ISOGENIES OF SUPERSINGULAR ELLIPTIC CURVES OVER FINITE FIELDS AND OPERATIONS IN ELLIPTIC COHOMOLOGY

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Abstract. In this paper we investigate stable operations in supersingular elliptic cohomology using isogenies of supersingular elliptic curves over finite fields. Our main results provide a framework in which we give a conceptually simple new proof of an elliptic cohomology version of the Morava change of rings theorem and also gives models for explicit stable operations in terms of isogenies and morphisms in certain enlarged isogeny categories. We are particularly inspired by number theoretic work of G. Robert, whose work we reformulate and generalize in our setting.

Introduction

In previous work we investigated *supersingular* reductions of elliptic cohomology [5], stable operations and cooperations in elliptic cohomology [3, 4, 6, 8] and in [9, 10] gave some applications to the Adams spectral sequence based on elliptic (co)homology. In this paper we investigate stable operations in supersingular elliptic cohomology using isogenies of supersingular elliptic curves over finite fields; this is similar in spirit to our earlier work [6] on isogenies of elliptic curves over the complex numbers although we give a largely self contained account. Indeed, the promised Part II of [6] is essentially subsumed into the present work together with [8, 9, 10]. A major inspiration for this work lies in the paper of Robert [29], which also led to the related work of [11]; we reformulate and generalize Robert's results in the language of the present paper.

Throughout, p will be a prime which we will usually assume to be greater than 3, although much of the algebraic theory works as well for the cases $p = 2, 3$ provided appropriate adjustments are made. However, the precise implications for elliptic cohomology at the primes 2 and 3 appear to be more delicate and we may return to this in future work.

1991 *Mathematics Subject Classification.* 55N20, 55N22, 55S05 (secondary 14H52, 14L05).

Key words and phrases. elliptic cohomology, supersingular elliptic curve, isogeny.

Glasgow University Mathematics Department preprint no. 98/39 (Version 6: 2/03/1999).

I would like to acknowledge the contribution of K. Buzzard, I. Connell, J. Cremona, R. Odoni, N. Strickland, G. Robert and J. Tate to my understanding of supersingular elliptic curves over finite fields.

1. Elliptic curves over finite fields

General references for this section are [18, 31] while [21, 22] provide more abstract formulations. We will be interested in elliptic curves $\mathcal{E}$ defined over a subfield $\Bbbk \subseteq \overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$, indeed, we will usually take $\Bbbk = \overline{\mathbb{F}}_p$. In fact, we will impose further structure by requiring that a sort of 'orientation' for a curve is also prescribed as part of the data. We will also usually assume that $p > 3$, although most of the algebraic details have analogues for the primes 2 and 3.

We adopt the viewpoint of [21, 22], defining an *oriented elliptic curve* to be a connected 1-dimensional abelian group scheme $\mathcal{E}$ over $\Bbbk$ equipped with a nowhere vanishing invariant holomorphic 1-form $\omega \in \Omega^1(\mathcal{E})$. We will not distinguish two such oriented curves $(\mathcal{E}_1, \omega_1)$, $(\mathcal{E}_2, \omega_2)$ if there is an isomorphism of abelian varieties $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ for which $\varphi^*\omega_2 = \omega_1$. The notation $\underline{\mathcal{E}}$ will signify an isomorphism class of such objects $(\mathcal{E}, \omega)$. We will sometimes abuse notation and write $\underline{\mathcal{E}} = (\mathcal{E}, \omega)$. We will refer to $\mathcal{E}$ as the underlying elliptic curve of $\underline{\mathcal{E}}$.

A morphism (or rather an equivalence class of morphisms) of abelian varieties $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ for which $\varphi^*\omega_2 \neq 0$ gives rise to a morphism $\varphi\colon \underline{\mathcal{E}_1} \longrightarrow \underline{\mathcal{E}_2}$. As $\Omega^1(\mathcal{E}_1)$ is 1-dimensional over $\overline{\mathbb{F}}_p$, there is a unique $\lambda \in \Bbbk^\times$ for which $\varphi^*\omega_2 = \lambda\omega_1$. We will discuss categories in which elliptic curves are the objects in greater detail later.

If $p > 3$, associated to an oriented elliptic curve $\underline{\mathcal{E}}$ is a non-singular cubic

$$y^2 = 4x^3 - ax - b \tag{1.1}$$

whose projectivisation $\mathcal{E}$ is a non-singular Weierstaraß cubic. This happens since there are (non-unique) meromorphic functions X, Y with poles of orders 2 and 3 at $\mathrm{O} = [0, 1, 0]$ satisfying the following relations:

$$\begin{cases} Y^2 = 4X^3 - aX - b \quad \text{for some } a, b \in \Bbbk, \\ \omega = dX/Y. \end{cases} \tag{1.2}$$

Conversely, a Weierstraß cubic yields an abstract elliptic curve with the nonvanishing invariant 1-form dX/Y where X, Y are the first two projective coordinate functions. We will freely switch between these two equivalent notions of elliptic curve.

A modular form f of weight n defined over $\Bbbk$ is a rule which assigns to each oriented elliptic curve $\underline{\mathcal{E}} = (\mathcal{E}, \omega)$ over $\Bbbk$ a section $f(\underline{\mathcal{E}})\omega^{\otimes n}$ of the

bundle $\Omega^1(\mathcal{E})^{\otimes n}$, such that each separable isomorphism $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ with $\varphi^*\omega_2 = \lambda\omega_1$ satisfies

$$\varphi^*(f(\underline{\mathcal{E}_2})\omega_2^{\otimes n}) = f(\underline{\mathcal{E}_1})\omega_1^{\otimes n},$$

which implies

$$f(\underline{\mathcal{E}_2}) = \lambda^{-n} f(\underline{\mathcal{E}_1}).$$

This is formally equivalent to f being a modular form of weight n in the familiar classical sense of [31].

If we write Equation (1.3) in a form consistent with the notation of [31] III§1,

$$\mathcal{E}\colon y^2 = 4x^3 - \frac{1}{12}c_4(\underline{\mathcal{E}})x - \frac{1}{216}c_6(\underline{\mathcal{E}}), \tag{1.3}$$

the functions c_4, c_6 are examples of such modular forms with weights 4 and 6 respectively. Another example of weight 12 is provided by the non-vanishing discriminant function Δ for which

$$\Delta(\underline{\mathcal{E}}) = \frac{c_4(\underline{\mathcal{E}})^3 - c_6(\underline{\mathcal{E}})^2}{1728}.$$

Notice that this curve is actually defined over the finite subfield $\mathbb{F}_p(c_4(\underline{\mathcal{E}}), c_6(\underline{\mathcal{E}})) \subseteq \overline{\mathbb{F}}_p$ and hence any finite subfield containing it. The *j-invariant* of this curve is

$$j(\underline{\mathcal{E}}) = \frac{c_4(\underline{\mathcal{E}})^3}{\Delta(\underline{\mathcal{E}})} \in \mathbb{F}_p(c_4(\underline{\mathcal{E}}), c_6(\underline{\mathcal{E}})).$$

The function j is a modular form of weight 0 and only depends on $\mathcal{E}$, so we may write $j(\mathcal{E})$.

The next result is well known [18, 31]. But note that further information is required to determine the isomorphism class over a finite field containing $\mathbb{F}_p(c_4(\underline{\mathcal{E}}), c_6(\underline{\mathcal{E}}))$.

Theorem 1.1. *The invariant $j(\mathcal{E})$ is a complete isomorphism invariant of the curve $\mathcal{E}$ over the algebraically closed field $\overline{\mathbb{F}}_p$.*

Another important invariant is the *Hasse invariant* $\mathrm{Hasse}(\underline{\mathcal{E}})$ which is a homogeneous polynomial of weight $p-1$ in $c_4(\underline{\mathcal{E}}), c_6(\underline{\mathcal{E}})$ which have given weights 4 and 6 respectively. The oriented elliptic curve $\underline{\mathcal{E}} = (\mathcal{E}, \omega)$ is said to be *supersingular* if $\mathrm{Hasse}(\underline{\mathcal{E}}) = 0$; again this notion only depends on $\mathcal{E}$ and not the 1-form ω.

Given $\mathcal{E}$ defined over $\Bbbk \subseteq \overline{\mathbb{F}}_p$, we can consider $\mathcal{E}(\Bbbk')$, the set of points defined over an extension field $\Bbbk' \supseteq \Bbbk$. We usually regard $\mathcal{E}(\overline{\mathbb{F}}_p)$ as 'the' set of points of $\mathcal{E}$; thus whenever $\Bbbk \subseteq \Bbbk' \subseteq \overline{\mathbb{F}}_p$, we have

$$\mathcal{E}(\Bbbk) \subseteq \mathcal{E}(\Bbbk') \subseteq \mathcal{E}(\overline{\mathbb{F}}_p).$$

We will also use the notation

$$\mathcal{E}[n] = \ker[n]_{\mathcal{E}} \colon \mathcal{E}(\overline{\mathbb{F}}_p) \longrightarrow \mathcal{E}(\overline{\mathbb{F}}_p),$$

where $[n]_{\mathcal{E}} \colon \mathcal{E} \longrightarrow \mathcal{E}$ is the multiplication by n morphism. Actually, this notation is potentially misleading when $p \mid n$ and should be restricted to the case $p \nmid n$. In Section 4, we will also discuss the general case.

For the elliptic curve $\underline{\mathcal{E}} = (\mathcal{E}, \omega)$, if meromorphic functions X, Y are chosen as in Equation (1.2), there is a local parameter at O, namely $-2X/Y$, vanishing to order 1 at O. In terms of the corresponding Weierstraß form of Equation (1.3), this is the local parameter at $\mathrm{O} = [0, 1, 0]$ given by $t_{\underline{\mathcal{E}}} = -2x/y$. When referring to the elliptic curve $\underline{\mathcal{E}}$, we will often use the notation $(\mathcal{E}, c_4(\underline{\mathcal{E}}), c_6(\underline{\mathcal{E}}), t_{\underline{\mathcal{E}}})$ to indicate that it has Weierstraß form as in Equation (1.3) and local parameter $t_{\underline{\mathcal{E}}}$. We refer to this data as a Weierstraß realisation of the elliptic curve $\underline{\mathcal{E}} = (\mathcal{E}, \omega)$.

The local parameter $t_{\underline{\mathcal{E}}}$ has an associated formal group law $F_{\underline{\mathcal{E}}}$ induced from the group structure map $\mu \colon \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$ by taking its local expansion

$$\mu^* t_{\underline{\mathcal{E}}} = F_{\underline{\mathcal{E}}}(t'_{\underline{\mathcal{E}}}, t''_{\underline{\mathcal{E}}})$$

where $t'_{\underline{\mathcal{E}}}, t''_{\underline{\mathcal{E}}}$ are the local functions on $\mathcal{E} \times \mathcal{E}$ induced from $t_{\underline{\mathcal{E}}}$ by projection onto the two factors. Thus we have a formal group law

$$F_{\underline{\mathcal{E}}}(Z', Z'') \in \Bbbk[[Z', Z'']]$$

if $\mathcal{E}$ is defined over $\Bbbk$. The coefficients of $F_{\underline{\mathcal{E}}}$ lie in the $\mathbb{F}_p$-algebra generated by the coefficients $c_4(\underline{\mathcal{E}}), c_6(\underline{\mathcal{E}})$ and the coefficient of $Z'^r Z''^s$ is a linear combination (with coefficients independent of $\underline{\mathcal{E}}$) of the monomials $c_4(\underline{\mathcal{E}})^i c_6(\underline{\mathcal{E}})^j$ for which $4i + 6j + 1 = r + s$; in particular, only odd degree terms in Z', Z'' occur.

Given two elliptic curves $\underline{\mathcal{E}}$ and $\underline{\mathcal{E}'}$ together with an isomorphism $\alpha \colon \mathcal{E} \longrightarrow \mathcal{E}'$ of abelian varieties, there is a new formal group law $F^{\alpha}_{\underline{\mathcal{E}}}$ defined by

$$F^{\alpha}_{\underline{\mathcal{E}}}(t'_{\underline{\mathcal{E}}}, t''_{\underline{\mathcal{E}}}) = \alpha^* F_{\underline{\mathcal{E}'}}(t'_{\underline{\mathcal{E}'}}, t''_{\underline{\mathcal{E}'}}).$$

Lemma 1.2. *Let $\underline{\mathcal{E}} = (\mathcal{E}, \omega)$ be an oriented elliptic curve and $\alpha \colon \mathcal{E} \longrightarrow \mathcal{E}$ an automorphism of abelian varieties, then $F^{\alpha}_{\underline{\mathcal{E}}} = F_{\underline{\mathcal{E}}}$. Hence $F_{\underline{\mathcal{E}}}$ depends only on the elliptic curve $\mathcal{E}$ and not on any particular Weierstraß realisation of it.*

Proof. From [18, 31], the possible absolute automorphism groups are

- $\mathbb{Z}/6$ if $j(\mathcal{E}) \equiv 0 \bmod (p)$;
- $\mathbb{Z}/4$ if $j(\mathcal{E}) \equiv 1728 \bmod (p)$;
- $\mathbb{Z}/2$ otherwise.

In all cases, provided that $\mathbb{F}_{p^2} \subseteq \Bbbk$, $\mathrm{Aut}_{\Bbbk}\,\mathcal{E} = \mathrm{Aut}\,\mathcal{E}$, the absolute automorphism group.

In the case when $j(\mathcal{E}) \equiv 0$, the Weierstraß form is

$$y^2 = x^3 - \frac{1}{216} c_6(\underline{\mathcal{E}})$$

and then

$$F_{\underline{\mathcal{E}}}(X, Y) = \sum_{i+j \equiv 1 \bmod(6)} a_{i,j} X^i Y^j.$$

An automorphism of order 6 is given by

$$(x, y) \mapsto (\zeta_6^2 x, \zeta_6^3 y)$$

where ζ_6 is a chosen primitive 6th root of unity in $\overline{\mathbb{F}}_p$ and so

$$t_{\underline{\mathcal{E}}} \mapsto -t_{\underline{\mathcal{E}}}.$$

The result is now easily verified.

In the case when $j(\mathcal{E}) \equiv 1728$, the Weierstraß form is

$$y^2 = x^3 - \frac{1}{12} c_4(\underline{\mathcal{E}}) x,$$

hence

$$F_{\underline{\mathcal{E}}}(X, Y) = \sum_{i+j \equiv 1 \bmod(4)} a_{i,j} X^i Y^j.$$

An automorphism of order 4 is given by

$$(x, y) \mapsto (\zeta_4^2 x, \zeta_4^3 y)$$

where ζ_4 is a chosen primitive 4th root of unity in $\overline{\mathbb{F}}_p$ and so

$$t_{\underline{\mathcal{E}}} \mapsto -t_{\underline{\mathcal{E}}}.$$

Thus again the result is easily verified.

Finally, in the last case, an automorphism of order 2 is given by

$$(x, y) \mapsto (x, -y)$$

and hence

$$t_{\underline{\mathcal{E}}} \mapsto -t_{\underline{\mathcal{E}}}.$$

Once again the result easily follows. □

Given a Weierstraß realisation $\mathcal{E}$ of $\underline{\mathcal{E}}$, defined over $\Bbbk$, for $u \in \Bbbk$, the curve

$$\mathcal{E}^u \colon y^2 = 4x^3 - \frac{u^2 c_4(\underline{\mathcal{E}})}{12} x - \frac{u^3 c_6(\underline{\mathcal{E}})}{216}$$

is the *u-twist* of $\mathcal{E}$. For $v \in \Bbbk$ with $v^2 = u$, there is a *twisting isomorphism* $\theta_v \colon \mathcal{E} \longrightarrow \mathcal{E}_0$ which is the completion of the affine map

$$\varphi_v \colon (x, y) \mapsto (v^2 x, v^3 y).$$

The effect of this on 1-forms is given by

$$\theta_v^*\left(\frac{dx}{y}\right) = v^{-1}\omega.$$

Theorem 1.3. *For each oriented elliptic curve $\underline{\mathcal{E}} = (\mathcal{E}, \omega)$ defined over $\Bbbk$, there is a twisting isomorphism $\underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}_0}$, defined over $\Bbbk$ or a quadratic extension $\Bbbk'$ of $\Bbbk$, where $\underline{\mathcal{E}_0} = (\mathcal{E}_0, dx/y)$ is a Weierstraß elliptic curve of one of the following types.*

- *If $j(\mathcal{E}) \equiv 0 \bmod (p)$,*
$$\mathcal{E}_0\colon y^2 = 4x^3 - 4;$$
- *if $j(\mathcal{E}) \equiv 1728 \bmod (p)$,*
$$\mathcal{E}_0\colon y^2 = 4x^3 - 4x;$$
- *if $j(\mathcal{E}) \not\equiv 0, 1728 \bmod (p)$,*
$$\mathcal{E}_0\colon y^2 = 4x^3 - \frac{27j(\mathcal{E})}{j(\mathcal{E}) - 1728}x - \frac{27j(\mathcal{E})}{j(\mathcal{E}) - 1728}.$$

Proof. The above forms are taken from Husemoller [18]. Given any Weierstraß realisation $\mathcal{E}$ of $\underline{\mathcal{E}}$, it is easy to see that $\mathcal{E}$ has the form $\mathcal{E}_0^u$ for some $u \in \Bbbk$, where $\mathcal{E}_0$ has one of the stated forms depending on $j(\mathcal{E})$. Then there is a twisting isomorphism $\theta_v\colon \mathcal{E} \longrightarrow \mathcal{E}_0$ for $v \in \overline{\Bbbk}$ satisfying $v^2 = u$. □

In each of the above cases, the isomorphism $\theta_v\colon \mathcal{E} \cong \mathcal{E}_0$ is defined using suitable choices of twisting parameter u. Although this is ambiguous by elements of the automorphism groups $\operatorname{Aut}\mathcal{E} \cong \operatorname{Aut}\mathcal{E}_0$, we have the following consequence of Lemma 1.2.

Proposition 1.4. *The formal group law $F_{\underline{\mathcal{E}}}$ only depends on $\underline{\mathcal{E}}$, and not on the isomorphism $\mathcal{E} \cong \mathcal{E}_0$, hence is an invariant of $\underline{\mathcal{E}}$.*

We also have the following useful consequence of the fact that $j(\mathcal{E}) \in \mathbb{F}_{p^2}$, see [31] Chapter V Theorem 3.1.

Proposition 1.5. *The coefficients of $F_{\underline{\mathcal{E}_0}}$ lie in the subfield $\mathbb{F}_p(j(\mathcal{E})) \subseteq \mathbb{F}_{p^2}$.*

2. Categories of isogenies over finite fields and their progeny

For elliptic curves $\underline{\mathcal{E}_1}$ and $\underline{\mathcal{E}_2}$ defined over a field $\Bbbk$, an *isogeny* (defined over $\Bbbk$) is a non-trivial morphism of abelian varieties $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$. A *separable isogeny* is an isogeny which is a separable morphism. This is equivalent to the requirement that $\varphi^*\omega_2 \neq 0$ where ω_2 is the non-vanishing invariant 1-form on $\mathcal{E}_2$. An isogeny φ is finite and the *separable degree* of φ is defined by

$$\deg_{\mathrm{s}}\varphi = |\ker\varphi|.$$

If φ is separable then $\deg_s \varphi = \deg \varphi$, the usual notion of degree.

Associated to the oriented elliptic curve $\underline{\mathcal{E}}$ over $\overline{\mathbb{F}}_p$ defined by Equation (1.3), are the p^kth power curve

$$\mathcal{E}^{(p^k)}\colon y^2 = 4x^3 - \frac{1}{12}c_4(\underline{\mathcal{E}})^{(p^k)}x - \frac{1}{216}c_6(\underline{\mathcal{E}})^{(p^k)}$$

and the $1/p^k$th power curve

$$\mathcal{E}^{(1/p^k)}\colon y^2 = 4x^3 - \frac{1}{12}c_4(\underline{\mathcal{E}})^{(1/p^k)}x - \frac{1}{216}c_6(\underline{\mathcal{E}})^{(1/p^k)}$$

where for $a \in \overline{\mathbb{F}}_p$, $a^{(1/p^k)} \in \overline{\mathbb{F}}_p$ is the unique element satisfying

$$(a^{(1/p^k)})^{(p^k)} = a.$$

Properties of these curves can be found in [31]. In particular, given an elliptic curve $\underline{\mathcal{E}}$, there is a canonical choice of invariant 1-forms $\omega^{(p^k)}$ and $\omega^{(1/p^k)}$ so that the assignments

$$\underline{\mathcal{E}} = (\mathcal{E}, \omega) \rightsquigarrow (\mathcal{E}^{(p^k)}, \omega^{(p^k)}) = \underline{\mathcal{E}}^{(p^k)},$$
$$\underline{\mathcal{E}} = (\mathcal{E}, \omega) \rightsquigarrow (\mathcal{E}^{(1/p^k)}, \omega^{(1/p^k)}) = \underline{\mathcal{E}}^{(1/p^k)}$$

extend to functors on the category of isogenies; these powering operations on 1-forms can easily be seen in terms of Weierstraß forms where they take the canonical 1-form dx/y on

$$\mathcal{E}\colon y^2 = 4x^3 - \frac{1}{12}c_4(\underline{\mathcal{E}})x - \frac{1}{216}c_6(\underline{\mathcal{E}})$$

to dX/Y on each of the curves

$$\mathcal{E}^{(p^k)}\colon y^2 = 4x^3 - \frac{1}{12}c_4(\underline{\mathcal{E}})^{(p^k)}x - \frac{1}{216}c_6(\underline{\mathcal{E}})^{(p^k)},$$
$$\mathcal{E}^{(1/p^k)}\colon y^2 = 4x^3 - \frac{1}{12}c_4(\underline{\mathcal{E}})^{(1/p^k)}x - \frac{1}{216}c_6(\underline{\mathcal{E}})^{(1/p^k)}.$$

Proposition 2.1. *An isogeny $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ has unique factorizations*

$$\varphi = \mathrm{Fr}^k \circ \varphi_s = {}_s\varphi \circ \mathrm{Fr}^k$$

where the morphisms ${}_s\varphi\colon \mathcal{E}_1^{(p^k)} \longrightarrow \mathcal{E}_2$, $\varphi_s\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2^{(p^{1/k})}$ are separable and the morphisms denoted Fr^k are the evident iterated Frobenius morphisms $\mathrm{Fr}^k\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_1^{(p^k)}$, $\mathrm{Fr}^k\colon \mathcal{E}_2^{(p^{1/k})} \longrightarrow \mathcal{E}_2$.

A special case of this is involved in the following.

Proposition 2.2. *For an elliptic curve $\mathcal{E}$ defined over $\Bbbk$, the iterated Frobenius $\mathrm{Fr}^2\colon \mathcal{E} \longrightarrow \mathcal{E}^{(p^2)}$ factors as*

$$\mathrm{Fr}^2\colon \mathcal{E} \xrightarrow{[p]_{\mathcal{E}}} \mathcal{E} \xrightarrow{\lambda} \mathcal{E}^{(p^2)},$$

where λ is a separable isomorphism defined over $\Bbbk$. In particular, if $\mathcal{E}$ is defined over $\mathbb{F}_{p^2}$ then $\mathcal{E}^{(p^2)} = \mathcal{E}$ and $\lambda \in \operatorname{Aut}\mathcal{E}$.

Now let $\mathcal{E}_1$ and $\mathcal{E}_2$ be defined over $\overline{\mathbb{F}}_p$ and let $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ be a separable isogeny; then there is a finite field $\Bbbk \subseteq \overline{\mathbb{F}}_p$ such that $\mathcal{E}_1$, $\mathcal{E}_2$ and φ are all defined over $\Bbbk$. Later we will make use of this together with properties of zeta functions of elliptic curves over finite fields to determine when two curves over $\overline{\mathbb{F}}_p$ are isogenous.

Associated to an isogeny $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ between two elliptic curves defined over $\Bbbk$ there is a *dual isogeny* $\widehat{\varphi}\colon \mathcal{E}_2 \longrightarrow \mathcal{E}_1$ satisfying the identities

$$\widehat{\varphi} \circ \varphi = [\deg \varphi]_{\mathcal{E}_1}, \quad \varphi \circ \widehat{\varphi} = [\deg \varphi]_{\mathcal{E}_2},$$

where $[n]_{\mathcal{E}}$ denotes the multiplication by n morphism on the elliptic curve $\mathcal{E}$. Localizing the category of separable isogenies of elliptic curves over finite fields by forcing every isogeny $[n]_{\mathcal{E}}$ to be invertible results in a groupoid since every other regular isogeny also becomes invertible. Using the theory of p-primary Tate modules, we will modify this construction to define a larger category which also captures significant p-primary information.

Let $\underline{\mathcal{E}}$ be elliptic curve over $\overline{\mathbb{F}}_p$ with a Weierstraß form as in Equation (1.3) with its associated local coordinate function $t_{\underline{\mathcal{E}}} = -2x/y$ and its formal group law $F_{\underline{\mathcal{E}}}(X,Y)$. We say that an isogeny $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is *strict* if

$$\varphi^* t_{\underline{\mathcal{E}_2}} \equiv t_{\underline{\mathcal{E}_1}} \bmod ({t_{\underline{\mathcal{E}_1}}}^2).$$

This condition is equivalent to the requirement that $\varphi^*\omega_2 = \omega_1$, hence a strict isogeny is separable.

For a separable isogeny $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ there is a unique factorization of the form

$$\varphi\colon \mathcal{E}_1 \xrightarrow{\rho} \mathcal{E}_1/\ker\varphi \xrightarrow{\varphi'} \mathcal{E}_2 \tag{2.1}$$

where φ' is an isomorphism, and ρ is a strict isogeny. The quotient elliptic curve $\mathcal{E}_1/\ker\varphi$ is characterized by this property and is constructed explicitly by Vélu [34] who also gives a calculation of $\rho^* t_{(\mathcal{E}_1/\ker\varphi,\omega)}$, where ω is the 1-form induced by the quotient map.

We will denote by **Isog** the category of elliptic curves over $\overline{\mathbb{F}}_p$ with isogenies $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ as its morphisms. **Isog** has the subcategory **SepIsog** whose morphisms are the separable isogenies. These categories have full subcategories $\mathbf{Isog}_{\mathrm{ss}}$ and $\mathbf{SepIsog}_{\mathrm{ss}}$ whose objects are the supersingular curves.

These categories can be localized to produce groupoids. This can be carried out using dual isogenies and twisting. For the Weierstraß cubic $\mathcal{E}$ defined by Equation (1.3), and a natural number n prime to p, the factorization of $[n]_{\mathcal{E}}$ given by Equation (2.1) has the form

$$[n]_{\mathcal{E}}\colon \mathcal{E} \longrightarrow \mathcal{E}^{n^2} \xrightarrow{\widetilde{[n]}} \mathcal{E}$$

where

$$\mathcal{E}^{n^2}\colon y^2 = 4x^3 - \frac{1}{12}n^4 c_4(\underline{\mathcal{E}})x - \frac{1}{216}n^6 c_6(\underline{\mathcal{E}})$$

is the twist of $\mathcal{E}$ by $n^2 \in {\overline{\mathbb{F}}_p}^\times$ and $\widetilde{[n]}$ is the map given by

$$[x, y, 1] \mapsto [x/n^2, y/n^3, 1].$$

If we invert all such isogenies $[n]_\mathcal{E}$, then as an isogeny $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is a morphism of abelian varieties,

$$\varphi \circ [n]_{\mathcal{E}_1} = [n]_{\mathcal{E}_2} \circ \varphi,$$

hence φ inherits an inverse

$$\varphi^{-1} = [n]_{\mathcal{E}_1}^{-1} \circ \widehat{\varphi} = \widehat{\varphi} \circ [n]_{\mathcal{E}_2}^{-1}.$$

The resulting localized category of isogenies will be denoted $\mathbf{Isog}^\times$ and the evident localized supersingular category $\mathbf{Isog}_{\mathrm{ss}}^\times$. We can also consider the subcategories of separable morphisms, and localize these by inverting the separable isogenies $[n]_\mathcal{E}$, i.e., those for which $p \nmid n$. The resulting categories $\mathbf{SepIsog}^\times$ and $\mathbf{SepIsog}_{\mathrm{ss}}^\times$ are all full subcategories of $\mathbf{Isog}^\times$ and $\mathbf{Isog}_{\mathrm{ss}}^\times$.

Given $\underline{\mathcal{E}_1} = (\mathcal{E}_1, \omega_1)$, $\underline{\mathcal{E}_2} = (\mathcal{E}_2, \omega_2)$, we extend the action of a separable isogeny $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ to the morphism

$$\underline{\varphi} = (\varphi, {\varphi^*}^{-1})\colon \underline{\mathcal{E}_1} \longrightarrow \underline{\mathcal{E}_2}.$$

Hence if $\varphi^*\omega_2 = \lambda\omega$, then

$$\underline{\varphi}(\mathbf{x}, \omega_1) = (\varphi(\mathbf{x}), \lambda^{-1}\omega_2).$$

We will often just write φ for $\underline{\varphi}$ when no ambiguity is likely to result. Using this construction, we define modified versions of the above isogeny categories as follows. $\underline{\mathbf{SepIsog}}$ is the category with objects the oriented elliptic curves over $\overline{\mathbb{F}}_p$ and morphisms $(\varphi, \lambda^{-1}{\varphi^*}^{-1})\colon (\mathcal{E}_1, \omega_1) \longrightarrow (\mathcal{E}_2, \omega_2)$ where $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is a separable isogeny and $\lambda \in {\overline{\mathbb{F}}_p}^\times$. Thus $\underline{\mathbf{SepIsog}}$ is generated by morphisms of the form $\underline{\varphi}$ together with the 'twisting' morphisms $\underline{\lambda}\colon (\mathcal{E}, \omega) \longrightarrow (\mathcal{E}, \omega)$ given by $\underline{\lambda} = (\mathrm{Id}_\mathcal{E}, \lambda^{-1})$ which commute with all other morphisms. We can localize this category to form $\underline{\mathbf{SepIsog}}^\times$ with morphisms obtained in an obvious fashion from those of $\mathbf{SepIsog}^\times$ together with the $\underline{\lambda}$. There are also evident full subcategories $\underline{\mathbf{SepIsog}}_{\mathrm{ss}}$ and $\underline{\mathbf{SepIsog}}_{\mathrm{ss}}^\times$ whose objects involve only supersingular elliptic curves.

We end this section with a discussion of two further pieces of structure possessed by our isogeny categories, both being actions by automorphisms of these categories. First observe there is an action of the group of units

$\overline{\mathbb{F}}_p{}^{\times}$ (or more accurately, the multiplicative group scheme $\mathbb{G}_{\mathrm{m}}$) on **Isog** and its subcategories described above, given by

$$\lambda\cdot(\mathcal{E},\omega) = (\mathcal{E}^{\lambda^2},\lambda\omega),$$
$$\lambda\cdot\varphi = \varphi^u$$

where $\lambda\in\overline{\mathbb{F}}_p{}^{\times}$, $\varphi\colon(\mathcal{E}_1,\omega_1)\longrightarrow(\mathcal{E}_2,\omega_2)$ is an isogeny and φ^u is the evident composite

$$\varphi^u\colon(\mathcal{E}_1^{\lambda^{-2}},\lambda^{-1}\omega_1)\longrightarrow(\mathcal{E}_1,\omega_1)\xrightarrow{\varphi}(\mathcal{E}_2,\omega_2)\longrightarrow(\mathcal{E}_2^{\lambda^2},\lambda\omega_2).$$

The second action is induced by the Frobenius morphisms Fr^k and their inverses. Namely,

$$\mathrm{Fr}^k\cdot(\mathcal{E},\omega) = (\mathcal{E}^{p^k},\omega^{p^k}),$$
$$\mathrm{Fr}^k\cdot\varphi = \varphi^{(p^k)}$$

where for an isogeny $\varphi\colon(\mathcal{E}_1,\omega_1)\longrightarrow(\mathcal{E}_2,\omega_2)$, $\varphi^{(p^k)}$ is the composite

$$\varphi^{(p^k)}\colon(\mathcal{E}_1^{(1/p^k)},\omega_1^{(1/p^k)})\xrightarrow{\mathrm{Fr}^{-k}}(\mathcal{E}_1,\omega_1)\xrightarrow{\varphi}(\mathcal{E}_2,\omega_2)\xrightarrow{\mathrm{Fr}^k}(\mathcal{E}_2^{(p^k)},\omega_2^{(p^k)}).$$

If $\varphi(x,y) = (\varphi_1(x,y),\varphi_2(x,y))$, then

$$\varphi^{(p^k)(x,y)} = (\varphi_1(x^{1/p^k},y^{1/p^k})^{p^k},\varphi_2(x^{1/p^k},y^{1/p^k})^{p^k}).$$

similar considerations apply to the inverse Frobenius morphism Fr^{-k}.

3. Recollections on elliptic cohomology

A general reference on elliptic cohomology is provided by the foundational paper of Landweber, Ravenel & Stong [25], while aspects of the level 1 theory which we use can be found in Landweber [24] as well as our earlier papers [4, 5, 6].

Let $p>3$ be a prime. We will denote by $E\ell\ell_*$ the graded ring of modular forms for $\mathrm{SL}_2(\mathbb{Z})$, meromorphic at infinity and with q-expansion coefficients lying in the ring of p-local integers $\mathbb{Z}_{(p)}$. Here $E\ell\ell_{2n}$ consists of the modular forms of weight n. We have

Theorem 3.1. *As a graded ring,*

$$E\ell\ell_* = \mathbb{Z}_{(p)}[Q,R,\Delta^{-1}],$$

where $Q \in E\ell\ell_8$, $R \in E\ell\ell_{12}$ *and* $\Delta = (Q^3 - R^2)/1728 \in E\ell\ell_{24}$ *have the* q*-expansions*

$$\begin{aligned} Q(q) =& E_4 = 1 + 240 \sum_{1 \leqslant r} \sigma_3(r) q^r, \\ R(q) =& E_6 = 1 - 504 \sum_{1 \leqslant r} \sigma_5(r) q^r, \\ \Delta(q) =& q \prod_{n \geqslant 1} (1 - q^n)^{24}. \end{aligned}$$

The element $A = E_{p-1} \in E\ell\ell_{2(p-1)}$ is particularly important for our present work. We have

$$A(q) = 1 - \frac{2(p-1)}{B_{p-1}} \sum_{1 \leqslant r} \sigma_{p-2}(r) q^r \equiv 1 \bmod (p).$$

We also have $B = E_{p+1} \in E\ell\ell_{2(p+1)}$ with q-expansion

$$B(q) = 1 - \frac{2(p+1)}{B_{p+1}} \sum_{1 \leqslant r} \sigma_p(r) q^r.$$

Finally, we recall that there is a canonical formal group law $F_{E\ell\ell}(X, Y)$ defined over $E\ell\ell_*$ whose p-series satisfies

$$\begin{aligned} [p]_{F_{E\ell\ell}}(X) &= pX + \cdots + u_1 X^p + \cdots + u_2 X^{p^2} + (\text{higher order terms}) \\ &\equiv u_1 X^p + \cdots + u_2 X^{p^2} + (\text{higher order terms}) && \bmod (p) \\ &\equiv u_2 X^{p^2} + (\text{higher order terms}) && \bmod (p, u_1). \end{aligned} \tag{3.1}$$

Combining results of [24] and [11], we obtain the following in which $\left(\frac{-1}{p}\right)$ is the Legendre symbol.

Theorem 3.2. *The sequence* p, A, B *is regular in the ring* $E\ell\ell_*$*, in which the following congruences are satisfied:*

$$\begin{aligned} u_1 &\equiv A && \bmod (p); \\ u_2 &\equiv \left(\frac{-1}{p}\right) \Delta^{(p^2-1)/12} \equiv -B^{(p-1)} && \bmod (p, A). \end{aligned}$$

With the aid of this Theorem together with Landweber's Exact Functor Theorem, in both its original form [23] and its generalization due to Yagita [37], we can define *elliptic cohomology* and its *supersingular reduction* by

$$\begin{aligned} E\ell\ell^*(\ ) &= E\ell\ell^* \underset{BP^*}{\otimes} BP^*(\ ) \\ {}^{\mathrm{ss}}E\ell\ell^*(\ ) &= (E\ell\ell/(p, A))^*(\ ) \cong E\ell\ell^*/(p, A) \underset{P(2)^*}{\otimes} P(2)^*(\ ), \end{aligned}$$

where as usual, for any graded group M_* we set $M^n = M_{-n}$. The structure of the coefficient ring ${}^{\mathrm{ss}}E\ell\ell_*$ was described in [5] and depends on the factorization of A mod (p). In fact, ${}^{\mathrm{ss}}E\ell\ell_*$ is a product of 'graded fields' and the forms of the simple factors of A mod (p) are related to the possible j-invariants of supersingular elliptic curves over $\overline{\mathbb{F}}_p$.

Using the definition of supersingular elliptic curves as pairs $(\mathcal{E},\omega)$, an element $f \in {}^{\mathrm{ss}}E\ell\ell_{2n}$ can be viewed as a family of sections of bundles $\Omega^1(\mathcal{E})^{\otimes n}$ assigning to $(\mathcal{E},\omega)$ the section $f(\mathcal{E},\omega)\omega^{\otimes n}$. A separable isomorphism $\varphi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ for which $\varphi^*\omega_2 = \lambda\omega_1$ satisfies

$$\varphi^* f(\mathcal{E}_2,\omega_2)\omega_2^{\otimes n} = f(\mathcal{E}_1,\omega_1)\omega_1^{\otimes n}$$

and so

$$\varphi^* f(\mathcal{E}_2,\omega_2) = \lambda^{-n} f(\mathcal{E}_1,\omega_1).$$

This is formally equivalent to f being a modular form of weight n in the traditional sense.

The ring $E\ell\ell_*/(p)$ is universal for Weierstraß elliptic curves defined over $\overline{\mathbb{F}}_p$ while ${}^{\mathrm{ss}}E\ell\ell_*$ is universal for those which are supersingular, in the sense of the following result.

Proposition 3.3. *The projectivisation $\mathcal{E}$ of the cubic*

$$y^2 = 4x^3 - ax - b$$

defined over $\overline{\mathbb{F}}_p$ is an elliptic curve if and only if there is a ring homomorphisms $\theta\colon E\ell\ell_*/(p) \longrightarrow \overline{\mathbb{F}}_p$ for which

$$\theta(Q) = 12a, \quad \theta(R) = -216b.$$

For such an elliptic curve, $\mathcal{E}$ is supersingular if and only if $\theta(A) = 0$.

The first part amounts to the well known fact that the discriminant of $\mathcal{E}$ is $(a^3 - b^2)/1728$, whose non-vanishing is equivalent to the nonsingularity of $\mathcal{E}$. The second part of this result is equivalent to the statement that $\theta(A) = \mathrm{Hasse}(\underline{\mathcal{E}})$, a result which can be found in [18, 31] together with further equivalent conditions.

Next we discuss some cooperation algebras. In [6], we gave a description of the cooperation algebra $\Gamma^0_* = E\ell\ell_*E\ell\ell$ as a ring of functions on the category of isogenies of elliptic curves defined over $\mathbb{C}$. We will be interested in the supersingular cooperation algebra

$${}^{\mathrm{ss}}\Gamma^0_* = {}^{\mathrm{ss}}E\ell\ell_*E\ell\ell = {}^{\mathrm{ss}}E\ell\ell_* \underset{E\ell\ell_*}{\otimes} E\ell\ell_*E\ell\ell \cong {}^{\mathrm{ss}}E\ell\ell_*(E\ell\ell).$$

The ideal $(p,A) \lhd E\ell\ell_*$ is invariant under the Γ^0_*-coaction on $E\ell\ell_*$ and hence ${}^{\mathrm{ss}}\Gamma^0_*$ can be viewed as the quotient of Γ^0_* by the ideal generated by the image of (p,A) in Γ^0_* under either the left or equivalently the right unit map

$E\ell\ell_* \longrightarrow \Gamma^0_*$. The pair $({}^{\mathrm{ss}}E\ell\ell_*, {}^{\mathrm{ss}}\Gamma^0_*)$ therefore inherits the structure of a Hopf algebroid over $\mathbb{F}_p$.

The Hopf algebroid structure on $({}^{\mathrm{ss}}E\ell\ell_*, {}^{\mathrm{ss}}\Gamma^0_*)$ implies that $\mathrm{Spec}_{\mathbb{F}_p}\, {}^{\mathrm{ss}}\Gamma^0_* = \mathbf{Alg}_{\mathbb{F}}({}^{\mathrm{ss}}\Gamma^0_*, \overline{\mathbb{F}}_p)$ is a groupoid, or at least this is so if the grading is ignored. By the discussion of Devinatz [15] section 1 (see also our Section 6), the grading is equivalent to an action of $\mathbb{G}_{\mathrm{m}}$ which here is derived from the twisting action discussed in Section 1. Let ${}^{\mathrm{ss}}\mathbf{SEllFGL}$ denote the category of supersingular elliptic curves over $\overline{\mathbb{F}}_p$ with the morphism set

$${}^{\mathrm{ss}}\mathbf{SEllFGL}(\underline{\mathcal{E}_1}, \underline{\mathcal{E}_2}) = \{f \colon F_{\underline{\mathcal{E}_1}} \longrightarrow F_{\underline{\mathcal{E}_2}} : f \text{ a strict isomorphism of formal group laws}\}.$$

Notice that there is an action of $\mathbb{G}_{\mathrm{m}}$ on this extending the twisting action on curves, and also an action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. These actions are compatible with the composition and inversion maps. ${}^{\mathrm{ss}}\mathbf{SEllFGL}$ is also a 'formal scheme' in the sense used by Devinatz [15], thus it can be viewed as a pro-scheme and we can consider continuous functions ${}^{\mathrm{ss}}\mathbf{SEllFGL} \longrightarrow \overline{\mathbb{F}}_p$ where the codomain is given the discrete topology.

Theorem 3.4. *There is a natural isomorphism of groupoids with $\mathbb{G}_{\mathrm{m}}$-action,*

$$\mathrm{Spec}_{\overline{\mathbb{F}}_p}\, \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma^0_* \cong {}^{\mathrm{ss}}\mathbf{SEllFGL}.$$

Moreover, $\overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma^0_{2n}$ can be identified with the set of all continuous functions ${}^{\mathrm{ss}}\mathbf{SEllFGL} \longrightarrow \overline{\mathbb{F}}_p$ of weight n and ${}^{\mathrm{ss}}\Gamma^0_{2n} \subseteq \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma^0_{2n}$ can be identified with the subset of Galois invariant functions.

The proof is straightforward, given the existence of identification of $E\ell\ell_* E\ell\ell$ as

$$E\ell\ell_* E\ell\ell = E\ell\ell_* \underset{MU_*}{\otimes} MU_* MU \underset{MU_*}{\otimes} E\ell\ell_*,$$

and the universality of MU_*MU for strict isomorphisms of formal group laws due to Quillen [1, 28]. We will require a modified version of his result.

Recall from [1, 28] that

$$MU_*MU = MU_*[b_k : k \geqslant 1]$$

with the convention that $b_0 = 1$, and that the coaction is determined by the formula

$$\sum_{k \geqslant 0} \psi b_k T^{k+1} = \sum_{r \geqslant 0} 1 \otimes b_r \Big(\sum_{s \geqslant 0} b_s \otimes 1\, T^{s+1}\Big)^{r+1}.$$

This coaction corresponds to composition of power series with leading term T. We can also form the algebras $MU_*[u, u^{-1}]$ and $MU_*[u, u^{-1}][b_0, b_0^{-1}, b_k :$

$k \geqslant 1]$ in which $|u| = |b_0| = 0$ and there is a coaction corresponding to composition of power series with invertible leading term,

$$\sum_{k\geqslant 0} \psi b_k T^{k+1} = \sum_{r\geqslant 0} 1 \otimes b_r (\sum_{s\geqslant 0} b_s \otimes 1\, T^{s+1})^{r+1}.$$

This also defines a Hopf algebroid $(MU_*[u,u^{-1}], MU_*[u,u^{-1}][b_0, b_0^{-1}, b_k : k \geqslant 1])$ whose right unit is given by

$$\eta_R(xu^n) = \eta_R(x)u^{d+n}b_0^n,$$

where $x \in MU_{2d}$ and $\eta_R(x)$ is the image of x under the usual right unit $MU_* \longrightarrow MU_*MU$. There is a ring epimorphism $MU_*[u,u^{-1}][b_0, b_0^{-1}, b_k : k \geqslant 1] \longrightarrow MU_*MU$ under which $u, b_0 \mapsto 1$ and which induces a morphism of Hopf algebroids

$$(MU_*[u,u^{-1}], MU_*[b_0, b_0^{-1}, b_k : k \geqslant 1]) \longrightarrow (MU_*, MU_*MU).$$

Setting

$$\Gamma_* = E\ell\ell_*[u,u^{-1}] \underset{MU_*[u,u^{-1}]}{\otimes} MU_*[u,u^{-1}][b_0, b_0^{-1}, b_k : k \geqslant 1] \underset{MU_*[u,u^{-1}]}{\otimes} E\ell\ell_*[u,u^{-1}],$$

we can form the evident Hopf algebroid $(E\ell\ell_*[u,u^{-1}], \Gamma_*)$ and the induced morphism of Hopf algebroids

$$(E\ell\ell_*[u,u^{-1}], \Gamma_*) \longrightarrow (E\ell\ell_*, \Gamma_*^0).$$

Similarly, we can define Hopf algebroid $({}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}], {}^{\mathrm{ss}}\Gamma_*)$ with

$${}^{\mathrm{ss}}\Gamma_* = {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}] \underset{MU_*[u,u^{-1}]}{\otimes} MU_*[b_0, b_0^{-1}, b_k : k \geqslant 1] \underset{MU_*[u,u^{-1}]}{\otimes} {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}].$$

Now let ${}^{\mathrm{ss}}\mathbf{EllFGL}$ denote the category whose objects are the supersingular oriented elliptic curves over $\overline{\mathbb{F}}_p$ with morphisms being the isomorphisms of their formal group laws; this category is a topological groupoid with $\mathbb{G}_{\mathrm{m}}$-action, containing ${}^{\mathrm{ss}}\mathbf{SEllFGL}$. Using the canonical Weierstraß realizations given in Theorem 1.3, we have the following result.

Theorem 3.5. *There is a natural isomorphism of groupoids with $\mathbb{G}_{\mathrm{m}}$-action,*

$$\mathrm{Spec}_{\overline{\mathbb{F}}_p}\, \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_* \cong {}^{\mathrm{ss}}\mathbf{EllFGL}.$$

Moreover, $\overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_{2n}$ can be identified with the set of all continuous functions ${}^{\mathrm{ss}}\mathbf{EllFGL} \longrightarrow \overline{\mathbb{F}}_p$ of weight n and ${}^{\mathrm{ss}}\Gamma_{2n} \subseteq \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_{2n}$ can be identified with the subset of Galois invariant functions.

Later we will give a different interpretation of ${}^{\mathrm{ss}}\Gamma_*^0$ in terms of the supersingular category of isogenies.

4. Tate modules

In this section we discuss Tate modules of elliptic curves over finite fields. While the definition and properties of the Tate module $\mathcal{T}_\ell\mathcal{E}$ for primes $\ell \neq p$ can be found for example in [18, 31], we require the details for $\ell = p$. Suitable references are provided by [35, 36, 16, 17]. Actually, it is surprisingly difficult to locate full details of this material for abelian varieties in the literature, which seems to have originally appeared in unpublished papers of Tate *et al.*

In this section $\Bbbk$ will be a perfect field of characteristic $p > 0$ and $\mathbb{W}(\Bbbk)$ its ring of *Witt vectors,* endowed with its usual structure of a local ring (if $\Bbbk$ is finite it is actually a complete discrete valuation ring). The absolute Frobenius automorphism $x \mapsto x^p$ on $\Bbbk$ lifts uniquely to an automorphism $\sigma\colon \mathbb{W}(\Bbbk) \longrightarrow \mathbb{W}(\Bbbk)$; we will often use the notation $x^{(p)} = \sigma(x)$ for this. Let $\mathbb{D}_\Bbbk$ be the *Dieudonné algebra*

$$\mathbb{D}_\Bbbk = \mathbb{W}(\Bbbk)\left\langle \mathrm{F}, \mathrm{V}\right\rangle,$$

i.e., the non-commutative $\mathbb{W}(\Bbbk)$-algebra generated by the elements F, V subject to the relations

$$\begin{aligned} \mathrm{FV} &= \mathrm{VF} = p, \\ \mathrm{F}a &= a^{(p)}\mathrm{F}, \\ a\mathrm{V} &= \mathrm{V}a^{(p)}, \end{aligned}$$

for $a \in \mathbb{W}(\Bbbk)$. Let $\mathbf{Mod}^{\mathrm{f.l.}}_{\mathbb{D}_\Bbbk}$ be the category of finite length $\mathbb{D}_\Bbbk$-modules and $\mathbf{CommGpSch}_\Bbbk[p]$ be the category of finite commutative group schemes over $\Bbbk$ with rank of the form p^d.

Theorem 4.1. *There is an anti-equivalence of categories*

$$\begin{aligned} \mathbf{CommGpSch}_\Bbbk[p] &\longleftrightarrow \mathbf{Mod}^{\mathrm{f.l.}}_{\mathbb{D}_\Bbbk} \\ G &\leftrightsquigarrow \mathrm{M}(G). \end{aligned}$$

Moreover, if $\operatorname{rank} G = p^s$*, then* $\mathrm{M}(G)$ *has length* s *as a* $\mathbb{W}(\Bbbk)$*-module.*

This result can be extended to $\mathbf{DivGp}_\Bbbk$, the category of *p-divisible groups* over $\Bbbk$.

Theorem 4.2. *There is an anti-equivalence of categories*

$$\begin{aligned} \mathbf{DivGp}_\Bbbk &\longleftrightarrow \mathbf{Mod}^{\mathrm{f.l.}}_{\mathbb{D}_\Bbbk} \\ G &\leftrightsquigarrow \mathrm{M}(G). \end{aligned}$$

Moreover, if $\operatorname{rank} G = p^s$*,* $\mathrm{M}(G)$ *is a free* $\mathbb{W}(\Bbbk)$*-module of rank* s*.*

A p-divisible group G of rank p^s is a collection of finite group schemes G_n ($n \geqslant 0$) with $\operatorname{rank} G_n = p^{ns}$ and exact sequences of abelian group schemes

$$0 \longrightarrow G_n \xrightarrow{j_n} G_{n+1} \longrightarrow G_1 \longrightarrow 0$$

for $n \geqslant 0$. The extension of the result to such groups is accomplished by setting

$$\mathrm{M}(G) = \varprojlim_n \mathrm{M}(G_n)$$

where the limit is taken over the inverse system of maps $\mathrm{M}(j_n)\colon \mathrm{M}(G_{n+1}) \longrightarrow \mathrm{M}(G_n)$. The main types of examples we will be concerned with here are the following.

If F is a 1-dimensional formal group law over $\Bbbk$ of height h, then the p^n-series of F has the form

$$[p^n]_F(X) \equiv uX^{p^{nh}} \bmod (X^{p^{nh}+1}) \tag{4.1}$$

where $u \in \Bbbk^\times$. We have an associated p-divisible group $\ker[p^\infty]_F$ of rank p^h with

$$(\ker[p^\infty]_F)_n = \ker[p^n]_F = \operatorname{Spec}(\Bbbk[[X]]/([p^n]_F(X))).$$

Let $\mathcal{E}$ be an elliptic curve defined over $\Bbbk$. Then there is are finite group schemes $\mathcal{E}[p^n] = \ker[p^n]_{\mathcal{E}}$ ($n \geqslant 0$) which together constitute a p-divisible group $\mathcal{E}[p^\infty]$ of rank p^2.

In particular, if $F_{\mathcal{E}}$ is the formal group law associated to the local parameter $t_{\mathcal{E}}$ coming from a Weierstraß equation for $\mathcal{E}$, we have

Lemma 4.3. *There is a compatible family of isomorphisms of group schemes over* $\Bbbk$

$$\mathcal{E}[p^n] \cong \ker[p^n]_{F_{\mathcal{E}}} \quad (n \geqslant 0).$$

Hence there is an isomorphism of divisible groups

$$\mathcal{E}[p^\infty] \cong \ker[p^\infty]_{F_{\mathcal{E}}} \quad (n \geqslant 0).$$

Proof. This is essentially shown by Silverman in [31], Chapter VII, Proposition 2.2. If $\mathcal{E}$ is given by a Weierstraß equation (1.3), then in terms of the local parameter $t = -x/y$, we can express y in the form $y = -1/w(t)$ for some power series $w(t) \in \Bbbk[[t]]$ satisfying

$$w(t) \equiv t^3 \bmod (t^4).$$

By Equation (4.1), the assignment

$$t \mapsto \left(\frac{t}{w(t)}, \frac{-2}{w(t)}\right)$$

extends to a $\Bbbk$-algebra homomorphism

$$\Bbbk[[t]]/([p^n]_{F_{\mathcal{E}}}(t)) \longrightarrow \mathcal{E}(\Bbbk[[t]]/(t^{p^{nh}})$$

where h is the height of $F_{\mathcal{E}}$, known to be 1 or 2. This induces a homomorphism of $\Bbbk$-schemes

$$\mathcal{E}(\Bbbk[[t]]/(t^{p^{nh}})[p^n] \cong \ker[p^n]_{F_{\mathcal{E}}}$$

and Silverman's argument applied to the complete local ring $R = \Bbbk[[t]]/(t^{p^{nh}})$ shows this to be an isomorphism.

An alternative approach to proving this makes use of the Serre-Tate theory described in Katz [20] Theorem 1.2.1 together with Silverman [31] Chapter VII Proposition 2.2. □

We can now define the *Tate module* of the elliptic curve $\mathcal{E}$ to be

$$\mathcal{T}_p\mathcal{E} = \mathrm{M}(\mathcal{E}[p^\infty]) \cong \mathrm{M}(\ker[p^\infty]_{F_{\mathcal{E}}}).$$

Proposition 4.4. *The Tate module $\mathcal{T}_p\mathcal{E}$ is a free topological $\mathbb{W}(\Bbbk)$-module of rank 2.*

In its strongest form, the following result from [35] is due to J. Tate, although never formally published by him; weaker variants were established earlier by Weil and others; a proof appears in [36].

Theorem 4.5. *Let $\mathcal{E}$ and $\mathcal{E}'$ be elliptic curves over $\mathbb{F}_{p^d}$. Then the natural map*

$$\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}') \longrightarrow \mathrm{Hom}_{\mathbb{D}_{\mathbb{F}_{p^d}}}(\mathcal{T}_p\mathcal{E}', \mathcal{T}_p\mathcal{E})$$

is injective and the induced map

$$\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}') \otimes \mathbb{Z}_p \longrightarrow \mathrm{Hom}_{\mathbb{D}_{\mathbb{F}_{p^d}}}(\mathcal{T}_p\mathcal{E}', \mathcal{T}_p\mathcal{E})$$

is an isomorphism.

Since $\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}')$ is a free abelian group of finite rank, $\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}') \otimes \mathbb{Z}_p$ agrees with its p-adic completion $\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}')\widehat{\ }_p$. Indeed, this finiteness also implies that for large enough d,

$$\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}') = \mathrm{Hom}_{\overline{\mathbb{F}}_p}(\mathcal{E}, \mathcal{E}'),$$

and interpreting $\mathbb{F}_{p^\infty}$ as $\overline{\mathbb{F}}_p$, Theorem 4.5 holds in that case too.

The above definition of $\mathcal{T}_p\mathcal{E}$ is different in essence from that of the Tate modules

$$\mathcal{T}_\ell\mathcal{E} = \varprojlim_n \mathcal{E}[\ell^n]$$

for primes $\ell \neq p$. However, for any $\Bbbk$-algebra S, we may follow Fontaine [16] Chapitre V and consider

$$\mathcal{T}'_p\mathcal{E}(S) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{E}[p^\infty](S)).$$

Fontaine shows that the functor $\mathcal{T}'_p\mathcal{E}$ satisfies

$$\mathcal{T}'_p\mathcal{E}(S) = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{D}_\Bbbk}(\mathbb{Q}_p/Z_p \underset{\mathbb{Z}_p}{\otimes} \mathcal{T}_p\mathcal{E}[p^\infty](S), \mathrm{CW}_\Bbbk(S))$$

in his notation and terminology. From this it can be deduced that the case $\ell = p$ of the following result holds, the case where $\ell \neq p$ being covered in [31, 18].

Theorem 4.6. *Let $\mathcal{E}$ and $\mathcal{E}'$ be elliptic curves over $\mathbb{F}_{p^d}$ and for a prime ℓ let*

$$\mathcal{T}'_\ell\mathcal{E} = \begin{cases} \varprojlim_n \mathcal{E}[\ell^n] & \text{if } \ell \neq p, \\ \mathcal{T}'_p\mathcal{E} & \text{if } \ell = p. \end{cases}$$

Then the natural map

$$\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}') \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^d})}(\mathcal{T}'_\ell\mathcal{E}, \mathcal{T}'_\ell\mathcal{E}')$$

is injective and the induced map

$$\mathrm{Hom}_{\mathbb{F}_{p^d}}(\mathcal{E}, \mathcal{E}') \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^d})}(\mathcal{T}'_\ell\mathcal{E}, \mathcal{T}'_\ell\mathcal{E}')$$

is an isomorphism.

If $\mathcal{E}$ is a supersingular elliptic curve defined over $\overline{\mathbb{F}}_p$, its absolute endomorphism ring $\mathrm{End}\,\mathcal{E} = \mathrm{End}_{\overline{\mathbb{F}}_p}\mathcal{E}$ is a maximal order in a quaternion division algebra over $\mathbb{Q}$. By p-adically completing $\mathrm{End}\,\mathcal{E}$, we obtain a non-commutative $\mathbb{W}(\mathbb{F}_{p^2})$-algebra of rank 2,

$$\mathcal{O}_\mathcal{E} = \mathrm{End}\,\mathcal{E} \otimes \mathbb{Z}_p.$$

Proposition 4.7. *The division algebra $\mathrm{End}\,\mathcal{E} \otimes \mathbb{Q}$ is unramified except at p and ∞.*

If $\mathcal{E}$ is defined over $\mathbb{F}_{p^d}$, then as a $\mathbb{W}(\mathbb{F}_{p^2})$-algebra, the p-adic completion $\mathcal{O}_\mathcal{E}$ is given by

$$\mathcal{O}_\mathcal{E} = \mathbb{W}(\mathbb{F}_{p^2})\left\langle \mathrm{Fr}^{(d)} \right\rangle,$$

where $\mathrm{Fr}^{(d)}$ is the relative Frobenius map $\mathrm{Fr}^{(d)}\colon \mathcal{E} \longrightarrow \mathcal{E}^{(p^d)}$ which satisfies the relations

$$\mathrm{Fr}^{(d)^2} = up^d$$
$$\mathrm{Fr}^{(d)}\,\alpha = \alpha^{(p^d)}\,\mathrm{Fr}^{(d)} \quad (\alpha \in \mathbb{W}(\mathbb{F}_{p^2}),$$

where u is a unit in $\mathbb{W}(\mathbb{F}_{p^2})$. When $d = 1$, $\mathcal{O}_{\mathcal{E}} = \mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle$ is also isomorphic to the $\mathbb{W}(\mathbb{F}_{p^2})$-algebra $\mathbb{D}_{\mathbb{F}_{p^2}}$ with S *corresponding to the Frobenius element* F *and agreeing with* Fr *up to a unit in $\mathbb{W}(\mathbb{F}_{p^2})$.*

Proof. See [35], Chapters 2 & 4. □

Notice that $\mathcal{O}_{\mathcal{E}}$ has a natural p-adic topology extending that of $\mathbb{Z}_p$. Moreover, every element $\alpha \in \mathcal{O}_{\mathcal{E}}$ has a unique Teichmüller expansion

$$\alpha = \alpha_0 + \alpha_1 S \quad (\alpha_0 \in \mathbb{W}(\mathbb{F}_{p^2}),\ \alpha_0^{p^2} = \alpha_0). \tag{4.2}$$

As consequence of Proposition 4.7, the formal group law $F_{\underline{\mathcal{E}}}$ becomes a *formal $\mathbb{W}(\mathbb{F}_{p^2})$-module* as defined in Hazewinkel [17]. We set $\operatorname{End} F_{\underline{\mathcal{E}}} = \operatorname{End}_{\overline{\mathbb{F}}_p} F_{\underline{\mathcal{E}}}$.

Proposition 4.8. *The natural homomorphism $\operatorname{End}\mathcal{E} \longrightarrow \operatorname{End} F_{\underline{\mathcal{E}}}$ extends to an isomorphism of $\mathbb{W}(\mathbb{F}_{p^2})$-algebras $\mathcal{O}_{\mathcal{E}} \longrightarrow \operatorname{End} F_{\underline{\mathcal{E}}}$.*

Proof. Extending to a map on the p-completion is straightforward, the fact that the resulting map is an isomorphism uses Lemma 4.3 together with Tate's Theorem 4.5. See also Katz [19] §IV. □

Corollary 4.9. *The Tate module $\mathcal{T}_p\mathcal{E}$ is a module over the $\mathbb{Z}_p$-algebra $(\mathbb{W}(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \mathbb{W}(\mathbb{F}_{p^2}))\langle \mathrm{S}\rangle$, where*

$$\mathrm{S}(\alpha \otimes \beta) = \alpha^{(p)} \otimes \beta^{(p)}\mathrm{S}.$$

Proof. Elements of $\mathbb{W}(\mathbb{F}_{p^2}) \subseteq \operatorname{End}\mathcal{E}$ induce morphisms of $\mathcal{T}_p\mathcal{E}$. By the definition of the Frobenius operation F in [14] Chapter III §5, we get the stated intertwining formula. □

Using Corollary 4.9, we can deduce more on the structure of $\mathcal{T}_p\mathcal{E}$. Let $\Gamma = \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$ and $\mathrm{H} = \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$, hence $\Gamma/\mathrm{H} \cong \mathbb{Z}/2$. From [16] Chapitre III Proposition 2.1 (with the obvious extension to the infinite dimensional situation), the multiplication map

$$\mathbb{W}(\overline{\mathbb{F}}_p) \underset{\mathbb{W}(\mathbb{F}_{p^2})}{\otimes} (\mathcal{T}_p\mathcal{E})^{\mathrm{H}} \longrightarrow \mathcal{T}_p\mathcal{E}$$

is an isomorphism of $\mathbb{W}(\overline{\mathbb{F}}_p)$-modules. In fact it is an isomorphism of (topological) left Γ-modules and indeed of $\mathbb{W}(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \mathbb{W}(\mathbb{F}_{p^2})$-modules. Moreover, viewed as a module over the right hand factor of

$$\mathbb{W}(\mathbb{F}_{p^2}) \cong 1 \underset{\mathbb{Z}_p}{\otimes} \mathbb{W}(\mathbb{F}_{p^2}) \subseteq \mathbb{W}(\overline{\mathbb{F}}_p) \underset{\mathbb{Z}_p}{\otimes} \mathbb{W}(\mathbb{F}_{p^2})$$

it is free of rank 2. From this we can deduce that for any pair of supersingular elliptic curves $\mathcal{E}, \mathcal{E}'$, $\operatorname{Hom}_{\mathbb{D}_{\mathbb{F}_{p^d}}}(\mathcal{T}_p\mathcal{E}', \mathcal{T}_p\mathcal{E})$ is a free module of rank 1 over $\mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle$, hence is a free module of rank 2 over $\mathbb{W}(\mathbb{F}_{p^2})$.

The ring $\mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle$ is familiar to topologists as the absolute endomorphism ring of the universal Lubin-Tate formal group law height 2, agreeing with that of the natural orientation in Morava $K(2)$-theory. Its group of units is

$$\mathbb{S}_2 = \{\alpha_0 + \alpha_1\mathrm{S} \in \mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle : \alpha_0, \alpha_1 \in \mathbb{W}(\mathbb{F}_{p^2}),\ \alpha_0 \not\equiv 0 \bmod (p)\},$$

while

$$\mathbb{S}_2^0 = \{\alpha_0 + \alpha_1\mathrm{S} \in \mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle : \alpha_0, \alpha_1 \in \mathbb{W}(\mathbb{F}_{p^2}),\ \alpha_0 \equiv 1 \bmod (p)\}$$

is its group of *strict units,* known to topologists as the *Morava stabilizer group.* Let $\mathbb{B}_2$ be the rationalization of $\mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle$ which is a 4-dimensional central division algebra over $\mathbb{Q}_p$. Adopting notation of [8], we will also introduce the following closed subgroup of the group of units $\widetilde{\mathbb{S}}_2 = \mathbb{B}_2^\times$:

$$\widetilde{\mathbb{S}}_2^0 = \bigcup_{r\in\mathbb{Z}} \mathbb{S}_2^0\,\mathrm{S}^r.$$

Notice also that

$$\widetilde{\mathbb{S}}_2 = \bigcup_{r\in\mathbb{Z}} \mathbb{S}_2\,\mathrm{S}^r.$$

Then $\mathbb{S}_2 \lhd \widetilde{\mathbb{S}}_2$ and $\mathbb{S}_2^0 \lhd \widetilde{\mathbb{S}}_2^0$, i.e., these are closed normal subgroups.

We can rationalize the Tate module $\mathcal{T}_p\mathcal{E}$, to give

$$\mathcal{V}_p\mathcal{E} = \mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes} \mathcal{T}_p\mathcal{E},$$

which is a 2-dimensional vector space over the fraction field $\mathrm{B}(\Bbbk) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{W}(\Bbbk)$. In fact, $\mathcal{V}_p\mathcal{E}$ is a module over the rationalization

$$\mathbb{B}_\Bbbk = \mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes} \mathbb{D}_\Bbbk.$$

We can generalize Tate's Theorem to give the following which we only state for curves defined over $\overline{\mathbb{F}}_p$.

Theorem 4.10. *Let $\underline{\mathcal{E}_1}$ and $\underline{\mathcal{E}_2}$ be elliptic curves over $\overline{\mathbb{F}}_p$. Then the natural map*

$$\mathbf{Isog}^\times(\underline{\mathcal{E}_1}, \underline{\mathcal{E}_2}) \longrightarrow \mathrm{Hom}_{\mathbb{B}_{\overline{\mathbb{F}}_p}}(\mathcal{V}_p\mathcal{E}_2, \mathcal{V}_p\mathcal{E}_1)$$

is injective and has image contained in $\mathrm{InvtHom}_{\mathbb{B}_{\overline{\mathbb{F}}_p}}(\mathcal{V}_p\mathcal{E}_2, \mathcal{V}_p\mathcal{E}_1)$*, the open subset of invertible homomorphisms, and the induced map*

$$\mathbf{Isog}^\times(\underline{\mathcal{E}_1}, \underline{\mathcal{E}_2})\widehat{}_p \longrightarrow \mathrm{InvtHom}_{\mathbb{B}_{\overline{\mathbb{F}}_p}}(\mathcal{V}_p\mathcal{E}_2, \mathcal{V}_p\mathcal{E}_1)$$

is a homeomorphism.

The natural map

$$\mathbf{SepIsog}^\times(\underline{\mathcal{E}_1}, \underline{\mathcal{E}_2}) \longrightarrow \mathrm{Hom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2, \mathcal{T}_p\mathcal{E}_1)$$

is injective with image contained in the open set of invertible homomorphisms and the induced map

$$\mathbf{SepIsog}^{\times}(\underline{\mathcal{E}_1},\underline{\mathcal{E}_2})\widehat{\ }_p \longrightarrow \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2,\mathcal{T}_p\mathcal{E}_1)$$

is a homeomorphism.

Similar results hold for the supersingular isogeny categories.

Since each separable isogeny induces an invertible homomorphism of Tate modules, the image of $\mathbf{SepIsog}(\mathcal{E}_1,\mathcal{E}_2)$ in $\mathrm{Hom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2,\mathcal{T}_p\mathcal{E}_1)$ is a dense subset of $\mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2,\mathcal{T}_p\mathcal{E}_1)$.

Notice also that $\alpha \in \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E},\mathcal{T}_p\mathcal{E})$ with $\alpha = \alpha_0+\alpha_1 S$ as in Equation (4.2), has the well defined effect $\alpha^*\omega = \alpha_0\omega$ on 1-forms, since this is certainly true for elements of the dense subgroup $\mathrm{End}\,\mathcal{E}^{\times} \subseteq \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E},\mathcal{T}_p\mathcal{E})$.

5. Thickening the isogeny categories

We are led by Theorem 4.10 to define the following 'thickenings' of our isogeny categories. Starting with the category **Isog** and its subcategories, we enlarge each to a subcategory of $\widetilde{\mathbf{Isog}}$, which has the same objects but as the set of morphisms $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$,

$$\widetilde{\mathbf{Isog}}(\mathcal{E}_1,\mathcal{E}_2) = \mathrm{Hom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2,\mathcal{T}_p\mathcal{E}_1) - \{0\},$$

$$\widetilde{\mathbf{Isog}}^{\times}(\mathcal{E}_1,\mathcal{E}_2) = \mathrm{InvtHom}_{\mathbb{B}_{\overline{\mathbb{F}}_p}}(\mathcal{V}_p\mathcal{E}_2,\mathcal{V}_p\mathcal{E}_1),$$

$$\widetilde{\mathbf{SepIsog}}^{\times}(\mathcal{E}_1,\mathcal{E}_2) = \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2,\mathcal{T}_p\mathcal{E}_1),$$

and similarly for the supersingular categories.

We can make similar constructions for the categories with objects oriented elliptic curves, and set

$$\widetilde{\underline{\mathbf{SepIsog}}}^{\times}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2)) = \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2,\mathcal{T}_p\mathcal{E}_1),$$

and if $\mathcal{E}_1,\mathcal{E}_2$ are supersingular,

$$\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2)) = \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2,\mathcal{T}_p\mathcal{E}_1),$$

These morphism sets all have a natural profinite topologies, and composition of morphisms is continuous. These categories are 'formal schemes' in the sense of Devinatz [15] and we will make use of this in Section 6. Their object sets have the form $\mathrm{Spec}_{\mathbb{F}_p}{}^{\mathrm{ss}}E\ell\ell_*$, while the morphism set of a pair of objects can be identified with the limit of the pro-system obtained by factoring out by the open neighbourhoods of the identity morphisms in the

set homomorphisms between the associated Tate modules. For example,

$$\widetilde{\mathbf{Isog}}(\mathcal{E}_1,\mathcal{E}_2) = \varprojlim_U \mathrm{Hom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}_2, \mathcal{T}_p\mathcal{E}_1/U),$$

where the limit is taken over the basic neighbourhoods U of 0 in $\mathcal{T}_p\mathcal{E}_1$ which are just the finite index subgroups. We can describe representing algebras for some of these formal schemes. We will do this for the supersingular category $\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}$. Recall Theorem 3.5.

Theorem 5.1. *There is an equivariant isomorphism of topological groupoids with $\mathbb{G}_{\mathrm{m}}$-action,*

$$\mathrm{Spec}^{\mathrm{f}}_{\overline{\mathbb{F}}_p}\, \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_* \cong \widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}.$$

Moreover, $\overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_{2n}$ can be identified with the set of all continuous functions $\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}} \longrightarrow \overline{\mathbb{F}}_p$ of weight n and ${}^{\mathrm{ss}}\Gamma_{2n} \subseteq \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_{2n}$ can be identified with the subset of Galois invariant functions.

Proof. This follows from an argument similar to that of [2]; see also [7] for a similar generalization. The idea is to consider locally constant functions

$$\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2)) \cong \mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle \longrightarrow \overline{\mathbb{F}}_p,$$

where $(\mathcal{E}_1,\omega_1)$, $(\mathcal{E}_2,\omega_2)$ are given elliptic curves. The space of all such functions is determined as in [2], and turns out to be spanned by monomials in generalized Teichmüller functions relative to expansions in terms of powers of S. These Teichmüller functions are in fact the images (up to powers of u) of the elements $D_r \in E\ell\ell_* E\ell\ell$ of [6], equation (9.8) under the natural map $E\ell\ell_* E\ell\ell \longrightarrow \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_{2n}$. □

For later use we provide a useful example of such a Galois invariant continuous function on $\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}$, namely

$$\mathrm{ind}\colon \widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}} \longrightarrow \overline{\mathbb{F}}_p\,; \quad \mathrm{ind}((\mathcal{E},\omega) \xrightarrow{\varphi} (\mathcal{E}',\omega')) = \deg\varphi \bmod (p), \tag{5.1}$$

where $\deg\varphi \bmod (p)$ is calculated by choosing a separable isogeny $\varphi_0\colon (\mathcal{E},\omega) \longrightarrow (\mathcal{E}',\omega')$ which approximates φ in $\mathrm{Hom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p\mathcal{E}', \mathcal{T}_p\mathcal{E})$ in the sense that $\varphi \equiv \varphi_0 \bmod (S)$. Clearly ind is locally constant, hence continuous, as well as Galois invariant. Also, for a composable pair of morphisms φ,θ,

$$\mathrm{ind}(\varphi\theta) = \mathrm{ind}(\varphi)\,\mathrm{ind}(\theta).$$

Proposition 5.2. *The function* ind *correspond to an element of ${}^{\mathrm{ss}}\Gamma_0$ which is group-like under the coaction ψ and antipode χ in the sense that*

$$\psi(\mathrm{ind}) = \mathrm{ind}\otimes\mathrm{ind}, \quad \chi(\mathrm{ind}) = \mathrm{ind}^{-1}.$$

6. Some results on the cohomology of groupoids

An excellent reference for the following material is Devinatz [15] which contains the most thorough discussion we are aware of on the continuous cohomology of topological groupoid schemes of the type under consideration. We simply sketch the required details from [15] §1, amending them slightly to suit our needs. A different perspective on the cohomology of categories is provided by Baues & Wirsching [12].

Let $\mathcal{C}$ be a groupoid, $\Bbbk$ a commutative unital ring. Then $\mathbf{Mod}^{\mathrm{c}}_{\Bbbk}$ will denote the category of *complete Hausdorff* $\Bbbk$*-modules* defined in [15] Definition 1.1. Similarly, $\mathbf{Alg}^{\mathrm{c}}_{\Bbbk}$ will denote the category of *complete commutative* $\Bbbk$*-algebras.* We will refer to morphisms in these two categories as *continuous homomorphisms* of complete modules or algebras.

A *cogroupoid object* in $\mathbf{Alg}^{\mathrm{c}}_{\Bbbk}$ is then a pair (A, Γ) of objects in $\mathbf{Alg}^{\mathrm{c}}_{\Bbbk}$ together with the usual structure maps $\eta_R, \eta_L, \varepsilon, \psi, \chi$ for a Hopf algebroid over $\Bbbk$ except that in all relevant diagrams the completed tensor product $\underset{\Bbbk}{\widehat{\otimes}}$ has to be used; such data $(A, \Gamma, \eta_R, \eta_L, \varepsilon, \psi, \chi)$ will be referred to as constituting a *complete Hopf algebroid.* Equivalently, $\mathcal{C} = \mathrm{Spec}^{\mathrm{f}}_{\Bbbk} \Gamma$ is an (affine) *formal groupoid scheme.*

We can also consider the category $\mathbf{Comod}^{\mathrm{c}}_{\Gamma}$ of complete (left) Γ-comodules with morphisms being comodule morphisms also lying in $\mathbf{Mod}^{\mathrm{c}}_{\Bbbk}$. For two objects M, N in $\mathbf{Comod}^{\mathrm{c}}_{\Gamma}$, we will denote the set of morphisms $M \longrightarrow N$ by $\mathrm{Hom}_{\Gamma}(M, N)$. Then the functor $\mathbf{Comod}^{\mathrm{c}}_{\Gamma} \longrightarrow \mathbf{Mod}_{\Bbbk}$ given by

$$M \rightsquigarrow \mathrm{Hom}_{\Gamma}(A, M)$$

is left exact and its right derived functors form a graded functor $\mathrm{Ext}^*_{\Gamma}(A, \ )$.

Given a continuous morphism of complete Hopf algebroids $f\colon (A, \Gamma) \longrightarrow (B, \Sigma)$ and a Γ-comodule M, there is an induced map

$$H^*f\colon \mathrm{Ext}^*_{\Gamma}(A, M) \longrightarrow \mathrm{Ext}^*_{\Sigma}(B, f^*M)$$

where $f^*M = B \underset{A}{\widehat{\otimes}} M$ with the Σ-comodule structure described in [15].

Now recall from [15] definition 1.14 the notion of a *natural equivalence* $\tau\colon f \longrightarrow g$ between two continuous morphisms $f, g\colon (A, \Gamma) \longrightarrow (B, \Sigma)$ of complete Hopf algebroids. In particular, such a τ induces a continuous homomorphism of complete Σ-comodules $\tau^*\colon g^*M \longrightarrow f^*M$, in turn inducing a map

$$H^*\tau\colon \mathrm{Ext}^*_{\Sigma}(B, g^*M) \longrightarrow \mathrm{Ext}^*_{\Sigma}(B, f^*M).$$

We will require Devinatz's important Proposition 1.16.

Proposition 6.1. *Let $\tau\colon f \longrightarrow g$ be a natural equivalence between continuous morphisms $f, g\colon (A, \Gamma) \longrightarrow (B, \Sigma)$ of complete Hopf algebroids. Then*

for any continuous Γ*-comodule* M*,*

$$H^*(\tau^*) \circ H^*g = H^*f\colon \operatorname{Ext}^*_\Gamma(A,M) \longrightarrow \operatorname{Ext}^*_\Sigma(B, f^*M).$$

Using Devinatz's notion of *equivalence* of two complete Hopf algebroids we can deduce

Corollary 6.2. *If* $f\colon (A,\Gamma) \longrightarrow (B,\Sigma)$ *is an equivalence of complete Hopf algebroids, then*

$$H^*f\colon \operatorname{Ext}^*_\Gamma(A,M) \longrightarrow \operatorname{Ext}^*_\Sigma(B, f^*M)$$

is an isomorphism.

All of the above can also be reworked with graded $\Bbbk$-modules and algebras. As Devinatz observes, provided that Γ is concentrated in even degrees, this is equivalent to introducing actions of the multiplicative group scheme $\mathbb{G}_{\mathrm{m}}$ which factor through the quotient scheme $\mathbb{G}_{\mathrm{m}}/\mu_2$ where $\mu_d \subseteq \mathbb{G}_{\mathrm{m}}$ is subscheme of dth roots of unity. In the situations we will consider this applies and indeed locally the actions of $\mathbb{G}_{\mathrm{m}}$ factor through $\mathbb{G}_{\mathrm{m}}/\mu_{2n}$ where may take some of the values $n = 2, 4, 6$. An alternative approach to this is to view a $\mathbb{Z}$-graded module as a $\mathbb{Z}/2$-graded view module with action of $\mathbb{G}_{\mathrm{m}}$, and we will prefer this approach. We then define and element x in degree n to be of *weight* $\mathrm{wt}x = n/2$; this means that if $\alpha \in \mathbb{G}_{\mathrm{m}}$, then

$$\alpha \cdot x = \begin{cases} \alpha^{\mathrm{wt}x}x & \text{if } n \text{ even,} \\ -\alpha^{\mathrm{wt}x-1}x & \text{if } n \text{ odd.} \end{cases}$$

7. The connectivity of category of supersingular isogenies

We will show that the category of isogenies of supersingular elliptic curves $\mathbf{Isog}_{\mathrm{ss}}$ is connected in the sense that there is a morphism between any given pair of objects.

Theorem 7.1. *Two elliptic curves* $\mathcal{E}$*,* $\mathcal{E}'$ *defined over a finite field* $\mathbb{F}_{p^d}$ *are isogenous over* $\mathbb{F}_{p^d}$ *if and only if* $|\mathcal{E}(\mathbb{F}_{p^d})| = |\mathcal{E}'(\mathbb{F}_{p^d})|$*.*

Proof. See [18] Chapter 3 Theorem 8.4. □

Theorem 7.2. *An elliptic curve* $\mathcal{E}$ *defined over a finite field* $\mathbb{F}_{p^d}$ *satisfies*

- $|\mathcal{E}(\mathbb{F}_{p^d})| = 1 + p^d$ *if* d *is odd;*
- $|\mathcal{E}(\mathbb{F}_{p^{2m}})| = (1 \pm p^m)^2$ *if* $\operatorname{End}_{\overline{\mathbb{F}}_p}\mathcal{E} = \operatorname{End}_{\mathbb{F}_{p^d}}\mathcal{E}$*.*

Proof. The list of all possible orders of $|\mathcal{E}(\mathbb{F}_{p^d})|$ appears in [35], while [30] gives a complete list of the actual groups $\mathcal{E}(\mathbb{F}_{p^d})$ that can occur. □

Theorem 7.3. *The isogeny categories* $\mathbf{Isog}_{\mathrm{ss}}$ *and* $\widetilde{\mathbf{Isog}}$ *are connected.*

Proof. We will show that any two supersingular elliptic curves $\mathcal{E}$, $\mathcal{E}'$ defined over $\overline{\mathbb{F}}_p$ are isogenous. We may assume that $\mathcal{E}$ and $\mathcal{E}'$ are both defined over some finite field and then by Theorems 7.1 and 7.2 we need only show that they become isogenous over some larger finite field. We begin by enlarging the common field of definition to $\mathbb{F}_{p^{2m}}$ for which

$$\mathrm{End}_{\overline{\mathbb{F}}_p}\,\mathcal{E} = \mathrm{End}_{\mathbb{F}_{p^{2m}}}\,\mathcal{E}, \quad \mathrm{End}_{\overline{\mathbb{F}}_p}\,\mathcal{E}' = \mathrm{End}_{\mathbb{F}_{p^{2m}}}\,\mathcal{E}'.$$

Thus $|\mathcal{E}(\mathbb{F}_{p^{2m}})|$ and $|\mathcal{E}'(\mathbb{F}_{p^{2m}})|$ both have the form $(1 \pm p^m)^2$. If these are equal then the curves are isogenous over $\mathbb{F}_{p^{2m}}$. Otherwise we may assume that

$$|\mathcal{E}(\mathbb{F}_{p^{2m}})| = 1 + 2p^m + p^{2m}, \quad |\mathcal{E}'(\mathbb{F}_{p^{2m}})| = 1 - 2p^m + p^{2m}.$$

If the Weierstraß equation of $\mathcal{E}$ is

$$\mathcal{E}\colon y^2 = 4x^3 - ax - b,$$

taking a quadratic non-residue u in $\mathbb{F}_{p^{2m}}$ allows us to define a twisted curve by

$$\mathcal{E}^u\colon y^2 = 4x^3 - u^2ax - u^3b,$$

which becomes isomorphic to $\mathcal{E}$ over $\mathbb{F}_{p^{4m}}$. If

$$\begin{aligned} N_0 &= |\{t \in \mathbb{F}_{p^{2m}} : 4t^3 - at - b = 0\}|, \\ N_1 &= |\{t \in \mathbb{F}_{p^{2m}} : 4t^3 - at - b \neq 0 \text{ is a quadratic residue}\}|, \end{aligned}$$

then

$$1 + N_0 + 2N_1 = 1 + 2p^m + p^{2m}.$$

But as

$$4x^3 - u^2ax - u^3b = u^3(4(u^{-1}x)^3 - a(u^{-1}x) - b),$$

we find that

$$\begin{aligned} |\mathcal{E}^u(\mathbb{F}_{p^{2m}})| &= 1 + N_0 + 2(p^{2m} - N_0 - N_1) \\ &= 1 - N_0 - 2N_1 + 2p^{2m} \\ &= 1 - 2p^m + p^{2m}. \end{aligned}$$

Hence,

$$|\mathcal{E}'(\mathbb{F}_{p^{2m}})| = |\mathcal{E}^u(\mathbb{F}_{p^{2m}})|$$

and so these are isogenous curves over $\mathbb{F}_{p^{2m}}$, implying that $\mathcal{E}'$ is isogenous to $\mathcal{E}$ over $\mathbb{F}_{p^{2m}}$.

We could have also used the fact $j(\mathcal{E}^u) = j(\mathcal{E})$ to obtain an isomorphism $\mathcal{E}^u \cong \mathcal{E}$ over $\overline{\mathbb{F}}_p$, but the argument given is more explicit about the field of definition of such an isomorphism.

The connectivity of $\widetilde{\mathbf{Isog}}$ now follows from Tate's Theorem 4.5. □

Corollary 7.4. *The groupoids* $\mathbf{Isog}_{ss}^{\times}$ *and* $\widetilde{\mathbf{Isog}}_{ss}^{\times}$ *are connected.*

The following deeper fact about supersingular curves over finite fields, which is a consequence of Theorem 12.1, allows us to show the connectivity of $\widetilde{\mathbf{SepIsog}}_{ss}^{\times}$.

Theorem 7.5. *For any prime $p > 3$, there is a supersingular elliptic curve $\mathcal{E}_0$ defined over $\mathbb{F}_p$. If $p > 11$, this can be chosen to satisfy $j(\mathcal{E}) \not\equiv 0, 1728 \bmod (p)$.*

Proposition 7.6. *The separable isogeny categories* $\mathbf{SepIsog}_{ss}^{\times}$ *and* $\widetilde{\mathbf{SepIsog}}_{ss}^{\times}$ *are connected as are the associated categories of isogenies of oriented elliptic curves* $\underline{\mathbf{SepIsog}}_{ss}^{\times}$ *and* $\underline{\widetilde{\mathbf{SepIsog}}}_{ss}^{\times}$.

Proof. Choose a supersingular curve $\mathcal{E}_0$ defined over $\mathbb{F}_p$ as in Theorem 7.5.

Given a supersingular curve $\mathcal{E}$ defined over $\overline{\mathbb{F}}_p$ there is an isogeny $\varphi\colon \mathcal{E} \longrightarrow \mathcal{E}_0$. By Proposition 2.1 we have a factorization

$$\varphi = \mathrm{Fr}^k \circ \varphi_s$$

where $\varphi_s\colon \mathcal{E} \longrightarrow \mathcal{E}_0^{(1/p^k)}$ is separable. But $\mathcal{E}_0^{(1/p^k)} = \mathcal{E}_0$ since $\mathcal{E}_0$ is defined over $\mathbb{F}_p$, hence $\varphi_s\colon \mathcal{E} \longrightarrow \mathcal{E}_0$ is a separable isogeny connecting $\mathcal{E}$ to $\mathcal{E}_0$. Thus $\mathbf{SepIsog}_{ss}^{\times}$ is connected.

The connectivity of $\widetilde{\mathbf{SepIsog}}_{ss}^{\times}$ now follows from Tate's Theorem 4.5.

The results for $\underline{\mathbf{SepIsog}}_{ss}^{\times}$ and $\underline{\widetilde{\mathbf{SepIsog}}}_{ss}^{\times}$ follow by twisting. □

These results have immediate implications for the cohomology of the groupoids $\mathbf{Isog}_{ss}^{\times}$ and $\mathbf{SepIsog}_{ss}^{\times}$, however, for our purposes with $\underline{\widetilde{\mathbf{SepIsog}}}_{ss}^{\times}$ we need to take the topological structure into account and consider an appropriate continuous cohomology. We will discuss this further in the following sections.

8. Splittings of a quotient of the supersingular category of isogenies

In this section we introduce some quotient categories of $\underline{\widetilde{\mathbf{SepIsog}}}_{ss}^{\times}$. The first is perhaps more 'geometric', while the second is a 'p-typical' approximation.

Our first quotient category is $\boldsymbol{\mathcal{C}} = \underline{\widetilde{\mathbf{SepIsog}}}_{ss}^{\times}/\mathbf{Aut}$, where $\mathbf{Aut}$ denotes the automorphism subgroupoid scheme of $\underline{\widetilde{\mathbf{SepIsog}}}_{ss}^{\times}$ which is defined by

taking the collection of automorphism groups of all the objects of $\underline{\widetilde{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}$,

$$\mathbf{Aut} = \coprod_{(\mathcal{E},\omega)} \operatorname{Aut}\mathcal{E}.$$

Notice that the automorphism group of $(\mathcal{E},\omega)$ only depends on $\mathcal{E}$ and so we can safely write $\operatorname{Aut}\mathcal{E}$ for this. The objects of $\boldsymbol{\mathcal{C}}$ are the objects of $\widetilde{\mathbf{SepIsog}}_{\mathrm{ss}}^{\times}$, whereas the morphism sets are double cosets of the form

$$\boldsymbol{\mathcal{C}}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2)) = \operatorname{Aut}\mathcal{E}_2 \backslash \underline{\widetilde{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2))/\operatorname{Aut}\mathcal{E}_1.$$

If we denote the twisting automorphism corresponding to $t \in \operatorname{Aut}\mathcal{E} \subseteq \overline{\mathbb{F}}_p{}^{\times}$ by $\tau_t\colon \mathcal{E} \longrightarrow \mathcal{E}^{t^2}$, where

$$\tau_t(x,y) = (t^2x, t^3y),$$

then an element of $\boldsymbol{\mathcal{C}}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2))$ is an equivalence class of morphisms in $\underline{\widetilde{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}$ of the form

$$\tau_t \circ \varphi \circ \tau_{s^{-1}} \quad (s \in \operatorname{Aut}(\mathcal{E}_1), t \in \operatorname{Aut}(\mathcal{E}_2)),$$

for some fixed morphism $\varphi\colon (\mathcal{E}_1,\omega_1) \longrightarrow (\mathcal{E}_2,\omega_2)$.

Our second quotient category is $\boldsymbol{\mathcal{C}}_0 = \widetilde{\mathbf{SepIsog}}_{\mathrm{ss}}^{\times}/\boldsymbol{\mu}_{p^2-1}$, where $\boldsymbol{\mu}_{p^2-1}$ denotes the étale subgroupoid scheme of $\underline{\widetilde{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}$ generated by all twistings by elements in the kernel of the (p^2-1)-power map $\mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{G}_{\mathrm{m}}$, whose points over $\overline{\mathbb{F}}_p$ form the group

$$\mu_{p^2-1}(\overline{\mathbb{F}}_p) = \{t \in \overline{\mathbb{F}}_p{}^{\times} : t^{p^2-1} = 1\}.$$

Notice that $\mathbf{Aut}$ is a subgroupoid scheme of $\boldsymbol{\mu}_{p^2-1}$. The objects of $\boldsymbol{\mathcal{C}}_0$ are equivalence classes $[\mathcal{E},\omega]$ of objects of $\underline{\widetilde{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}$, whereas the morphism set $\boldsymbol{\mathcal{C}}_0([\mathcal{E}_1,\omega_1],[\mathcal{E}_2,\omega_2])$ is a double coset of the form

$$\mu_{p^2-1} \backslash \underline{\widetilde{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2))/\mu_{p^2-1},$$

this is the equivalence class consisting of morphisms in $\underline{\widetilde{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}$ of the form

$$\tau_t(x,y) = (t^2x, t^3y),$$

then an element of $\boldsymbol{\mathcal{C}}((\mathcal{E}_1,\omega_1),(\mathcal{E}_2,\omega_2))$ is an equivalence class of morphisms in $\widetilde{\mathbf{SepIsog}}_{\mathrm{ss}}^{\times}$ of the form

$$\tau_t \circ \varphi \circ \tau_{s^{-1}} \quad (s,t \in \mu_{p^2-1}),$$

for some fixed morphism $\varphi\colon (\mathcal{E}_1,\omega_1) \longrightarrow (\mathcal{E}_2,\omega_2)$.

The set of objects in $\boldsymbol{\mathcal{C}}_0$ is represented by the invariant subring

$${}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}} \subseteq {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}],$$

where the action of μ_{p^2-1} is given by

$$t \cdot xu^n = t^{d+n}x \quad (x \in {}^{\mathrm{ss}}E\ell\ell_{2d},\ t \in \mu_{p^2-1}(\overline{\mathbb{F}}_p)).$$

Furthermore, the set of morphisms of $\boldsymbol{\mathcal{C}}_0$ is represented by the algebra

$${}^{\mathrm{ss}}\Gamma_*^{\mu_{p^2-1}} = {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}} \underset{\varepsilon}{\otimes} {}^{\mathrm{ss}}\Gamma_* \underset{\varepsilon}{\otimes} {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}},$$

where the tensor products are formed using the idempotent ring homomorphism

$$\varepsilon\colon {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}] \longrightarrow {}^{\mathrm{ss}}E\ell\ell_*$$

obtained by averaging over the action of μ_{p^2-1} whose image is ${}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}}$.

Theorem 8.1. *There is a natural isomorphism of groupoids with $\mathbb{G}_{\mathrm{m}}$-action,*

$$\mathrm{Spec}_{\overline{\mathbb{F}}_p}\, \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_*^{\mu_{p^2-1}} \cong \boldsymbol{\mathcal{C}}_0.$$

Moreover, $\overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_{2n}^{\mu_{p^2-1}}$ can be identified with the set of continuous functions $\boldsymbol{\mathcal{C}}_0 \longrightarrow \overline{\mathbb{F}}_p$ of weight n and ${}^{\mathrm{ss}}\Gamma_{2n}^{\mu_{p^2-1}} \subseteq \overline{\mathbb{F}}_p \otimes {}^{\mathrm{ss}}\Gamma_{2n}^{\mu_{p^2-1}}$ with the subset of Galois invariant functions.

The natural morphism of topological groupoids $\widetilde{\varepsilon}\colon \widetilde{\underline{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times} \longrightarrow \boldsymbol{\mathcal{C}}_0$ is induced by the natural morphism of Hopf algebroids $\varepsilon\colon ({}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}}, {}^{\mathrm{ss}}\Gamma_*^{\mu_{p^2-1}}) \longrightarrow ({}^{\mathrm{ss}}E\ell\ell_*, {}^{\mathrm{ss}}\Gamma_*)$ under which u goes to 1. Furthermore, $\widetilde{\varepsilon}$ is an equivalence of topological groupoids.

In the latter part of this result, the topological structure has to be taken into account when discussing equivalences of groupoids, with all the relevant maps required to be continuous. This fact will be used to prove some cohomological results in Section 9. Notice that $\boldsymbol{\mu}_{p^2-1}$ is an étale group scheme and $\widetilde{\varepsilon}$ is an étale morphism.

By Proposition 7.6, $\widetilde{\underline{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}$ is connected, hence so are the quotient categories $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_0$. The following stronger result holds.

Theorem 8.2. *Let $\boldsymbol{\mathcal{E}}_0$ be an object of either of these categories. Then there is a continuous map $\sigma\colon \boldsymbol{\mathcal{C}} \longrightarrow \mathrm{Obj}\,\boldsymbol{\mathcal{C}}$ or $\sigma_0\colon \boldsymbol{\mathcal{C}}_0 \longrightarrow \mathrm{Obj}\,\boldsymbol{\mathcal{C}}_0$ for which $\mathrm{dom}\,\sigma(\boldsymbol{\mathcal{E}}) = \boldsymbol{\mathcal{E}}_0$ and $\mathrm{codom}\,\sigma(\boldsymbol{\mathcal{E}}) = \boldsymbol{\mathcal{E}}$. Hence there are splittings of topological categories*

$$\boldsymbol{\mathcal{C}} \cong \mathrm{Obj}\,\boldsymbol{\mathcal{C}} \rtimes \mathrm{Aut}_{\boldsymbol{\mathcal{C}}}\,\boldsymbol{\mathcal{E}}_0, \quad \boldsymbol{\mathcal{C}}_0 \cong \mathrm{Obj}\,\boldsymbol{\mathcal{C}}_0 \rtimes \mathrm{Aut}_{\boldsymbol{\mathcal{C}}_0}\,\boldsymbol{\mathcal{E}}_0.$$

Proof. We verify this for $\boldsymbol{\mathcal{C}}$, the proof for $\boldsymbol{\mathcal{C}}_0$ being similar. Choose an object $(\mathcal{E}_0,\omega_0)$ of $\boldsymbol{\mathcal{C}}$ and set $\alpha_0 = j(\mathcal{E}_0)$.

First note that for each $\alpha \in \overline{\mathbb{F}}_p$, the subcategory of $\widetilde{\underline{\mathbf{SepIsog}}}_{\mathrm{ss}}^{\times}$ consisting of objects $(\mathcal{E},\omega)$ with $j(\mathcal{E}) = \alpha$ is either empty or forms a closed and open set U_α in the natural (Zariski) topology on the space of all such elliptic

curves. In each of the non-empty sets U_α, we may chose an element $(\mathcal{E}_\alpha, \omega_\alpha)$. Then for each $(\mathcal{E}, \omega)$ with $j(\mathcal{E}) = \alpha$, there is a non-unique isomorphism in **SepIsog** $\varphi_{(\mathcal{E},\omega)} \colon (\mathcal{E}_\alpha, \omega_\alpha) \longrightarrow (\mathcal{E}, \omega)$. Given a second such isomorphism φ', the composite $\varphi^{-1} \circ \varphi'$ is in $\operatorname{Aut} \mathcal{E}_\alpha$. Passing to the quotient category $\boldsymbol{\mathcal{C}}$ we see that the image of the subcategory generated by U_α is connected since all such isomorphisms φ have identical images.

Now for every α with U_α non-empty, we may choose a separable isogeny $\varphi_\alpha \colon (\mathcal{E}_0, \omega_0) \longrightarrow (\mathcal{E}_\alpha, \omega_\alpha)$. Again, although this is not unique, on passing to the image set $\overline{U}_\alpha$ in $\boldsymbol{\mathcal{C}}$ we obtain a unique such morphism between the images in $\boldsymbol{\mathcal{C}}$. Forming the composite $\varphi_{(\mathcal{E},\omega)} \circ \varphi_\alpha$ and passing to $\boldsymbol{\mathcal{C}}$ gives a continuous map $\overline{U}_\alpha \longrightarrow \boldsymbol{\mathcal{C}}$ with the desired properties, and then patching together these maps over the finitely many supersingular j-invariants for the prime p establishes the result. □

Corollary 8.3. *There are equivalences of topological categories*

$$\boldsymbol{\mathcal{C}} \simeq \operatorname{Aut}_{\boldsymbol{\mathcal{C}}} \boldsymbol{\mathcal{E}}_0, \quad \boldsymbol{\mathcal{C}}_0 \simeq \operatorname{Aut}_{\boldsymbol{\mathcal{C}}_0} \boldsymbol{\mathcal{E}}_0.$$

Corollary 8.4. *There is an equivalence of Hopf algebroids*

$$({}^{\mathrm{ss}}E\ell\ell_*[u, u^{-1}]^{\mu_{p^2-1}}, {}^{\mathrm{ss}}\Gamma_*^{\mu_{p^2-1}}) \longrightarrow (K(2)_*, K(2)_*K(2)).$$

9. Some equivalences of Hopf algebroids

Now that we possess the machinery developed in Section 6, we are in a position to state and prove our promised cohomological results. Our goal is to reprove the following result of [9], theorem 4.1.

Theorem 9.1. *There is an equivalence of Hopf algebroids*

$$(E\ell\ell_*/(p, A), E\ell\ell_*E\ell\ell/(p, A)) \longrightarrow (K(2)_*, K(2)_*K(2)),$$

inducing an isomorphism

$$\operatorname{Ext}^{*,*}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p, A)) \cong \operatorname{Ext}^{*,*}_{K(2)_*K(2)}(K(2)_*, K(2)_*).$$

First we will use a further result of Devinatz [15].

Proposition 9.2. *The natural morphism of Hopf algebroids*

$$({}^{\mathrm{ss}}E\ell\ell_*[u, u^{-1}], {}^{\mathrm{ss}}\Gamma_*) \longrightarrow ({}^{\mathrm{ss}}E\ell\ell_*, {}^{\mathrm{ss}}\Gamma_*^0)$$

induces an isomorphism of Ext *groups.*

Proof. See the discussion of [15] Construction 2.7, in particular the remarks between equations (2.8) and (2.9). □

By [26] proposition 1.3d, we have

$$\begin{aligned}\mathrm{Ext}^{*,*}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p,A)) &\cong \mathrm{Ext}^{*,*}_{{}^{\mathrm{ss}}\Gamma^0_*}({}^{\mathrm{ss}}E\ell\ell_*, {}^{\mathrm{ss}}E\ell\ell_*)\\ &\cong \mathrm{Ext}^{*,*}_{{}^{\mathrm{ss}}\Gamma_*}({}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}], {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]).\end{aligned}$$

By Theorem 8.1, there is an isomorphism

$$\mathrm{Ext}^{*,*}_{{}^{\mathrm{ss}}\Gamma_*}({}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}], {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]) \cong \mathrm{Ext}^{*,*}_{{}^{\mathrm{ss}}\Gamma_*^{\mu_{p^2-1}}}({}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}}, {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}}).$$

Finally, by Corollary 8.4 there is an isomorphism

$$\mathrm{Ext}^{*,*}_{{}^{\mathrm{ss}}\Gamma_*^{\mu_{p^2-1}}}({}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}}, {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^{\mu_{p^2-1}}) \cong \mathrm{Ext}^{*,*}_{K(2)_*K(2)}(K(2)_*, K(2)_*).$$

10. Isogenies and stable operations in supersingular elliptic cohomology

In this section we explain how the category $\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}$ naturally provides a model for a large part of the stable operation algebra of supersingular elliptic cohomology ${}^{\mathrm{ss}}E\ell\ell^*(\ )$. In fact, it turns out that the subalgebra ${}^{\mathrm{ss}}E\ell\ell^*E\ell\ell = {}^{\mathrm{ss}}E\ell\ell^*(E\ell\ell)$ can be described as a subalgebra of the 'twisted topologized category algebra' of $\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}$ with coefficients in ${}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]$. Such a clear description is not available for $E\ell\ell^*E\ell\ell = E\ell\ell^*(E\ell\ell)$, although an analogous result for the stable operation algebra $K(1)^*(E(1))$ is well known with the later being a twisted topological group algebra. More generally, Morava and his interpreters have given analogous descriptions of $K(n)^*(E(n))$ for $n \geqslant 1$.

By [5], ${}^{\mathrm{ss}}E\ell\ell_*$ and ${}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]$ are products of 'graded fields', hence

$$\begin{aligned}{}^{\mathrm{ss}}E\ell\ell^*E\ell\ell &= \mathrm{Hom}_{{}^{\mathrm{ss}}E\ell\ell_*}({}^{\mathrm{ss}}E\ell\ell_*(E\ell\ell), {}^{\mathrm{ss}}E\ell\ell_*),\\ {}^{\mathrm{ss}}E\ell\ell[u,u^{-1}]^*E\ell\ell &= \mathrm{Hom}_{{}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]}({}^{\mathrm{ss}}E\ell\ell[u,u^{-1}]_*(E\ell\ell), {}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}].\end{aligned}$$

The set of all morphisms originating at a particular object $\underline{\mathcal{E}_0} = (\mathcal{E}_0, \omega_0)$ of $\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}$, with defined over $\mathbb{F}_p$ (such curves always exist by Theorem 12.1), is

$$\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}(\underline{\mathcal{E}_0}, *) = \coprod_{\substack{\underline{\mathcal{E}} \text{ isogenous}\\ \text{to } \underline{\mathcal{E}_0}}} \widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}(\underline{\mathcal{E}_0}, \underline{\mathcal{E}}),$$

is noncanonically a product of the form

$$\widetilde{\underline{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}(\underline{\mathcal{E}_0}, *) = \coprod_{(\mathcal{E}_0/N, \omega_0)} \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p(\mathcal{E}_0/N), \mathcal{T}_p\mathcal{E}_0) \times {\overline{\mathbb{F}}_p}^{\times},$$

where N ranges over the finite subgroups of $\mathcal{E}_0$ of order prime to p.

Using similar methods to those of [4, 8] we can construct stable operations T_n $(p \nmid n)$ in the cohomology theory ${}^{\mathrm{ss}}E\ell\ell_*[u,u^{-1}]^*(\ )$ and these actually restrict to operations in ${}^{\mathrm{ss}}E\ell\ell^*_*(\ )$. These operations (together with Adams-like operations originating on the factor of $\overline{\mathbb{F}}_p$) generate a Hecke-like algebra.

There is also a further set of operations coming from elements of the component

$$\underline{\widetilde{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}(\underline{\mathcal{E}_0}, \underline{\mathcal{E}_0}) = \mathrm{InvtHom}_{\mathbb{D}_{\overline{\mathbb{F}}_p}}(\mathcal{T}_p(\mathcal{E}_0), \mathcal{T}_p\mathcal{E}_0) \times {\overline{\mathbb{F}}_p}^{\times}.$$

These operations correspond to the Hecke-like algebra of [8] associated to the Morava stabilizer group $\mathbb{S}_2$.

Combining these two families of operations gives rise to a composite Hecke-like algebra which in turn generates subalgebras of the operation algebras in the cohomology theories ${}^{\mathrm{ss}}E\ell\ell_*[u, u^{-1}]^*(\ )$ and ${}^{\mathrm{ss}}E\ell\ell^*_*(\ )$.

11. Relationship with work of Robert

Robert [29] discussed the action of Hecke operators T_n $(p \nmid n)$ on the ring of holomorphic modular forms modulo the supersingular ideal generated by p, A, in effect studying the action of the étale part of the category $\underline{\widetilde{\mathbf{SepIsog}}}^{\times}_{\mathrm{ss}}$. We begin with some comments on Robert's work which provides a classical Hecke operator perspective on ours. We denote by $\mathrm{B}\colon {}^{\mathrm{ss}}E\ell\ell_* \longrightarrow {}^{\mathrm{ss}}E\ell\ell_*$ the operator $\mathrm{B}(F) = BF$, which raises weight by $p+1$ and degree by $2(p+1)$. The following operator commutativity formula holds for all primes $\ell \neq p$:

$$\mathrm{BT}_\ell = \ell \mathrm{T}_\ell \mathrm{B}. \tag{11.1}$$

This is actually a more general result than Robert proves since he only works with holomorphic modular forms, but our result of [11], discussed earlier in Theorem 3.2, gives in the ring ${}^{\mathrm{ss}}E\ell\ell_*$

$$B^{p-1} = -\left(\frac{-1}{p}\right)\Delta^{(p^2-1)/12}$$

and this allows us to localize with respect to powers of Δ or equivalently of B.

Recall the Hecke algebra

$$\mathbf{H}_p = \mathbb{F}_p[\mathrm{T}_\ell, \psi^\ell : \text{primes } \ell \neq p]/(\text{relations}),$$

where the relations are the usual ones satisfied by Hecke operations, as described in Theorem 7 of [4]. We follow Robert in introducing certain twistings of a module M. For each natural number a let $M[a]$ denote underlying $\mathbb{F}_p$-module M with the twisted Hecke action

$$\mathrm{T}_\ell \cdot m = \mathrm{T}_\ell^{[a]} m = \ell^a \mathrm{T}_\ell m.$$

When ${}^{\mathrm{ss}}E\ell\ell_*$, this agrees with Robert's action

$$\mathrm{T}_\ell^{[a]} F = \ell^a \mathrm{T}_\ell F,$$

at least when restricted to the holomorphic part, and then $M[a] \cong M[a + p - 1]$ as $\mathbf{H}_p$-modules since $\ell^{p-1} \equiv 1 \bmod (p)$. We view multiplication by

B as giving rise to homomorphisms of graded $\mathbf{H}_p$-modules $\mathrm{B}\colon {}^{\mathrm{ss}}E\ell\ell_*[a] \longrightarrow {}^{\mathrm{ss}}E\ell\ell_*[a-1]$ for $a \in \mathbb{Z}$, uniformly raising degrees by $2(p+1)$.

More generally, if M_* is a right comodule over the Hopf algebroid $({}^{\mathrm{ss}}E\ell\ell_*, {}^{\mathrm{ss}}\Gamma^0_*)$ with coproduct $\gamma\colon M \longrightarrow M \otimes_{{}^{\mathrm{ss}}E\ell\ell_*} {}^{\mathrm{ss}}\Gamma^0_*$, then associated to each $a \in \mathbb{Z}$ there is a twisted comodule $M_*[a]$ with coproduct

$$\gamma^{[a]}m = \sum_i m_i \otimes t_i \,\mathrm{ind}^a,$$

where ind is defined in Equation 5.1, see also Propsoition5.2, and $\gamma m = \sum_i m_i \otimes t_i$.

Recall that for any left ${}^{\mathrm{ss}}E\ell\ell_*$-linear map $\theta\colon {}^{\mathrm{ss}}\Gamma^0_* \longrightarrow {}^{\mathrm{ss}}E\ell\ell_*$ there is an operation $\overline{\theta}$ on M_* given by

$$\overline{\theta}m = \sum_i m_i \otimes \theta(t_i).$$

By [4, 8], this construction gives rise to an induced $\mathbf{H}_p$-module structures on M_* and $M_*[a]$ agreeing with that generalizing Robert's discussed above. In fact, these extend to module structures over the associated *twisted Hecke algebra* containing ${}^{\mathrm{ss}}E\ell\ell_*$ and $\mathbf{H}_p$ as discussed in [8]. Also there are homomorphisms $\mathrm{B}\colon M_*[a] \longrightarrow M_*[a-1]$ of modules over the twisted Hecke algebra and induced from multiplication by B. The following is our analogue of [29], lemme 6.

Theorem 11.1. *For $a \in \mathbb{Z}$, $\mathrm{B}\colon M_*[a] \longrightarrow M_*[a-1]$ defines an isomorphism of $({}^{\mathrm{ss}}E\ell\ell_*, {}^{\mathrm{ss}}\Gamma^0_*)$-comodules.*

Proof. We make use of the description of $E\ell\ell_*E\ell\ell_{(p)}$ from [6] and view modular forms as functions on the space of oriented lattices in $\mathbb{C}$. As will see, the argument used by Robert to determine $\mathrm{T}_\ell B$ for a prime $\ell \neq p$ is based on a congruence in $E\ell\ell_*E\ell\ell_{(p)}$.

By [29], equation 19 and théorème B/lemme 7, we have the following. If for some $r = 0, 1, \dots \ell - 1$,

$$L = \langle \tau, 1\rangle \subseteq \langle \tau', 1\rangle, \quad \tau' = \frac{\tau + r}{\ell}, \quad q' = e^{2\pi i \tau'},$$

then taking q-expansions over $\mathbb{Z}_{(p)}$ gives

$$B(q') - B(q) \equiv -12 \sum_{\substack{1 \leqslant s \leqslant \ell-1 \\ 1 \leqslant k}} \sum \left(\frac{q^k (q')^s}{(1 - q^k (q')^s)^2} + \frac{q^k (q')^{-s}}{(1 - q^k (q')^{-s})^2} \right) \bmod (p).$$

On the other hand, if

$$L = \langle \tau, 1\rangle \subseteq \left\langle \tau, \frac{1}{\ell} \right\rangle, \quad \tau' = \ell\tau, \quad q' = e^{2\pi i \tau'},$$

then taking q-expansions over $\mathbb{Z}_{(p)}$ yields

$$\ell^{p+1}B(q') - B(q) \equiv$$
$$\sum_{1\leqslant s\leqslant \ell-1}\left((\ell^2-1)B(q) - 12\ell^2\sum_{1\leqslant k}\left(\frac{q^k(q')^s}{(1-q^k(q')^s)^2} + \frac{q^k(q')^{-s}}{(1-q^k(q')^{-s})^2}\right)\right) \bmod (p).$$

In the terminology of [6], the coefficient of each monomial $q^u(q')^v$ is a stably numerical polynomial in ℓ. Indeed, using the integrality criterion of [6] theorem 6.3 for generalized modular forms to lie in $E\ell\ell_*E\ell\ell_{(p)}$, together with the fact that every lattice inclusion of index not divisible by p factors into a sequence of lattice inclusions of prime index, we can obtain similar formulæ for all lattice inclusions of (not necessarily prime) index not divisible by p.

The precise interpretation of what is going on here is that there are functions $F_0, F_1 \in E\ell\ell_*E\ell\ell_{(p)}$ on inclusions of lattices such that for any inclusion of lattices $L \subseteq L'$ of degree $[L';L]$ not divisible by p,

$$B(L') - [L';L]B(L) = pF_0(L\subseteq L') + A(L)F_1(L\subseteq L'). \tag{11.2}$$

From this it follows that in the ring $E\ell\ell_*E\ell\ell_{(p)}$,

$$\eta_R(B) - \eta_L(B)\,\mathrm{ind} \equiv 0 \bmod (p, A_1), \tag{11.3}$$

Notice that under the reduction map $E\ell\ell_*E\ell\ell_{(p)} \longrightarrow {}^{\mathrm{ss}}\Gamma^0_*$, the index function $(L\subseteq L') \mapsto [L';L]$ goes to ind. This can be seen as follows. For any supersingular elliptic curve $\mathcal{E}$ defined over $\mathbb{F}_{p^2}$ there is an imaginary quadratic number field K in which p is unramified and so there is a lift α of $j(\mathcal{E})$ contained in the ring of integers $\mathcal{O}_K$. Then there is an elliptic curve $\widetilde{\mathcal{E}}$ defined over $\mathcal{O}_K$ with $j(\widetilde{\mathcal{E}}) = \alpha$ and reduction modulo p induces an isomorphism $\widetilde{\mathcal{E}}[n] \longrightarrow \mathcal{E}[n]$ for $p \nmid n$. Since a strict separable isogeny of degree n, $\varphi\colon \mathcal{E} \longrightarrow \mathcal{E}'$ (defined over an extension of $\mathbb{F}_{p^2}$) is determined by $\ker\varphi \subseteq \mathcal{E}[n]$, it can be lifted to a strict separable isogeny of degree n, $\widetilde{\varphi}\colon \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{E}}'$, where $\ker\widetilde{\varphi}$ is the preimage of $\ker\varphi$ under reduction (here the lift will need to be defined over an extension of $\mathcal{O}_K$). Then $\mathrm{ind}\,\varphi \equiv n \bmod (p)$. Hence if we realize $\widetilde{\mathcal{E}}$ in the form $\mathbb{C}/L$ then $\widetilde{\mathcal{E}}'$ can be realized in the form $\mathbb{C}/L'$ where $L \subseteq L'$ has index n.

Since $\mathrm{ind}^{p-1} = c_1$ (i.e., the constant function taking value 1) we obtain

$$\eta_R(B^{p-1}) - \eta_L(B^{p-1}) \equiv 0 \bmod (p, A_1), \tag{11.4}$$

which implies that $B^{p-1} \in {}^{\mathrm{ss}}E\ell\ell_*$ is coaction primitive.

By Equation (11.2), we have

$$\begin{aligned}\gamma\mathrm{B}(m) = \gamma(mB) &= \sum_i m_i \otimes t_i\eta_R B\\ &= \sum_i m_i \otimes Bt_i \,\mathrm{ind}\\ &= \sum_i m_i B \otimes t_i \,\mathrm{ind}\\ &= \sum_i \mathrm{B}(m_i) \otimes t_i \,\mathrm{ind}\\ &= \mathrm{B}\gamma^{[1]}m,\end{aligned}$$

where we have viewed M_* as a right ${}^{\mathrm{ss}}E\ell\ell_*$-module and used ${}^{\mathrm{ss}}E\ell\ell_*$-bimodule tensor products.

The determination of $\mathrm{T}_\ell B$ now follows from our definition of the Hecke operators of [6], equation 6.5 as does the following generalization of Robert's formula valid for all primes $\ell \neq p$:

$$\mathrm{T}_\ell(BF) \equiv \ell B\mathrm{T}_\ell(F) \quad (F \in {}^{\mathrm{ss}}E\ell\ell_*).$$

□

Our results are more general than those of Robert since they allow us to use generalized isogenies rather than just isogenies to define ${}^{\mathrm{ss}}E\ell\ell_*$-linear maps ${}^{\mathrm{ss}}\Gamma^0_* \longrightarrow \overline{\mathbb{F}}_p$ and hence operations $\overline{\varphi}$ on ${}^{\mathrm{ss}}\Gamma^0_*$-comodules. Explicit operations of this type were defined in [8] using Hecke operators derived from the space of double cosets $\langle \mu_{p^2-1}, p\rangle \backslash \widetilde{\mathbb{S}}_2 / \langle \mu_{p^2-1}, p\rangle$ and its associated Hecke algebra. In fact $\langle \mu_{p^2-1}, p\rangle \backslash \widetilde{\mathbb{S}}_2 / \langle \mu_{p^2-1}, p\rangle$ is homeomorphic to $\mathbb{S}^0_2 \rtimes \mathbb{Z}/2$ as a space. For each supersingular elliptic curve $(\mathcal{E}, \omega)$ we can identify $\mathbb{W}(\mathbb{F}_{p^2})\langle \mathrm{S}\rangle$ with $\widetilde{\mathbf{Isog}}((\mathcal{E},\omega),(\mathcal{E},\omega))$ and following [8] obtain for each $\alpha \in \mathbb{S}_2 \rtimes \mathbb{Z}/2$ a ${}^{\mathrm{ss}}E\ell\ell_*$-linear map $\alpha_*\colon {}^{\mathrm{ss}}\Gamma^0_* \longrightarrow {}^{\mathrm{ss}}E\ell\ell_*$ and hence an operation $\overline{\alpha}$ on ${}^{\mathrm{ss}}\Gamma^0_*$-comodules. This can be further generalized by associating to each positive integer d and each separable isogeny $(\mathcal{E},\omega) \xrightarrow{\varphi} (\mathcal{E}',\omega)$ of degree d the corresponding element $\alpha^\varphi \in \widetilde{\mathbf{Isog}}((\mathcal{E}',\omega'),(\mathcal{E}',\omega'))$ and then symmetrizing over all of these to form a ${}^{\mathrm{ss}}E\ell\ell_*$-linear map

$$\alpha^d_*\colon {}^{\mathrm{ss}}\Gamma^0_* \longrightarrow {}^{\mathrm{ss}}E\ell\ell_*; \quad (\alpha^d_* F)(\mathcal{E},\omega) = \frac{1}{d}\sum_\varphi \alpha^\varphi_*.$$

We will return to this in greater detail in future work.

Robert analyzes the holomorphic part of ${}^{\mathrm{ss}}E\ell\ell_{2n}$ as a $\mathbf{H}_p$-module, in particular he determines when the *Eisenstein modules* Ei_k embed, where Ei_k is the 1-dimensional $\mathbb{F}_p$-module on the generator e_k for which

$$\mathrm{T}_\ell e_k = (1+\ell^{k-1})e_k.$$

Thus Ei_k is an eigenspace for each Hecke operator T_ℓ, and there is an isomorphism of $\mathbf{H}_p$-modules

$$\mathrm{Ei}_{2k} \cong \mathbb{F}_p\{\widetilde{E}_{2k}\} \subseteq {}^{\mathrm{ss}}E\ell\ell_{4k}; \quad e_{2k} \mapsto \widetilde{E}_{2k},$$

where $\widetilde{E}_{2k}$ is the reduction of one of the following elements of $(E\ell\ell_{2k})_{(p)}$:

$$\begin{cases} E_{2k} & \text{if } (p-1) \mid 2k, \\ (B_{2k}/4k)E_{2k} & \text{if } (p-1) \nmid 2k. \end{cases}$$

In particular, Ei_0 is the 'trivial' module for which

$$\mathrm{T}_\ell e_0 = (1+\ell^{-1})e_0.$$

Robert gives conditions on when there is an occurrence of Ei_k in ${}^{\mathrm{ss}}E\ell\ell_{2n}$, at least in the holomorphic part. Since localization with respect to powers of Δ is equivalent to that with respect to powers of B by the main result of [11] we can equally well apply his results to ${}^{\mathrm{ss}}E\ell\ell_{2n}$, obtaining the following version of [29] théorème 3.

Theorem 11.2. *For a prime $p \geqslant 5$ and an even integer k, there is an embedding of $\mathbf{H}_p$-modules $\mathrm{Ei}_k \longrightarrow {}^{\mathrm{ss}}E\ell\ell_{2n}$ if and only if one of the following congruences holds:*

$$n \equiv k \bmod (p^2-1), \quad n \equiv pk \bmod (p^2-1).$$

Notice that in particular, the trivial module Ei_0 occurs precisely in degrees $2n$ where $(p^2-1) \mid 2n$. The Ext groups of ${}^{\mathrm{ss}}E\ell\ell_*$ over ${}^{\mathrm{ss}}\Gamma^0_*$ were investigated in [9, 10], and the results show that Robert's conditions are weaker than needed to calculate Ext^0. Of course, his work ignores the effect of operations coming from the 'connected' part of $\underline{\mathbf{SepIsog}}^{\times}_{\mathrm{ss}}$.

12. The existence of supersingular curves over $\mathbb{F}_p$

For every prime $p > 3$ with $p \not\equiv 1 \bmod (12)$, it is easily seen there are supersingular elliptic curves defined over $\mathbb{F}_p$ since the Hasse invariant then has Q or R as a factor. The following stronger result (probably due to Deuring) also holds and a sketch of its proof can be found in [11]; Cox [13] contains an accessible account of related material.

Recall that the endomorphism ring of a supersingular elliptic curve $\mathcal{E}$ over $\overline{\mathbb{F}}_p$ with $j(\mathcal{E}) \equiv 0, 1728 \bmod (p)$ contains an imaginary quadratic number ring

$$\begin{cases} \mathbb{Z}[\omega] & \text{if } j(\mathcal{E}) \equiv 0 \bmod (p), \\ \mathbb{Z}[i] & \text{if } j(\mathcal{E}) \equiv 1728 \bmod (p). \end{cases}$$

By Theorem 1.3, such elliptic curves are isomorphic to Weierstraß curves defined over $\mathbb{F}_p$.

Let $K = \mathbb{Q}(\sqrt{-p})$ and $\mathcal{O}_K$ be its ring of integers which is its unique maximal order.

Theorem 12.1. *For any prime $p > 11$, there are supersingular elliptic curves $\mathcal{E}$ defined over $\mathbb{F}_p$ and with $j(\mathcal{E}) \not\equiv 0, 1728 \bmod (p)$ and having $\mathcal{O}_K \subseteq \operatorname{End}\mathcal{E}$.*

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