ITERATES OF THE SUSPENSION MAP AND MITCHELL'S FINITE SPECTRA WITH A_k-FREE COHOMOLOGY

GREG ARONE

June 16, 1998

ABSTRACT. We study certain cross-effects of the unstable homotopy of spheres. These cross-effects were constructed by Weiss, for different purposes, in the context of "Orthogonal calculus". We show that Mithchell's finite spectra with A_k-free cohomology (constructed in [Mt85]) arise naturally as stabilizations of Weiss' cross-effects. In particular, we find that after a suitable Bousfield localization, our cross-effects, which capture meaningful information about the unstable homotopy of spheres, are homotopy equivalent to the infinite loop spaces associated with Mitchell's spectra. This last result is a partial generalization of the main result of Mahowald and the author in [AM97].

0. INTRODUCTION

Let X be a topological space. We are interested in studying the difference between the homotopy type of X and that of S^2X, the double suspension of X (for technical reasons the statements turn out to be a little cleaner if one works with double suspensions rather than single suspensions). One naive way to compare X and S^2X would be by means of the map $X \to S^2X$ given by taking the smash product of the identity map on X with the inclusion $S^0 \to S^2$. Of course, this map is null-homotopic and thus is unlikely to provide useful information about the difference between X and S^2X. A much better idea is to consider the Freudental (double) suspension map $w_1 : X \to \Omega^2 S^2 X$. Let $F_1(X)$ be the homotopy fiber of this map. We regard $F_1(X)$ as measuring the difference between the homotopy type of X and that of S^2X. We would like to iterate the process and find higher analogues of the suspension map and higher iterated differences of the suspension map. Thus at the second stage we would like to find a suitable way to compare $F_1(X)$ with $F_1(S^2X)$. Obviously, there exists a natural map $F_1(X) \to \Omega^2 F_1(S^2X)$. However, this map turns out to be null-homotopic, just as the map $X \to S^2X$ is null-homotopic. It follows from the

1991 *Mathematics Subject Classification.* 55P40, 55P42, 55P65.
Key words and phrases. spheres, periodic homotopy, orthogonal calculus.
The author is partially supported by the NSF

remarkable analysis performed in [W95], that there exists a natural transformation $w_2 : F_1(X) \to \Omega^4 F_1(S^2X)$, extending the naive map $F_1(X) \to \Omega^2 F_1(S^2X)$, which deserves to be called the analogue of the map $X \to \Omega^2 S^2 X$.

Remark 0.1. the map $F_1(X) \to \Omega^4 F_1(S^2X)$ had been constructed in [C83], in the case of X being an odd-dimensional sphere localized at 2. The construction in [C83] uses specific calculations in unstable homotopy of spheres. However, Weiss' calculus tells us that the map exists for very general reasons.

Let $F_2(X)$ be the homotopy fiber of w_2. Iterating Weiss' construction, one obtains a sequence of functors $F_0, \dots, F_m, \dots$ (where $F_0(X) = X$) equipped with natural maps

$$w_m : F_{m-1}(X) \to \Omega^{2m} F_{m-1}(S^2X)$$

such that there are fibration sequences

$$\begin{array}{l} F_1(X) \to F_0(X) \stackrel{w_1}{\to} \Omega^2 F_0(S^2X) \\ F_2(X) \to F_1(X) \stackrel{w_2}{\to} \Omega^4 F_1(S^2X) \\ \vdots \\ F_m(X) \to F_{m-1}(X) \stackrel{w_m}{\to} \Omega^{2m} F_{m-1}(S^2X) \\ \vdots \end{array}$$

We think of $F_m(-)$ as the m-th iterated difference, or the m-th cross-effect, of the double suspension map.

Our next step is to investigate the layers in the Goodwillie towers of the functors $F_m(X)$, first for a general space X and then for X an odd-dimensional sphere localized at a prime. We will assume some familiarity with Goodwillie's theory of "Taylor towers". The references for this material are [G90, G92, G3]. See also [Jo95] for an exposition of some of the material in [G3]. Let $D_n(X)$ be the n-th layer in the Goodwillie tower of the identity. By layers of the Goodwillie tower we will usually mean not the infinite loop space that is the actual homotopy fiber in the tower, but the associated spectrum. We recall the description of $D_n(X)$ from [Jo95, AM97]

$$D_n(X) \simeq \mathrm{Map}_* \left(K_n, \Sigma^\infty X^{\wedge n}\right)_{h\Sigma_n}$$

where K_n is (the double suspension of) the geometric realization of the poset of (non-trivial) partitions of a set with n elements.

For a general functor F let D_nF be the n-th layer in the Goodwillie tower of F and let P_nF be the n-th "Taylor polynomial" of F in the sense of Goodwillie. Finally, let $\mathrm{U}(n)$ be the unitary group on n letters. We think of Σ_n as a subgroup of $\mathrm{U}(n-1)$ via the reduced standard representation. The following theorem is essentially [W95, Example 5.7].

Theorem 0.2. There is a natural equivalence

$$D_nF_m(X) \simeq \mathrm{Map}_*\left(K_n, \Sigma^\infty X^{\wedge n}\right) \wedge_{h\Sigma_n} \left(\mathrm{U}(n-1)/\,\mathrm{U}(n-m-1)_+\right)$$

A couple of notational remarks are in order. First, if G is a finite group, X and Y are spaces with an action of G on the right and the left respectively then by $X \wedge_{hG} Y$ we mean $X \wedge EG_+ \wedge_G Y$. This still makes sense if X is a spectrum with an action of G. Second, if $k < 0$ then $\mathrm{U}(n)/\,\mathrm{U}(k)$ is the empty space. In other words, D_nF_m is non-trivial only if $n > m$, and the bottom non-trivial layer of F_m is

$$D_{m+1}F_m(X) \simeq \mathrm{Map}_*\left(K_{m+1}, \Sigma^\infty X^{\wedge m+1}\right) \wedge_{\Sigma_{m+1}} \left(\mathrm{U}(m)_+\right)$$

(here we may use strict orbits instead of homotopy orbits, because the action of Σ_{m+1} on $\mathrm{U}(m)$ is free).

It is easy to see that the map w_m induces an equivalence on the bottom non-trivial layers, and that the mapping telescope of w_m consists only of the bottom layers. We can summarize this in the following diagram, where the last column lists the mapping telescopes of the raws:

$$\begin{array}{ccccccc}
X & \overset{w_1}{\to} & \Omega^2 S^2 X & \overset{w_1}{\to} & \Omega^4 S^4 X & \to \cdots & \Omega^\infty D_1F_0(X) \\
F_1(X) & \overset{w_2}{\to} & \Omega^4 F_1(S^2X) & \overset{w_2}{\to} & \Omega^8 F_1(S^4X) & \to \cdots & \Omega^\infty D_2F_1(X) \\
 & & \vdots & & \vdots & & \\
F_{m-1}(X) & \overset{w_m}{\to} & \Omega^{2m} F_{m-1}(S^2X) & \overset{w_m}{\to} & \Omega^{4m} F_{m-1}(S^4X) & \to \cdots & \Omega^\infty D_mF_{m-1}(X) \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

Our next goal is to analyze in further detail the layers of $F_m(X)$ in the special case when X is a sphere localized at p and to relate them to familiar objects in homotopy theory. For the rest of the introduction, X stands for an odd-dimensional sphere, and cohomology is always taken with mod p coefficients. The layers of F_0 are the spectra

$$\mathrm{Map}_*\left(K_n, \Sigma^\infty X^{\wedge n}\right)_{h\Sigma_n}$$

and these were studied extensively in in [AM97] and [AD97] in the case of X being an odd-dimensional sphere. The plan is to extend those results to the spectra

$$\mathrm{Map}_*\left(K_n, \Sigma^\infty X^{\wedge n}\right) \wedge_{h\Sigma_n} \left(\mathrm{U}(n-1)/\,\mathrm{U}(n-m-1)_+\right) \tag{1}$$

by induction on m.

The following was proved in [AM97] (in a somewhat different formulation)

Theorem 0.3. Let X be an odd-dimensional sphere. Let $n > 1$. If n is not a power of a prime then the spectrum

$$\mathrm{Map}_*\left(K_n, \Sigma^\infty X^{\wedge n}\right)_{h\Sigma_n}$$

is contractible. Suppose $n = p^k$ for some prime p. Then the integral cohomology of this spectrum is all p-torsion. The mod p cohomology is freely generated over A_{k-1} by a polynomial algebra on k-generators. More precisely, there is an isomorphism of A_{k-1}-modules (up to a degree shift depending on the dimension of X)

$$\mathrm{H}^*\left(\mathrm{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right)_{h\Sigma_{p^k}}; \mathbb{F}_p\right) \cong A_{k-1} \otimes P$$

Here P is a free module on one generator over the polynomial algebra

$$\mathbb{F}_p[d_0, \ldots, d_{k-1}]$$

where $|d_j| = 2p^k - 2p^j$.

Here A_{k-1} is the sub-algebra of the mod p Steenrod algebra generated by the elements $Sq^1, Sq^2, Sq^4, \ldots, Sq^{2^{k-1}}$ if $p = 2$ and by the elements $\beta, P^1, P^p, \ldots, P^{p^{k-2}}$ if p is odd. The polynomial generators d_j above should be thought of as the Dickson polynomials evaluated at the polynomial generators of $\mathrm{H}^*\left(\mathrm{B}\left(\mathbb{Z}/p\mathbb{Z}\right)^k; \mathbb{F}_p\right)$ (squares of the polynomial generators of $\mathrm{H}^*\left(\mathrm{B}\left(\mathbb{Z}/p\mathbb{Z}\right)^k; \mathbb{F}_p\right)$ if $p = 2$). By the Dickson polynomials we mean the polynomials that give the generators of the Dickson algebra of invariants of $\mathbb{F}_p[y_1, \ldots, y_k]^{\mathrm{GL}_k(\mathbb{F}_p)}$ (see [Wil83]). The generators d_j can also be thought of as the Chern classes of the reduced regular representation of $(\mathbb{Z}/p\mathbb{Z})^k$ and thus $d_j = c_{p^k - p^j}$.

The following theorem will be proved as part (b) of theorem 2.2

Theorem 0.4. Let X be an odd-dimensional sphere. The spectrum

$$\mathrm{Map}_*\left(K_n, \Sigma^\infty X^{\wedge n}\right) \wedge_{h\Sigma_n} \left(\mathrm{U}(n-1)/\,\mathrm{U}(n-m-1)_+\right)$$

is contractible if n is not a power of a prime. Suppose $n = p^k$, then there is an isomorphism (up to a dimension shift) of A_{k-1}-modules

$$\mathrm{H}^*\left(\mathrm{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{h\Sigma_{p^k}} \left(\mathrm{U}(p^k-1)/\,\mathrm{U}(p^k-m-1)_+\right); \mathbb{F}_p\right) \cong$$
$$\cong A_{k-1} \otimes E \otimes P$$

where E is a free module on one generator over the exterior algebra

$$\Lambda < \overline{c_i} \mid p^k - m \leq i \leq p^k - 1 \text{ and } i \text{ is not of the form } p^k - p^j >$$

(here $|\overline{c_i}| = 2i - 1$), and P is a free module on one generator over the polynomial algebra

$$\mathbb{F}_p[c_{p^k - p^j} \mid p^j > m]$$

Thus the exterior generators are precisely the Chern classes in $\mathrm{U}(p^k-1)/\,\mathrm{U}(p^k-m-1)$ that are not Dickson classes, and the polynomial generators are the Dickson classes in $\mathrm{B}\,\mathrm{U}(p^k-m-1)$.

Consider the bottom layer of $F_m(X)$ that is non-trivial for odd-dimensional spheres. It follows from theorem 0.4 that it is the p^k-th layer, where k is the smallest integer such that $m \leq p^k - 1$. Moreover, by theorem 0.4, the cohomology of this layer, as an A_{k-1}-module, is given by

$$\mathrm{H}^*\left(\mathrm{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{h\Sigma_{p^k}} \left(\mathrm{U}(p^k-1)/\,\mathrm{U}(p^k-m-1)_+\right); \mathbb{F}_p\right) \cong$$
$$\cong A_{k-1} \otimes E$$

where E is a free module on one genrator over the exterior algebra

$$\Lambda < \overline{c_i} \mid p^k - m \leq i \leq p^k - 1 \text{ and } i \neq p^k - p^j >$$

in other words, for the cohomology of the bottom non-trivial layer, the polynomial part is trivial. In particular, the mod p cohomology of the bottom layer is a *finite* A_{k-1}-free module. It is not clear if for a general m the bottom layer is in fact homotopy equivalent to a finite spectrum. However, if we take $m = p^k - 1$, then the bottom layer is homotopy equivalent to

$$\mathrm{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{\Sigma_{p^k}} \left(\mathrm{U}(p^k-1)_+\right)$$

(the point is that we can use strict orbits instead of homotopy orbits, as was noticed earlier in the paper). Clearly, this is a finite spectrum, whose mod p cohomology is an A_{k-1} free module (if X is an odd-dimensional sphere). The existence of such spectra was at one time an important question in homotopy theory. The question was motivated by the "chromatic philosophy" in homotopy theory, as such spectra are natural candidates to be spectra of type k. The first one to construct such spectra was S. Mitchell in [Mt85]. It turns out that our finite spectrum above and Mitchell's A_{k-1} free spectrum are "essentially the same". We formulate this imprecise statement as a "pre-theorem"

Pre-Theorem 0.5. Let X be an odd-dimensional sphere. The spectrum

$$\mathrm{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{\Sigma_{p^k}} \left(\mathrm{U}(p^k-1)_+\right)$$

is very closely related to the A_{k-1}-free spectrum constructed in [Mt85]. In fact, our spectra are more or less Thom spectra over Mitchell's spectra.

We will give more precise statements in section 2. For the time being, we pretend that instead of "very closely related" we have "equivalent". Thus Mitchell's spectra play a role in unstable homotopy theory. The infinite loop space associated with Mitchell's A_{k-1}-free spectrum can be thought of as the "principal part" (i.e., the bottom layer in the Goodwillie tower) of $F_{p^k-1}(X)$, when X is an odd-dimensional sphere.

These results have an interpretation in terms of the chromatic filtration of homotopy theory. As explained in [AM97], the cohomological properties of the layers of the functors F_m, together with the fact that when evaluated at spheres the tower converges exponentially faster than in general, imply the following theorem

Theorem 0.6. Let X be an odd-dimensional sphere localized at p. The map

$$F_m(X) \to P_{p^k} F_m(X)$$

induces an equivalence in "v_i-periodic homotopy" for $i \leq k$.

Taking k to be the smallest integer such that $P_{p^k} F_m(X)$ in non-trivial (for X an odd-sphere localized at p) we obtain the following special case as a corollary:

Corollary 0.7. Let X be an odd-dimensional sphere localized at a prime p. Let m be a non-negative integer an let k be the smallest integer such that $m \leq p^k - 1$. Then $F_m(X)$ is trivial in v_i-periodic homotopy for $i < k$ and the composed (weak) map

$$F_m(X) \to P_{p^k} F_m(X) \overset{\simeq}{\leftarrow} \Omega^\infty D_{p^k} F_m(X)$$

induces an equivalence in v_k-periodic homotopy.

The corollary says that a certain unstable object, namely $F_m(X)$ where X is an odd-dimensional sphere, is equivalent in v_k periodic homotopy (where k is the smallest integer such that $m \leq p^k - 1$) to the infinite loop space of a spectrum of type k. This spectrum is always finite in mod p cohomology, and if $m = p^k - 1$ then it actually is a finite spectrum (the Mitchell spectrum).

It is easy to see, for instance, that for $m = 1$ and any p, $D_p F_1(X)$ is the spectrum realizing one copy of A_0, and in fact it is the mod p Moore spectrum. By coincidence, if $m = 2$, $p = 2$, $D_4 F_2(X)$ turns out to be the spectrum whose cohomology realizes one copy of A_1. However, for all other values of m and p, the cohomology of the bottom non-trivial layer has more than one copy of A_{k-1}.

Remark 0.8. For $m = 0$, corollary 0.7 is essentially Serre's theorem to the effect that if X is an odd-dimensional sphere then the map $X \to \Omega^\infty \Sigma^\infty X$ is a rational homotopy equivalence (v_0-periodic homotopy is, essentially, rational homotopy). For $m = 1$ the corollary is due to Mahowald ([M82]). For $p = 2$ and $m = 2$ the corollary is due to Mahowald and Thompson ([MT94, Theorem 1.5]).

As a concluding remark, notice that since the map w_m induces an equivalence on the m-th layers, we obtain the following corollary

Corollary 0.9. Let X be an odd-dimensional sphere localized at p. Let $m = p^k$. The map $w_{p^k} : F_{p^k - 1} \to \Omega^{2p^k} F_{p^k - 1}(S^2 X)$ induces an equivalence in v_k-periodic homotopy.

On the other hand, if X is an odd sphere localized at p and m is not a power of p, then w_m seems to be far from being an equivalence. Preliminary calculations suggest that in this case w_m induces the zero map in homology and in all the Morava K-theories. In fact, preliminary calculations suggest the following conjecture

Conjecture 0.10. If X is an odd-dimensional sphere localized at p, and m is not a power of p then the map

$$w_m : F_{m-1}(X) \to \Omega^{2m} F_{m-1}(S^2 X)$$

is zero on homotopy groups.

In any case, it seems that for X an odd sphere, the values of m for which F_{m-1} and w_m are most interesting are powers of primes.

The rest of the paper is organized as follows: In section 1 we review the relevant points in Weiss' orthogonal calculus and explain why theorem 0.2 is implicit in [W95]. In section 2 we discuss the relationship with Mitchell's spectra and make the "pretheorem" precise.

Acknowledgements: It is obvious that I owe a lot to Weiss' paper [W95]. I would like to thank Goodwillie for suggesting that Weiss' ideas should help my efforts on Mahowald's program to construct higher iterates of the suspension map.

It is a pleasure to acknowledge my immeasurable debt to M. Mahowald. His suggestions were a crucial motivating force for this work.

1. Weiss' calculus

Let $\mathcal{F}$ be the category whose objects are finite-dimensional complex vector spaces with positive-definite inner-product and morphisms are linear maps respecting the inner product. Weiss' calculus [W95] is concerned with continuous functors from $\mathcal{F}$ to spaces (=compactly generated topological spaces with non-degenerate basepoint). In fact [W95] works with real rather than complex vector spaces, but since we want to work with double suspensions we will use complex vector spaces. Typical examples of functors that we will be interested in are $V \mapsto \mathrm{U}(V)$, $V \mapsto \mathrm{B\,U}(V)$ and $V \mapsto \Omega^V S^V X$. In the last example, X is a fixed based space, S^V is the one-point compactification of V and Ω^V stands for continuous maps from S^V.

Let $G : \mathcal{F} \to \mathrm{Spaces}_*$ be a functor. Define $G_1(V)$ to be the homotopy fiber of the map $G(V) \to G(V \oplus \mathbb{C})$. Clearly, G_1 is again a functor of $\mathcal{F}$. An important insight of [W95] is that the natural map $G_1(V) \to G_1(V \oplus \mathbb{C})$ lifts to a natural map $G_1(V) \to \Omega^2 G_1(V \oplus \mathbb{C})$, where the map $\Omega^2 G_1(V \oplus \mathbb{C}) \to G_1(V \oplus \mathbb{C}))$ is given by evaluation at zero. In fact, a little more is true:

Lemma 1.1. Let G_1 be as above. Let W be an object of $\mathcal{F}$. There exists a functor $\mathcal{F} \to \mathrm{Spaces}_*$ given on objects by $V \mapsto \Omega^V G_1(W \oplus V)$ such that the natural map $G_1(W) \to \Omega^V G_1(W \oplus V)$ lifts the map $G_1(W) \to G_1(W \oplus V)$.

Now let X be a space and take $G(V) = \Omega^V S^V X$. Let $F_1(X)$ be the homotopy fiber of the map $X \to \Omega^2 S^2 X$. In the language of the previous paragraph, $F_1(X) = G_1(\mathbb{C}^0)$. Similarly, $\Omega^2 F_1(S^2 X)$ is identified with $G_1(\mathbb{C})$. By lemma 1.1, there is a natural map $G_1(\mathbb{C}^0) \to \Omega^2 G_1(\mathbb{C})$. Rewriting this map in terms of F_1, we obtain that F_1 comes equipped with a natural map $F_1(X) \to \Omega^4 F_1(S^2 X)$ (lifting the obvious map $F_1(X) \to \Omega^2 F_1(S^2X))$. Let $F_2(X)$ be the homotopy fiber of this map. Repeating the argument above, one easily finds that $F_2(X)$ comes equipped with a natural map $F_2(X) \to \Omega^6 F_2(S^2X)$. Letting $F_3(X)$ be the homotopy fiber of this map and continuing inductively, one obtains the sequence of functors $F_m(X)$ together with fibration sequences $F_m(X) \to F_{m-1}(X) \to \Omega^{2m} F_{m-1}(S^2X)$ promised in the introduction.

Let C_n be the n-th Goodwillie derivative of $F_0(X) = X$. Thus C_n is a spectrum with an action of the symmetric group Σ_n, and the n-th layer of the Goodwillie tower of $F_0(X)$ is the (infinite loop space associated with) the spectrum

$$(C_n \wedge X^{\wedge n})_{h\Sigma_n}$$

Our next task is to describe the layers of the Goodwillie tower of the functor F_m in term of the layers of the Goodwillie tower of the functor F_0. Of course the description that we are looking for is of the form $(C_n^m \wedge X^{\wedge n})_{h\Sigma_n}$ for some Σ_n-spectrum C_n^m. Moreover, the natural map $F_{m-1}(X) \to \Omega^{2m} F_{m-1}(S^2X)$ induces a map on the layers

$$(C_n^{m-1} \wedge X^{\wedge n})_{h\Sigma_n} \to \Omega^{2m}(C_n^{m-1} \wedge X^{\wedge n} \wedge S^{2n})_{h\Sigma_n}$$

which is determined by some Σ_n-equivariant map

$$C_n^{m-1} \wedge X^{\wedge n} \to \Omega^{2m} C_n^{m-1} \wedge X^{\wedge n} \wedge S^{2n}$$

(where the action of Σ_n is trivial on the Ω^{2m}, and is given by the standard complex representation on S^{2n}). We will describe this Σ_n-equivariant map.

Lemma 1.2. The n-th layer of the Goodwillie tower of $F_m(X)$ is contractible for $n \leq m$. Assume that $n > m$. Then

$$D_n F_m(X) \simeq (C_n \wedge X^{\wedge n}) \wedge_{h\Sigma_n} (\mathrm{U}(n-1)/\,\mathrm{U}(n-m-1)_+)$$

Moreover, on the level of the n-th layers, the fibration sequence $F_m(X) \to F_{m-1}(X) \to \Omega^{2m} F_{m-1}(S^2X)$ is induced by the following Σ_n-equivariant fibration/cofibration sequence of spectra:

$$\mathrm{U}(n-1)/\,\mathrm{U}(n-m-1)_+ \to \mathrm{U}(n-1)/\,\mathrm{U}(n-m)_+ \to \mathrm{U}(n-1)/\,\mathrm{U}(n-m) \ltimes S^{2(n-m)}$$

where $\mathrm{U}(n-1)/\,\mathrm{U}(n-m) \ltimes S^{2(n-m)}$ is the Thom complex of the tautological $n-m$-dimensional complex bundle over $\mathrm{U}(n-1)/\,\mathrm{U}(n-m)$.

Proof. This is the content of [W95, Example 5.7], except that we use complex rather than real vector spaces. □

In particular, the bottom non-trivial layer of F_m is the $m+1$-th layer, and it is given by $C_{m+1} \wedge_{\Sigma_{m+1}} ((\mathrm{U}(m)_+) \wedge X^{\wedge m+1})$. (here we may use strict orbits instead of homotopy orbits, because the action of Σ_{m+1} on $\mathrm{U}(m)$ is free). Recalling from [Jo95, AM97] that C_{m+1} (the $m+1$-th Goodwillie derivative of the identity) is given by $C_{m+1} = \mathrm{Map}_*(K_{m+1}, \Sigma^\infty S^0)$, we obtain that the bottom layer of $F_m(X)$ is given by the infinite loop space of

$$\mathrm{Map}_*(K_{m+1}, \Sigma^\infty X^{\wedge m+1}) \wedge_{\Sigma_{m+1}} \mathrm{U}(m)_+$$

as stated in the introduction.

2. Cohomology and relation with Mitchell's spectra

Throughout this section, X stands for an odd-dimensional sphere. Our goal is to further analyze the layers of $F_m(X)$ in this case. Thus we want to study the spectra $\mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n}) \wedge_{h\Sigma_n} (\mathrm{U}(n-1)/\mathrm{U}(n-m-1)_+)$. In particular, we will be interested in their cohomology. We recall the following facts from [AM97, AD97]:

Theorem 2.1. Let $n > 1$, X and odd sphere. The spectrum

$$\mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

is contractible rationally. Moreover, it is contractible mod p unless n is a power of p.

It follows immediately from theorem 2.1, lemma 1.2 and induction on m that the same statement holds for all the layers of F_m for all m. Thus, if X is an odd sphere then the spectrum

$$D_n F_m(X) \cong \mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n}) \wedge_{h\Sigma_n} (\mathrm{U}(n-1)/\mathrm{U}(n-m-1)_+)$$

is contractible rationally for all $n > 1$ and is contractible mod p unless n is a power of p. We may, therefore, take $n = p^k$ and concentrate on the mod p homotopy type of

$$\mathrm{Map}_*(K_{p^k}, \Sigma^\infty X^{\wedge p^k}) \wedge_{h\Sigma_{p^k}} (\mathrm{U}(p^k-1)/\mathrm{U}(p^k-m-1)_+)$$

Start with the case $m = 0$. In this case, what one gets is the familiar spectrum $\mathrm{Map}_*(K_{p^k}, \Sigma^\infty X^{\wedge p^k})_{h\Sigma_{p^k}}$. We recall from [AD97] that there is a smaller model for this spectrum. To describe this model, let T_k be (the double suspension of) the geometric realization of the category of (strict, non-zero) vector subspaces of $\mathbb{F}_p^k$, (which is of course the same as the category of strict non-zero subgroups of $(\mathbb{Z}/p\mathbb{Z})^k$). Extend the action of $\mathrm{GL}_k(\mathbb{F}_p)$ on T_k to an action of the affine group $\mathrm{Aff}_k(\mathbb{F}_p) := \mathrm{GL}_k(\mathbb{F}_p) \ltimes (\mathbb{Z}/p\mathbb{Z})^k$ by letting $(\mathbb{Z}/p\mathbb{Z})^k$ act trivially. There is a map $T_k \to K_{p^k}$ determined by sending a subgroup P of $(\mathbb{Z}/p\mathbb{Z})^k$ to the partition determined by the quotient map $(\mathbb{Z}/p\mathbb{Z})^k \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^k/P$. It is easily checked that this map is equivariant with respect to the group inclusion $\mathrm{Aff}_k(\mathbb{F}_p) \hookrightarrow \Sigma_{p^k}$. The main result of [AD97] is that this map induces a mod p equivalence (only if X is an odd-dimensional sphere, of course)

$$\mathrm{Map}_*(K_{p^k}, \Sigma^\infty X^{\wedge p^k})_{h\Sigma_{p^k}} \to \mathrm{Map}_*(T_k, \Sigma^\infty X^{\wedge p^k})_{h\,\mathrm{Aff}_k(\mathbb{F}_p)} \tag{2}$$

Recall that the subgroup $(\mathbb{Z}/p\mathbb{Z})^k$ of $\mathrm{Aff}_k(\mathbb{F}_p)$ acts trivially on T_k. It follows that

$$\mathrm{Map}_*(T_k, \Sigma^\infty X^{\wedge p^k})_{h\,\mathrm{Aff}_k(\mathbb{F}_p)} \simeq \mathrm{Map}_*\left(T_k, \Sigma^\infty (X^{\wedge p^k})_{h(\mathbb{Z}/p\mathbb{Z})^k}\right)_{h\,\mathrm{GL}_k(\mathbb{F}_p)}$$

Moreover, since X is a sphere, $X^{\wedge p^k}_{h(\mathbb{Z}/p\mathbb{Z})^k}$ is a Thom space over $\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k$. Let us denote it $(\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k)^\gamma$. Thus, there is an equivalence

$$\mathrm{Map}_*\left(T_k, \Sigma^\infty X^{\wedge p^k}\right)_{h\,\mathrm{Aff}_k(\mathbb{F}_p)} \simeq \mathrm{Map}\left(T_k, \Sigma^\infty \left(\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k\right)^\gamma\right)_{h\,\mathrm{GL}_k(\mathbb{F}_p)}$$

Now recall that T_k is (non-equivariantly) homotopy equivalent to a wedge of spheres, and its only non-trivial homology group realizes the Steinberg representation of $\mathrm{GL}_k(\mathbb{F}_p)$. Since the Steinberg representation is projective and self-dual (in characteristic p), the cohomology of this spectrum is the image of the cohomology of the Thom space $\left(\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k\right)^\gamma$ under the action of the Steinberg idempotent. The image of $\mathrm{H}^*\left(\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k\right)$ itself under the action of the Steinberg idempotent is well known. It was first calculated by Mitchell and Priddy in [MtP83], and then again in a more "algebraic" fashion (at the prime 2) by Carlise and Kuhn in [CK89]. In particular, it is well-known that $St\,\mathrm{H}^*\left(\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k\right)$ is free over A_{k-1}. Same holds if one replaces $\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k$ with the Thom complex of any multiple of the regular representation. The calculation is a modification of the well-known calculations of Mitchell and Priddy cited above. A detailed account will appear either in the final version of [AD97] or in a separate note. In fact, as stated in theorem 0.3, there is an isomorphism, up to a dimension shift, of A_{k-1}-modules

$$\mathrm{H}^*\left(\mathrm{Map}_*\left(T_k, \Sigma^\infty\left(\mathrm{B}(\mathbb{Z}/p\mathbb{Z})^k\right)^\gamma\right)_{h\,\mathrm{GL}_k(\mathbb{F}_p)}\right) \cong A_{k-1} \otimes P$$

where P is a free module on one generator over the (doubled, if $p = 2$) Dickson algebra $\mathbb{F}_p[d_0, \ldots, d_{k-1}]$.

This finishes the discussion of the case $m = 0$. To extend it to all values of m, i.e., to analyze the spectra

$$\mathrm{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{h\Sigma_{p^k}} \left(\mathrm{U}(p^k-1)/\,\mathrm{U}(p^k-m-1)_+\right)$$

where X is an odd-dimensional sphere, we use lemma 1.2 and induction on m.

Theorem 2.2. Let X be an odd-dimensional sphere. The spectrum

$$\mathrm{Map}_*\left(K_n, \Sigma^\infty X^{\wedge n}\right) \wedge_{h\Sigma_n} \left(\mathrm{U}(n-1)/\,\mathrm{U}(n-m-1)_+\right)$$

is contractible if n is not a power of a prime. Suppose $n = p^k$, then:

(a) The map $T_k \to K_{p^k}$ induces an equivalence

$$\mathrm{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{h\Sigma_{p^k}} \left(\mathrm{U}(p^k-1)/\,\mathrm{U}(p^k-m-1)_+\right) \to$$

$$\to \operatorname{Map}_*\left(T_k, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{h\,\mathrm{Aff}_k(\mathbb{F}_p)} \left(\mathrm{U}(p^k-1)/\,\mathrm{U}(p^k-m-1)_+\right)$$

(b) There is an isomorphism (up to a dimension shift) of A_{k-1}-modules

$$\mathrm{H}^*\left(\operatorname{Map}_*\left(K_{p^k}, \Sigma^\infty X^{\wedge p^k}\right) \wedge_{h\Sigma_{p^k}} \left(\mathrm{U}(p^k-1)/\,\mathrm{U}(p^k-m-1)_+\right); \mathbb{F}_p\right) \cong$$
$$\cong A_{k-1} \otimes E \otimes P$$

where E is a free module on one generator on the exterior algebra

$$\Lambda < \overline{c_i} \mid p^k - m \leq i \leq p^k - 1 \text{ and } i \text{ is not of the form } p^k - p^j >$$

with $|\overline{c_i}| = 2i-1$ and P is a free module on one generator over the polynomial algebra $\mathbb{F}_p[c_{p^k-p^j} \mid p^j > m]$.

(c) If m is not a power of p then the map w_m induces the zero map on the mod p cohomology of the layers. If $m = p^j$ then w_m induces the inclusion of the ideal generated by d_j.

Proof. The proof is by induction on m, using lemma 1.2 for the induction step. For $m = 0$ part (a) is the equivalence (2) and parts (b) and (c) are given by theorem 0.3. Part (a) for a general m follows from part (a) for $m-1$. Part (c) for a general m follows from part (b) for $m-1$ and elementary considerations about characteristic classes. Part (b) for m follows from part (b) for $m-1$ and part (c) for m. □

Taking $m = p^k - 1$ in part (a) of theorem 2.2, we find that the spectrum $D_{p^k}F_{p^k-1}(X)$ is equivalent to the image under the Steinberg idempotent of the suspension spectrum of a certain Thom space over the homogeneous space $(\mathbb{Z}/p\mathbb{Z})^k \backslash \mathrm{U}(p^k-1)$. This is almost precisely Mitchell's construction in [Mt85] of a finite A_{k-1}-free spectrum, except he does not take a Thom space and that he uses the special orthogonal rather than the unitary group, but this makes very little difference. This is what we meant in the "pretheorem" in the introduction.

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(G. Arone) Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Il 60637, USA

E-mail address: arone@math.uchicago.edu