2 we set up a general scheme for approximating G-spaces (G a finite group) and then in §3 study its homological properties. Section 4 describes a particular approximation to the partition complex, and §5 identifies it more or less as a Tits building. (To get the Tits building, it is actually necessary to work in a relative sense with respect to the approximation of a one-point space.) Section 6 has proofs of the main homological results and §7 deals with symmetric powers. The next two sections establish the stable self-duality of the Tits building and then combine this with earlier results to study layers in the Goodwillie tower of the identity. Finally, §10 has some algebraic material which is needed to deal with the Spanier-Whitehead duality which appears in 1.8.

1.18. Notation and terminology. As a technical convenience, we take the term *space* on its own to mean *simplicial set* [15] [6]. A map of spaces is an *equivalence* or *weak equivalence* if it becomes an ordinary weak equivalence of topological spaces after geometric realization. The union of a space X with a disjoint basepoint is X_+; the suspension spectrum associated to a pointed space X is $\Sigma^\infty X$.

If G is a finite group, a map $f : X \to Y$ of G-spaces is said to be a *G-equivalence* or *weak G-equivalence* if f induces a weak equivalence $X^K \to Y^K$ for each subgroup K of G. If M is a G-module, f is said to be *an* $\mathrm{H}_*^G(-;M)$*-equivalence* if it induces an isomorphism $\mathrm{H}_*(X_{\mathrm{h}G};M) \to \mathrm{H}_*(Y_{\mathrm{h}G};M)$. Note that the coefficients here are local coefficients. We will sometimes encounter G-spectra, by which we mean ordinary spectra with an action of G (these are called naive G-spectra in [14]). For instance, if X is a pointed space then $\Sigma^\infty X^{\wedge n}$ is a Σ_n-spectrum. If X is a G-spectrum, the spectrum $X_{\tilde{\mathrm{h}}G}$ is defined by stabilizing the usual Borel construction [14, I, 3.7]. A map of G-spectra is said to be an $\mathrm{H}_*^G(-;M)$-equivalence if the induced map of Borel constructions gives an isomorphism on homology with local coefficients in M. (The local coefficient homology of a stable Borel construction is defined by stabilizing the reduced local coefficient homology of unstable Borel constructions.)

The equivariant cohomology $\mathrm{H}_G^*(X;M)$ of a G-space is the cohomology of $X_{\mathrm{h}G}$ with local coefficients in M. Equivariant cohomology of a G spectrum, reduced equivariant cohomology of a pointed G-space, etc., are defined in obvious ways. Note that we use the canonical antiautomorphism $g \mapsto g^{-1}$ of G to switch back and forth if necessary between right G-modules and left G-modules, so that the same module might be used as coefficients both for equivariant homology and for equivariant cohomology.

If X is a spectrum or a pointed space, then $X^\# = \mathrm{Map}_*(X, \mathcal{S}^0)$ denotes the Spanier-Whitehead dual of X; here $\mathcal{S}^0$ is the sphere spectrum. If X is a G-spectrum, so is $X^\#$.

If $\mathcal{P}$ is a poset, then $|\mathcal{P}|$ denotes the space associated to $\mathcal{P}$; this is the simplicial set whose nondegenerate m-simplices correspond to $(m+1)$-tuples of elements of $\mathcal{P}$ which are totally ordered by the partial order relation. More precisely, this is the nerve [3, XI §2] of the category whose objects are the elements of $\mathcal{P}$, and in which there is exactly one morphism $x \to y$ if $y \leq x$. (Note the reversal, which is just for convenience [4, 2.10]). If $\mathcal{P}$ has a maximal element or a minimal element then $|\mathcal{P}|$ is contractible [4, 2.6].

We are grateful to N. Kuhn for reading through an initial version of this manuscript and making some very useful suggestions. In particular, he helped us to formulate some of our results in a much better way.

2. Approximations

Suppose that G is a finite group and that X is a G-space (in other words, a simplicial set with an action of G). Recall from [5, 1.3] that a *collection* $\mathcal{C}$ of subgroups of G is a set of subgroups closed under conjugation. In this section we construct for each such $\mathcal{C}$ an approximation $X_{\mathcal{C}}$ of X and a G-map $a_{\mathcal{C}} : X_{\mathcal{C}} \to X$. The space $X_{\mathcal{C}}$ is in a sense the best approximation to X which can be built up from the orbits G/K, $K \in \mathcal{C}$. We give examples of the approximations, and describe some properties of the construction.

2.1. *Remark.* A *family* $\mathcal{F}$ of subgroups of G is a collection which is closed under inheritance, in the sense that if $K \in \mathcal{F}$ and $H \subset K$ then

$H \in \mathcal{F}$. Some of the collections we look at are families, and some of them aren't.

For the rest of this section, $\mathcal{C}$ denotes a fixed collection of subgroups of G.

Constructing the approximation. For a G-space X, let $\mathrm{Iso}(X)$ denote the collection of all subgroups of G which appear as isotropy subgroups of simplices of X. A G-space X is said to *have $\mathcal{C}$-isotropy* if $\mathrm{Iso}(X) \subset \mathcal{C}$. A G-map $X \to Y$ is said to be a *$\mathcal{C}$-equivalence* if the induced map $X^K \to Y^K$ is an equivalence of spaces for each $K \in \mathcal{C}$.

2.2. Definition. Suppose that X is a G-space. A G-map $f : Y \to X$ is said to be a *$\mathcal{C}$-approximation to X* if Y has $\mathcal{C}$-isotropy and f is a $\mathcal{C}$-equivalence.

2.3. Proposition. *Let X be a G-space. Then there exists a functorial $\mathcal{C}$-approximation to X, denoted $a_{\mathcal{C}} : X_{\mathcal{C}} \to X$. Any two $\mathcal{C}$-approximations to X are canonically G-equivalent.*

The uniqueness part of the proposition depends on the following well-known result (see for instance [4, 4.1]).

2.4. Lemma. *Suppose that the G-map $f : X \to Y$ is a $\mathcal{C}$-equivalence for $\mathcal{C} = \mathrm{Iso}(X) \cup \mathrm{Iso}(Y)$. Then f is a G-equivalence.*

Proof of 2.3. We sketch one construction, which is related to ideas from [7] in the same way as in [5] the subgroup decomposition is related to the centralizer decomposition. Let $\mathbf{O}_*(\mathcal{C})$ be the category whose objects consists of pairs $(G/K, c)$, where $K \in \mathcal{C}$ and $c \in G/K$; a map $(G/K, c) \to (G/K', c')$ is a G-map $f : G/K \to G/K'$ such that $f(c) = c'$. For any G-space X there is a functor from $\mathbf{O}_*(\mathcal{C})^{\mathrm{op}}$ to the category of spaces given by

$$\Phi_X^{\mathcal{C}}(G/K, c) = \mathrm{Map}_G(G/K, X) \cong X^K \ .$$

We define $X_{\mathcal{C}}$ to be the homotopy colimit [3] of $\Phi_X^{\mathcal{C}}$. Evaluating elements of $\Phi_X^{\mathcal{C}}(G/K, c)$ at c leads to a map $a_{\mathcal{C}} : X_{\mathcal{C}} \to X$. This map is G-equivariant, where the action of G on $X_{\mathcal{C}}$ is induced by the action of G on $\mathbf{O}_*(\mathcal{C})$ defined by letting $g \in G$ send $(G/K, c)$ to $(G/K, gc)$. It is clear by inspection that $\mathrm{Iso}(X_{\mathcal{C}}) \subset \mathcal{C}$. It is also easy to see that for any subgroup $K \subset G$, $(X_{\mathcal{C}})^K$ is the homotopy colimit of $\Phi_X^{\mathcal{C}}$ over the full subcategory of $\mathbf{O}_*(C)^{\mathrm{op}}$ containing the objects $(G/H, c)$ with the property that K fixes c. If $K \in \mathcal{C}$, then this subcategory has $(G/K, eK)$ as a terminal object, and so the homotopy colimit of $\Phi_X^{\mathcal{C}}$ over this subcategory is equivalent to $\Phi_X^{\mathcal{C}}(G/K, eK) = X^K$. The construction of $X_{\mathcal{C}}$

is clearly functorial. If $Y \to X$ is another $\mathcal{C}$-approximation of X then there is a commutative diagram

$$\begin{array}{ccc} Y_{\mathcal{C}} & \longrightarrow & X_{\mathcal{C}} \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

in which by 2.4 the upper arrow and the left vertical arrow are G-equivalences. $\square$

2.5. *Remark.* In later sections we will mostly be interested in the collection $\mathcal{E}$ all nontrivial elementary abelian p-subgroups of G. The $\mathcal{E}$-approximation to a G-space X is closely related to the Henn's work in [7]. Henn builds an approximation to the Borel construction on X; we essentially work with the fibres over $\mathrm{B}G$ and build $X_{\mathcal{E}}$ as the corresponding approximation to X itself.

We will now give some examples and properties of the $\mathcal{C}$-approximation construction.

2.6. Elementary examples. If $\mathcal{C}$ is the collection of all subgroups of G, then $X_{\mathcal{C}}$ is G-equivalent to X. If $\mathcal{C} = \{G\}$, then $X_{\mathcal{C}}$ is X^G. If $\mathcal{C} = \{\{e\}\}$, then $X_{\mathcal{C}}$ is $\mathrm{E}G \times X$. If $\mathcal{C}$ is the collection of all nontrivial subgroups of G, then $X_{\mathcal{C}}$ is G-equivalent to the *G-singular set* $\mathrm{Sing}_G(X)$, i.e., to the subspace of X consisting of all simplices which have a nontrivial isotropy subgroup (to see this note that $\mathrm{Sing}_G(X)$ has $\mathcal{C}$-isotropy and that the inclusion $\mathrm{Sing}_G(X) \to X$ is a $\mathcal{C}$-equivalence).

2.7. The universal space $\mathrm{E}\mathcal{C}$. Let $*$ be the trivial one-point G-space.

2.8. Definition. The *universal space for $\mathcal{C}$*, denoted $\mathrm{E}\mathcal{C}$, is the $\mathcal{C}$-approximation $(*)_{\mathcal{C}}$ to $*$. The *classifying space for $\mathcal{C}$*, denoted $\mathrm{B}\mathcal{C}$, is the quotient $(\mathrm{E}\mathcal{C})/G$.

2.9. *Remark.* If $\mathcal{C} = \{\{e\}\}$, then $\mathrm{E}\mathcal{C}$ is $\mathrm{E}G$ and $\mathrm{B}\mathcal{C} = \mathrm{B}G$. More generally, if $\mathcal{C}$ is a family of subgroups of G, then $\mathrm{E}\mathcal{C}$ is the ordinary universal space [14] of $\mathcal{C}$ and $\mathrm{B}\mathcal{C}$ is the associated classifying space.

2.10. *Homotopy type of* $\mathrm{E}\mathcal{C}$. The collection $\mathcal{C}$ can be treated as a poset under subgroup inclusion, and the associated space (1.18) is denoted $|\mathcal{C}|$. (This is not the same as $\mathrm{B}\mathcal{C}$!) The action of G on $\mathcal{C}$ by conjugation induces an action of G on $|\mathcal{C}|$. The space $\mathrm{E}\mathcal{C}$ is isomorphic to the nerve of the category $\mathbf{O}_*(\mathcal{C})^{\mathrm{op}}$, and so there is a map $\mathrm{E}\mathcal{C} \to |\mathcal{C}|$ induced by the functor $\mathrm{E}\mathcal{C} \to \mathcal{C}$ which sends $(G/K, c)$ to the isotropy subgroup G_c. This map is G-equivariant and is a weak equivalence of spaces [5, 3.8] [4, 4.12]. In particular $\mathrm{E}\mathcal{C}$ is weakly equivalent to a finite complex.

2.11. *Fixed point sets of* $\mathrm{E}\,\mathcal{C}$*.* If K is a subgroup of G we let $\mathcal{C}\downarrow K$ denote the set $\{H\in\mathcal{C}\mid H\subset K\}$ and $K\downarrow\mathcal{C}$ the set $\{H\in\mathcal{C}\mid K\subset H\}$. Both of these sets are sub-posets of $\mathcal{C}$; the first one is also a collection of subgroups of K. For any subgroup K of G the fixed point set $(\mathrm{E}\,\mathcal{C})^K$ is equivalent to $|K\downarrow\mathcal{C}|$ [4, 2.14]. Note that if $K\in\mathcal{C}$ then $K\downarrow\mathcal{C}$ has K as a minimal element and so $(\mathrm{E}\,\mathcal{C})^K$ is contractible, as it must be.

By construction $\mathrm{Iso}(\mathrm{E}\,\mathcal{C})=\mathcal{C}$. If $\mathcal{C}'$ is a collection of subgroups of G with $\mathcal{C}'\subset\mathcal{C}$, then there is a natural G-inclusion $\mathrm{E}\,\mathcal{C}'\to\mathrm{E}\,\mathcal{C}$.

2.12. Building $X_{\mathcal{C}}$ for general X. Suppose that K is a subgroup of G, and that $\mathcal{C}'=\mathcal{C}\downarrow K$. If X is a K-space with $\mathcal{C}'$-isotropy, then $G\times_K X$ is a G-space with $\mathcal{C}$-isotropy. If the K-map $Y\to X$ is a $\mathcal{C}'$-equivalence, then $G\times_K Y\to G\times_K X$ is a $\mathcal{C}$-equivalence. Both of these statements are easy to prove by inspection. If follows that if $Y\to X$ is a $\mathcal{C}'$-approximation of X as K-space, then $G\times_K Y\to G\times_K X$ is a $\mathcal{C}$-approximation of $G\times_K X$ as a G-space.

In particular $(G/K)_{\mathcal{C}}$ can be identified up to G-equivalence with $G\times_K(*)_{\mathcal{C}'}=G\times_K\mathrm{E}\,\mathcal{C}'$, and so $(G/K)_{\mathcal{C}}$ is weakly equivalent as a space to the finite complex $G\times_K|\mathcal{C}'|$. It is clear that $(G/K\times\Delta[n])_{\mathcal{C}}$ is isomorphic to $(G/K)_{\mathcal{C}}\times\Delta[n]$. Since the functor $X\mapsto X_{\mathcal{C}}$ commutes with homotopy pushouts and directed colimits, this calculation for the G-cells $G/K\times\Delta[n]$ gives an inductive approach to understanding the G-homotopy type of $X_{\mathcal{C}}$ for any G-space X.

2.13. Preservation of equivalences. Say that a G-map $f:X\to Y$ is *a* (G,n)*-equivalence* if it induces an n-equivalence $X^K\to Y^K$ for each subgroup K of G; more generally, f is a $(\mathcal{C},n)$-equivalence if it induces an n-equivalence $X^K\to Y^K$ for each $K\in\mathcal{C}$. The following statement is easy to prove by the inductive method used in the proof of [4, 4.1].

2.14. Proposition. *If $X\to Y$ is a $(\mathcal{C},n)$-equivalence, then the induced map $X_{\mathcal{C}}\to Y_{\mathcal{C}}$ is a (G,n)-equivalence.*

2.15. Preservation of dimension. It follows from 2.7 and 2.12 that if X is a finite G-space, that is, one with a finite number of nondegenerate simplices, then $X_{\mathcal{C}}$ is equivalent as a space to a finite complex. Moreover, there is a integer $d_{\mathcal{C}}$ such that the homotopy dimension of $X_{\mathcal{C}}$ is bounded above by $\dim(X)+d_{\mathcal{C}}$. The integer $d_{\mathcal{C}}$ can be taken to be the dimension of $|\mathcal{C}|$.

Note however that $X_{\mathcal{C}}$ is not necessarily G-equivalent to a finite complex, even if $X=*$ (see 2.6).

3. Homological properties of approximations

Let G be a finite group, $\mathcal{C}$ a collection of subgroups of G, and M a fixed $\mathbb{F}_p[G]$-module. In this section we study the faithfulness of the $\mathcal{C}$-approximation of a G-space (§2) from the point of view of equivariant M-homology. In particular, we look for ways to identify G-spaces X such that the map $a_{\mathcal{C}} : X_{\mathcal{C}} \to X$ or its Spanier-Whitehead dual $a_{\mathcal{C}}^{\#} : X^{\#} \to X_{\mathcal{C}}{}^{\#}$ are $\mathrm{H}_*^G(-;M)$-equivalences (1.18).

Testing with a forward arrow. Before looking at whether the map $X_{\mathcal{C}} \to X$ is an $\mathrm{H}_*^G(-;M)$-equivalence, we will establish some terminology.

3.1. Definition. If X is a G-space, the collection $\mathcal{C}$ is *M-ample for X* if the map $X_{\mathcal{C}} \to X$ is an $\mathrm{H}_*^G(-;M)$-equivalence.

If there is more than one group around, we will sometime emphasize the role of G by saying that $\mathcal{C}$ is (G,M)-ample for X.

3.2. Proposition. *Suppose that X is a G-space and that, for each $K \in \mathrm{Iso}(X)$, $\mathcal{C}$ is (G,M)-ample for G/K. Then $\mathcal{C}$ is (G,M)-ample for X.*

This is proved by a Mayer-Vietoris argument (2.12).

3.3. Proposition. *Let K be a subgroup of G. If $\mathcal{C} \downarrow K$ is (K,M)-ample for $*$, then $\mathcal{C}$ is (G,M)-ample for G/K.*

Proof. See [7, 2.8]. Let $\mathcal{C}' = \mathcal{C} \downarrow K$. Since $(G/K)_{\mathcal{C}} \cong G \times_K (*)_{\mathcal{C}'}$ (cf. 2.12) the result follows from the form of Shapiro's lemma which says if Y is a K-space, then $(G \times_K Y)_{\mathrm{h}G}$ is equivalent to $Y_{\mathrm{h}K}$. □

The following proposition is the starting point we will use in applying the previous two results.

3.4. Proposition. *Let $\mathcal{E}$ be the collection of nontrivial elementary abelian p-subgroups of G. If p divides the order of the kernel of the action map $G \to \mathrm{Aut}(M)$, then $\mathcal{E}$ is M-ample for $*$.*

Proof. By the remarks in 2.10, $\mathcal{E}$ is M-ample for $*$ if and only if the map $|\mathcal{E}| \to *$ is an $\mathrm{H}_*^G(-;M)$-equivalence. The proposition then follows from [5, §8]. □

3.5. *Remark.* See [5, 1.4] for examples involving other collections.

Testing with a reversed arrow. Our second measure of the effectiveness of $\mathcal{C}$-approximation uses Spanier-Whitehead duality.

3.6. Definition. Let X be a G-space and M a G-module. The collection $\mathcal{C}$ is *reverse M-ample for X* (or *reverse (G,M)-ample for X*) if the map $(X_{\mathcal{C}})_+^{\#} \leftarrow X_+^{\#}$ induced by $X_{\mathcal{C}} \to X$ is an $\mathrm{H}_*^G(-;M)$-equivalence.

3.7. Proposition. *Suppose that X is a G-space with finite skeleta such that, for each $K \in \mathrm{Iso}(X)$, $\mathcal{C}$ is reverse (G,M)-ample for G/K. Assume in addition that the spaces X and $X_{\mathcal{C}}$ have the homotopy type of finite complexes. Then $\mathcal{C}$ is reverse (G,M)-ample for X.*

Proof. Let Z be the cofibre of the map $X_{\mathcal{C}} \to X$ and $Z(n)$ the cofibre of the map $(\mathrm{sk}_n X)_{\mathcal{C}} \to \mathrm{sk}_n X$. (Here $\mathrm{sk}_n X$ is the n-skeleton of X.) Let d_Z be the homotopy dimension of Z (as a space, not as a G-space); by assumption, d_Z is finite. Let $D = Z^{\#}$, $D(n) = Z(n)^{\#}$. By an inductive Mayer-Vietoris argument (cf. 2.12) it is easy to see that $\mathcal{C}$ is reverse (G,M)-ample for $\mathrm{sk}_n X$, or equivalently that $\mathrm{H}_*^G(D(n);M)$ vanishes; we want to prove that $\mathrm{H}_*^G(D;M)$ also vanishes. The homology groups of the spectra D and $D(n)$ are concentrated in dimensions ≤ 0. The group $\mathrm{H}_i\, D$ vanishes unless $0 \geq i \geq -d_Z$, and $\mathrm{H}_i\, D(n)$ vanishes unless $0 \geq i \geq -(n+d_{\mathcal{C}})$ (see 2.15). In fact, if $n >> d_Z$, then by 2.14 there are isomorphisms

$$\mathrm{H}_i\, D(n) = \begin{cases} \mathrm{H}_i\, D & i \geq -d_Z \\ 0 & -d_Z > i > -n \\ 0 & i < -(n+d_{\mathcal{C}}+1) \end{cases}$$

Now there is a spectral sequence

$$E^2_{i,j}(D) = \mathrm{H}_i(G; \mathrm{H}_j(D;M)) \Rightarrow \mathrm{H}^G_{i+j}(D;M)$$

which maps to parallel spectral sequences $E^r_{i,j}(D(n))$. The E^2-page $E^2_{i,j}(D)$ is concentrated in the horizontal band $0 \geq j \geq -d_Z$, while $E^2_{i,j}(D(n))$ is concentrated in an upper band $0 \geq j \geq -d_Z$ and a lower band $(-n) \geq j \geq -(n+d_{\mathcal{C}}+1)$; the map $D \to D(n)$ induces an isomorphism on E^2-pages in the upper band. Since $E^r(D(n)$ converges to zero, we can conclude that $E^r(D)$ converges to zero if we can pick n in such a way that no nontrivial differential in $E^r(D(n))$ jumps from the lower band to the upper band (remember that these are fourth quadrant homology spectral sequences, so differentials move up and to the left). Choose n_1 and n_2 so that $n_1 > n_2 + d_{\mathcal{C}} + 1 >> d_Z$. Then the natural map $E^2(D(n_1)) \to E^2(D(n_2))$ is an isomorphism on upper bands but zero on lower bands, since the lower bands in these two E^2-pages lie in different strips in the plane. By naturality all differentials in $E^r(D(n_1))$ from the lower band to the upper band are zero, which as above implies that $E^r(D)$ converges to zero. $\square$

3.8. Proposition. *Let K be a subgroup of G. If $\mathcal{C} \downarrow K$ is reverse (K, M)-ample for $*$, then $\mathcal{C}$ is reverse (G, M)-ample for G/K.*

Proof. Given 10.2, this follows from the proof of 3.3 □.

3.9. Proposition. *Let $\mathcal{E}$ be the collection of nontrivial elementary abelian p-subgroups of G. Suppose that p divides the order of G, that M is a finite $\mathbb{F}_p[G]$-module, and that every element of order p in G acts trivially on M. Then $\mathcal{E}$ is reverse M-ample for $*$.*

This will be proved below in §10.

4. An approximation to the partition complex

In this section we apply the approximation machinery of §2 in a special case. The finite group in question is the symmetric group Σ_n, with its collection $\mathcal{E}$ of nontrivial elementary abelian subgroups. The Σ_n-space X to be approximated is a model for the partition complex $\mathbf{P}_n$, where we use the term *model* to mean that there is a Σ_n-map $X \to \mathbf{P}_n$ which is a weak equivalence of spaces. Let M denote the sign representation $\mathbb{F}_p^{\pm}$ of Σ_n on $\mathbb{F}_p$. We find a model X such that $\mathcal{E}$ is both M-ample and reverse M-ample for X. It will later turn out that if $n = p^k$, the space $X_{\mathcal{E}}$ can be described in terms of the Tits building $\mathbf{T}_k$.

For the rest of this section we let Σ denote Σ_n and M the Σ-module $\mathbb{F}_p^{\pm}$.

4.1. A model for $\mathbf{P}_n$. For each equivalence relation λ on $\mathbf{n}$, let K_λ denote the subgroup of Σ consisting of all $\sigma \in \Sigma$ which preserve λ in the strong sense that for each $j \in \mathbf{n}$, $\sigma(j) \sim_\lambda j$. Let $\mathcal{P}$ denote the collection consisting of all the subgroups K_λ for nontrivial equivalence relations λ. Up to conjugacy, the elements of $\mathcal{P}$ are exactly the subgroups of Σ of the form $\Sigma_{n_1} \times \cdots \times \Sigma_{n_j}$ with $j > 1$, $\sum n_i = n$, and $n_i > 1$ for at least one i. The association $\lambda \mapsto K_\lambda$ is bijective and order preserving, and so $|\mathcal{P}|$ is isomorphic as a Σ-space to the partition complex $\mathbf{P}_n$.

By 2.10, there is a Σ-map $\mathrm{E}\,\mathcal{P} \to |\mathcal{P}| = \mathbf{P}_n$ which is an equivalence of spaces. The space $\mathrm{E}\,\mathcal{P}$ is the model for $\mathbf{P}_n$ that we will work with.

4.2. Another description. There is another way to construct $\mathrm{E}\,\mathcal{P}$ which gives the same result up to Σ-equivalence. Let $\mathcal{F}$ denote the family of all *non-transitive* subgroups of Σ, i.e. the family of all subgroups of Σ which do not act transitively on the set $\mathbf{n}$, and let $\mathcal{F}^\circ \subset \mathcal{F}$ be the subcollection obtained by deleting the trivial subgroup $\{e\}$. Since $\mathcal{P} \subset \mathcal{F}^\circ$, there is a Σ-inclusion $\mathrm{E}\,\mathcal{P} \to \mathrm{E}\,\mathcal{F}^\circ$.

4.3. Lemma. *The inclusion $\mathrm{E}\,\mathcal{P} \to \mathrm{E}\,\mathcal{F}^\circ$ is a Σ-equivalence.*

Proof. By 2.4, it is enough to show that for each $H \in \mathcal{F}^\circ$ the map $(\mathrm{E}\,\mathcal{P})^H \to (\mathrm{E}\,\mathcal{F}^\circ)^H$ is an equivalence. By 2.11 (see [4, 2.12]), this amounts to showing that the inclusion $|H \downarrow \mathcal{P}| \to |H \downarrow \mathcal{F}^\circ|$ is an equivalence, or, what is the same thing (2.11), that $|H \downarrow \mathcal{P}|$ is contractible. Let $\lambda(H)$ be the equivalence relation on $\mathbf{n}$ in which the equivalence classes are the H-orbits, and let $K = K_{\lambda(H)}$. Then $K \in \mathcal{P}$ and K is a minimal element of $H \downarrow \mathcal{P}$, so the result follows. □

4.4. *Remark.* By the remarks in 2.7 there is a commutative diagram of Σ-spaces

$$\begin{array}{ccc} \mathrm{E}\,\mathcal{P} & \longrightarrow & \mathrm{E}\,\mathcal{F}^\circ \\ \big\downarrow & & \big\downarrow \\ |\mathcal{P}| & \longrightarrow & |\mathcal{F}^\circ| \end{array}$$

in which the vertical arrows are weak equivalences. By 4.3 the upper arrow is a Σ-equivalence, and so the lower arrow is a weak equivalence. The construction in the proof of 4.3 gives for each $H \in \mathcal{F}^\circ$ an element $K_{\lambda(H)} \in \mathcal{P}$. The assignment $H \mapsto K_{\lambda(H)}$ is order-preserving and Σ-equivariant; it induces a Σ-map $|\mathcal{F}^\circ| \to |\mathcal{P}|$ which is a left inverse to the lower arrow in the above square, and so is a weak equivalence.

The approximation. Let $\mathcal{E}'$ denote the collection of subgroups of Σ given by $\mathcal{E} \cap \mathcal{F}^\circ$. This is the collection obtained from $\mathcal{E}$ by deleting all elementary abelian p-subgroups of Σ which act transitively on $\mathbf{n}$. In terms of $\mathcal{E}'$, we can now give a preliminary calculation of the $\mathcal{E}$-approximation $(\mathrm{E}\,\mathcal{P})_{\mathcal{E}} \simeq (\mathrm{E}\,\mathcal{F}^\circ)_{\mathcal{E}}$.

4.5. Lemma. *The natural map* $\mathrm{E}(\mathcal{E}') \to \mathrm{E}\,\mathcal{F}^\circ$ *is an* $\mathcal{E}$*-approximation.*

Proof. By construction the Σ-space $\mathrm{E}(\mathcal{E}')$ has $\mathcal{E}'$-isotropy, and so a fortiori it has $\mathcal{E}$-isotropy. It remains to show that the indicated map is an $\mathcal{E}$-equivalence. However, the V-fixed points of both spaces involved are contractible if $V \in \mathcal{E}'$ and empty if $V \in \mathcal{E} \setminus \mathcal{E}'$. □

Ampleness and reverse ampleness. Recall that M is the sign representation of Σ on $\mathbb{F}_p$.

4.6. Lemma. *If K is an element of $\mathcal{P}$, then the collection $\mathcal{E} \downarrow K$ of subgroups of K is (K, M)-ample for the trivial K-space $*$.*

Proof. The collection $\mathcal{E} \downarrow K$ is the collection of nontrivial elementary abelian p-subgroups of K. If p divides the order of K it is clear that p also divides the order of the kernel of the action map $K \to \mathrm{Aut}(M)$, and so by 3.4 the collection $\mathcal{E} \downarrow K$ is (K, M)-ample for $*$. If p does not divide the order of K then p must be odd and by calculation the group $\mathrm{H}_i(K;M)$ vanishes for $i \geq 0$. By inspection the poset $\mathcal{E} \downarrow K$ is empty. For trivial reasons, then, the map $|\mathcal{E} \downarrow K| \to *$ is an $\mathrm{H}_*^K(-;M)$-equivalence, and so (by the proof of 3.4) $\mathcal{E} \downarrow K$ is (K, M)-ample for the space $*$. $\square$

4.7. Proposition. *The collection $\mathcal{E}$ is M-ample for* $\mathrm{E}\,\mathcal{P}$.

Proof. This follows from 4.6, 3.3, 3.2 and the fact that $\mathrm{Iso}(\mathrm{E}\,\mathcal{P}) = \mathcal{P}$. $\square$

4.8. Lemma. *If K is an element of $\mathcal{P}$, then $\mathcal{E} \downarrow K$ is reverse (K, M)-ample for the trivial K-space $*$.*

Proof. This is very similar to the proof of 4.6; it depends on 3.9. $\square$

4.9. Proposition. *The collection $\mathcal{E}$ is reverse M-ample for* $\mathrm{E}\,\mathcal{P}$.

Proof. This follows from 3.7, 3.8, and 4.8. It is necessary to check that the Σ-spaces $\mathrm{E}(\mathcal{E})$ and $*$ have finite skeleta, but this is clear by inspection. It is also necessary to check that $\mathrm{E}\,\mathcal{P}$, $(\mathrm{E}\,\mathcal{P})_{\mathcal{E}}$ and $(*)_{\mathcal{E}}$ all have the homotopy types of finite complexes. By 2.10 and 4.5, these spaces are equivalent, respectively, to the finite simplicial complexes $|\mathcal{P}|$, $|\mathcal{E}'|$, and $|\mathcal{E}|$. $\square$

4.10. *Remark.* Similar arguments show that the collection $\mathcal{E}$ is both M-ample and reverse M-ample for the trivial Σ-space $*$.

5. Computing with the approximation

In this section we draw some specific conclusions from the approximation construction in §4. We continue to use the notation of that section.

We are going to concentrate on the commutative square

$$\begin{array}{ccc} |\mathcal{E}'| & \longrightarrow & |\mathcal{E}| \\ \downarrow & & \downarrow \\ |\mathcal{P}| & \longrightarrow & * \end{array} \tag{5.1}$$

of Σ-spaces. The left vertical arrow here requires a little comment: it is constructed by composing the inclusion $\mathcal{E}' \to \mathcal{F}^\circ$ with the poset retraction $\mathcal{F}^\circ \to \mathcal{P}$ from 4.4. Recall that M denotes the Σ-module $\mathbb{F}_p^{\pm}$. We are interested in proving the following three statements.

5.2. Proposition. *Both of the vertical arrows in 5.1 are $\mathrm{H}_*^{\Sigma}(-;M)$-equivalences.*

5.3. Proposition. *If n is not a power of p, then the homotopy cofibre of the upper horizontal arrow in 5.1 is contractible.*

Note that the homotopy cofibre of the lower horizontal arrow in 5.1 is $\mathbf{P}_n^{\diamondsuit}$ (see 4.1).

5.4. Proposition. *Suppose that $n = p^k$. Let C be the homotopy cofibre of the upper horizontal arrow in 5.1. Then there is a Σ-map*

$$C \to \Sigma_+ \wedge_{\mathrm{Aff}_{k,p}} \mathbf{T}_k^{\diamondsuit}$$

which is a weak equivalence of spaces. Under this equivalence the map $C \to \mathbf{P}_n^{\diamondsuit}$ induced by the square 5.1 corresponds to the map

$$\Sigma_+ \wedge_{\mathrm{Aff}_{k,p}} \mathbf{T}_k^{\diamondsuit} \to \mathbf{P}_n^{\diamondsuit}$$

induced by the $\mathrm{Aff}_{k,p}$-equivariant inclusion $\mathbf{T}_k \to \mathbf{P}_n$.

Proof of 5.2. It is enough show that in the square

$$\begin{array}{ccc} |\mathcal{E}'| & \longrightarrow & |\mathcal{E}| \\ \downarrow & & \downarrow \\ |\mathcal{F}^\circ| & \longrightarrow & * \end{array}$$

both of the vertical arrows are $\mathrm{H}_*^{\Sigma}(-;M)$-equivalences. Consider the square

$$\begin{array}{ccc} \mathrm{E}(\mathcal{E}') & \longrightarrow & \mathrm{E}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{E}\,\mathcal{F}^\circ & \longrightarrow & * \end{array}$$

Both vertical arrow are $\mathcal{E}$-approximations, the right by construction and the left by 4.5. By 4.7 and 4.10, both vertical arrows are $\mathrm{H}_*^{\Sigma}(-;M)$-equivalences. The proposition follows from the construction of 2.10, which produces a Σ-map from the second square to the first giving a weak equivalence at each of the four corners. $\square$

Proof of 5.3. This is obvious; $\mathcal{E}' = \mathcal{E}$ because for cardinality reasons no elementary abelian subgroup of Σ can act transitively on $\mathbf{n}$. $\square$

Suppose that $n = p^k$. It is not hard to see that up to conjugacy there is a unique elementary abelian subgroup of Σ which acts transitively on $\mathbf{n}$. Such a subgroup is obtained using an identification of $(\mathbb{F}_p)^k$ with $\mathbf{n}$ and then letting this additive group act on itself by translation. Let Δ denote a chosen one of these subgroups. The normalizer $N(\Delta)$ of Δ in Σ is isomorphic to the semidirect product $\mathrm{Aut}(\Delta) \ltimes \Delta \cong \mathrm{Aff}_{k,p}$, and the quotient $N(\Delta)/\Delta$ is isomorphic to $\mathrm{Aut}(\Delta) \cong \mathrm{GL}_{k,p}$.

5.5. Lemma. *Suppose that $n = p^k$. Let Δ and N be as above. Then there is a homotopy pushout square of Σ-spaces*

$$\begin{array}{ccc} \Sigma \times_N |\mathcal{E}' \downarrow \Delta| & \longrightarrow & \Sigma \times_N |\mathcal{E} \downarrow \Delta| \\ \Big\downarrow & & \Big\downarrow \\ |\mathcal{E}'| & \longrightarrow & |\mathcal{E}| \end{array} .$$

Proof. By inspection this is a pushout square, and the upper horizontal arrow is an inclusion. □

Proof of 5.4. Consider the square of 5.5. The poset $\mathcal{E}' \downarrow \Delta$ is the poset of all subgroups of Δ other than the trivial subgroup and Δ itself, and so $|\mathcal{E}' \downarrow \Delta|$ is the Tits building $\mathbf{T}_k$. The poset $\mathcal{E} \downarrow \Delta$ is the poset of all subgroups of Δ other than the trivial subgroup. This poset has Δ as a maximal element, and so $|\mathcal{E} \downarrow \Delta|$ is contractible; in fact, $|\mathcal{E} \downarrow \Delta|$ is combinatorially the cone on $|\mathcal{E}' \downarrow \Delta|$. Both horizontal maps in this square thus have homotopy cofibre $\Sigma_+ \wedge_N \mathbf{T}_k^\diamondsuit$. It is easy to check that the map from this space to $\mathbf{P}_n^\diamondsuit$ given by the 5.1 is the obvious one. □

6. The homological results

In this section we give proofs of the main homological results from the introduction, namely, 1.1, 1.2, 1.5, 1.7, and 1.8. All of these results follow directly from the calculations in §5. We continue to denote the symmetric group Σ_n by Σ.

Proof of 1.1. Let M be the Σ-module $\mathbb{F}_p^\pm$. By 5.2, both of the vertical arrows in the square 5.1 are $\mathrm{H}_*^\Sigma(-;M)$-equivalences. By a long exact sequence argument, the induced map of horizontal homotopy cofibres is also an $\mathrm{H}_*^\Sigma(-;M)$-equivalence. The proof is finished by identifying the map between these cofibres (5.3, 5.4), and, if $n = p^k$, by applying Shapiro's lemma. □

Proof of 1.2. If G is a finite group, X is a G-space, and M is a G-module, then there is a Serre sequence

$$E^2_{i,j} = \mathrm{H}_i(G; \tilde{\mathrm{H}}_j(X^\diamondsuit; M)) \Rightarrow \tilde{\mathrm{H}}^G_{i+j}(X^\diamondsuit; M) .$$

If X is spherical and has homotopy dimension d, the spectral sequence collapses at E^2 into isomorphisms

$$\tilde{\mathrm{H}}^G_i(X^\diamondsuit; M) \cong \mathrm{H}_{i-d-1}(G; \tilde{\mathrm{H}}_d(X; M)) .$$

The theorem is proved by starting with 1.1 and applying this collapse observation to the spherical space $\mathbf{P}_n$ and, if $n = p^k$, to the spherical space $\mathbf{T}_k$. □

Proof of 1.5. If M is a module over $\mathrm{Aff}_{k,p}$, then associated to the group extension $\Delta \to \mathrm{Aff}_{k,p} \to \mathrm{GL}_{k,p}$ is a Serre spectral sequence

$$E^2_{i,j} = \mathrm{H}_i(\mathrm{GL}_{k,p}; \mathrm{H}_j(\Delta; M)) \Rightarrow \mathrm{H}_{i+j}(\mathrm{Aff}_{k,p}; M)\,.$$

In the special case $M = \mathbb{F}_p \otimes \mathrm{St}_k^{\pm}$, the action of Δ on M is trivial, and so the spectral sequence takes the form

$$E^2_{i,j} = \mathrm{H}_i(\mathrm{GL}_{k,p}; \mathrm{H}_j(\Delta; \mathbb{F}_p) \otimes \mathrm{St}_k^{\pm}) \Rightarrow \mathrm{H}_{i+j}(\mathrm{Aff}_{k,p}; M)\,,$$

where the action of $\mathrm{GL}_{k,p}$ on $\mathrm{H}_j(\Delta; \mathbb{F}_p) \otimes \mathrm{St}_k^{\pm}$ is a diagonal one. This action can be identified with the diagonal action of $\mathrm{GL}_{k,p}$ on $\mathrm{H}_j(\Delta; \mathbb{F}_p^{\pm}) \otimes \mathrm{St}_k$. Since $\mathbb{F}_p \otimes \mathrm{St}_k$ is projective as a module over $\mathrm{GL}_{k,p}$, the group $\mathrm{H}_i(\mathrm{GL}_{k,p}; N \otimes \mathrm{St}_k)$ vanishes for any $i > 0$ and any $\mathbb{F}_p[\mathrm{GL}_{k,p}]$-module N. By formula 1.4, then, the above spectral sequence collapses into isomorphisms

$$\mathrm{H}_i(\mathrm{Aff}_{k,p}; \mathbb{F}_p \otimes \mathrm{St}_k^{\pm}) \cong \epsilon_k^{\mathrm{St}} \cdot \mathrm{H}_i(\Delta; \mathbb{F}_p^{\pm})$$

and the proof is finished by combining these isomorphisms with 1.2. □

Proof of 1.7. Since ℓ is odd, the action of Σ on $\mathrm{H}_{\ell n}(S^{\ell n}; \mathbb{F}_p)$ gives the Σ-module $\mathbb{F}_p^{\pm}$. Essentially by the Thom isomorphism theorem (or by a Serre spectral sequence argument) there are natural isomorphisms

$$\tilde{\mathrm{H}}_i((S^{\ell n} \wedge \mathbf{P}_n^{\diamondsuit})_{\tilde{\mathrm{h}}\Sigma}; \mathbb{F}_p) \cong \tilde{\mathrm{H}}^{\Sigma}_{i-\ell n}(\mathbf{P}_n^{\diamondsuit}; \mathbb{F}_p^{\pm})$$

as well as similar natural isomorphisms for $\tilde{\mathrm{H}}_*((S^{\ell n} \wedge \mathbf{T}_k^{\diamondsuit})_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}}; \mathbb{F}_p)$. The theorem is proved by combining these isomorphisms with 1.1. □

Proof of 1.8. Let M denote the Σ-module $\mathbb{F}_p^{\pm}$. Consider the square

$$\begin{array}{ccc} |\mathcal{E}'|_+^{\#} & \longleftarrow & |\mathcal{E}|_+^{\#} \\ \uparrow & & \uparrow \\ |\mathcal{P}|_+^{\#} & \longleftarrow & (*)_+^{\#} \end{array} \qquad (6.1)$$

obtained by applying Spanier-Whitehead duality to 5.1. The line of argument in the proof of 5.2 shows that the vertical arrows in this square are $\mathrm{H}^{\Sigma}_*(-; M)$-equivalences; it is only necessary to replace the appeal to 4.7 by an appeal to 4.9. By a long exact sequence argument, it follows that the induced map of horizontal homotopy fibres is an $\mathrm{H}^{\Sigma}_*(-; M)$-equivalence. The homotopy fibre of the lower horizontal map is $(\mathbf{P}_n^{\diamondsuit})^{\#}$. Let C be the homotopy fibre of the upper map. If n is not a power of p then C is contractible (5.3). If $n = p^k$ then by 5.4 C is equivalent to $(\Sigma_+ \wedge_{\mathrm{Aff}_{k,p}} \mathbf{T}_k^{\diamondsuit})^{\#}$, which by 10.2 is in turn equivalent to $\Sigma_+ \wedge_{\mathrm{Aff}_{k,p}} (\mathbf{T}_k^\diamondsuit)^\#$. In any case, the kind of Thom isomorphism that figures in the proof of 1.7 shows that the induced map

$$(S^{\ell n} \wedge (\mathbf{P}_n^\diamondsuit)^\#)_{\tilde{\mathrm{h}}\Sigma} \to (S^{\ell n} \wedge C)_{\tilde{\mathrm{h}}\Sigma}$$

gives an isomorphism on mod p homology. If n is not a power of p the target of this map is contractible, while if $n = p^k$ it follows from the stable form of Shapiro's lemma that the target is equivalent to $(S^{\ell n} \wedge (\mathbf{T}_k^\diamondsuit)^\#)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}}$. □

6.2. *Remark.* Theorem 1.8 leads to a cohomological version of 1.2. Suppose that $n = p^k$. The spectrum $(\mathbf{P}_n^\diamondsuit)^\#$ is spherical; its only nontrivial mod p homology group is in dimension $(2-n)$ and is isomorphic to $\mathbb{F}_p \otimes \mathrm{Lie}_n^\pm$ as a module over Σ_n. Similarly, $(\mathbf{T}_k^\diamondsuit)^\#$ is spherical; its only nontrivial mod p homology group is in dimension $1-k$ and is isomorphic to $\mathrm{Hom}(\mathrm{St}_k, \mathbb{F}_p)$ as a module over $\mathrm{Aff}_{k,p}$. By the collapsing spectral sequence argument in the above proof of 1.2, the mod p homology equivalence of 1.8 gives isomorphisms

$$\mathrm{H}_i(\mathrm{Aff}_{k,p}; \mathrm{Hom}(\mathrm{St}_k^\pm, \mathbb{F}_p)) \cong \mathrm{H}_{i+n-k-1}(\Sigma_n; \mathbb{F}_p \otimes \mathrm{Lie}_n)\ . \tag{6.3}$$

If G is a finite group and M is a module over $\mathbb{F}_p[G]$, there is a natural isomorphism between $\mathrm{H}^*(G; \mathrm{Hom}(M, \mathbb{F}_p))$ and the $\mathbb{F}_p$-dual of $\mathrm{H}_*(G; M)$ (see the proof of 10.1). By this duality, 6.3 gives isomorphisms

$$\mathrm{H}^i(\mathrm{Aff}_{k,p}; \mathbb{F}_p \otimes \mathrm{St}_k^\pm) \cong \mathrm{H}^{i+n-k-1}(\Sigma_n; \mathbb{F}_p \otimes \mathrm{Lie}_n^*)\ .$$

There is another consequence of 6.3. By the self-duality of $\mathbb{F}_p \otimes \mathrm{St}_k$ described in §8, $\mathrm{Hom}(\mathrm{St}_k^\pm; \mathbb{F}_p)$ is isomorphic as an $\mathrm{Aff}_{k,p}$-module to $\mathbb{F}_p \otimes \mathrm{St}_k^\pm$. In combination with 1.2, then, 6.3 gives isomorphisms

$$\mathrm{H}_{i-n+k+1}(\Sigma_n; \mathbb{F}_p \otimes \mathrm{Lie}_n^*) \cong \mathrm{H}_{i+n-k-1}(\Sigma_n; \mathbb{F}_p \otimes \mathrm{Lie}_n)\ .$$

It is not hard to see that the groups involved are not all zero (1.13), and it follows that if n is a power of p then $\mathbb{F}_p \otimes \mathrm{Lie}_n^*$ and $\mathbb{F}_p \otimes \mathrm{Lie}_n$ are not isomorphic as Σ_n-modules.

The above arguments also show that if n is not a power of p then the groups $\mathrm{H}^*(\Sigma_n; \mathbb{F}_p \otimes \mathrm{Lie}_n^*)$ and $\mathrm{H}_*(\Sigma_n; \mathbb{F}_p \otimes \mathrm{Lie}_n)$ vanish.

7. Symmetric powers

The goal of this section is to prove Theorems 1.11 and 1.13. There are three auxiliary propositions. We continue to use the notation of section 4, so that in particular $\Sigma = \Sigma_n$, $\mathcal{F}^\circ$ is the collection of all nontrivial subgroups of Σ which do not act transitively on $\mathbf{n}$, and $\mathcal{P}$ is the collection of subgroups of Σ described in 4.1. In the first two statements below, Σ acts on the sphere $S^{\ell n}$ by permuting the factors of $S^{\ell n} = (S^\ell)^{\wedge n}$.

7.1. Proposition. *The space* $\mathrm{SP}^n(S^\ell)/\mathrm{SP}^{n-1}(S^\ell)$ *is homeomorphic to the quotient space* $S^{\ell n}/\Sigma$

Let $\tilde{\mathcal{P}}$ be the collection of subgroups of Σ given by adding to $\mathcal{P}$ the trivial subgroup $\{e\}$. Recall from 2.7 that $\mathrm{B}\tilde{\mathcal{P}}$ is defined to be the quotient space $(\mathrm{E}\,\tilde{\mathcal{P}})/\Sigma$.

7.2. Proposition. *In the stable range with respect to* ℓ *there is an equivalence* $S^{\ell n}/\Sigma \simeq S^\ell \wedge \mathrm{B}\tilde{\mathcal{P}}^\diamondsuit$

The following proposition is interesting because it relates an orbit space ($\mathrm{B}\tilde{\mathcal{P}}$) on the left to a homotopy orbit space on the right. In the statement, Σ acts on $S^n = (S^1)^{\wedge n}$ by permuting smash factors.

7.3. Proposition. *There is an equivalence of spaces*

$$S^1 \wedge \mathrm{B}\tilde{\mathcal{P}}^\diamondsuit \simeq S^1 \wedge (S^n \wedge \mathrm{E}\,\mathcal{P}^\diamondsuit)_{\tilde{\mathrm{h}}\Sigma}$$

Theorem 1.11 is now proved by combining 7.1, 7.2, and 7.3. Theorem 1.13 is proved by first checking that it is possible to pass to the limit in ℓ with 1.11 and then copying the spectral sequence calculation from the proof of 1.2.

7.4. *Remark.* Recall from 4.1 that $\mathcal{F}$ is the family of nontransitive subgroups of Σ. It is clear that $\tilde{\mathcal{P}} \subset \mathcal{F}$, and the argument of 4.3 shows that the induced inclusion $\mathrm{E}\,\tilde{\mathcal{P}} \to \mathrm{E}\,\mathcal{F}$ is a Σ-equivalence. It follows that the quotient map $\mathrm{B}\tilde{\mathcal{P}} \to \mathrm{B}\mathcal{F}$ is an equivalence. Combining this with 7.1 and 7.2, and passing to the limit in ℓ gives the following calculation of Lesh.

7.5. Corollary. [13] *Let* $\mathcal{S}^0$ *be the stable zero sphere, and let* $\mathcal{F}$ *be the family of nontransitive subgroups of* Σ. *Then there is an equivalence*

$$\mathrm{SP}^n(\mathcal{S}^0)/\mathrm{SP}^{n-1}(\mathcal{S}^0) \simeq \Sigma^\infty\, \mathrm{B}\mathcal{F}^\diamondsuit\ .$$

Proposition 7.1 is clear by inspection. We now go on to the proofs of 7.2 and 7.3. The first proof depends on a few lemmas.

7.6. Lemma. *Suppose that* G *is a finite group, that* X *is a* G*-space, and that* $\mathcal{C}'$ *is a collection of subgroups of* G. *Let* $\mathcal{C} = \mathcal{C}' \cup \{G\}$. *Then the natural* G*-map* $X^G \to X$ *gives rise to a homotopy pushout diagram of* G*-spaces*

$$\begin{array}{ccc} (X^G)_{\mathcal{C}'} & \longrightarrow & (X^G)_{\mathcal{C}} \\ \downarrow & & \downarrow \\ X_{\mathcal{C}'} & \longrightarrow & X_{\mathcal{C}} \end{array}$$

Proof. Assume $G \notin \mathcal{C}'$, since otherwise the statement is trivial. By 2.4, it is only necessary to check that for each $K \in \mathcal{C}$ the square becomes a homotopy pushout square when the fixed point functor $(-)^K$ is applied. This is obvious if $K \in \mathcal{C}'$, since the horizontal arrows are $\mathcal{C}'$-equivalences. The remaining case is $K = G$. The G-fixed point sets of the spaces on the left are empty, since these spaces have $\mathcal{C}'$-isotropy. Since $G \in \mathcal{C}$, the G-fixed points of the spaces on the right are both equivalent to X^G. □

7.7. *Remark.* In the situation of 7.6, it is easy to see that $(X^G)_{\mathcal{C}'}$ is isomorphic to $X^G \times (*)_{\mathcal{C}'}$, and similarly that $(X^G)_{\mathcal{C}}$ is isomorphic to $X^G \times (*)_{\mathcal{C}}$. This last product is G-equivalent to X^G, because $*$ has $\mathcal{C}$-isotropy.

The following lemma can be proved by the inductive technique used to prove [4, 4.1]. The notion of (G, n)-equivalence is from 2.13.

7.8. Lemma. *Suppose that G is a finite group. If $X \to Y$ is a (G, n)-equivalence of G-spaces, then the quotient map $X/G \to Y/G$ is an n-equivalence of spaces.*

Proof of 7.2. Let $X = S^{\ell n}$ and let $\mathcal{C} = \tilde{\mathcal{P}} \cup \{\Sigma\}$. Consider the map of squares

$$
\begin{array}{ccc}
(X^\Sigma)_{\tilde{\mathcal{P}}} & \longrightarrow & (X^\Sigma)_{\mathcal{C}} \\
\downarrow & & \downarrow \\
X_{\tilde{\mathcal{P}}} & \longrightarrow & X_{\mathcal{C}}
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccc}
(X^\Sigma)_{\tilde{\mathcal{P}}} & \longrightarrow & (X^\Sigma)_{\mathcal{C}} \\
\downarrow & & \downarrow \\
(*)_{\tilde{\mathcal{P}}} & \longrightarrow & Y
\end{array} .
$$

Here the map $X_{\tilde{\mathcal{P}}} \to (*)_{\tilde{\mathcal{P}}}$ is induced by $X \to *$ and Y is defined so that the left-hand square is a homotopy pushout square. The map $X_{\mathcal{C}} \to Y$ is well-defined because (7.6) the right hand square is a homotopy pushout square. If K is a subgroup of Σ then X^K is a sphere $S^{o(K)\ell}$, where $o(K)$ is the number of orbits of the action of K on $\mathbf{n}$. If $K \in \tilde{\mathcal{P}}$, then $o(K) \geq 2$, so that $X^K \to (*)^K = *$ is a $(2\ell - 1)$-equivalence. By 2.14, the map $X_{\tilde{\mathcal{P}}} \to (*)_{\tilde{\mathcal{P}}}$ is a $(\Sigma, 2\ell - 1)$-equivalence. It follows that $X_{\mathcal{C}} \to Y$ is also a $(\Sigma, 2\ell - 1)$-equivalence. Since X has $\mathcal{C}$-isotropy, the map $X_{\mathcal{C}} \to X$ is a Σ-equivalence. The fixed point set X^Σ is S^ℓ, so by 7.7 the right-hand square can be identified up to Σ-equivalence as the square

$$
\begin{array}{ccc}
S^\ell \times \mathrm{E}\,\tilde{\mathcal{P}} & \longrightarrow & S^\ell \\
\downarrow & & \downarrow \\
\mathrm{E}\,\tilde{\mathcal{P}} & \longrightarrow & Y
\end{array} .
$$

It is easy to argue by naturality that the upper arrow and the left-hand vertical arrow are the obvious projections. This implies that Y is Σ-equivalent to the join $S^\ell \# \mathrm{E}\,\tilde{\mathcal{P}}$, which is itself Σ-equivalent to $S^\ell \wedge \mathrm{E}\,\tilde{\mathcal{P}}^\diamond$. Since X is Σ-equivalent to Y in the stable range with respect to ℓ, it follows from 7.8 that X/Σ is equivalent to $Y/\Sigma \simeq S^\ell \wedge \mathrm{B}\tilde{\mathcal{P}}^\diamond$ in the same stable range. □

The proof of 7.3 also depends on a few lemmas. Recall (2.6) that if G is a finite group and X is a G-space, $\mathrm{Sing}_G(X)$ is the G-subspace of X consisting of all simplicies which have a nontrivial isotropy subgroup.

7.9. Lemma. *Suppose that G is a finite group, X is a pointed G-space, and $Y \subset X$ is any G-subspace of X containing* $\mathrm{Sing}_G(X)$. *Then the diagram*

$$\begin{array}{ccc} Y_{\tilde{\mathrm{h}}G} & \longrightarrow & X_{\tilde{\mathrm{h}}G} \\ \downarrow & & \downarrow \\ Y/G & \longrightarrow & X/G \end{array}$$

is a homotopy pushout diagram.

Proof. The statement is clear if $X = Y$. The fact that $\mathrm{Sing}_G(X) \subset Y$ implies that X is obtained from Y by adding free G-cells of the form $C_k = (G \times \Delta[k], G \times \partial\Delta[k])$. For any one of these cells C_k the map $(C_k)_{\mathrm{h}G} \to C_k/G$ is clearly an equivalence of pairs. The lemma is proved by induction on the number of added cells, if this number is finite, and then in general by passage to a sequential colimit. □

7.10. Lemma. *If $k \geq 2$ the quotient S^k/Σ_k is contractible.*

Proof. We will give a topological argument. Take S^1 to be the unit circle in the complex plane, and use the basepoint $1 \in S^1$ to obtain inclusions $\mathrm{SP}^i(S^1) \to \mathrm{SP}^j(S^1)$, $i < j$. Then S^k/Σ_k is homeomorphic to $\mathrm{SP}^k(S^1)/\mathrm{SP}^{k-1}(S^1)$ (see 7.1), so to obtain the desired contractibility it is enough to show that the map $S^1 = \mathrm{SP}^1(S^1) \to \mathrm{SP}^k(S^1)$ is an equivalence. Now $\mathrm{SP}^k(S^1)$ is equivalent to $\mathrm{SP}^k(\mathbb{C} \setminus \{0\})$, which, by the fundamental theorem of algebra, is homeomorphic to the space of monic polynomials of degree k with complex coefficients and nonzero constant term. The result follows easily from the fact that using the coefficients of the polynomials as coordinates gives a homeomorphism $\mathrm{SP}^k(\mathbb{C} \setminus \{0\}) \cong \mathbb{C}^{n-1} \times (\mathbb{C} \setminus \{0\})$. □

7.11. Lemma. *If X is a pointed Σ-space with* $\mathrm{Iso}(X) \subset \mathcal{P} \cup \{\Sigma\}$, *then the quotient $(X \wedge S^n)/\Sigma$ is contractible.*

Proof. The statement is clear if $X = *$. By induction on the number of Σ-cells of X (and eventual passage to a sequential colimit) it is enough to show that if $(Y \wedge S^n)/\Sigma$ is contractible and X is obtained from Y by adding a single Σ-cell of the form $\Sigma/K \times (\Delta[m], \partial\Delta[m])$, $K \in \mathcal{P} \cup \{\Sigma\}$, then $(X \wedge S^n)/\Sigma$ is contractible. There is a (homotopy) pushout diagram

$$\begin{array}{ccc} \partial\Delta[m]_+ \wedge (S^n/K) & \longrightarrow & Y/\Sigma \\ \downarrow & & \downarrow \\ \Delta[m]_+ \wedge (S^n/K) & \longrightarrow & X/\Sigma \end{array}$$

where K is of the form $\Sigma_{n_1} \times \cdots \times \Sigma_{n_j}$, with $\sum n_i = n$ and at least one integer n_i greater than one. By 7.10 the quotient $S^n/K \cong S^{n_1}/\Sigma_{n_1} \wedge \cdots \wedge S^{n_j}/\Sigma_{n_j}$ is contractible, and the lemma follows. $\square$

7.12. Lemma. *The fixed point inclusion $S^1 = (S^n)^\Sigma \to S^n$ induces a Σ-equivalence*

$$\mathrm{E}\tilde{\mathcal{P}}^\diamondsuit \wedge S^1 \to \mathrm{E}\tilde{\mathcal{P}}^\diamondsuit \wedge S^n \,.$$

Proof. As in 7.4, it is enough it is enough to prove the result with $\mathrm{E}\,\tilde{\mathcal{P}}$ replaced by $\mathrm{E}\,\mathcal{F}$. We check that for each subgroup K of Σ the fixed point map $(\mathrm{E}\mathcal{F}^\diamondsuit \wedge S^1)^K \to (\mathrm{E}\mathcal{F}^\diamondsuit \wedge S^n)^K$ is an equivalence. If K acts transitively on $\mathbf{n}$, this is the map $(\emptyset^\diamondsuit) \wedge S^1 \to (\emptyset^\diamondsuit) \wedge (S^n)^K = (\emptyset^\diamondsuit) \wedge S^1$ and so it is a homeomorphism. If K does not act transitively on $\mathbf{n}$, the space $(\mathrm{E}\mathcal{F})^K$ is contractible, hence $(\mathrm{E}\mathcal{F}^\diamondsuit)^K$ is contractible, and hence both domain and range of the above fixed point map are contractible. $\square$

Proof of 7.3. Let X denote the pointed Σ-space $\mathrm{E}\,\tilde{\mathcal{P}}^\diamondsuit$. Then $\mathrm{Sing}_\Sigma(X)$ is $\mathrm{E}\,\mathcal{P}^\diamondsuit$. It is clear that $\mathrm{Sing}_\Sigma(X \wedge S^n)$ is contained in $\mathrm{Sing}_\Sigma(X) \wedge S^n$, and so by 7.9 there is a homotopy pushout diagram

$$\begin{array}{ccc} (\mathrm{E}\,\mathcal{P}^\diamondsuit \wedge S^n)_{\tilde{\mathrm{h}}\Sigma} & \longrightarrow & (X \wedge S^n)_{\tilde{\mathrm{h}}\Sigma} \\ \downarrow & & \downarrow \\ (\mathrm{E}\,\mathcal{P}^\diamondsuit \wedge S^n)/\Sigma & \longrightarrow & (X \wedge S^n)/\Sigma \end{array} .$$

Since $\mathrm{Iso}(\mathrm{E}\,\mathcal{P}^\diamondsuit) = \mathcal{P} \cup \{\Sigma\}$, the lower left-hand space is contractible by 7.11. The space X is contractible because $\mathrm{E}\,\tilde{\mathcal{P}}$ is contractible, where this last follows (cf. 2.10) from the fact that the poset $\tilde{\mathcal{P}}$ contains the minimal element $\{e\}$; it follows that the upper right-hand space is also contractible. The homotopy pushout diagram then shows that $(X \wedge S^n)/\Sigma$ is equivalent to $S^1 \wedge (\mathrm{E}\,\mathcal{P}^\diamondsuit \wedge S^n)_{\tilde{\mathrm{h}}\Sigma}$. By 7.12, $(X \wedge S^n)/\Sigma$ is equivalent to $\mathrm{B}\tilde{\mathcal{P}} \wedge S^1$. $\square$

8. DUALITY

It is known that mod p reduction of the Steinberg module is self-dual. In this section we describe an explicit duality isomorphism and use it to construct a geometric duality map for the suspension spectrum of the Tits building.

Fix $n = p^k$, and assume $k > 2$; the cases with $k \le 2$ are simpler but require some adjustments in the notation. Let Δ be the elementary abelian p-group $(\mathbb{F}_p)^k$, $\mathbf{T}_k$ the associated Tits building, and G the group $\mathrm{GL}_{k,p} = \mathrm{Aut}(\Delta)$. Let C_* denote the reduced normalized simplicial chain complex of $\mathbf{T}_k^{\diamondsuit}$ with coefficients in $\mathbb{F}_p$. We will use the simplicial model for $\mathbf{T}_k^{\diamondsuit}$ from [15, 27.6] (adjusted in an evident way to omit baspoint identifications), so that the nondegenerate m-simplices of $\mathbf{T}_k^{\diamondsuit}$, $m > 0$, correspond bijectively to the nondegenerate $(m-1)$-simplices of $\mathbf{T}_k$; there are two zero-simplices. Thus

$$C_m \cong \begin{cases} \bigoplus_I \mathbb{F}_p[G/P_I] & 0 < m \le k-1 \\ 0 & m > k-1 \end{cases}$$

where I ranges through the set of ordered partitions of k with $(m+1)$ constituents and P_I is the parabolic subgroup of G associated with I. Recall that if $I = \langle i_1, i_2, \ldots, i_{m+1}\rangle$ with $\sum i_j = k$ and $i_j \ge 1$, then P_I is the subgroup of G which preserves the flag

$$(\mathbb{F}_p)^{i_1} \subset (\mathbb{F}_p)^{i_1+i_2} \subset \cdots \subset (\mathbb{F}_p)^{\sum i_j} = \Delta\ .$$

In particular, $C_{k-1} \cong \mathbb{F}_p[G/B]$, where B is the Borel subgroup of upper triangular matrices, and $C_{k-2} \cong \bigoplus_{i=1}^{k-1} \mathbb{F}_p[G/P_i]$, where P_i is the parabolic subgroup associated with the partition $\langle 1, \ldots, 2, \ldots, 1\rangle$ with $(k-1)$ constituents and 2 in the i-th place. The boundary map $\partial_{k-1} \colon C_{k-1} \to C_{k-2}$ is the sum of components $(-1)^i d_i$, where the map $d_i \colon \mathbb{F}_p[G/B] \to \mathbb{F}_p[G/P_i]$ is induced by the inclusion $B \to P_i$. Note that there are only $(k-1)$ terms in the sum for ∂_{k-1} because one of the simplicial face operators induces the zero map on normalized chains. The homology of C_* is concentrated in degree $k-1$ and $\mathrm{H}_{k-1}(C_*) \cong \bigcap_{i=1}^{k-1} \ker(d_i) \cong \tilde{\mathrm{H}}_{k-2}(\mathbf{T}_k; \mathbb{F}_p)$ is the mod p Steinberg representation of G.

Let C^* be the cochain complex dual to C_*. Each group C_j is a direct sum of permutation modules and so has a basis preserved by G. It follows that $C^j = \hom(C_j, \mathbb{F}_p)$ is isomorphic to C_j as a module over G. The cohomology of C^* is concentrated in degree $k-1$ and $\mathrm{H}^{k-1}(C^*)$ is abstractly isomorphic to $\mathrm{H}_{k-1}(C_*)$, at least as a vector space over $\mathbb{F}_p$. The group $\mathrm{H}^{k-1}(C^*)$ is the dual of the mod p Steinberg module. The coboundary map $\partial^{k-1} \colon C^{k-2} \to C^{k-1}$ is given by the alternating sum of maps d^i, where d^i is dual to d_i. The map d^i can be interpreted as a transfer associated with the inclusion $B \to P_i$.

We use permutation bases to identify C_j with C^j. To emphasize this, we will use the notation $C(j)$ for both of these groups. Thus we view the boundary and cobundary maps ∂_j and ∂^j as maps $C(j) \to C(j-1)$ and $C(j-1) \to C(j)$ respectively.

Consider the composite homomorphism

$$S\colon\ \mathrm{H}_{k-1}(C_*) \to C(k-1) \to \mathrm{H}^{k-1}(C^*)$$

where the first map is the inclusion of the kernel of ∂_{k-1} and the second map is projection to the cokernel of ∂^{k-1}. Clearly, S is a G-equivariant map.

8.1. Lemma. *The map S is an isomorphism between the mod p Steinberg module and its dual.*

Proof. Since the source and the target of S are finite dimensional vector spaces over $\mathbb{F}_p$ of the same dimension, it is enough to show that S is surjective. To do this, we consider the maps $e_i\colon C(k-1) \to C(k-1)$ defined by $e_i = d^i d_i$ for $i = 1, \ldots, k-1$. Let w_i be the transposition $(i, i+1)$ in the symmetric group Σ_k, which is the Weyl group of the standard split (B, N)-pair structure on G. It is not hard to see that e_i can be identified with $1 + f_{w_i}$, where f_{w_i} is the G-endomorphism of $\mathbb{F}_p[G/B]$ referred to in [10, 2.7]. Since $(f_{w_i})^2 = -f_{w_i}$, it follows that $(e_i)^2 = e_i$; this also follows directly from the fact that the index of B in P_i is congruent to 1 mod p. The endomorphisms f_{w_i} satisfy braid relations due to Iwahori [10, 2.4], and (as a consequence of the fact that $(f_{w_i})^2 = -f_{w_i}$) the maps e_i satisfy the same relations:

$$\begin{aligned} e_i e_j &= e_j e_i \qquad \text{if } |i-j| \geq 2 \\ e_i e_{i+1} e_i &= e_{i+1} e_i e_{i+1} \end{aligned} .$$

To prove that S is surjective it is enough to prove that for each $u \in C(k-1)$ there exists an element v in the image of ∂^{k-1} such that $u + v \in \ker(\partial_{k-1})$. Let $u \in C(k-1)$. Let $\bar{e}_i = 1 - e_i$ and let $\bar{E}_i = \bar{e}_1 \bar{e}_2 \cdots \bar{e}_i$. Define

$$w = \bar{E}_{k-1} \bar{E}_{k-2} \cdots \bar{E}_1 u$$

and let $v = w - u$.

Clearly, v is in the subspace of $C(k-1)$ generated by the images of the maps e_i, and thus v is in the image of ∂^{k-1}. (To derive this last conclusion, note that $d^j d_i = 0$ if $i \neq j$, so that $e_i = \pm \partial^{k-1} d_i e_i$.) The idempotents $\bar{e}_i$ satisfy the same braid relations that the idempotents e_i do, and from this it is easy to see that $\bar{e}_i w = w$ and thus that $e_i w = 0$

for $i = 1, \dots, k-1$. By inspection, $e_i x = 0$ only if $d_i x = 0$, so it follows that $w \in \ker(\partial_{k-1})$. $\square$

Our next step will be to construct a map of spectra that realizes the map S on homology. Consider the map

$$\alpha\colon \Sigma^\infty \mathbf{T}_k^\diamondsuit \longrightarrow \Sigma^\infty(S^{k-1} \wedge (G/B)_+)$$

given by collapsing the $(k-2)$-skeleton of $\mathbf{T}_k^\diamondsuit$ to a point. The mod p homology of both spectra is concentrated in dimension $(k-1)$ and it is clear that the induced homology map in this dimension is the inclusion of the Steinberg module in $C(k-2)$. In the case of the Spanier-Whitehead dual map

$$\alpha^\#\colon \left(S^{k-1} \wedge (G/B)_+\right)^\# \longrightarrow \left(\mathbf{T}_k^\diamondsuit\right)^\#$$

the mod p homology of both spectra is concentrated in degree $(-k+1)$ and the induced homology map in this dimension is the projection of $C(k-1)$ on the dual of the Steinberg module. By 10.2 there is a G-equivariant map

$$\beta\colon \Sigma^\infty(S^{k-1} \wedge (G/B)_+) \longrightarrow S^{2(k-1)} \wedge \left(S^{k-1} \wedge (G/B)_+\right)^\#$$

inducing on mod p homology the isomorphism $C_{k-1} \cong C^{k-1}$.

Now let S_{top} be the composed map

$$(S^{2(k-1)} \wedge \alpha^\#) \cdot \beta \cdot \alpha \;:\; \Sigma^\infty \mathbf{T}_k^\diamondsuit \longrightarrow S^{2(k-1)} \wedge \left(\mathbf{T}_k^\diamondsuit\right)^\# \;.$$

8.2. Theorem. *The map S_{top} is a $\mathrm{GL}_{k,p}$-equivariant map that induces an isomorphism in mod p homology.*

Proof. It is clear that S_{top} is equivariant and that it induces the algebraic map S on the only non-trivial homology group. The theorem follows from lemma 8.1. $\square$

9. Layers in the Goodwillie tower of the identity

In this section we prove Theorems 1.16 and 1.17. We begin with an immediate consequence of Theorem 8.2. Let $\Delta = (\mathbb{F}_p)^k$, and observe that the map S_{top} from 8.2 is equivariant with respect to $\mathrm{Aff}_{k,p} = \mathrm{GL}_{k,p} \ltimes \Delta$, where Δ as usual acts trivially on the Tits building and its dual.

9.1. Proposition. *For any $\mathrm{Aff}_{k,p}$-spectrum $\mathcal{W}$, the map*

$$\left(\Sigma^\infty \mathbf{T}_k^\diamondsuit \wedge \mathcal{W}\right)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}} \to \left(S^{2(k-1)} \wedge (\mathbf{T}_k^\diamondsuit)^\# \wedge \mathcal{W}\right)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}}$$

induced by S_{top} is a mod p homology isomorphism.

Now suppose that $n = p^k$ and embed $\mathrm{Aff}_{k,p}$ in Σ_n in the usual way (§1, §5). Let X be a based space, so that $X^{\wedge n}$ has a natural action of Σ_n, and apply 9.1 to $\mathcal{W} = \Sigma^\infty X^{\wedge n}$ with the induced action of $\mathrm{Aff}_{k,p}$. Since $(\mathbf{T}_k^\diamondsuit)^\# \wedge \mathcal{W} \sim \mathrm{Map}_*(\mathbf{T}_k^\diamondsuit, \mathcal{W})$, the statement becomes the following.

9.2. Corollary. *For any based space X, the map*

$$\left(\Sigma^\infty \mathbf{T}_k^\diamondsuit \wedge X^{\wedge n}\right)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}} \to S^{2(k-1)} \wedge \mathrm{Map}_*\left(\mathbf{T}_k^\diamondsuit, \Sigma^\infty X^{\wedge n}\right)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}}$$

induced by S_{top} is a mod p homology isomorphism.

9.3. *Remark.* If $\mathcal{W}$ is a spectrum with an action of $\mathrm{Aff}_{k,p}$, there is an equivalence $\mathcal{W}_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}} \sim (\mathcal{W}_{\tilde{\mathrm{h}}\Delta})_{\tilde{\mathrm{h}}\,\mathrm{GL}_{k,p}}$ (cf. [4, 8.5]). Let X be as in 9.2, and let $\mathcal{U}$ denote $\Sigma^\infty (X^{\wedge n})_{\tilde{\mathrm{h}}\Delta}$. It follows that the map of 9.2 can be interpreted as the mod p homology isomorphism

$$(\Sigma^\infty \mathbf{T}_k^\diamondsuit \wedge \mathcal{U})_{\tilde{\mathrm{h}}\,\mathrm{GL}_{k,p}} \to S^{2(k-1)} \wedge \mathrm{Map}_*(\mathbf{T}_k^\diamondsuit, \mathcal{U})_{\tilde{\mathrm{h}}\,\mathrm{GL}_{k,p}} \tag{9.4}$$

induced by S_{top}. Let $B \subset \mathrm{GL}_{k,p}$ be the group of upper-triangular matrices. By way in which S_{top} is constructed in §8, the map in 9.4 factors through the spectrum

$$(S^{k-1} \wedge (\mathrm{GL}_{k,p}/B)_+ \wedge \mathcal{U})_{\tilde{\mathrm{h}}\,\mathrm{GL}_{k,p}} \sim S^{k-1} \wedge \mathcal{U}_{\tilde{\mathrm{h}}B}\ . \tag{9.5}$$

This implies that after p-completion the spectra in 9.2 are retracts of the p-completion of the spectrum in 9.5.

Proof of 1.16. Let X be an odd-dimensional sphere. In this case, by theorems 1.7 and 1.8 there are mod p equivalences

$$\left(\Sigma^\infty \mathbf{T}_k^\diamondsuit \wedge X^{\wedge n}\right)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}} \to \left(\Sigma^\infty \mathbf{P}_n^\diamondsuit \wedge X^{\wedge n}\right)_{\tilde{\mathrm{h}}\Sigma_n}$$

and

$$\mathrm{Map}_*(\mathbf{P}_n^\diamondsuit, \Sigma^\infty X^{\wedge n})_{\tilde{\mathrm{h}}\Sigma_n} \to \mathrm{Map}_*\left(\mathbf{T}_k^\diamondsuit, \Sigma^\infty X^{\wedge n}\right)_{\tilde{\mathrm{h}}\,\mathrm{Aff}_{k,p}}\ .$$

Combining these equivalences with 9.2 we obtain 1.16. □

The Steinberg idempotent $\epsilon_k^{\mathrm{St}} \in \mathbb{F}_p[\mathrm{GL}_{k,p}]$ (1.4) lifts to an element $\tilde{\epsilon}_k^{\mathrm{St}} \in \mathbb{Z}[\mathrm{GL}_{k,p}] = \pi_0\Sigma^\infty(\mathrm{GL}_{k,p})_+$ and in this way can be made to act up to homotopy on any G-spectrum. A straightforward telescope construction shows that after p-completion a G-spectrum $\mathcal{W}$ can be split as a wedge $\mathcal{W}_1 \vee \mathcal{W}_2$, where $\mathrm{H}_*(\mathcal{W}_1;\mathbb{F}_p) \cong \epsilon_k^{\mathrm{St}} \cdot \mathrm{H}_*(\mathcal{W};\mathbb{F}_p)$. The spectrum $\mathcal{W}_1$ is well-defined up to homotopy and we will denote it $\epsilon_k^{\mathrm{St}} \cdot \mathcal{W}$. A homology calculation using the sphericity of the Tits building shows that up to p-completion there is an equivalence

$$\epsilon_k^{\mathrm{St}} \cdot \mathcal{W} \underset{p}{\sim} S^{1-k} \wedge (\mathbf{T}_k^\diamondsuit \wedge \mathcal{W})_{\tilde{\mathrm{h}}\,\mathrm{GL}_{k,p}}\ .$$

In view of the remarks above in 9.3, Theorem 1.16 gives the following statement.

9.6. Corollary. *Let X be an odd-dimensional sphere and let $n = p^k$. Then up to p-completion there is an equivalence*

$$S^k \wedge D_n(X) \underset{p}{\sim} \epsilon_k^{\mathrm{St}} \cdot \Sigma^\infty (X^{\wedge n})_{\tilde{\mathrm{h}}\Delta} \ .$$

Note that since X is a sphere, $(X^{\wedge p^k})_{\tilde{\mathrm{h}}\Delta}$ is the Thom space of a vector bundle over $\mathrm{B}\Delta$.

Proof of 1.17. By Theorem 1.13 there is an equivalence of spectra

$$\left(\Sigma^\infty \mathbf{P}_n^\diamondsuit \wedge S^n\right)_{\tilde{\mathrm{h}}\Sigma} \simeq \mathrm{SP}^n(\mathcal{S}^0)/\,\mathrm{SP}^{n-1}(\mathcal{S}^0)$$

On the other hand, by theorem 1.16 the left-hand side is equivalent after p-completion to $S^{2(k-1)+1} \wedge D_{p^k}(S^1)$. This proves 1.17. □

9.7. *Remark.* That corollary 9.6 and theorem 1.17 are true had been suggested by M. Mahowald to the first-named author a few years prior to the writing of this paper. Theorem 1.17 and Corollary 9.6 (in the case $X = S^1$) have been demonstrated by N. Kuhn, using different methods. He shows that the mod p cohomologies of the spectra involved are isomorphic as modules over the Steenrod algebra. In this particular case it turns out that the homotopy types of the spectra are determined up to p-completion by their mod p cohomology modules.

10. Reverse ampleness

The purpose of this section is to prove 3.9. We will follow a line of argument similar to the one used in [5] to prove the results in section 8 of that paper. We first describe a kind of acyclicity which implies reverse ampleness, prove a general acyclicity theorem, and then apply it to the collection of nontrivial elementary abelian subgroups. For the rest of the section G denotes a particular finite group.

We first need some constructions from [5]. Recall that a *coefficient system* $\mathcal{H}$ for G [5, 4.1] is a functor from the category of $\mathbb{F}_p[G]$-modules to the category of vector spaces over $\mathbb{F}_p$ which preserves arbitrary direct sums. If X is a G-space, the *Bredon homology* of X with coefficients in $\mathcal{H}$, denoted $\mathrm{HBr}_*^G(X;\mathcal{H})$, is the homology of the chain complex $\mathrm{CBr}_*^G(X;\mathcal{H})$ obtained by the following three-step process:

1. Apply the free $\mathbb{F}_p$-module functor dimensionwise to X, to obtain a simplicial $\mathbb{F}_p[G]$-module $\mathbb{F}_p[X]$.
2. Apply $\mathcal{H}$ dimensionwise to $\mathbb{F}_p[X]$ to obtain a simplicial vector space $\mathcal{H}(\mathbb{F}_p[X])$.
3. Normalize $\mathcal{H}(\mathbb{F}_p[X])$ to obtain $\mathrm{CBr}_*^G(X;\mathcal{H})$.

A G-space X is said to be *acyclic for* $\mathcal{H}$ if the map $X \to *$ induces an isomorphism $\mathrm{HBr}^G_*(X;\mathcal{H}) \to \mathrm{HBr}^G_*(*;\mathcal{H})$.

If M is a module over $\mathbb{F}_p[G]$, let $\mathcal{H}^i_M$ denote the coefficient system for G which assigns to an $\mathbb{F}_p[G]$-module A the $\mathbb{F}_p$-module $\mathrm{H}^i(G; A \otimes \mathrm{Hom}(M,\mathbb{F}_p))$. The following proposition is an analog of [5, 6.2].

10.1. Proposition. *Suppose that M is a finite $\mathbb{F}_p[G]$-module. If the G-space $|\mathcal{C}|$ is acyclic for the coefficient systems $\mathcal{H}^i_M$, $i \geq 0$, then $\mathcal{C}$ is reverse M-ample for $*$.*

10.2. Lemma. *Suppose that K is a subgroup of G and that X is a K-space. Then there is a G-map*

$$G_+ \wedge_K (X^\#_+) \to (G \times_K X)^\#_+$$

which is an equivalence of spectra.

Proof. There is a G-module isomorphism $\mathbb{Z}[G] \to \mathrm{Hom}(\mathbb{Z}[G],\mathbb{Z})$ which takes the basis element e of $\mathbb{Z}[G]$ to the map $\mathbb{Z}[G] \to \mathbb{Z}$ obtained by sending $\sum_{x\in G} c_x x$ to c_e. This extends easily to a G-map $\Sigma^\infty G_+ \to G^\#_+$ which is a weak equivalence of spectra; note that both of these spectra are wedges of $\mathcal{S}^0$; their respective zero-dimensional homology groups are isomorphic respectively to $\mathbb{Z}[G]$ and $\mathrm{Hom}(\mathbb{Z}[G],\mathbb{Z})$. The proof is finished by observing that because $\Sigma^\infty G_+$ is a finite K-spectrum there is for any K-spectrum Y an equivalence

$$\Sigma^\infty G_+ \wedge_K Y^\# \cong G^\#_+ \wedge_K Y^\# \to (\Sigma^\infty G_+ \wedge_K Y)^\# \ . \quad \square$$

10.3. *Remark.* An algebraic reflection of the above lemma is the fact that if K is a subgroup of G, then $\mathrm{H}^j(G; M \otimes \mathbb{Z}[G/K])$ is naturally isomorphic to $\mathrm{H}^j(K;M)$. We note for future reference that if $H \subset K$, then under this isomorphism the projection $G/H \to G/K$ induces the cohomological transfer map $\mathrm{H}^j(H;M) \to \mathrm{H}^j(K;M)$. For example, the map $\mathrm{H}^0(G; M \otimes \mathbb{Z}[G/H]) \to \mathrm{H}^0(G; M \otimes \mathbb{Z}[G/K])$ induced by $G/H \to G/K$ is the map $M^H \to M^K$ given by averaging an element of M^H over coset representatives of H in K.

Proof of 10.1. By 2.10, it is enough to show that under the given assumptions the map $\mathcal{S}^0 = (*)^\#_+ \to |\mathcal{C}|^\#_+$ induced by $|\mathcal{C}| \to *$ is an $\mathrm{H}^G_*(-;M)$-equivalence. For any $\mathbb{F}_p[G]$-module A there is a natural isomorphism $\mathrm{Hom}_G(A, \mathrm{Hom}(M;\mathbb{F}_p)) \cong \mathrm{Hom}(A \otimes_{\mathbb{F}_p[G]} M, \mathbb{F}_p)$; this implies that for any G-space or G-spectrum Y, there are isomorphisms

$$\mathrm{H}^*_G(Y; \mathrm{Hom}(M,\mathbb{F}_p)) \cong \mathrm{Hom}(\mathrm{H}^G_*(Y;M),\mathbb{F}_p) \ .$$

Note that left and right module structures are being implicitly switched here (1.18). Given this duality formula, it is enough to show that the map $\mathcal{S}^0 \to |\mathcal{C}|^\#_+$ induces an isomorphism on $\mathrm{H}^*_G(-;\mathrm{Hom}(M,\mathbb{F}_p))$.

Let K be a subgroup of G. By inspection (see 10.3 and 10.2) there are natural isomorphisms

$$\mathrm{HBr}_i^G(G/K;\mathcal{H}_M^j) \cong \begin{cases} \mathrm{H}_G^j((G/K)_+^{\#};\mathrm{Hom}(M,\mathbb{F}_p)) & i=0 \\ 0 & i>0 \end{cases} .$$

Dualizing the skeletal filtration of $|\mathcal{C}|$ thus gives a spectral sequence

$$E_2^{i,j} = \mathrm{HBr}_i^G(|\mathcal{C}|\,;\mathcal{H}_M^j) \Rightarrow \mathrm{H}_G^{j-i}(|\mathcal{C}|_+^{\#}\,;\mathrm{Hom}(M,\mathbb{F}_p)) \ .$$

This spectral sequence converges because the skeletal filtration of $|\mathcal{C}|$ is finite. Under the assumed acyclicity condition, the spectral sequence collapses onto the j-axis and gives the desired isomorphism. □

The next proposition is like [5, 6.8], but a little more awkward to formulate. In the statement, $\mathcal{H}$ is a coefficient system for G. If K is a subgroup of G, $\mathcal{H}|_K$ is the coefficient system for K which assigns to the K-module N the abelian group $\mathcal{H}(\mathbb{Z}[G]\otimes_{\mathbb{Z}[K]} N)$.

10.4. Proposition. *Let X be a G-space, K a subgroup of G of index prime to p, and Y a subspace of X which is closed under the action of K. Assume that Y is acyclic for $\mathcal{H}|_K$, and that for any $x\in X\setminus Y$ the following three conditions hold:*

1. *the map $\mathcal{H}(\mathbb{F}_p[G/K_x])\to\mathcal{H}(\mathbb{F}_p[G/G_x)$ is zero,*
2. *the map $\mathcal{H}(\mathbb{F}_p[G/K_x])\to\mathcal{H}(\mathbb{F}_p[G/G])=\mathcal{H}(\mathbb{F}_p)$ is zero, and*
3. *for any $y\in Y$ with $K_x\subset K_y$, the map $\mathcal{H}(\mathbb{F}_p[G/K_x])\to\mathcal{H}(\mathbb{F}_p[G/K_y])$ is zero.*

Then X is acyclic for $\mathcal{H}$.

Proof. Compare this with [5, proof of 6.8], but note that we use slightly different notation. For instance, what we denote $\mathrm{CBr}_*^G(X;\mathcal{H})$ is denoted $C_*^G(X;\mathcal{H})$ in [5].

The transfers associated to the maps $q: G\times_K X_n\to X_n$ provide a map $t:\mathrm{CBr}_*^G(X;\mathcal{H})\to\mathrm{CBr}^K(X;\mathcal{H}|_K)$ [5, 4.1]. By [5, 5.10] this map commutes with differentials, and there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{CBr}_*^G(X;\mathcal{H}) & \xrightarrow{t} & \mathrm{CBr}_*^K(X;\mathcal{H}|_K) & \xrightarrow{q} & \mathrm{CBr}_*^G(X;\mathcal{H}) \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow \\ \mathrm{CBr}_*^G(*;\mathcal{H}) & \longrightarrow & \mathrm{CBr}_*^K(*;\mathcal{H}|_K) & \longrightarrow & \mathrm{CBr}_*^G(*;\mathcal{H}) \end{array}$$

in which the lower arrows are induced by similar transfers and projections. The index assumption assures that the horizontal composites are isomorphisms. Let D be the graded submodule of $\mathrm{CBr}_*^K(X;\mathcal{H}|_K)$ which in dimension m is given by $\mathcal{H}|_K(\mathbb{F}_p[X_m\setminus Y_m])$. By assumption (3), D is actually a subcomplex; the quotient complex $Q=\mathrm{CBr}_*^K(X;\mathcal{H}|_K)/D$

is clearly isomorphic to $\mathrm{CBr}_*^K(Y;\mathcal{H}|_K)$. By assumption (1) the map q in the above diagram factors through a map $Q \to \mathrm{CBr}_*^G(X;\mathcal{H})$, and by assumption (2) the map v factors through a map $Q \to \mathrm{CBr}_*^K(*;\mathcal{H}|_K)$. This implies that the homology map $\mathrm{HBr}_*^G(X;\mathcal{H}) \to \mathrm{HBr}_*^G(*;\mathcal{H})$ is a retract of the isomorphism $\mathrm{HBr}_*^K(Y;\mathcal{H}|_K) \to \mathrm{HBr}_*^K(*;\mathcal{H}|_K)$, and the theorem follows from the fact that a retract of an isomorphism is an isomorphism. □

This is a variation on Webb's theorem (see [22] or [5, 6.0]), but in this cohomological situation the hypotheses are stronger.

10.5. Proposition. *Let X be a G-space, P a Sylow p-subgroup of G, and M a module over $\mathbb{F}_p[G]$. Suppose that for any nonidentity subgroup Q of P the fixed point set X^Q is contractible. Assume in addition that for each simplex $x \in X$ the order of G_x is divisible by p, and that every element of order p in G acts trivially on M. Then X is acyclic for the functors $\mathrm{H}^j(G; M \otimes -)$, $j \geq 0$.*

Proof. We will use the remarks in 10.3. Let Y be the P-subspace of X consisting of simplices which are fixed by a nonidentity element of P. By [5, 4.7] the map $Y \to *$ is a P-equivalence, and so by [5, 4.8] the space Y is acyclic for the functors $\mathrm{H}^j(P; M \otimes -)$. Let x be a simplex of X which is not in Y, so that $P_x = \{e\}$. We now check the three conditions of 10.4. For (1), the map

$$\mathrm{H}^j(\{e\}; M) = \mathrm{H}^j(P_x; M) \to \mathrm{H}^j(G_x; M)$$

is trivial for $j > 0$ because the domain group vanishes, and for $j = 0$ because it can be identified with the norm or transfer map

$$\sum_{g \in G_x} g : M \to M^{G_x} \ .$$

This norm map vanishes because M is a vector space over $\mathbb{F}_p$ and there is an element of order p in G_x which acts trivially on M. For similar reasons the map $\mathrm{H}^j(\{e\}; M) \to \mathrm{H}^j(G; M)$ vanishes (condition (2)), as well as the maps $\mathrm{H}^j(\{e\}; M) \to \mathrm{H}^j(P_y; M)$ for each $y \in Y$ (condition (3)). □

Proof of 3.9. By 10.1, it is enough to prove that the G-space $|\mathcal{E}|$ is acyclic for the coefficient systems $\mathcal{H}_M^i$. This will be a consequence of 10.5 if we can check that for every simplex $x \in |\mathcal{E}|$, G_x has order divisible by p, and that for every nonidentity p-subgroup Q of G, $|\mathcal{E}|^Q$ is contractible. Both of these conditions are verified in [5, §8], where the space $|\mathcal{E}|$ is denoted $X_{\mathcal{E}}^{\delta}$. □

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