

# HOMOLOGY AND COHOMOLOGY OF $E_\infty$ RING SPECTRA

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ABSTRACT. We show that every homology or cohomology theory on a category of  $E_\infty$  ring spectra is Topological André–Quillen homology or cohomology with appropriate coefficients. We show that the cotangent complex of  $MU$  is  $MU \wedge bu$ .

## INTRODUCTION

Homology and cohomology theories in various contexts provide some of the most effective tools in mathematics because of their computability, their usefulness for classification problems, and their close relationship to extensions and obstructions. In the context of homotopy theory, homology and cohomology typically refer to theories that satisfy the Eilenberg–Steenrod (and Milnor) axioms. Although originally phrased for topological spaces, these axioms make sense in the more general context of homotopy theory of closed model categories.

The Eilenberg–Steenrod axioms involve a category of pairs. For a closed model category  $\mathfrak{C}$ , an appropriate category of pairs is the category of arrows in  $\mathfrak{C}$ : A pair is a map  $A \rightarrow X$  in  $\mathfrak{C}$  and a map of pairs is a commutative square. Standard notation is to write  $(X, A)$  for the pair  $f: A \rightarrow X$  (with  $f$  understood) and to write  $X$  rather than  $(X, \emptyset)$  for the pair  $\emptyset \rightarrow X$  where  $\emptyset$  is the initial object of  $\mathfrak{C}$ . A map of pairs  $(X, A) \rightarrow (Y, B)$  is called a weak equivalence when the maps  $A \rightarrow B$  and  $X \rightarrow Y$  are weak equivalences in  $\mathfrak{C}$ . In this terminology, we understand cohomology theories as follows:

**Definition.** Let  $\mathfrak{C}$  be a closed model category. A cohomology theory on  $\mathfrak{C}$  consists of a contravariant functor  $h^*$  from the category of pairs to the category of graded abelian groups together with natural transformations of abelian groups  $\delta^n: h^n(A) \rightarrow h^{n+1}(X, A)$  for all  $n$ , satisfying the following axioms:

(i) (Homotopy) If  $(X, A) \rightarrow (Y, B)$  is a weak equivalence of pairs, then the induced map  $h^*(Y, B) \rightarrow h^*(X, A)$  is an isomorphism of graded abelian groups.

(ii) (Exactness) For any pair  $(X, A)$ , the sequence

$$\cdots \longrightarrow h^n(X, A) \longrightarrow h^n(X) \longrightarrow h^n(A) \xrightarrow{\delta^n} h^{n+1}(X, A) \longrightarrow \cdots$$

is exact.

(iii) (Excision) If  $A$  is cofibrant,  $A \rightarrow B$  and  $A \rightarrow X$  are cofibrations, and  $Y$  is the pushout  $X \cup_A B$ , then the map of pairs  $(X, A) \rightarrow (Y, B)$  induces an isomorphism of graded abelian groups  $h^*(Y, B) \rightarrow h^*(X, A)$ .

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- (iv) (Product) If  $\{X_\alpha\}$  is a set of cofibrant objects and  $X$  is the coproduct, then the natural map  $h^*(X) \rightarrow \prod h^*(X_\alpha)$  is an isomorphism.

A map of cohomology theories  $\phi: h^* \rightarrow k^*$  is a natural transformation of contravariant functors that makes the diagram

$$\begin{array}{ccc} h^n(A) & \xrightarrow{\delta^n} & h^{n+1}(X, A) \\ \phi_A \downarrow & & \downarrow \phi_{(X,A)} \\ k^n(A) & \xrightarrow{\delta^n} & k^{n+1}(X, A) \end{array}$$

commute for all  $(X, A)$ ,  $n$ .

A homology theory consists of a covariant functor  $h_*$  together with natural transformations  $\partial_n: h_{n+1}(X, A) \rightarrow h_n(A)$  satisfying completely analogous axioms, with the Product Axiom replaced by the following Direct Sum Axiom:

- (iv)' (Direct Sum) If  $\{X_\alpha\}$  is a set of cofibrant objects and  $X$  is the coproduct, then the natural map  $\bigoplus h_*(X_\alpha) \rightarrow h_*(X)$  is an isomorphism.

As a matter of pure algebra, the axioms above imply the stronger property that for  $A \rightarrow B \rightarrow X$ , the sequence

$$\dots \rightarrow h^n(X, B) \rightarrow h^n(X, A) \rightarrow h^n(B, A) \rightarrow h^{n+1}(X, B) \rightarrow \dots$$

is exact, where the last map is the composite  $h^n(B, A) \rightarrow h^n(B) \rightarrow h^{n+1}(X, B)$ . Likewise, they imply the stronger property that the natural map  $h^*(\coprod X_\alpha, \coprod A_\alpha) \rightarrow \prod h^*(X_\alpha, A_\alpha)$  is an isomorphism when each  $X_\alpha$  and  $A_\alpha$  is cofibrant. As a matter of pure homotopy theory, whether or not  $(h^*, \delta^*)$  or  $(h_*, \partial_*)$  satisfies the axioms for a particular model structure on  $\mathfrak{C}$  depends only on the weak equivalences and not on the cofibrations (see Section 9 for details).

The purpose of this paper is to study homology and cohomology theories on categories of  $E_\infty$  ring spectra, or equivalently, on the modern categories of EKMM commutative  $S$ -algebras [5], where the initial object is the sphere spectrum  $S$  and the coproduct is the modern symmetric monoidal smash product. Because the final object in the category of commutative  $S$ -algebras is the trivial (one-point) spectrum  $*$ , if we take  $\mathfrak{C}$  above to be the model category of commutative  $S$ -algebras, then there are no non-trivial cohomology theories: If  $A$  is any cofibrant commutative  $S$ -algebra and  $C \rightarrow *$  is a cofibrant approximation, then  $A \wedge C$  is contractible, and so by the Homotopy Axiom and the Exactness Axiom,  $h^*(A \wedge C, C) = 0$ , for any cohomology theory  $h^*$ . Then the Excision Axiom applied to the cofibrations  $S \rightarrow A$  and  $S \rightarrow C$  implies that the map

$$h^*(A) = h^*(A, S) \rightarrow h^*(A \wedge C, C) = 0$$

is an isomorphism. Now it easily follows from the Homotopy Axiom and the Exactness Axiom that  $h^*$  is zero on any pair. In order to have non-trivial homology and cohomology theories, we therefore need to consider categories of commutative  $S$ -algebras with non-trivial final objects. We do this by considering over-categories, and we work more generally with categories of commutative  $R$ -algebras for a cofibrant commutative  $S$ -algebra  $R$ . For a commutative  $R$ -algebra  $B$ , let  $\mathfrak{C}_R/B$  denote the category of commutative  $R$ -algebras lying over  $B$ . This is a closed model category with initial object  $R$ , final object  $B$ , and coproduct  $\wedge_R$ , the smash product over  $R$ .

Topological André–Quillen cohomology with various coefficients provides examples of cohomology theories on  $\mathfrak{C}_R/B$ . For a cofibrant commutative  $R$ -algebra  $A$  and a cofibrant commutative  $A$ -algebra  $X$ , Basterra [1] constructs the cotangent complex  $\mathbf{L}\Omega_A X$  as the derived commutative  $X$ -algebra indecomposables (of the derived augmentation ideal) of  $X \wedge_A X$ . The cotangent complex  $\mathbf{L}\Omega_A X$  is an  $X$ -module, and restricting to maps  $A \rightarrow X$  in  $\mathfrak{C}_R/B$ , we can regard

$$\mathbf{LAb}_A^B X = B \wedge_X \mathbf{L}\Omega_A X$$

as a functor from the homotopy category of  $\mathfrak{C}_R/B$  to the homotopy category of  $B$ -modules. Topological André–Quillen cohomology with coefficients in a  $B$ -module  $M$  is defined by

$$D_R^*(X, A; M) = \mathrm{Ext}_B^*(\mathbf{LAb}_A^B X, M)$$

(where  $\mathrm{Ext}$  is as in [5, IV.1.1]). Using the connecting homomorphism in the transitivity sequence [1, 4.4],  $D_R^*(-; M)$  becomes a cohomology theory on  $\mathfrak{C}_R/B$ , functorially in  $M$  in the homotopy category of  $B$ -modules. Our main result is the following theorem, which says in particular that every cohomology theory on  $\mathfrak{C}_R/B$  is isomorphic to  $D_R^*(-; M)$  for some  $B$ -module  $M$ .

**Theorem 1.** *Topological André–Quillen cohomology, viewed as a functor from the homotopy category of  $B$ -modules to the category of cohomology theories on  $\mathfrak{C}_R/B$ , is an equivalence of categories.*

A characterization of the category of homology theories on  $\mathfrak{C}_R/B$  is slightly trickier because of the failure of Brown’s Representability Theorem for homology theories [3]. Given a homology theory  $h_*$  on the category  $\mathfrak{M}_B$  of  $B$ -modules, we obtain a homology theory  $h_*^D$  on the category  $\mathfrak{C}_R/B$  by setting

$$h_*^D(X, A) = h_*(\mathbf{LAb}_A^B X, *).$$

This describes a functor from the category of homology theories on  $\mathfrak{M}_B$  to the category of homology theories on  $\mathfrak{C}_R/B$ . We prove that this functor is an equivalence of categories.

**Theorem 2.** *The category of homology theories on  $\mathfrak{C}_R/B$  is equivalent to the category of homology theories on  $\mathfrak{M}_B$ .*

As always, there is a close relationship between homology and cohomology theories and a category of “spectra” that we review in Section 1. For  $R = B$ , the category  $\mathfrak{C}_B/B$  is enriched over the category of based spaces, with tensors and cotensors, and so in particular we have a suspension functor. We denote this suspension functor by  $E$  to avoid confusion with suspension of the underlying  $B$ -module. A  $\mathfrak{C}_B/B$ -spectrum is a sequence of objects  $A_n$ ,  $n \geq 0$ , together with “structure maps”  $\sigma_n: EA_n \rightarrow A_{n+1}$ . A map of spectra is a collection of maps  $A_n \rightarrow A'_n$  that commute with the structure maps. We define the homotopy groups of a spectrum  $\underline{A} = \{A_n\}$  by

$$\pi_q \underline{A} = \mathrm{Colim} \tilde{\pi}_{q+n} A_n,$$

where  $\tilde{\pi}_* A = \mathrm{Ker}(\pi_* A \rightarrow \pi_* B)$ . Standard techniques [6, 10] allow us to prove in Section 7 that the category of  $\mathfrak{C}_B/B$ -spectra forms a closed model category, with weak equivalences the maps that induce isomorphisms on homotopy groups. The resulting homotopy category is called the “stable category” of  $\mathfrak{C}_B/B$ . In Section 3, we prove the following theorem:

**Theorem 3.** *Let  $B$  be a cofibrant commutative  $R$ -algebra. The stable category of  $\mathfrak{C}_B/B$  is equivalent to the homotopy category of  $B$ -modules.*

In fact, as explained in Section 3, the equivalence of homotopy categories arises from a Quillen equivalence. Although the technical hypotheses do not quite apply, this theorem is closely related to the title theorem of Schwede–Shipley [16] that stable categories are categories of modules. Theorem 3 is in marked contrast to the corresponding situation for simplicial commutative algebras studied by Schwede [15], where the stable category is equivalent to the homotopy category of modules over a ring spectrum that is generally very different from the ground ring; see also Theorem 2.8 below.

For an object  $A$  in  $\mathfrak{C}_R/B$ ,  $B \wedge_R A$  is naturally an object of  $\mathfrak{C}_B/B$  and we have an associated  $\mathfrak{C}_B/B$ -spectrum  $E^\infty(B \wedge_R A) = \{E^n(B \wedge_R A)\}$  called the “suspension spectrum”. Closely related to the previous theorems is the following:

**Theorem 4.** *Let  $B$  be a cofibrant commutative  $R$ -algebra, and  $A$  a cofibrant object in  $\mathfrak{C}_R/B$ . Under the equivalence of Theorem 3, the suspension spectrum  $E^\infty(B \wedge_R A)$  corresponds to the  $B$ -module  $\mathbf{LAb}_R^B A$ .*

The suspension for commutative  $S$ -algebras turns out to be closely related to delooping for spaces. When  $X$  is an  $E_\infty$  space, that is, a space with an action of an  $E_\infty$  operad, the work of May, Quinn, and Ray [13, IV§1] implies that the suspension spectrum  $\Sigma^\infty X_+$  is an  $E_\infty$  ring spectrum. We can make sense of the André–Quillen Cohomology and the cotangent complex of  $E_\infty$  ring spectra. Up to equivalence, these do not depend on the operad, and we can understand these in terms of an equivalent commutative  $S$ -algebra. The work of May and Thomason [12] shows that up to isomorphism in the stable category, there is a canonical spectrum associated to  $X$  whose zeroth space is the group completion of  $X$ ; it is any spectrum output by an “infinite loop space machine”. In Section 6, we reinterpret part of the proof of the previous theorem to prove the following delooping theorem:

**Theorem 5.** *For an  $E_\infty$  space  $X$ , the cotangent complex of the  $E_\infty$  ring spectrum  $\Sigma^\infty X_+$  is the extended  $\Sigma^\infty X_+$ -module  $(\Sigma^\infty X_+) \wedge Z$ , where  $Z$  is the spectrum associated to  $X$ .*

According to Lewis [8, §IX], the Thom spectrum  $M$  obtained from a map of  $E_\infty$  spaces  $X \rightarrow BF$  naturally has the structure of an  $E_\infty$  ring spectrum (see also Mahowald [9]), and the diagonal map

$$M \longrightarrow M \wedge X_+$$

is a map of  $E_\infty$  ring spectra. The derived extension of scalars to  $E_\infty$   $M$ -algebras,

$$M \wedge M \longrightarrow M \wedge X_+ \cong M \wedge \Sigma^\infty X_+$$

induces the Thom isomorphism and is a weak equivalence. The cotangent complex commutes with extension of scalars [1, 4.5], and we obtain the following corollary of the previous theorem:

**Corollary.** *Let  $\alpha: X \rightarrow BF$  be a map of  $E_\infty$  spaces,  $M$  the associated  $E_\infty$  ring Thom spectrum, and  $Z$  the spectrum associated to  $X$ . Then the cotangent complex  $\mathbf{L}\Omega_S M$  is the  $M$ -module  $M \wedge Z$ .*

As a special case, we obtain the following result. In it  $bu$  denotes the spectrum with zeroth  $E_\infty$  space  $BU$ ; it is equivalent to  $\Sigma^2 ku$ , where  $ku$  is connective  $K$ -theory.

**Corollary.** *The cotangent complex  $\mathbf{L}\Omega_S MU$  of  $MU$  is the  $MU$ -module  $MU \wedge bu$ .*

We have stated these results in terms of EKMM commutative  $S$ -algebras, but because of the homotopy invariant nature of Theorems 1 and 2, they hold in the context of commutative symmetric spectra and commutative orthogonal spectra [10], or in any Quillen equivalent modern category of  $E_\infty$  ring spectra.

More generally, we have stated results in terms of commutative  $S$ -algebras, but results analogous to Theorems 1–4 hold for any sort of operadic algebras in EKMM  $S$ -modules with slight modifications of the arguments below; see Section 8 for details.

## 1. REDUCED THEORIES

Although the most natural statement of Theorems 1 and 2 is in terms of cohomology and homology theories defined on pairs, the most natural proof is in terms of reduced cohomology and homology theories. The purpose of this section is to record some basic facts about the relationship of cohomology theories on pairs, reduced cohomology theories, “Omega weak spectra” (see Definition 1.4 below), and spectra. These results hold quite generally and their arguments depend very little on the specifics of the category  $\mathfrak{C}_R/B$  of commutative  $R$ -algebras over  $B$ . Since the arguments are familiar from other contexts, we omit many of the details.

We begin with the definition of reduced theories. These are defined for the homotopy categories of “pointed closed model categories” (closed model categories where the initial object is the final object); these categories have the extra structure of a “suspension functor” [14, I§2] and of “cofiber sequences” [14, I§3].

**Definition 1.1.** Let  $\mathfrak{C}$  be a pointed closed model category. A reduced cohomology theory on  $\mathfrak{C}$  consists of a contravariant functor  $h^*$  from the homotopy category  $\mathrm{Ho}\mathfrak{C}$  to the category of graded abelian groups together with a natural isomorphism of abelian groups  $\sigma: h^n(X) \rightarrow h^{n+1}(\Sigma X)$  (the suspension isomorphism) for all  $n$ , satisfying the following axioms:

- (i) (Exactness) If  $X \rightarrow Y \rightarrow Z$  is part of a cofibration sequence, then

$$h^n(Z) \longrightarrow h^n(Y) \longrightarrow h^n(X)$$

is exact for all  $n$ .

- (ii) (Product) If  $\{X_\alpha\}$  is a set of objects and  $X$  is the coproduct in  $\mathrm{Ho}\mathfrak{C}$ , then the natural map  $h^*(X) \rightarrow \prod h^*(X_\alpha)$  is an isomorphism.

A reduced homology theory consists of a covariant functor  $h_*$  together with natural isomorphisms  $\sigma: h_{n+1}(\Sigma X) \rightarrow h_n(X)$  satisfying an analogous exactness axiom and the following Direct Sum Axiom:

- (ii)' (Direct Sum) If  $\{X_\alpha\}$  is a set of objects and  $X$  is the coproduct in  $\mathrm{Ho}\mathfrak{C}$ , then the natural map  $\bigoplus h_*(X_\alpha) \rightarrow h_*(X)$  is an isomorphism.

A map of reduced cohomology theories or of reduced homology theories is a natural transformation that commutes with the suspension isomorphisms.

Whenever the final object in a closed model category is cofibrant, there is a close relationship between cohomology theories and reduced cohomology theories on the under-category of the final object. Writing  $B$  for the final object and  $\mathfrak{C}\backslash B$  for the under-category of  $B$  (for  $\mathfrak{C} = \mathfrak{C}_R/B$ , the under-category  $\mathfrak{C}\backslash B$  is  $\mathfrak{C}_B/B$ ), a cohomology theory  $h$  on  $\mathfrak{C}$  leads to a reduced cohomology theory  $\tilde{h}^*$  on  $\mathfrak{C}\backslash B$  with

$\tilde{h}^*(X) = h^*(X, B)$  and the suspension isomorphism (for  $X$  cofibrant) obtained from the connecting homomorphism  $h^n(X, B) \rightarrow h^{n+1}(CX, X)$  and the inverse of the excision isomorphism  $h^{n+1}(\Sigma X, B) \rightarrow h^{n+1}(CX, X)$ , where  $X \rightarrow CX$  is a Quillen cone. Conversely, given a reduced cohomology theory  $\tilde{h}^*$  on  $\mathfrak{C} \setminus B$ , we obtain a cohomology theory on  $\mathfrak{C}$  by setting  $h^n(X, A)$  to be  $\tilde{h}^*$  of the homotopy pushout  $B \cup_A X$ . In general, we have the following proposition:

**Proposition 1.2.** *Let  $\mathfrak{C}$  be a closed model category with final object  $B$ , and let  $B' \rightarrow B$  be a cofibrant approximation (an acyclic fibration with  $B'$  cofibrant). The following categories are equivalent:*

- (i) *The category of cohomology theories on  $\mathfrak{C}$ .*
- (ii) *The category of cohomology theories on  $\mathfrak{C}/B'$ .*
- (iii) *The category of reduced cohomology theories on  $(\mathfrak{C}/B') \setminus B'$ .*

*The analogous result holds for homology theories.*

The homotopy category  $\mathrm{Ho}[\mathfrak{C}_B/B]$  of  $\mathfrak{C}_B/B$  together with a skeleton of the full subcategory of finite cell commutative  $B$ -algebras over  $B$  satisfy the hypotheses of Brown [2, §2] for a “homotopy category”. Brown’s Abstract Representability Theorem [2, 2.8] therefore applies to show that certain functors are representable. The following is the relevant special case; a standard homological algebra argument (the Barratt–Whitehead “ladder” argument) reduces its hypotheses to those of the Representability Theorem.

**Proposition 1.3.** *Let  $h$  be a contravariant functor from  $\mathrm{Ho}[\mathfrak{C}_B/B]$  to abelian groups that satisfies hypotheses (i) and (ii) in the definition of reduced cohomology theory. Then there exists an object  $X_h$  in  $\mathrm{Ho}[\mathfrak{C}_B/B]$  and a natural isomorphism of functors  $h(-) \cong \mathrm{Ho}[\mathfrak{C}_B/B](-, X_h)$ .*

It follows that when  $h^*$  is a reduced cohomology theory on  $\mathfrak{C}_B/B$ , each functor  $h^n$  is representable by an object  $X_{h^n}$ . If we write  $E^{\mathbf{L}}$  for the left derived functor of  $E$ , the suspension functor on  $\mathrm{Ho}[\mathfrak{C}_B/B]$ , and  $\Omega^{\mathbf{R}}$  for its right adjoint, the loop functor on  $\mathrm{Ho}[\mathfrak{C}_B/B]$ , then the suspension isomorphism

$$\begin{aligned} \mathrm{Ho}[\mathfrak{C}_B/B](-, X_{h^n}) &\cong h^n(-) \longrightarrow h^{n+1}(E^{\mathbf{L}}-) \cong \mathrm{Ho}[\mathfrak{C}_B/B](E^{\mathbf{L}}-, X_{h^{n+1}}) \\ &\cong \mathrm{Ho}[\mathfrak{C}_B/B](-, \Omega^{\mathbf{R}}X_{h^{n+1}}) \end{aligned}$$

induces (by the Yoneda Lemma) an isomorphism (in  $\mathrm{Ho}[\mathfrak{C}_B/B]$ ),

$$X_{h^n} \longrightarrow \Omega^{\mathbf{R}}X_{h^{n+1}}.$$

This leads to the following definition:

**Definition 1.4.** Let  $\mathfrak{C}$  be a pointed closed model category. An Omega weak spectrum in  $\mathfrak{C}$  consists of objects  $X_0, X_1, \dots$ , and isomorphisms  $\tilde{\sigma}_n: X_n \rightarrow \Omega X_{n+1}$  in  $\mathrm{Ho}\mathfrak{C}$ , where  $\Omega$  denotes the (Quillen) loop functor on  $\mathrm{Ho}\mathfrak{C}$ . A map of Omega weak spectra from  $\underline{X} = \{X_n\}$  to  $\underline{Y} = \{Y_n\}$  consists of maps  $X_n \rightarrow Y_n$  in  $\mathrm{Ho}\mathfrak{C}$  that commute with the structure maps  $\tilde{\sigma}_n$ .

The rule  $h^n(-; \underline{X}) = \mathrm{Ho}[\mathfrak{C}_B/B](-, X_n)$  defines a functor from the category of Omega weak spectra to the category of reduced cohomology theories on  $\mathfrak{C}_B/B$ . The Yoneda Lemma implies that this functor is a full embedding, and the discussion above implies that every reduced cohomology theory is isomorphic to one associated to an Omega weak spectrum. In summary, we have the following proposition:

**Proposition 1.5.** *The category of reduced cohomology theories on  $\mathfrak{C}_B/B$  is equivalent to the category of Omega weak spectra in  $\mathfrak{C}_B/B$ .*

Finally, we need a general result about the relationship between Omega weak spectra and spectra in a simplicial or topological pointed model category  $\mathfrak{C}$ . In this context, when  $X$  is cofibrant, the tensor  $X \wedge I_+$  is a cylinder object, and so the tensor  $EX = X \wedge S^1$  represents the (Quillen) suspension functor on the homotopy category. Likewise, when  $Y$  is fibrant, the cotensor of  $Y$  with  $S^1$ ,  $\Omega Y$ , represents the (Quillen) loop functor on the homotopy category. A spectrum  $\underline{X}$  is defined as a sequence of objects  $X_0, X_1, \dots$  and maps  $\sigma: EX_n \rightarrow X_{n+1}$  in  $\mathfrak{C}$ . We say that a spectrum  $\underline{X}$  is “cofibrant” if each  $X_n$  is cofibrant and each structure map  $EX_n \rightarrow X_{n+1}$  is a cofibration. In the case of interest the cofibrant spectra in  $\mathfrak{C}_B/B$  are the cofibrant objects in the stable model structure (see Theorem 3.1 below); quite generally, the cofibrant spectra are the cofibrant objects in some model structure on the category of spectra, q.v. [6, 1.13–14].

We write  $\tilde{\sigma}$  for the adjoint of the structure map  $X_n \rightarrow \Omega X_{n+1}$ . A spectrum  $\underline{X}$  is called an Omega spectrum when each  $X_n$  is fibrant and each adjoint structure map is a weak equivalence. Then by neglect of structure (passing from  $\mathfrak{C}$  to  $\text{Ho } \mathfrak{C}$ ), an Omega spectrum becomes an Omega weak spectrum. The following lemma is the standard observation that every map of Omega weak spectra can be rectified to a map of spectra (though typically not uniquely).

**Lemma 1.6.** *Let  $\mathfrak{C}$  be a simplicial or topological pointed closed model category.*

- (i) *Every Omega weak spectrum is (non-canonically) isomorphic to the underlying Omega weak spectrum of a cofibrant Omega spectrum.*
- (ii) *Let  $\underline{X}$  and  $\underline{Y}$  be Omega spectra and suppose that  $\underline{X}$  is cofibrant. Any map of Omega weak spectra  $f: \underline{X} \rightarrow \underline{Y}$  is represented by a map of spectra (generally not uniquely).*

*Proof.* Given an arbitrary Omega weak spectrum  $\underline{X}$ , each  $X_n$  is isomorphic in  $\text{Ho } \mathfrak{C}$  to an object  $X'_n$  that is fibrant, and using these isomorphisms, we get an isomorphic Omega weak spectrum  $\underline{X}'$ . We choose a cofibrant Omega spectrum  $\underline{X}''$  as follows: We choose  $X''_0$  to be a cofibrant approximation of  $X'_0$ . Then since  $X''_0$  is cofibrant and  $\Omega X'_1$  is fibrant, we can choose a map  $X''_0 \rightarrow \Omega X'_1$  representing the composite map in  $\text{Ho } \mathfrak{C}$

$$X''_0 \longrightarrow X'_0 \longrightarrow \Omega X'_1.$$

We factor the adjoint map  $EX''_0 \rightarrow X'_1$  as a cofibration followed by an acyclic fibration

$$EX''_0 \longrightarrow X''_1 \longrightarrow X'_1.$$

Continuing in this fashion constructs a cofibrant Omega spectrum  $\underline{X}''$  and an isomorphism of Omega weak spectra  $\underline{X}'' \rightarrow \underline{X}'$ .

For (ii), since  $X_0$  is cofibrant and  $Y_0$  is fibrant, we can choose a map  $X_0 \rightarrow Y_0$  representing  $f_0$ . The hypothesis SM7 that  $\mathfrak{C}$  is a simplicial model category or the analogous hypothesis that  $\mathfrak{C}$  is a topological model category implies that the induced map of simplicial sets or of spaces

$$\mathfrak{C}(X_1, Y_1) \longrightarrow \mathfrak{C}(EX_0, Y_1)$$

is a fibration and identifies the induced map on components as  $\text{Ho } \mathfrak{C}(X_1, Y_1) \rightarrow \text{Ho } \mathfrak{C}(EX_0, Y_1)$ . Since the given map  $f_1$  in  $\text{Ho } \mathfrak{C}$  maps to the same component as

the map adjoint to the composite  $X_0 \rightarrow Y_0 \rightarrow \Omega Y_1$ , there exists a map  $X_1 \rightarrow Y_1$  in  $\mathfrak{C}$  that represents  $f_1$  and makes the diagram

$$\begin{array}{ccc} EX_0 & \longrightarrow & X_1 \\ & \searrow & \downarrow \\ & & Y_1 \end{array}$$

commute in  $\mathfrak{C}$ . Applying this argument inductively to each map  $X_n \rightarrow Y_n$  constructs a compatible map  $X_{n+1} \rightarrow Y_{n+1}$ , and the map of spectra  $\underline{X} \rightarrow \underline{Y}$ .  $\square$

## 2. NUCAS, INDECOMPOSABLES, AND STABILIZATION

The functors  $\mathbf{L}\Omega_A X$  and  $\mathbf{LAb}_A^B X$  mentioned in the introduction are somewhat awkward to work with formally because they are composites of both left and right derived functors. However, the technical trick introduced in [1] of working with non-unital commutative algebras (“nucas”) alleviates this problem by providing a point-set (left adjoint) functor whose left derived functor is a model for  $\mathbf{LAb}^B$ . The first half of this section consists of a brief overview of the theory of nucas from [1]. Another technical advantage of the category of nucas is that every object is fibrant, and this makes the construction of a “stabilization” functor  $\mathbb{S}$  from cofibrant spectra to Omega spectra easier. The second half of this section constructs this functor and explores its basic properties.

**Definition 2.1.** Let  $B$  be a commutative  $R$ -algebra. A non-unital commutative  $B$ -algebra (or  $B$ -nuca) consists of a  $B$ -module  $N$  together with an associative and commutative multiplication  $\mu: N \wedge_B N \rightarrow N$ . A map of  $B$ -nucas is a map of  $B$ -modules  $N \rightarrow N'$  that commutes with the multiplications. We write  $\mathfrak{N}_B$  for the category of  $B$ -nucas.

The category of  $B$ -nucas may also be described as the category of algebras in  $B$ -modules over the operad  $\mathcal{C}\widetilde{om}$  with  $\mathcal{C}\widetilde{om}(k) = *$  for  $k > 0$  and  $\mathcal{C}\widetilde{om}(0)$  empty. We have a free  $B$ -nuca functor

$$\mathbb{N}M = \bigvee_{k>0} M^{(k)}/\Sigma_k$$

for a  $B$ -module  $M$  (where  $M^{(k)} = M \wedge_B \cdots \wedge_B M$ ), and we have a functor  $K$  from  $\mathfrak{N}_B$  to  $\mathfrak{C}_B/B$  defined by formally adding a unit: On the underlying  $B$ -modules,

$$KN = B \vee N,$$

with the multiplication extended from  $N$  to  $KN$  by the usual multiplication on  $B$  and the  $B$ -action maps of  $B$  on  $N$ . The functor  $K$  is the left adjoint of the “augmentation ideal” functor  $I$  from  $\mathfrak{C}_B/B$  to  $\mathfrak{N}_B$  defined by setting  $I(A)$  to be the (point-set) fiber of the augmentation  $A \rightarrow B$ . According to Basterra [1, 1.1,2.2], the adjunction  $(K, I)$  is a Quillen equivalence.

**Proposition 2.2.** *The category  $\mathfrak{N}_B$  is a topological pointed closed model category with weak equivalences and fibrations the maps that are weak equivalences and fibrations (resp.) of the underlying  $B$ -modules. The adjunction  $(K, I)$  is a Quillen equivalence between  $\mathfrak{N}_B$  and  $\mathfrak{C}_B/B$ .*

Since  $K$  preserves all weak equivalences, it is harmless to use the same notation for the derived functor. We denote the right derived functor of  $I$  by  $I^{\mathbf{R}}$ . As in any Quillen adjunction, the derived functor  $I^{\mathbf{R}}$  preserves the Quillen loop functor, but since the derived functor  $K$  is an inverse equivalence, it also preserves the Quillen loop functor, and we obtain the following proposition:

**Proposition 2.3.** *The derived functors  $K$  and  $I^{\mathbf{R}}$  induce an equivalence between the category of Omega weak spectra in  $\mathfrak{N}_B$  and the category of Omega weak spectra in  $\mathfrak{C}_B/B$ .*

Basterra [1, §3] constructs an “indecomposables” functor  $Q$  from  $\mathfrak{N}_B$  to  $\mathfrak{M}_B$  that is left adjoint to the “zero multiplication” functor  $Z$  from  $\mathfrak{M}_B$  to  $\mathfrak{N}_B$ . Precisely, the indecomposables functor  $QN$  is defined as the (point-set) pushout

$$QN = * \cup_{(N \wedge_B N)} N$$

in the category of  $B$ -modules of the multiplication  $N \wedge_B N \rightarrow N$  over the trivial map  $N \wedge_B N \rightarrow *$ . The functor  $Z$  simply assigns a  $B$ -module  $M$  the trivial map  $M \wedge_B M \rightarrow M$  for its multiplication. Since  $Z$  clearly preserves weak equivalences and fibrations, we have the following observation of [1]:

**Proposition 2.4.** *The functors  $(Q, Z)$  between  $\mathfrak{N}_B$  and  $\mathfrak{M}_B$  form a Quillen adjunction.*

Again, since  $Z$  preserves all weak equivalences, it is harmless to denote its derived functor by the same notation. We write  $Q^{\mathbf{L}}$  for the left derived functor of  $Q$ .

The derived functors  $Q^{\mathbf{L}}$  and  $I^{\mathbf{R}}$  are needed in [1, 4.1] to defined the cotangent complex. For a cofibrant commutative  $R$ -algebra  $A$  and a cofibration of commutative  $R$ -algebras  $A \rightarrow X$ , the cotangent complex of  $X$  relative to  $A$  is defined to be the  $X$ -module

$$\mathbf{L}\Omega_A X = Q^{\mathbf{L}} I^{\mathbf{R}}(X \wedge_A X),$$

where  $I^{\mathbf{R}}$  and  $Q^{\mathbf{L}}$  are understood in terms of  $\mathfrak{C}_X/X$  and  $\mathfrak{N}_X$ . When  $A$  is not cofibrant or  $A \rightarrow X$  is not a cofibration, the cotangent complex is constructed by choosing a cofibrant approximation  $A' \rightarrow X'$  of  $A \rightarrow X$ , and then using the derived extension of scalars functor  $X \wedge_{X'}^{\mathbf{L}} (-)$ :

$$\mathbf{L}\Omega_A X = X \wedge_{X'}^{\mathbf{L}} \mathbf{L}\Omega_{A'} X' = X \wedge_{X'}^{\mathbf{L}} (Q^{\mathbf{L}} I^{\mathbf{R}}(X' \wedge_{A'} X')).$$

Because the category this construction lands in depends on its inputs, a discussion of functoriality would be somewhat involved, but (as mentioned in the introduction), for  $(X, A)$  a pair in  $\mathfrak{C}_R/B$ , the construction we consider,

$$\mathbf{LAb}_A^B X = B \wedge_{X'}^{\mathbf{L}} \mathbf{L}\Omega_{A'} X'$$

assembles to a functor from the category of pairs in  $\mathfrak{C}_R/B$  to the homotopy category of  $B$ -modules, and this functoriality suffices for our work. The last fact we need about the cotangent complex is the following version of [1, 4.4–5]:

**Proposition 2.5.** *Assume that  $B$  is a cofibrant commutative  $R$ -algebra. If  $A$  is cofibrant and  $A \rightarrow X$  is a cofibration in  $\mathfrak{C}_R/B$ , then  $\mathbf{LAb}_A^B X$  is isomorphic to  $Q^{\mathbf{L}} I^{\mathbf{R}}(B \wedge_A X)$ , naturally in  $A$  and cofibrations  $A \rightarrow X$ .*

Next we move on to the stabilization functor. In other contexts, this functor is typically denoted by  $Q$ , but here we denote it by  $\mathbb{S}$  to avoid confusion with the indecomposables functor. We have the notion of a spectrum in the category of

$B$ -nucas, as discussed in the previous section. The stabilization functor  $\mathbb{S}$  turns out to be a functor from spectra in  $\mathfrak{N}_B$  to Omega spectra in  $\mathfrak{N}_B$ .

**Definition 2.6.** For a  $\mathfrak{N}_B$ -spectrum  $\underline{X}$ , we define a  $\mathfrak{N}_B$ -spectrum  $\mathbb{S}\underline{X} = \{S_n\underline{X}\}$  as follows: Let

$$S_n\underline{X} = \mathrm{Tel}_{k \geq n} \Omega^{k-n} X_k,$$

the telescope over the adjoint structure maps. We have a map of telescopes

$$\mathrm{Tel}_{k \geq n+1} E\Omega^{k-n} X_k \longrightarrow \mathrm{Tel}_{k \geq n+1} \Omega^{k-(n+1)} X_k$$

induced by sending  $E\Omega^{k-n} X_k$  to  $\Omega^{k-(n+1)} X_k$  using the counit of the suspension, loop adjunction applied to the innermost factor of  $\Omega$ , and we have a map of telescopes

$$\mathrm{Tel}_{k \geq n} E\Omega^{k-n} X_k \longrightarrow \mathrm{Tel}_{k \geq n+1} E\Omega^{k-n} X_k$$

induced by collapsing down the map  $EX_n \rightarrow E\Omega X_{n+1}$ . We define the structure map  $\sigma: ES_n\underline{X} \rightarrow S_{n+1}\underline{X}$  to be the composite

$$\begin{aligned} E(\mathrm{Tel}_{k \geq n} \Omega^{k-n} X_k) &\cong \mathrm{Tel}_{k \geq n} E\Omega^{k-n} X_k \longrightarrow \\ &\mathrm{Tel}_{k \geq n+1} E\Omega^{k-n} X_k \longrightarrow \mathrm{Tel}_{k \geq n+1} \Omega^{k-(n+1)} X_k. \end{aligned}$$

The construction  $\mathbb{S}$  assembles in the obvious way to a functor from  $\mathfrak{N}_B$ -spectra to  $\mathfrak{N}_B$ -spectra. Since the composite map

$$EX_n \xrightarrow{E\tilde{\sigma}} E\Omega X_{n+1} \longrightarrow X_{n+1}$$

is the structure map  $\sigma$ , the inclusion of  $X_n$  into the telescope defining  $S_n\underline{X}$  defines a natural transformation  $\underline{X} \rightarrow \mathbb{S}\underline{X}$ . The main fact we need about  $\mathbb{S}$  is the following proposition. In it, the homotopy groups of a spectrum  $\underline{X}$  are defined by

$$\pi_q \underline{X} = \mathrm{Colim} \pi_{q+n} X_n.$$

**Proposition 2.7.** *For any spectrum  $\underline{X}$  in  $\mathfrak{N}_B$ ,  $\mathbb{S}\underline{X}$  is an Omega spectrum in  $\mathfrak{N}_B$  and the natural transformation  $\underline{X} \rightarrow \mathbb{S}\underline{X}$  induces an isomorphism on homotopy groups.*

*Proof.* The telescope in the category of  $B$ -nucas is naturally weakly equivalent to the telescope in the category of  $B$ -modules, and so the usual map

$$\Omega(\mathrm{Tel}_{k \geq n+1} \Omega^{k-(n+1)} X_k) \longrightarrow \mathrm{Tel}_{k \geq n+1} \Omega^{k-n} X_k$$

is a weak equivalence. The composite of the adjoint structure map and the map above,

$$\mathrm{Tel}_{k \geq n} \Omega^{k-n} X_k \longrightarrow \Omega(\mathrm{Tel}_{k \geq n+1} \Omega^{k-(n+1)} X_k) \longrightarrow \mathrm{Tel}_{k \geq n+1} \Omega^{k-n} X_k$$

is a homotopy equivalence, and so the adjoint structure map  $S_n\underline{X} \rightarrow \Omega S_{n+1}\underline{X}$  is a weak equivalence. Thus  $\mathbb{S}\underline{X}$  is an Omega spectrum. We have

$$\pi_{q+n} S_n \underline{X} \cong \mathrm{Colim}_{k \geq n} \pi_{q+k} X_k,$$

and the map  $X_n \rightarrow S_n \underline{X}$  induces on homotopy groups the inclusion of  $\pi_{q+n} X_n$  into this colimit system. Under this identification, the map  $\underline{X} \rightarrow \mathbb{S}\underline{X}$  induces on homotopy groups the map

$$\mathrm{Colim}_n \pi_{q+n} X_n \longrightarrow \mathrm{Colim}_n \mathrm{Colim}_{k \geq n} \pi_{q+k} X_k,$$

which is clearly an isomorphism.  $\square$

For a  $B$ -nuca  $N$ , let  $E^\infty N$  be the ‘‘suspension spectrum’’ which has as its  $n$ -th object the  $n$ -fold suspension  $E^n N$  and structure maps the identity map

$$E(E^n N) \longrightarrow E^{n+1} N.$$

In the case when  $N$  is free, i.e.,  $N = \mathbb{N}M$  for some  $B$ -module  $M$ , we have a canonical isomorphism

$$E^n \mathbb{N}M \cong \mathbb{N} \Sigma^n M,$$

and in particular, we have canonical maps  $\Sigma^n M \rightarrow S_n E^\infty \mathbb{N}M$  for all  $n$ . The following result on the suspension spectra of free  $B$ -nucas represents the fundamental difference between the context of commutative  $S$ -algebras and simplicial commutative algebras.

**Theorem 2.8.** *Assume that  $B$  is a cofibrant commutative  $R$ -algebra. If  $M$  is a cofibrant  $B$ -module, then the canonical maps  $\Sigma^n M \rightarrow S_n E^\infty \mathbb{N}M$  are weak equivalences for all  $n$ .*

*Proof.* The general case follows from the case  $n = 0$ , where we are studying the map

$$\pi_q \mathbb{N}M \cong \pi_{q+k} \Sigma^k \mathbb{N}M \longrightarrow \pi_{q+k} \mathbb{N} \Sigma^k M.$$

The map  $\Sigma^k \mathbb{N}M \rightarrow \mathbb{N} \Sigma^k M$  takes the wedge summand  $\Sigma^k M^{(m)}/\Sigma_m$  to the corresponding wedge summand  $(\Sigma^k M)^{(m)}/\Sigma_m$  via the diagonal map on the sphere  $S^k$ . The proposition is an immediate consequence of the following lemma.  $\square$

**Lemma 2.9.** *Let  $B$  be a cofibrant commutative  $R$ -algebra, let  $M$  be a cofibrant  $B$ -module, and let  $x$  be an element of  $\pi_q(M^{(m)}/\Sigma_m)$  for some  $m > 1$  and some integer  $q$ . Then for some  $k$ , the composite map*

$$\pi_q(M^{(m)}/\Sigma_m) \cong \pi_{q+k} \Sigma^k(M^{(m)}/\Sigma_m) \longrightarrow \pi_{q+k}((\Sigma^k M)^{(m)}/\Sigma_m)$$

*sends  $x$  to zero.*

*Proof.* By [5, III.5.1], for any cofibrant  $B$ -module  $N$  (e.g.,  $M$ ,  $\Sigma^k M$ ), the map

$$E\Sigma_{m+} \wedge_{\Sigma_m} N^{(m)} \longrightarrow N^{(m)}/\Sigma_m$$

is a weak equivalence. The cellular filtration of the  $\Sigma_m$ -CW complex  $E\Sigma_m$  induces an increasing filtration on  $E\Sigma_{m+} \wedge_{\Sigma_m} N^{(m)}$  and on the homotopy groups of  $N^{(m)}/\Sigma_m$ ; clearly, this filtration is trivial (zero) below the zero level. The map

$$\delta: E\Sigma_{m+} \wedge_{\Sigma_m} \Sigma M^{(m)} \longrightarrow E\Sigma_{m+} \wedge_{\Sigma_m} (\Sigma M)^{(m)}$$

(induced by the diagonal  $S^1 \rightarrow (S^1)^{(m)}$ ) preserves the filtration. Since  $m > 1$ , the map  $\Sigma M^{(m)} \rightarrow (\Sigma M)^{(m)}$  is null homotopic, and so  $\delta$  induces the zero map on the  $E^1$ -term of the homotopy group spectral sequence associated to the filtration. It follows that the map on homotopy groups induced by  $\delta$  strictly lowers filtration level. The map  $\pi_q \delta^k$  is the map

$$\pi_q(M^{(m)}/\Sigma_m) \cong \pi_{q+k} \Sigma^k(M^{(m)}/\Sigma_m) \longrightarrow \pi_{q+k}((\Sigma^k M)^{(m)}/\Sigma_m)$$

in the statement, which therefore takes every element in filtration level  $n$  to an element of filtration level  $n - k$ . Taking  $k$  greater than the minimum filtration level of  $x$ , the map must send  $x$  to zero.  $\square$

We can illustrate the previous lemma and theorem in the case  $R = S$  and  $B = H\mathbb{F}_2$  since in this case,  $\pi_*\mathbb{N}M$  is easy to describe in terms of  $\pi_*M$ . Specifically,  $\pi_*\mathbb{N}M$  is the polynomial  $\mathbb{F}_2$ -nuca on the free allowable Dyer-Lashof (“DL”) module on  $\pi_*M$  modulo the relation that the square (in the algebra structure) is equal to the squaring operation (in the DL structure) on each element. The suspension map  $\sigma: \pi_*\mathbb{N}M \rightarrow \pi_{*+1}\mathbb{N}\Sigma M$  kills decomposables (in the algebra structure) and is a map of DL-modules. An element of the form  $Q_s x$  for  $x \in \pi_q\mathbb{N}M$  is therefore killed by the map  $\sigma^k: \pi_q\mathbb{N}M \rightarrow \pi_{q+k}\mathbb{N}\Sigma^k M$  for  $k = s - q + 1$  because  $\sigma^{k-1}(Q_s x) = Q_s(\sigma^{k-1}x) = (\sigma^{k-1}x)^2$  is decomposable. From this it is easy to see directly that the map  $\pi_*M \rightarrow \text{Colim } \pi_{*+k}\mathbb{N}\Sigma^k M$  is an isomorphism.

### 3. PROOF OF THEOREMS 3 AND 4

In this section, we prove Theorems 3 and 4 of the introduction. The arguments take advantage of the technical simplifications the category of  $B$ -nucas provide and use Theorem 2.8 above as the key step. They also make use of the following theorem, proved in Section 7:

**Theorem 3.1.** *Let  $\mathfrak{C}$  be  $\mathfrak{M}_B$ ,  $\mathfrak{N}_B$ , or  $\mathfrak{C}_B/B$ . Then the category  $\text{Sp}(\mathfrak{C})$  of  $\mathfrak{C}$ -spectra is a topological closed model category with:*

- (i) *Cofibrations the maps  $\underline{X} \rightarrow \underline{Y}$  with  $X_0 \rightarrow Y_0$  a cofibration and each  $EY_n \cup_{EX_n} X_{n+1} \rightarrow Y_{n+1}$  a cofibration,*
- (ii) *Fibrations the maps  $\underline{X} \rightarrow \underline{Y}$  with each  $X_n \rightarrow Y_n$  a fibration and each  $X_n \rightarrow Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$  a weak equivalence, and*
- (iii) *Weak equivalences the maps that induce an isomorphism on homotopy groups.*

In the statement, “E” and “ $\Omega$ ” denote the (point-set) tensor and cotensor with the based space  $S^1$  in the pointed topological category  $\mathfrak{C}$ , and “cofibration” means a cofibration in the model structure on  $\mathfrak{C}$  (which was called a “q-cofibration” in [1] and [5]). We have defined homotopy groups for  $\mathfrak{C}_B/B$ -spectra and  $\mathfrak{N}_B$ -spectra above, and the definition is the same for  $\mathfrak{M}_B$ -spectra:

$$\pi_q \underline{X} = \text{Colim } \pi_{q+n} X_n.$$

To avoid confusion, we use the term “stable equivalence” for weak equivalence in the model structure on  $\text{Sp}(\mathfrak{C})$  above, and we call the homotopy category of this model structure the “stable category” of  $\mathfrak{C}$ .

We have the following easy consequence of the characterization of fibrations:

**Proposition 3.2.** *An object is fibrant in one of the model structures in Theorem 3.1 if and only if it is an Omega spectrum.*

The following proposition is also clear:

**Proposition 3.3.** *A map of Omega spectra  $\underline{X} \rightarrow \underline{Y}$  is a stable equivalence if and only if it is a weak equivalence  $X_0 \rightarrow Y_0$ .*

For any of the model categories  $\mathfrak{C}$  in Theorem 3.1, we have a Quillen adjunction between  $\mathfrak{C}$  and the category  $\text{Sp}(\mathfrak{C})$  of  $\mathfrak{C}$ -spectra with left adjoint the suspension spectrum functor (that sends an object  $X$  to the suspension spectrum  $E^\infty X = \{E^n X\}$ ) and the zeroth object functor (that sends a spectrum  $\underline{X}$  to the zeroth object  $X_0$ ). The previous propositions applied to  $\mathfrak{C} = \mathfrak{M}_B$  imply that this is a Quillen equivalence.

**Proposition 3.4.** *Let  $\mathfrak{C}$  be one of the categories in Theorem 3.1. The suspension spectrum functor and the zeroth object functor are a Quillen adjunction between the model category  $\mathfrak{C}$  and the model category  $\mathrm{Sp}(\mathfrak{C})$  of  $\mathfrak{C}$ -spectra. In the case of  $\mathfrak{C} = \mathfrak{M}_B$ , this Quillen adjunction is a Quillen equivalence.*

If we consider any adjunction that is enriched over the category of based spaces, the left adjoint preserves tensors and the right adjoint preserves cotensors, and so both functors extend to functors between the categories of spectra. It is easy to see that the induced functors on spectra remain adjoints. When in addition the categories are ones considered in Theorem 3.1 and the adjunction is a Quillen adjunction, the characterization of the cofibrations and fibrations in the categories of spectra imply that the induced adjunction on spectra is a Quillen adjunction.

When we combine the previous proposition with the observations of the last paragraph applied to the free, forgetful adjunction  $\mathfrak{M}_B \rightleftarrows \mathfrak{N}_B$  and to the  $K, I$  adjunction  $\mathfrak{N}_B \rightleftarrows \mathfrak{C}_B/B$ , we have the following sequence of Quillen adjunctions:

$$(3.5) \quad \mathfrak{M}_B \begin{array}{c} \xrightarrow{\Sigma^\infty} \\ \xleftarrow{(-)_0} \end{array} \mathrm{Sp}(\mathfrak{M}_B) \begin{array}{c} \xrightarrow{\mathbb{N}} \\ \xleftarrow{\quad} \end{array} \mathrm{Sp}(\mathfrak{N}_B) \begin{array}{c} \xleftarrow{K} \\ \xrightarrow{I} \end{array} \mathrm{Sp}(\mathfrak{C}_B/B)$$

The first and last of these Quillen adjunctions are Quillen equivalences, and we prove that the middle one is also a Quillen equivalence.

**Lemma 3.6.** *Assume that  $B$  is a cofibrant commutative  $R$ -algebra. Then the free, forgetful adjunction between  $\mathfrak{M}_B$ -spectra and  $\mathfrak{N}_B$ -spectra is a Quillen equivalence.*

*Proof.* Let  $M$  be a cofibrant  $B$ -module and let  $\underline{X}$  be a spectrum in  $\mathfrak{N}_B$  that is fibrant in the model structure above. It suffices to show that a map  $\Sigma^\infty M \rightarrow \underline{X}$  is a stable equivalence if and only if the adjoint map  $E^\infty \mathbb{N}M \rightarrow \underline{X}$  is a weak equivalence. By Proposition 2.7 and Propositions 3.2 and 3.3 above, it suffices to show that  $M \rightarrow X_0$  is a weak equivalence if and only if  $S_0 E^\infty \mathbb{N}M \rightarrow S_0 \underline{X}$  is a weak equivalence. The diagram

$$\begin{array}{ccc} M & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ S_0 E^\infty \mathbb{N}M & \longrightarrow & S_0 \underline{X} \end{array}$$

in  $\mathfrak{M}_B$  commutes, and the result follows from Theorem 2.8.  $\square$

As an immediate consequence, we obtain a Quillen equivalence between the category of  $B$ -modules and the category of  $\mathfrak{C}_B/B$ -spectra. This is sufficient to prove Theorem 3 as stated, but we would like to know that the equivalence of homotopy categories is induced by the functor that sends a  $B$ -module  $M$  to the  $\mathfrak{C}_B/B$ -spectrum

$$ZM = \{KZ\Sigma^n M\}$$

that represents André–Quillen cohomology with coefficients in  $M$ . Also, for Theorem 4, we need to know that the equivalence of homotopy categories takes the suspension spectrum  $E^\infty(B \wedge_R A)$  of a cofibrant object  $A$  of  $\mathfrak{C}_R/B$  to the  $B$ -module  $\mathbf{LAb}_R^B A$ . Both of these are consequences of the following theorem.

**Theorem 3.7.** *Let  $B$  be a cofibrant commutative  $R$ -algebra. Then the  $\underline{Q}, \underline{Z}$  adjunction between  $\mathrm{Sp}(\mathfrak{N}_B)$  and  $\mathrm{Sp}(\mathfrak{M}_B)$  is a Quillen equivalence.*

*Proof.* Since the Quillen adjunction  $Q, Z$  between  $\mathfrak{N}_B$  and  $\mathrm{Sp}(\mathfrak{M}_B)$  is enriched over the category of based spaces, as observed above, we obtain a Quillen adjunction  $\underline{Q}, \underline{Z}$  between the categories of spectra. According to MMSS [10, A.2.(ii)], for this to be a Quillen equivalence, we just need to show that one of the derived functors is an equivalence on the homotopy categories. If we write  $\underline{Q}^{\mathbf{L}}$  for the left derived functor of  $\underline{Q}: \mathrm{Sp}(\mathfrak{N}_B) \rightarrow \mathrm{Sp}(\mathfrak{M}_B)$  and  $\underline{N}^{\mathbf{L}}$  for the left derived functor of  $\underline{N}: \mathrm{Sp}(\mathfrak{M}_B) \rightarrow \mathrm{Sp}(\mathfrak{N}_B)$ , then the composite functor  $\underline{Q}^{\mathbf{L}} \circ \underline{N}^{\mathbf{L}}$  is naturally isomorphic to the derived functor of the composite and so is naturally isomorphic to the identity functor on  $\mathrm{Sp}(\mathfrak{M}_B)$ . By the previous lemma,  $\underline{N}^{\mathbf{L}}$  is an equivalence, and it follows that  $\underline{Q}^{\mathbf{L}}$  is an equivalence. We conclude that  $\underline{Q}, \underline{Z}$  is a Quillen equivalence.  $\square$

In the course of the previous argument, we also proved the following result:

**Proposition 3.8.** *If  $B$  is a cofibrant commutative  $R$ -algebra, then the derived functors of  $\underline{N}$  and  $\underline{Z}$  are naturally isomorphic functors  $\mathrm{HoSp}(\mathfrak{M}_B) \rightarrow \mathrm{HoSp}(\mathfrak{N}_B)$ , and the derived functors of  $\underline{Q}$  and the forgetful functor are naturally isomorphic functors  $\mathrm{HoSp}(\mathfrak{N}_B) \rightarrow \mathrm{HoSp}(\mathfrak{M}_B)$ .*

We can be more explicit about these natural isomorphisms. The first is induced by the natural transformation of point-set functors  $\underline{N} \rightarrow \underline{Z}$  that arises from the universal property of the free functor. When  $\underline{M}$  is a cofibrant object in  $\mathrm{Sp}(\mathfrak{M}_B)$ , the map  $\underline{NM} \rightarrow \underline{ZM}$  is a stable equivalence because the composite map

$$\underline{M} \longrightarrow \underline{NM} \longrightarrow \underline{ZM}$$

in  $\mathrm{Sp}(\mathfrak{M}_B)$  is the identity and the map  $\underline{M} \rightarrow \underline{NM}$  is a stable equivalence by Lemma 3.6. The other natural transformation is induced by the unit of the  $Q, Z$  adjunction, specifically, the natural transformation  $\mathrm{Id} \rightarrow \underline{Q}$  in  $\mathrm{Sp}(\mathfrak{M}_B)$ . This map is hard to study directly; instead, the argument implicit in the proof of Theorem 3.7 studies the (solid) zigzag

$$\begin{array}{ccccc} \underline{X}' & \longrightarrow & \underline{NX}' & \longrightarrow & \underline{QNX}' \\ & \searrow & \vdots & & \downarrow \\ & & \underline{X} & \cdots\cdots\cdots & \underline{QX} \end{array}$$

where  $\underline{X}$  is a cofibrant object in  $\mathrm{Sp}(\mathfrak{N}_B)$ , and  $\underline{X}' \rightarrow \underline{X}$  is a cofibrant approximation in  $\mathrm{Sp}(\mathfrak{M}_B)$ . The diagonal solid arrow is therefore a stable equivalence by assumption, and the composite horizontal map  $\underline{X}' \rightarrow \underline{QNX}'$  is the isomorphism (explicit) in the proof of the theorem. The remaining solid arrows are also stable equivalences: The top one  $\underline{X}' \rightarrow \underline{NX}'$  is a stable equivalence by the lemma and the right-hand one is a stable equivalence because, as a Quillen left adjoint,  $\underline{Q}$  preserves stable equivalences between cofibrant objects. This is the concrete argument that shows that the natural map  $\underline{X} \rightarrow \underline{QX}$  in  $\mathrm{Sp}(\mathfrak{M}_B)$  is a stable equivalence for  $\underline{X}$  cofibrant in  $\mathrm{Sp}(\mathfrak{N}_B)$ .

*Proof of Theorems 3 and 4.* The equivalence of the stable category of  $\mathfrak{C}_B/B$  and the homotopy category of  $B$ -modules we need for Theorem 3 follows from the three Quillen equivalences in (3.5), or better, by the outer two in (3.5) and the one in Theorem 3.7. Theorem 4 then follows by Proposition 2.5.  $\square$

## 4. PROOF OF THEOREM 1

In this section, we prove Theorem 1 from the introduction. By Propositions 1.2, 1.5, and 2.3, it suffices to consider the case when  $B$  is a cofibrant commutative  $R$ -algebra and show that the functor that sends a  $B$ -module  $M$  to the Omega weak spectrum in  $\mathfrak{N}_B$

$$\underline{Z}\Sigma^\infty M = \{Z\Sigma^n M\}$$

gives an equivalence between the homotopy category of  $B$ -modules and the category of Omega weak spectra in  $\mathfrak{N}_B$ . We write  $W(\mathfrak{N}_B)$  for the category of Omega weak spectra in  $\mathfrak{N}_B$ .

The work of the previous section suggests that we should be able to use the forgetful functor on the zeroth object as the inverse equivalence  $W(\mathfrak{N}_B) \rightarrow \text{Ho}\mathfrak{M}_B$ . The composite functor  $\text{Ho}\mathfrak{M}_B \rightarrow \text{Ho}\mathfrak{M}_B$  is the identity. The composite functor  $W(\mathfrak{N}_B) \rightarrow W(\mathfrak{N}_B)$  sends  $\underline{X}$  to  $\underline{Z}\Sigma^\infty X_0$ ; the proof of Theorem 1 is completed by showing that this functor is naturally isomorphic to the identity.

To construct an isomorphism in  $W(\mathfrak{N}_B)$  between  $\underline{X}$  and  $\underline{Z}\Sigma^\infty X_0$ , we have to take a detour through the category of spectra in  $\mathfrak{N}_B$ . To avoid confusion, we write  $W\underline{A}$  for the underlying Omega weak spectrum of an Omega spectrum  $\underline{A}$ . First we choose a cofibrant Omega spectrum  $\underline{A}_X$  and an isomorphism of Omega weak spectra  $W\underline{A}_X \rightarrow \underline{X}$  as in Lemma 1.6.(i). We write  $A_0\underline{X}$  for the zeroth object of  $\underline{A}_X$ , and we choose a cofibrant approximation  $M_X \rightarrow A_0\underline{X}$  in the category of  $B$ -modules. Then we have the following chain of stable equivalences in  $\text{Sp}(\mathfrak{N}_B)$ :

$$\underline{A}_X \longleftarrow \underline{N}\Sigma^\infty M_X \longrightarrow \underline{Z}\Sigma^\infty M_X \longrightarrow \underline{Z}\Sigma^\infty A_0\underline{X}.$$

(See the explanation following Proposition 3.8 for a proof that the first two maps are stable equivalences.) Using the functor  $\mathbb{S}$ , we then have the following chain of stable equivalences of Omega spectra:

$$\underline{A}_X \longrightarrow \mathbb{S}\underline{A}_X \longleftarrow \mathbb{S}(\underline{N}\Sigma^\infty M_X) \longrightarrow \mathbb{S}(\underline{Z}\Sigma^\infty A_0\underline{X}) \longleftarrow \underline{Z}\Sigma^\infty A_0\underline{X}.$$

Finally, applying  $W$ , we have the following chain of isomorphisms of Omega weak spectra:

$$\begin{aligned} \underline{X} \cong W\underline{A}_X &\longrightarrow W\mathbb{S}\underline{A}_X \longleftarrow W\mathbb{S}(\underline{N}\Sigma^\infty M_X) \longrightarrow \\ &W\mathbb{S}(\underline{Z}\Sigma^\infty A_0\underline{X}) \longleftarrow W\underline{Z}\Sigma^\infty A_0\underline{X} \cong \underline{Z}\Sigma^\infty X_0. \end{aligned}$$

Let  $\phi_X: \underline{X} \rightarrow \underline{Z}\Sigma^\infty X_0$  be the composite isomorphism.

The isomorphism  $\phi_X$  appears to depend on the choices made above, and it is far from obvious that  $\phi$  is a natural transformation. Let  $f: \underline{X} \rightarrow \underline{Y}$  be a map of Omega weak spectra; we must show that the diagram

$$\begin{array}{ccc} \underline{X} & \xrightarrow{\phi_X} & \underline{Z}\Sigma^\infty X_0 \\ f \downarrow & & \downarrow f_0 \\ \underline{Y} & \xrightarrow{\phi_Y} & \underline{Z}\Sigma^\infty Y_0 \end{array}$$

commutes in the category of Omega weak spectra. According to Lemma 1.6.(ii), we can choose a map of spectra  $\underline{A}_X \rightarrow \underline{A}_Y$  whose underlying map of Omega weak spectra is the composite

$$\underline{A}_X \cong \underline{X} \xrightarrow{f} \underline{Y} \cong \underline{A}_Y.$$

Likewise, since  $M_{\underline{Y}} \rightarrow A_0\underline{Y}$  is an acyclic fibration, we can choose a map of  $B$ -modules  $M_{\underline{X}} \rightarrow M_{\underline{Y}}$  making the diagram

$$\begin{array}{ccc} M_{\underline{X}} & \longrightarrow & A_0\underline{X} \\ \downarrow & & \downarrow \\ M_{\underline{Y}} & \longrightarrow & A_0\underline{Y} \end{array}$$

commute in  $\mathfrak{M}_B$ . Then the following diagram commutes in  $W(\mathfrak{N}_B)$ :

$$\begin{array}{cccccccccccc} \underline{X} & \longleftarrow & W\underline{A}_X & \longrightarrow & WS\underline{A}_X & \longleftarrow & WS(\mathbb{N}\Sigma^\infty M_{\underline{X}}) & \longrightarrow & WS(\mathbb{Z}\Sigma^\infty A_0\underline{X}) & \longleftarrow & W\underline{\mathbb{Z}\Sigma^\infty A_0\underline{X}} & \longrightarrow & \underline{\mathbb{Z}\Sigma^\infty X_0} \\ \downarrow f & & \downarrow f_0 \\ \underline{Y} & \longleftarrow & W\underline{A}_Y & \longrightarrow & WS\underline{A}_Y & \longleftarrow & WS(\mathbb{N}\Sigma^\infty M_{\underline{Y}}) & \longrightarrow & WS(\mathbb{Z}\Sigma^\infty A_0\underline{Y}) & \longleftarrow & W\underline{\mathbb{Z}\Sigma^\infty A_0\underline{Y}} & \longrightarrow & \underline{\mathbb{Z}\Sigma^\infty Y_0} \end{array}$$

The required naturality now follows. (The same argument applied to the identity map on  $\underline{X}$  shows that  $\phi_{\underline{X}}$  is in fact independent of the choices.) This completes the proof of Theorem 1.

## 5. PROOF OF THEOREM 2

In this section we prove Theorem 2. By Proposition 1.2, it suffices to consider the case when  $B$  is a cofibrant commutative  $R$ -algebra, and by Proposition 2.2 it suffices to prove the analogous theorem for reduced homology theories on the category  $\mathfrak{N}_B$ . The work of Section 3 reduces this to proving the following theorem:

**Theorem 5.1.** *The category of reduced homology theories on  $\mathfrak{N}_B$  is equivalent to the category of reduced homology theories on  $\mathrm{Sp}(\mathfrak{N}_B)$ .*

In this equivalence, the functor in one direction sends the reduced homology theory  $k_*$  in  $\mathrm{Sp}(\mathfrak{N}_B)$  to the theory  $k_*^E$  on  $\mathfrak{N}_B$  defined by

$$k_*^E(N) = k_*(E^{L^\infty}N).$$

where  $E^{L^\infty}N$  denotes the left derived functor of the suspension spectrum functor  $E^\infty$ . On the other hand, for a reduced homology theory  $h_*$  on  $\mathfrak{N}_B$ , we define a functor  $h_*^c$  from  $\mathrm{Sp}(\mathfrak{N}_B)$  to graded abelian groups by

$$h_q^c(\underline{X}) = \mathrm{Colim} h_{q+n}X_n.$$

**Lemma 5.2.** *For any  $\underline{X}$  in  $\mathrm{Sp}(\mathfrak{N}_B)$ , the map  $h_*^c(\underline{X}) \rightarrow h_*^c(\mathbb{S}\underline{X})$  is an isomorphism.*

*Proof.* The map  $h_q^c\underline{X} \rightarrow h_q^c\mathbb{S}\underline{X}$  is the map

$$\begin{aligned} \mathrm{Colim}_n h_{q+n}X_n &\longrightarrow \mathrm{Colim}_n h_{q+n}(\mathrm{Tel}_{j \geq n} \Omega^{j-n}X_j) \\ &\cong \mathrm{Colim}_n \mathrm{Colim}_{j \geq n} h_{q+n}(\Omega^{j-n}X_j). \end{aligned}$$

To see that this is an isomorphism, it suffices to show that the map

$$\mathrm{Colim}_n h_{q+n}X_n \longrightarrow \mathrm{Colim}_n h_{q+n}(\Omega^i X_{n+i})$$

is an isomorphism for all  $i$ . The maps

$$h_{q+n}(\Omega^i X_{n+i}) \cong h_{q+n+i}(E^{L^i}\Omega^i X_{n+i}) \longrightarrow h_{q+n+i}X_{n+i}$$

induce a map

$$\mathrm{Colim}_n h_{q+n}(\Omega^i X_{n+i}) \longrightarrow \mathrm{Colim}_n h_{q+n}X_n$$

inverse to the map above.  $\square$

It follows that  $h_*^c$  sends stable equivalences to isomorphisms and therefore induces a functor from the stable category  $\text{HoSp}(\mathfrak{N}_B)$  to the category of graded abelian groups. The suspension isomorphism for  $h_*$  induces a suspension isomorphism for  $h_*^c$ , and the Direct Sum Axiom for  $h_*$  implies the Direct Sum Axiom for  $h_*^c$ . This then defines a functor from the category of homology theories on  $\mathfrak{N}_B$  to the category of homology theories on  $\text{Sp}(\mathfrak{N}_B)$ .

In order to prove Theorem 5.1, it suffices to prove that these functors  $(-)^c$  and  $(-)^E$  are inverse equivalences. It is clear that when we start with a homology theory  $h_*$  on  $\mathfrak{N}_B$ , we have a natural isomorphism between  $h_*$  and the composite functor  $h_*^{cE}$ . To produce the natural isomorphism between the other composite and the identity, we use the following construction:

**Definition 5.3.** For a spectrum  $\underline{X}$ , let  $T_n \underline{X}$  be the spectrum with  $j$ -th object  $X_j$  for  $j \leq n$  and  $E^{j-n} X_n$  for  $j > n$ .

We have compatible natural maps  $T_n \underline{X} \rightarrow T_{n+1} \underline{X}$ , with  $\underline{X}$  the colimit. More usefully for the work below, the map  $\text{Tel } T_n \underline{X} \rightarrow \underline{X}$  is an objectwise weak equivalence (a weak equivalence on each object); this implies the following less precise result:

**Proposition 5.4.** *The natural map  $\text{Tel } T_n \underline{X} \rightarrow \underline{X}$  is a stable equivalence.*

If we write  $F_n X_n$  for the spectrum that has  $j$ -th object the initial object  $*$  for  $j < n$  and  $E^{j-n} X_n$  for  $j \geq n$ , then we have natural maps

$$E^n T_n \underline{X} \longleftarrow E^n F_n X_n \longrightarrow E^\infty X_n$$

that are isomorphisms on  $j$ -th objects for  $j \geq n$  and are therefore stable equivalences. Applying  $k_*$  and the suspension isomorphism, we obtain the following proposition:

**Proposition 5.5.** *For  $\underline{X}$  cofibrant,  $k_q T_n \underline{X} \cong k_{q+n} E^\infty X_n$ .*

The previous two propositions give us a natural (in both  $\underline{X}$  and  $k_*$ ) isomorphism

$$k_q^{Ec}(\underline{X}) = \text{Colim } k_{q+n}(E^\infty X_n) \longrightarrow k_q(\underline{X}).$$

Since this natural isomorphism commutes with the suspension isomorphisms, it is a natural isomorphism of homology theories. This completes the proof of Theorem 5.1.

## 6. PROOF OF THEOREM 5

In this section we prove Theorem 5 that interprets the cotangent complex of the suspension spectra of  $E_\infty$  spaces in terms of the associated spectra. As indicated in the introduction, up to equivalence, the definition of cotangent complex should not depend on the  $E_\infty$  operad involved. In order to take advantage of the work in previous sections, we work with the linear isometries operad  $\mathcal{L}$ , and consider the category of  $\mathcal{L}$ -spaces, the  $E_\infty$  spaces for the  $E_\infty$  operad  $\mathcal{L}$ . If  $X$  is an  $\mathcal{L}$ -space, then  $\Sigma^\infty X_+$  is an “ $\mathcal{L}$ -spectrum”, an  $E_\infty$  ring spectrum for the  $E_\infty$  operad  $\mathcal{L}$ . The category of  $\mathcal{L}$ -spectra is closely related to the category of commutative  $S$ -algebras: The functor  $S \wedge_{\mathcal{L}} (-)$  studied in EKMM [5, I§8] converts  $\mathcal{L}$ -spectra to weakly equivalent commutative  $S$ -algebras. We study the functor that takes an  $\mathcal{L}$ -space  $X$  to the commutative  $S$ -algebra  $\Sigma_{S^+}^\infty X = S \wedge_{\mathcal{L}} \Sigma^\infty X_+$  and we prove the following theorem.

**Theorem 6.1.** *For an  $\mathcal{L}$ -space  $X$ , the  $S$ -module  $\mathbf{LAb}_S^S \Sigma_{S^+}^\infty X \cong Q^{\mathbf{LIR}} \Sigma_{S^+}^\infty X$  is naturally isomorphic in the stable category to the spectrum associated to  $X$ .*

In the theorem, we are regarding  $\Sigma_{S^+}^\infty X$  as augmented over  $S = \Sigma_{S^+}^\infty *$  via the map induced by the trivial map  $X \rightarrow *$ , and  $\mathbf{LAb}_S^S$  is as in Section 2 (with  $A = B = S$ ). Since the cotangent complex of  $\Sigma_{S^+}^\infty X$  is weakly equivalent to the extended  $\Sigma_{S^+}^\infty X$ -module  $(\Sigma_{S^+}^\infty X) \wedge \mathbf{LAb}_S^S \Sigma_{S^+}^\infty X$ , Theorem 5 is an immediate consequence.

We analyze  $\mathbf{LAb}_S^S \Sigma_{S^+}^\infty X$  using the results of Section 3. Since the work of that section is phrased in terms of model structures and Quillen adjunctions, it is convenient to make a number of model category observations for  $\mathcal{L}$ -spaces. Since the category of  $\mathcal{L}$ -spaces is the category of algebras for a continuous monad, Quillen's small object argument and standard techniques prove the following proposition.

**Proposition 6.2.** *The category of  $\mathcal{L}$ -spaces is a topological closed model category with weak equivalences and fibrations the weak equivalences and (Serre) fibrations of the underlying spaces.*

May, Quinn, and Ray [13, IV.1.8] observe that the functor  $\Sigma^\infty(-)_+$  from  $\mathcal{L}$ -spaces to  $\mathcal{L}$ -spectra is left adjoint to the zero-th space functor  $\Omega^\infty$ . Since the functor  $\Omega^\infty$  preserves weak equivalences and fibrations, this is in fact a Quillen adjunction. Since the functor  $S \wedge_{\mathcal{L}}(-)$  from  $\mathcal{L}$ -spectra to commutative  $S$ -algebras is a Quillen left adjoint, we obtain the following result.

**Proposition 6.3.** *The functor  $\Sigma_{S^+}^\infty$  from the category of  $\mathcal{L}$ -spaces to the category of commutative  $S$ -algebras is a Quillen left adjoint.*

The previous propositions in particular give us a notion of cofibrant  $\mathcal{L}$ -space and prove that the suspension spectrum functor  $\Sigma_{S^+}^\infty$  takes cofibrant  $\mathcal{L}$ -spaces to cofibrant commutative  $S$ -algebras. Since in  $\mathcal{L}$ -spaces, the one-point  $\mathcal{L}$ -space is both the initial and final object, the category of  $\mathcal{L}$ -spaces is enriched over the category of based spaces. When we regard  $\Sigma_{S^+}^\infty$  as a functor into the category of commutative  $S$ -algebras lying over  $S$ , the functor  $\Sigma_{S^+}^\infty$  is enriched over based spaces. As a formal consequence we obtain the following proposition.

**Proposition 6.4.** *The functor  $\Sigma_{S^+}^\infty$  from the category of  $\mathcal{L}$ -spaces to the category of commutative  $S$ -algebras over  $S$  preserves the tensor with based spaces. In particular, it converts the suspension functor  $\mathbf{B}$  in the category of  $\mathcal{L}$ -spaces to the suspension functor  $\mathbf{E}$  in the category of commutative  $S$ -algebras over  $S$ .*

We use the notation  $\mathbf{B}$  for the suspension in  $\mathcal{L}$ -spaces because of the following theorem, which is well-known to experts. The theorem is closely related to the uniqueness theorem of May and Thomason [12] and the relationship between the homotopy category of  $\mathcal{L}$ -spaces and the homotopy category of connective spectra. For the statement, note that since the category of  $\mathcal{L}$ -spaces may be defined as the category of algebras for a continuous monad in based spaces, the loop space  $\Omega X$  is the underlying based space of cotensor of an  $\mathcal{L}$ -space  $X$  with the based space  $S^1$ .

**Theorem 6.5.** *The derived functor of  $\mathbf{B}$  is a (one-fold) delooping functor: If  $X$  is a cofibrant  $\mathcal{L}$ -space, then the unit of the suspension, loops adjunction,*

$$X \longrightarrow \Omega \mathbf{B} X$$

*is group completion.*

Since no proof of this theorem has appeared in the literature, we outline a proof at the end of the section.

The canonical maps  $\Sigma B^n X \rightarrow B^{n+1} X$  make  $\{B^n X\}$  a spectrum in the category of spaces, or in the terminology of Lewis–May [8], an “indexed prespectrum”. When  $X$  is cofibrant, this has the further property that the adjoint structure map  $B^n X \rightarrow \Omega B^{n+1} X$  is a weak equivalence for  $n > 0$  and is group completion for  $n = 0$ . Also when  $X$  is cofibrant, the structure maps are cofibrations, and so

$$Z = \operatorname{Colim} S^{-n} \wedge B^n X,$$

is a Lewis–May spectrum whose zeroth space is a group completion of  $X$ . Although this construction does not constitute an infinite loop space machine on  $\mathcal{L}$ -spaces, we have the following proposition; see Remark 6.10 below for further discussion.

**Proposition 6.6.** *If  $X$  is cofibrant,  $Z$  is a model for the spectrum associated to  $X$ .*

The Lewis–May spectrum  $Z$  is naturally isomorphic in the stable category to the  $S$ -module

$$Z_S = \operatorname{Colim} S_S^{-n} \wedge B^n X.$$

This is the composite of the “free  $\mathbb{L}$ -spectrum” functor of EKMM applied to  $Z$ ,  $\mathbb{L}Z = \mathcal{L}(1) \times Z$ , and the functor  $S \wedge_{\mathcal{L}} (-)$  from  $\mathbb{L}$ -spectra to  $S$ -modules. For the proof of Theorem 5, it is useful to reframe this in the context of the spectra in the category of  $S$ -modules, i.e., the  $\mathfrak{M}_S$ -spectra of Section 3. We have a  $\mathfrak{M}_S$ -spectrum  $\underline{Z}$  defined by

$$Z_n = S_S \wedge B^n X,$$

and we have a canonical natural map of  $\mathfrak{M}_S$ -spectra from  $\underline{Z}$  to the suspension  $\mathfrak{M}_S$ -spectrum of  $Z_S$  induced by the inclusion of  $S_S \wedge B^n X \cong \Sigma^n(S_S^{-n} \wedge B^n X)$  in the colimit system defining  $Z_S$ . An easy colimit argument shows that this map is a stable equivalence.

The  $\mathfrak{M}_S$ -spectrum  $\underline{Z}$  is one  $\mathfrak{M}_S$ -spectrum associated to  $\{B^n X\}$ , but we also have a different one that takes into account the action of the topological monoid  $\mathcal{L}(1)$  on the based spaces  $B^n X$ . The (Lewis–May) suspension spectrum functor  $\Sigma^\infty$  takes based  $\mathcal{L}(1)$ -spaces to  $\mathbb{L}$ -spectra; we write  $\Sigma_S^\infty$  for the composite functor  $S \wedge_{\mathcal{L}} \Sigma^\infty(-)$  that lands in  $S$ -modules. The purpose for introducing this construction is that we have a canonical natural isomorphism of  $S$ -modules

$$\Sigma_S^\infty(X_+) \cong \Sigma_{S^+}^\infty X.$$

We have a  $\mathfrak{M}_S$ -spectrum  $\underline{B}$  defined by

$$B_n = \Sigma_S^\infty B^n X.$$

The identity isomorphism of  $\Sigma^\infty B^n X$  in the category of Lewis–May spectra induces a map of  $\mathbb{L}$ -spectra

$$(\mathbb{L}S) \wedge (B^n X) \cong \mathbb{L}(\Sigma^\infty B^n X) \longrightarrow \Sigma^\infty B^n X$$

and a map of  $S$ -modules

$$S_S \wedge B^n X = S \wedge_{\mathcal{L}} \mathbb{L}S \wedge B^n X \longrightarrow S \wedge_{\mathcal{L}} \Sigma^\infty B^n X = \Sigma_S^\infty B^n X,$$

that induces a map of  $\mathfrak{M}_S$ -spectra  $\underline{Z} \rightarrow \underline{B}$ . Since each map displayed above is a weak equivalence, the map  $\underline{Z} \rightarrow \underline{B}$  is a stable equivalence. We now have everything needed for the proof of Theorem 6.1.

*Proof of Theorem 6.1.* It suffices to consider the case when  $X$  is a cofibrant  $\mathcal{L}$ -space and show that  $Q^{\mathbf{L}}I^{\mathbf{R}}\Sigma_{S^+}^{\infty}X$  is naturally isomorphic to  $Z_S$  in the homotopy category of  $S$ -modules. Applying Propositions 3.4 and 3.8, and combining with the work above, it suffices to show that the underlying  $\mathfrak{M}_S$ -spectrum of the derived suspension spectrum  $E^{\mathbf{L}\infty}I^{\mathbf{R}}\Sigma_{S^+}^{\infty}X$  is naturally isomorphic to  $\underline{B}$  in the homotopy category of  $\mathrm{Sp}(\mathfrak{M}_S)$ . Since  $I$  is part of a Quillen equivalence, its derived functor preserves suspension and  $E^{\mathbf{L}\infty}I^{\mathbf{R}}\Sigma_{S^+}^{\infty}X$  is naturally isomorphic to  $\underline{I}^{\mathbf{R}}E^{\infty}\Sigma_{S^+}^{\infty}X$  in the homotopy category of  $\mathrm{Sp}(\mathfrak{M}_S)$ . Since the underlying  $\mathfrak{M}_S$ -spectrum of  $\underline{I}^{\mathbf{R}}E^{\infty}\Sigma_{S^+}^{\infty}X$  is the homotopy fiber of the augmentation, it is naturally weak equivalent to the  $\mathfrak{M}_S$ -spectrum cofiber of the unit map:

$$E^{\infty}\Sigma_{S^+}^{\infty}X \cup_{E^{\infty}S} CE^{\infty}S \xrightarrow{\sim} \underline{I}^{\mathbf{R}}E^{\infty}\Sigma_{S^+}^{\infty}X.$$

Since the unit map is a cofibration, the cofiber is equivalent to the quotient,  $(E^{\infty}\Sigma_{S^+}^{\infty}X)/E^{\infty}S$ . Using the natural isomorphisms of  $S$ -modules

$$(E^n\Sigma_{S^+}^{\infty}X)/S \cong (\Sigma_{S^+}^{\infty}B^nX)/S \cong \Sigma_S^{\infty}(B^nX_+/S^0) \cong \Sigma_S^{\infty}B^nX,$$

we obtain our chain of natural isomorphisms in  $\mathrm{Ho}\mathrm{Sp}(\mathfrak{M}_S)$  between  $E^{\mathbf{L}\infty}I^{\mathbf{R}}\Sigma_{S^+}^{\infty}X$  and  $\underline{B}$  as the zigzag

$$\underline{I}^{\mathbf{R}}E^{\infty}\Sigma_{S^+}^{\infty}X \xleftarrow{\sim} E^{\infty}\Sigma_{S^+}^{\infty}X \cup_{E^{\infty}S} CE^{\infty}S \xrightarrow{\sim} (E^n\Sigma_{S^+}^{\infty}X)/S \cong \Sigma_S^{\infty}B^nX = \underline{B}.$$

□

We now go on to the proof of Theorem 6.5. We write  $L$  for the monad on based spaces associated to the operad  $\mathcal{L}$ . For a based space  $T$ ,  $LT$  is the quotient of the disjoint union of  $\mathcal{L}(n) \times_{\Sigma_n} T^n$  (cartesian power of  $T$ ) by an equivalence relation in terms of the basepoint and the operad degeneracies (operadic multiplications with  $\mathcal{L}(0) = *$ ), described in detail in [11, 2.4].

We study the suspension in  $\mathcal{L}$ -spaces in terms of geometric realization. For an  $\mathcal{L}$ -space  $X$  and a based simplicial set  $T_{\bullet}$ , let  $X \widehat{\otimes} T_{\bullet}$  be the simplicial  $\mathcal{L}$ -space which in degree  $n$  is the tensor of  $X$  with the based set  $T_n$ ; this is the coproduct of copies of  $X$  indexed on the non-basepoint simplexes of  $T_n$ . Writing  $S_{\bullet}^1$  for the simplicial model of the based circle with one vertex and one non-degenerate 1-simplex, the following lemma implies in particular that  $BX$  is the geometric realization of the simplicial  $\mathcal{L}$ -space  $X \widehat{\otimes} S_{\bullet}^1$ .

**Lemma 6.7.** *The geometric realization of a simplicial  $\mathcal{L}$ -space is naturally an  $\mathcal{L}$ -space. For any  $\mathcal{L}$ -space  $X$  and any based simplicial set  $T_{\bullet}$ , the map  $X \widehat{\otimes} |T_{\bullet}| \rightarrow |X \widehat{\otimes} T_{\bullet}|$  induced by the universal property of the tensor is an isomorphism of  $\mathcal{L}$ -spaces.*

*Proof.* The first statement is [11, 12.2]: Since cartesian products and colimits of spaces commute with geometric realization, for any simplicial based space  $Y_{\bullet}$ , we have a natural isomorphism  $L|Y_{\bullet}| \cong |LY_{\bullet}|$ . It is straight-forward to check that the composite of this isomorphism and the geometric realization of the  $\mathcal{L}$ -space structure map  $|LY_{\bullet}| \rightarrow |Y_{\bullet}|$  provides an  $\mathcal{L}$ -space structure map for  $|Y_{\bullet}|$ .

For the statement about tensors, consider first the free  $\mathcal{L}$ -space  $LX$ . The universal property of the free functor and the coproduct induce an isomorphism of simplicial  $\mathcal{L}$ -spaces

$$L(X \wedge T_{\bullet}) \cong (LX) \widehat{\otimes} T_{\bullet},$$

and applying the isomorphism of the previous paragraph, we get an isomorphism

$$(\mathbf{L}X) \widehat{\otimes} |T_\bullet| \cong \mathbf{L}(X \wedge |T_\bullet|) \cong \mathbf{L}|X \wedge T_\bullet| \cong |\mathbf{L}(X \wedge T_\bullet)| \cong |(\mathbf{L}X) \widehat{\otimes} T_\bullet|,$$

which is the statement for  $\mathbf{L}X$ . In the general case, the tensor  $X \widehat{\otimes} |T_\bullet|$  is constructed as the reflexive coequalizer

$$\mathbf{L}(\mathbf{L}X \wedge |T_\bullet|) \rightrightarrows \mathbf{L}(X \wedge |T_\bullet|) \longrightarrow X \widehat{\otimes} |T_\bullet|.$$

Commuting geometric realization with  $\mathbf{L}$  and the coequalizer, we see that the coequalizer displayed above is the geometric realization of the reflexive coequalizer

$$\mathbf{L}(\mathbf{L}X \wedge T_\bullet) \rightrightarrows \mathbf{L}(X \wedge T_\bullet) \longrightarrow X \widehat{\otimes} T_\bullet.$$

describing the tensor of the  $\mathcal{L}$ -space  $X$  with the based set  $T_\bullet$ .  $\square$

The tensor of the  $\mathcal{L}$ -space  $X$  with a finite based set defines a functor from finite based sets to based spaces that takes the trivial based set  $*$  to the trivial based set  $*$ . This constructs a  $\Gamma$ -space associated to  $X$ , and the previous lemma identifies the suspension  $\mathbf{B}X$  as the classifying space of this  $\Gamma$ -space. Segal [17] proved that when a  $\Gamma$ -space is “special”, the loop space of the classifying space is a group completion. In this case, special means that the map from  $X \amalg \cdots \amalg X \rightarrow X \times \cdots \times X$  is a weak equivalence. Theorem 6.5 is therefore an immediate consequence of the following lemma.

**Lemma 6.8.** *If  $X$  and  $Y$  are cofibrant  $\mathcal{L}$ -spaces, the map from the coproduct  $X \amalg Y$  to the cartesian product  $X \times Y$  is a weak equivalence.*

We prove this using a shortcut. Recall from [11, 9.6], the two-sided monadic bar construction  $B(\mathbf{L}, \mathbf{L}, X)$  which is the geometric realization of the simplicial  $\mathcal{L}$ -space

$$B_n = \mathbf{L} \underbrace{\mathbf{L} \cdots \mathbf{L}}_n X.$$

The iterated structure map  $\mathbf{L} \cdots \mathbf{L}X \rightarrow X$  induces a map of  $\mathcal{L}$ -spaces  $B(\mathbf{L}, \mathbf{L}, X) \rightarrow X$ . When  $X$  is cofibrant, we can choose a map of  $\mathcal{L}$ -spaces  $X \rightarrow B(\mathbf{L}, \mathbf{L}, X)$  and a homotopy from the composite  $X \rightarrow X$  to the identity through maps of  $\mathcal{L}$ -spaces. Choosing such a map for cofibrant  $Y$  as well, we obtain a diagram

$$\begin{array}{ccccc} X \amalg Y & \longrightarrow & B(\mathbf{L}, \mathbf{L}, X) \amalg B(\mathbf{L}, \mathbf{L}, Y) & \longrightarrow & X \amalg Y \\ \downarrow & & \downarrow & & \downarrow \\ X \times Y & \longrightarrow & B(\mathbf{L}, \mathbf{L}, X) \times B(\mathbf{L}, \mathbf{L}, Y) & \longrightarrow & X \times Y \end{array}$$

where the horizontal composites are homotopic to the identity (through maps of  $\mathcal{L}$ -spaces). The lemma therefore reduces to showing that the map

$$B(\mathbf{L}, \mathbf{L}, X) \amalg B(\mathbf{L}, \mathbf{L}, Y) \longrightarrow B(\mathbf{L}, \mathbf{L}, X) \times B(\mathbf{L}, \mathbf{L}, Y)$$

is a weak equivalence. Both the coproduct of  $\mathcal{L}$ -spaces and the cartesian product commute with geometric realization. Since this bar construction is the geometric realization of a proper simplicial space, we are reduced to showing that the map

$$B_n(\mathbf{L}, \mathbf{L}, X) \amalg B_n(\mathbf{L}, \mathbf{L}, Y) \longrightarrow B_n(\mathbf{L}, \mathbf{L}, X) \times B_n(\mathbf{L}, \mathbf{L}, Y)$$

is a weak equivalence for all  $n$ . The following lemma therefore completes the proof of Lemma 6.8.

**Lemma 6.9.** *If  $T$  and  $U$  are nondegenerately based, then the map  $L(T \vee U) \rightarrow LT \times LU$  is a homotopy equivalence.*

As always, “nondegenerately based” means that the inclusion of the base point is an unbased  $h$ -cofibration. The proof of Lemma 6.9 involves studying the double filtration on  $L(T \vee U)$  and  $LT \times LU$  of homogeneous degree in  $T$  and  $U$ : Let  $F^{m,n}L(T \vee U) \subset L(T \vee U)$  be the image of  $\mathcal{L}(m+n) \times T^m \times U^n$ , and let  $F^{m,n}(LT \times LU) \subset LT \times LU$  be  $F^m LT \times F^n LU$  where  $F^m LT \subset LT$  is the image of  $\mathcal{L}(m) \times T^m$ , and similarly for  $F^n LU$ . The map in Lemma 6.9 preserves this double filtration. Let  $W^{m,n}$  denote the subspace of  $T^m \times U^n$  consisting of those points where at most  $m+n-1$  coordinates are not the basepoint (equivalently when  $m, n > 0$ , the subset where at least one coordinate is the basepoint). Since we have assumed that  $T$  and  $U$  are nondegenerately based, the inclusion of  $W^{m,n}$  in  $T^m \times U^n$  is a  $\Sigma_m \times \Sigma_n$ -equivariant  $h$ -cofibration; moreover,  $F^{m,n}L(T \vee U)$  is formed from

$$F^{m-1,n}L(T \vee U) \cup_{F^{m-1,n-1}L(T \vee U)} F^{m,n-1}L(T \vee U)$$

as the pushout over the map

$$\mathcal{L}(m+n) \times_{\Sigma_m \times \Sigma_n} W^{m,n} \longrightarrow \mathcal{L}(m+n) \times_{\Sigma_m \times \Sigma_n} (T^m \times U^n).$$

Likewise,  $F^{m,n}(LT \times LU)$  is formed from

$$F^{m-1,n}(LT \times LU) \cup_{F^{m-1,n-1}(LT \times LU)} F^{m,n-1}(LT \times LU)$$

as the pushout over the map

$$(\mathcal{L}(m) \times \mathcal{L}(n)) \times_{\Sigma_m \times \Sigma_n} W^{m,n} \longrightarrow (\mathcal{L}(m) \times \mathcal{L}(n)) \times_{\Sigma_m \times \Sigma_n} (T^m \times U^n).$$

The maps

$$\mathcal{L}(m+n) \times_{\Sigma_m \times \Sigma_n} W^{m,n} \longrightarrow (\mathcal{L}(m) \times \mathcal{L}(n)) \times_{\Sigma_m \times \Sigma_n} W^{m,n}$$

and

$$\mathcal{L}(m+n) \times_{\Sigma_m \times \Sigma_n} (T^m \times U^n) \longrightarrow (\mathcal{L}(m) \times \mathcal{L}(n)) \times_{\Sigma_m \times \Sigma_n} (T^m \times U^n)$$

are homotopy equivalences since the maps  $\mathcal{L}(m+n) \rightarrow \mathcal{L}(m) \times \mathcal{L}(n)$  are equivariant homotopy equivalences. Since the map

$$F^{0,0}L(X \vee Y) \longrightarrow F^{0,0}(LT \times LU)$$

is the isomorphism  $\mathcal{L}(0) \rightarrow \mathcal{L}(0) \times \mathcal{L}(0)$ , an easy double induction shows that we have a homotopy equivalence on  $F^{m,n}$  for all  $m, n$ . Passing to the colimit, we see that the map  $L(T \vee U) \rightarrow LT \times LU$  is a homotopy equivalence. This completes the proof of Lemma 6.9.

*Remark 6.10.* Let  $X$  be a cofibrant  $\mathcal{L}$ -space, and let  $Z$  be as above. This remark explains in the terminology of May and Thomason [12], why  $Z$  is equivalent to the output of an “infinite loop space machine”. Let  $E$  be any infinite loop space machine for  $\mathcal{L}$ -spaces. Writing  $\mathbf{n}$  for the finite based set  $\{0, \dots, n\}$  (with 0 as basepoint), the collection  $\{X \hat{\otimes} \mathbf{n}\}$  forms a  $\Gamma$ -space in the category of  $\mathcal{L}$ -spaces, or an  $\mathcal{FL}$ -space [12, 3.1]. Applying the machine  $E$ , a “whiskering functor”, if necessary, and Segal’s machine, we obtain a bispectrum [12, 3.9ff], that is equivalent to  $Z$  in one direction and  $EX$  in the other.

## 7. PROOF OF THEOREM 3.1

This section is devoted to the proof of Theorem 3.1, the topological closed model structure on the categories of spectra in  $\mathfrak{C}_B/B$ ,  $\mathfrak{N}_B$ , and  $\mathfrak{M}_B$ . For convenience we repeat the definition of the cofibrations, fibrations, and weak equivalences; we say that a map of spectra  $\underline{X} \rightarrow \underline{Y}$  is:

- (i) A cofibration if  $X_0 \rightarrow Y_0$  is a cofibration and each  $EY_n \cup_{EX_n} X_{n+1} \rightarrow Y_{n+1}$  is a cofibration,
- (ii) A fibration if each  $X_n \rightarrow Y_n$  is a fibration and each  $X_n \rightarrow Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$  is a weak equivalence, and
- (iii) A stable equivalence if it induces an isomorphism of homotopy groups  $\pi_* \underline{X} \rightarrow \pi_* \underline{Y}$ , where  $\pi_q \underline{X} = \text{Colim}_n \tilde{\pi}_{q+n} X_n$  for  $\tilde{\pi}_* X = \pi_* X$  in  $\mathfrak{N}_B$  and  $\mathfrak{M}_B$ , and  $\tilde{\pi}_* X = \text{Ker}(\pi_* X \rightarrow \pi_* B)$  in  $\mathfrak{C}_B/B$ .

It is clear that the categories of spectra in  $\mathfrak{C}_B/B$ ,  $\mathfrak{N}_B$ , and  $\mathfrak{M}_B$  have all small limits and colimits. Likewise, it is clear from the definitions above that cofibrations, fibration, and weak equivalences in these spectra are closed under retracts and that weak equivalences have the two-out-of-three property. The proof of Theorem 3.1 therefore amounts to proving the factorization and lifting properties, and proving the topological version of SM7. The arguments are identical for all of the categories, and we use  $\mathfrak{C}$  to denote any of the categories  $\mathfrak{C}_B/B$ ,  $\mathfrak{N}_B$ , and  $\mathfrak{M}_B$  in what follows.

We begin with an alternative characterization of the acyclic fibrations.

**Lemma 7.1.** *A map  $\underline{X} \rightarrow \underline{Y}$  is an acyclic fibration in  $\text{Sp}(\mathfrak{C})$  if and only if each map  $X_n \rightarrow Y_n$  is an acyclic fibration in  $\mathfrak{C}$ .*

*Proof.* Since  $\Omega$  preserves fibrations and acyclic fibrations (even on non-fibrant objects), when each  $X_n \rightarrow Y_n$  is an acyclic fibration in  $\mathfrak{C}$ , the map  $\underline{X} \rightarrow \underline{Y}$  is a fibration and stable equivalence in  $\text{Sp}(\mathfrak{C})$ . Conversely, assume  $\underline{X} \rightarrow \underline{Y}$  is a fibration and stable equivalence; then each map  $X_n \rightarrow Y_n$  is a fibration, and so it suffices to show that each map  $X_n \rightarrow Y_n$  is a weak equivalence. Let  $\underline{W}$  be the fiber (for  $\mathfrak{C} = \mathfrak{C}_B/B$ , this means  $W_n = B \times_{Y_n} X_n$ ); then  $\underline{W}$  is an Omega spectrum. Since the sequential colimit (of abelian groups) is an exact functor, the levelwise long exact sequences of homotopy groups for  $W_n \rightarrow X_n \rightarrow Y_n$  induce a long exact sequence of homotopy groups for  $\underline{W} \rightarrow \underline{X} \rightarrow \underline{Y}$ . It follows that  $\pi_* W = 0$  and therefore that  $\tilde{\pi}_* W_n = 0$  for all  $n$ , since  $\underline{W}$  is an Omega spectrum. We see from the long exact sequence of homotopy groups for  $W_n \rightarrow X_n \rightarrow Y_n$  that  $X_n \rightarrow Y_n$  is a weak equivalence.  $\square$

The previous lemma allows us to prove the lifting property for cofibrations and acyclic fibrations: If the solid rectangle

$$\begin{array}{ccc} \underline{A} & \longrightarrow & \underline{X} \\ \downarrow & \nearrow & \downarrow \sim \\ \underline{B} & \longrightarrow & \underline{Y} \end{array}$$

is a commutative diagram with the left-hand map a cofibration and the right-hand map an acyclic fibration of  $\mathfrak{C}$ -spectra, we can construct the dashed arrow making the diagram commute as follows. We have that  $A_0 \rightarrow B_0$  is a cofibration and  $X_0 \rightarrow Y_0$  is an acyclic fibration, and so we use the lifting property in  $\mathfrak{C}$  to construct the required map  $B_0 \rightarrow X_0$ . Inductively, having constructed  $B_n \rightarrow X_n$ , since we have that  $EB_n \amalg_{EA_n} A_{n+1} \rightarrow B_{n+1}$  is a cofibration and  $X_{n+1} \rightarrow Y_{n+1}$  is an acyclic

fibration, we can use the lifting property in  $\mathfrak{C}$  to construct a map  $B_{n+1} \rightarrow X_{n+1}$ , compatible with the map  $EB_n \rightarrow EX_n \rightarrow X_{n+1}$  and making the required diagram commute. This proves the following proposition.

**Proposition 7.2.** *In  $\mathrm{Sp}(\mathfrak{C})$ , cofibrations have the left lifting property with respect to acyclic fibrations.*

We could construct the factorization for cofibrations and acyclic fibrations analogously, but in order to analyze the topological version of SM7, it is useful to construct them instead using Quillen’s small object argument. For this, we recall the set  $I_{\mathfrak{C}}$  of “generating cofibrations” and  $J_{\mathfrak{C}}$  of “generating acyclic cofibrations” in  $\mathfrak{C}$ , the definition of which is implicit in the construction of the model structure on  $\mathfrak{C}$  in EKMM [5, VII§4]. For  $\mathfrak{C} = \mathfrak{M}_B$ ,  $I$  is simply the set of “cells”

$$I_{\mathfrak{M}_B} = \{S_B^m \longrightarrow CS_B^m \mid m \in \mathbb{Z}\},$$

where  $S_B^m$  denotes the cofibrant sphere  $B$ -module [5, III§2] and  $J$  is the set of cylinders of spheres,

$$J_{\mathfrak{M}_B} = \{S_B^m \longrightarrow S_B^m \wedge I_+ \mid m \in \mathbb{Z}\}.$$

For  $\mathfrak{C} = \mathfrak{N}_B$ , the sets  $I$  and  $J$  are

$$\begin{aligned} I_{\mathfrak{N}_B} &= \mathbb{N}I_{\mathfrak{M}_B} = \{\mathbb{N}i \mid i \in I_{\mathfrak{M}_B}\} \\ J_{\mathfrak{N}_B} &= \mathbb{N}J_{\mathfrak{M}_B} = \{\mathbb{N}j \mid j \in J_{\mathfrak{M}_B}\}. \end{aligned}$$

For  $\mathfrak{C} = \mathfrak{C}_B/B$ , the sets  $I$  and  $J$  are only slightly more complicated:  $I$  is the set of diagrams of commutative  $B$ -algebras

$$\begin{array}{ccc} \mathbb{P}S_B^m & \xrightarrow{\quad} & \mathbb{P}CS_B^m \\ & \searrow & \swarrow \\ & B & \end{array}$$

where  $\mathbb{P}$  denotes the free commutative  $B$ -algebra, and the map  $S_B^m \rightarrow CS_B^m$  is always the usual inclusion. The set  $J$  has an entirely analogous description using the inclusion  $S_B^m \rightarrow S_B^m \wedge I_+$ . The fundamental property of the sets  $I_{\mathfrak{C}}$  and  $J_{\mathfrak{C}}$  is the following:

**Proposition 7.3.** *A map in  $\mathfrak{C}$  is an acyclic fibration if and only if it has the right lifting property with respect to the maps in  $I_{\mathfrak{C}}$ ; it is a fibration if and only if it has the right lifting property with respect to the maps in  $J_{\mathfrak{C}}$ .*

In order to describe sets of generating cofibrations and generating acyclic cofibrations for  $\mathrm{Sp}(\mathfrak{C})$ , we need one more piece of notation. For an object  $X$  in  $\mathfrak{C}$ , we let  $F_n X$  denote the  $\mathfrak{C}$ -spectrum with  $(F_n X)_j$  the initial object for  $j < n$  and  $E^{j-n} X$  for  $j \geq n$ ; maps of  $\mathfrak{C}$ -spectra from  $F_n X$  into a  $\mathfrak{C}$ -spectrum  $\underline{Y}$  are in one-to-one correspondence with maps in  $\mathfrak{C}$  from  $X$  to  $Y_n$ . We set  $I_{\mathrm{Sp}}$  to be the set of maps

$$I_{\mathrm{Sp}} = \{F_n f \mid f \in I_{\mathfrak{C}}, n = 0, 1, 2, \dots\}.$$

We have the following analogue of the first part of the previous proposition.

**Lemma 7.4.** *A map in  $\mathrm{Sp}(\mathfrak{C})$  is an acyclic fibration if and only if it has the right lifting property with respect to  $I_{\mathrm{Sp}}$ .*

The description of the set  $J_{\text{Sp}}$  is slightly more complicated. Certainly  $J_{\text{Sp}}$  should contain the maps  $F_n f$  for  $f \in J_{\mathcal{C}}$ , but these maps only generate the cofibrations that are levelwise weak equivalences. Another general sort of stable equivalence occurs in the following way: If  $f: S \rightarrow T$  is a map in  $I_{\mathcal{C}}$ , then for any map  $S \rightarrow X_n$ , we could attach the cell  $F_n f$  at the  $n$ -th level or attach the cell  $F_{n+1} E f$  at the  $(n+1)$ -st level, and the map

$$\underline{X} \amalg_{F_{n+1}(\text{ES})} F_{n+1}(\text{ET}) \longrightarrow \underline{X} \amalg_{(F_n S)} F_n T$$

is a stable equivalence. This map is not a cofibration, but we can make it a cofibration using a cylinder: If we denote by  $(-) \otimes I$  the tensor in  $\mathcal{C}$  of an object with the (unbased) unit interval, the map

$$\begin{aligned} \underline{X} \amalg_{F_{n+1}(\text{ES})} F_{n+1}(\text{ES} \otimes I) \amalg_{F_{n+1}(\text{ES})} F_{n+1}(\text{ET}) \\ \longrightarrow (\underline{X} \amalg_{F_n S} F_n T) \amalg_{F_{n+1}(\text{ET})} F_{n+1}(\text{ET} \otimes I) \end{aligned}$$

gives a version of the previous map that is a cofibration. With this as motivation, for  $f: S \rightarrow T$  in  $I_{\mathcal{C}}$ , let

$$\begin{aligned} \underline{S}_{n;f} &= F_n S \amalg_{F_{n+1}(\text{ES})} F_{n+1}(\text{ES} \otimes I) \amalg_{F_{n+1}(\text{ES})} F_n(\text{ET}), \\ \underline{T}_{n;f} &= F_n T \amalg_{F_{n+1}(\text{ET})} F_{n+1}(\text{ET} \otimes I), \end{aligned}$$

and let  $\lambda_{n;f}: \underline{S}_{n;f} \rightarrow \underline{T}_{n;f}$  be the map induced by  $f$ . Then on  $j$ -th objects,  $\lambda_{n;f}$  is the identity map (on  $*$ ) for  $j < n$ , is the map  $S \rightarrow T$  (from  $I_{\mathcal{C}}$ ) for  $j = n$ , and is the map

$$E^{j-n} S \otimes I \amalg_{E^{j-n} S} E^{j-n} T \longrightarrow E^{j-n} T \otimes I,$$

for  $j > n$ . This last map is easily seen to be the inclusion of a deformation retraction. (In fact, it is isomorphic to a map in  $J_{\mathcal{C}}$ .)

In particular, we have that  $\lambda_{n;f}$  is a cofibration and for  $j > n$  is a weak equivalence on  $j$ -th objects. It follows that  $\lambda_{n;f}$  is an acyclic cofibration. We set  $J_{\text{Sp}}$  to be the set of maps

$$J_{\text{Sp}} = \{F_n g \mid g \in J_{\mathcal{C}}, n = 0, 1, 2, \dots\} \cup \{\lambda_{n;f} \mid f \in I_{\mathcal{C}}, n = 0, 1, 2, \dots\}$$

In studying these maps, it is convenient to use the following notation: Let

$$M_n \underline{X} = X_n \times_{\Omega X_{n+1}} (\Omega X_{n+1})^I,$$

where  $(-)^I$  denotes the cotensor in  $\mathcal{C}$  with the unbased interval. The two endpoints of the interval induce two maps  $(\Omega X_{n+1})^I \rightarrow \Omega X_{n+1}$ . The construction of  $M_n \underline{X}$  uses one of these maps; the other gives us a map  $M_n \underline{X} \rightarrow \Omega X_{n+1}$ . Unwinding the definition of the maps  $\lambda_{n;f}$  and the universal property of tensors and cotensors leads to the following proposition.

**Proposition 7.5.** *Let  $f: S \rightarrow T$  be a map in  $I_{\mathcal{C}}$  and let  $\underline{X}$  be a  $\mathcal{C}$ -spectrum. Then maps in  $\text{Sp}(\mathcal{C})$  from  $\underline{T}_{n;f}$  to  $\underline{X}$  are in one-to-one correspondence with maps in  $\mathcal{C}$  from  $T$  to  $M_n \underline{X}$ . Maps in  $\text{Sp}(\mathcal{C})$  from  $\underline{S}_{n;f}$  to  $\underline{X}$  are in one-to-one correspondence with commutative diagrams in  $\mathcal{C}$ ,*

$$\begin{array}{ccc} S & \longrightarrow & M_n \underline{X} \\ f \downarrow & & \downarrow \\ T & \longrightarrow & \Omega X_{n+1}. \end{array}$$

We need one more observation about the map  $M_n \underline{X} \rightarrow \Omega X_{n+1}$  before moving on to the analogue for  $J_{\text{Sp}}$  of Lemma 7.4.

**Lemma 7.6.** *Let  $\underline{X} \rightarrow \underline{Y}$  be a map in  $\text{Sp}(\mathcal{C})$  and assume that the maps  $X_n \rightarrow Y_n$  are fibrations in  $\mathcal{C}$  for all  $n$ . Then the map*

$$M_n \underline{X} \longrightarrow M_n \underline{Y} \times_{\Omega Y_{n+1}} \Omega X_{n+1}$$

is a fibration in  $\mathcal{C}$ .

*Proof.* Abbreviate  $M_n \underline{X}$  to  $M$  and  $M_n \underline{Y} \times_{\Omega Y_{n+1}} \Omega X_{n+1}$  to  $N$ ; we have a commutative cube where the double-headed arrows are known to be fibrations and the dotted arrow is the map we want to show is a fibration.

$$\begin{array}{ccccc}
 M & \xrightarrow{\quad} & (\Omega X_{n+1})^I & & \\
 \downarrow & \searrow \text{dotted} & \downarrow & \searrow & \\
 & & N & \xrightarrow{\quad} & (\Omega Y_{n+1})^I \\
 \downarrow & & \downarrow & & \downarrow \\
 X_n \times \Omega X_{n+1} & \xrightarrow{\quad} & \Omega X_{n+1} \times \Omega X_{n+1} & & \\
 \downarrow & & \downarrow & & \downarrow \\
 Y_n \times \Omega X_{n+1} & \xrightarrow{\quad} & \Omega Y_{n+1} \times \Omega Y_{n+1} & & 
 \end{array}$$

The front and back (rectangular) faces are pullbacks, and the map

$$(\Omega X_{n+1})^I \longrightarrow (\Omega Y_{n+1})^I \times_{(\Omega Y_{n+1} \times \Omega Y_{n+1})} \Omega X_{n+1} \times \Omega X_{n+1}$$

is a fibration. Since fibrations in  $\mathcal{C}$  are characterized by the right lifting property with respect to  $I_{\mathcal{C}}$ , it follows that  $M \rightarrow N$  is a fibration.  $\square$

**Lemma 7.7.** *A map in  $\text{Sp}(\mathcal{C})$  is a fibration if and only if it has the right lifting property with respect to  $J_{\text{Sp}}$ .*

*Proof.* Given  $h: \underline{X} \rightarrow \underline{Y}$ , it follows from Proposition 7.3 that the maps  $X_n \rightarrow Y_n$  are fibrations for all  $n$  if and only if  $h$  has the right lifting property with respect to the set  $\{F_n g \mid g \in J_{\mathcal{C}}, n = 0, 1, 2, \dots\}$ . We can therefore restrict to the case when  $X_n \rightarrow Y_n$  is a fibration for all  $n$  and prove that the map  $X_n \rightarrow Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$  is a weak equivalence if and only if  $h$  has the right lifting property with respect to  $\{\lambda_{n,f} \mid f \in I_{\mathcal{C}}, n = 0, 1, 2, \dots\}$ . Proposition 7.5 implies that  $h$  having the right lifting property with respect to the maps  $\lambda_{n,f}$  for  $f \in I_{\mathcal{C}}$  (for fixed  $n$ ) is equivalent to

$$M_n \underline{X} \longrightarrow M_n \underline{Y} \times_{\Omega Y_{n+1}} \Omega X_{n+1}$$

having the right lifting property with respect to  $I_{\mathcal{C}}$ , which is equivalent to it being an acyclic fibration (by Proposition 7.3). By the previous lemma, the displayed map is a fibration, so being an acyclic fibration is equivalent to being a weak equivalence.  $\square$

Recall that for a set of maps  $A$  (e.g.,  $A = I_{\text{Sp}}$  or  $A = J_{\text{Sp}}$ ), a relative  $A$ -complex is a map  $\underline{X} \rightarrow \text{Colim } \underline{X}_n$  where  $\underline{X}_0 = \underline{X}$  and each  $\underline{X}_{n+1}$  is formed from  $\underline{X}_n$  as the pushout over a coproduct of maps in  $A$ . Lemmas 7.4 and 7.7 therefore give us the right lifting property of acyclic fibrations and fibrations with respect to

relative  $I_{\text{Sp}}$ -complexes and relative  $J_{\text{Sp}}$ -complexes (respectively). We have already observed that the maps in  $I_{\text{Sp}}$  and  $J_{\text{Sp}}$  are cofibrations and it follows that relative  $I_{\text{Sp}}$ -complexes and relative  $J_{\text{Sp}}$ -complexes are cofibrations. The remainder of the proof of following lemma requires a compactness argument that we give at the end of the section.

**Lemma 7.8.** *A relative  $I_{\text{Sp}}$ -complex is a cofibration. A relative  $J_{\text{Sp}}$ -complex is an acyclic cofibration.*

The following lemma constructs factorizations:

**Lemma 7.9.** *Let  $A = I_{\text{Sp}}$  or  $J_{\text{Sp}}$ . Any map  $\underline{X} \rightarrow \underline{Y}$  can be factored as a relative  $A$ -complex  $\underline{X} \rightarrow \underline{Z}$  and a map  $\underline{Z} \rightarrow \underline{Y}$  that has the right lifting property with respect to the maps in  $A$ .*

*Proof.* Proposition 7.5 (for  $A = J_{\text{Sp}}$ ) and the characterization of maps out of  $F_n$  (for  $A = I_{\text{Sp}}$ ) show that for  $\underline{A}$  the domain or codomain of a map in  $A$ , the set of maps of  $\mathcal{C}$ -spectra out of  $\underline{A}$  commutes with sequential colimits,

$$\text{Colim Sp}(\mathcal{C})(\underline{A}, \underline{X}_n) \cong \text{Sp}(\mathcal{C})(\underline{A}, \text{Colim } \underline{X}_n),$$

when the maps  $\underline{X}_n \rightarrow \underline{X}_{n+1}$  are cofibrations. We can now apply Quillen's small object argument to construct the required factorizations.  $\square$

The usual retract argument (factoring using the previous lemma and applying the lifting property of Proposition 7.2) then proves the following lemma, the converse of Lemma 7.8.

**Lemma 7.10.** *If a map in  $\text{Sp}(\mathcal{C})$  is a cofibration, then it is a retract of a relative  $I_{\text{Sp}}$ -complex; if it is an acyclic cofibration, then it is a retract of a relative  $J_{\text{Sp}}$ -complex.*

We have now assembled everything we need for the proof of the theorem.

*Proof of Theorem 3.1.* As observed above, the proof that classes of maps defined in the statement form a closed model structure is completed by proving the required factorization and lifting properties. Using the characterization of the acyclic cofibrations from 7.10, the lifting properties follow from Proposition 7.2 and Lemma 7.7. The factorization properties follow from Lemma 7.8 and Lemma 7.9.

It remains to prove the topological version of SM7: We need to show that when  $i: \underline{A} \rightarrow \underline{B}$  is a cofibration and  $p: \underline{X} \rightarrow \underline{Y}$  is a fibration, the map of spaces

$$\text{Sp}(\mathcal{C})(\underline{B}, \underline{X}) \longrightarrow \text{Sp}(\mathcal{C})(\underline{B}, \underline{Y}) \times_{\text{Sp}(\mathcal{C})(\underline{A}, \underline{Y})} \text{Sp}(\mathcal{C})(\underline{A}, \underline{X})$$

is a (Serre) fibration and is a weak equivalence if either  $i$  or  $p$  is a stable equivalence.

To show that the map is a fibration, it suffices to consider the case when  $i$  is a relative  $I_{\text{Sp}}$ -complex by Lemma 7.10, and for this, it suffices to consider the case when  $i$  is a map in  $I_{\text{Sp}}$ . Then  $i$  is a map  $F_n f: F_n S \rightarrow F_n T$  for some  $n$  and some  $f$  in  $I_{\mathcal{C}}$ , and we can identify the map in question with the map of spaces

$$\mathcal{C}(T, X_n) \longrightarrow \mathcal{C}(T, Y_n) \times_{\mathcal{C}(S, Y_n)} \mathcal{C}(S, X_n).$$

This is a fibration by the topological version of SM7 for  $\mathcal{C}$ . An entirely similar argument proves that this map is an acyclic fibration when  $p$  is an acyclic fibration.

Finally, we need to show that the map is an acyclic fibration when  $i$  is an acyclic cofibration. As in the previous paragraph, this reduces to the case when

$i$  in  $J_{\text{Sp}}$ . When  $i$  is  $F_n g$  for some  $g$  in  $J_{\mathcal{C}}$ , the argument reduces to  $\mathcal{C}$  just as in the previous paragraph. Now consider the other case, when  $i = \lambda_{n;f}$  for some  $f$  in  $I_{\mathcal{C}}$ . The argument is (as always) to go back over the proof of the lifting property of Lemma 7.7 taking into account the topology of the mapping spaces. Proposition 7.5 was stated in terms of a bijection of sets, but the argument refines to give an isomorphism of spaces; this allows us to identify the map in question with the map of spaces

$$\mathfrak{C}(T, M) \longrightarrow \mathfrak{C}(T, N) \times_{\mathfrak{C}(S, N)} \mathfrak{C}(S, N),$$

where we have used the notation in the proof of Lemma 7.6. Lemma 7.7 shows that when  $\underline{X} \rightarrow \underline{Y}$  is a fibration,  $M \rightarrow N$  is an acyclic fibration. It now follows from the topological version of SM7 in  $\mathfrak{C}$  that the map displayed above is an acyclic fibration.  $\square$

Finally, we complete the proof of Lemma 7.8 by proving that a relative  $J_{\text{Sp}}$ -complex is a stable equivalence. For this, it suffices to see that a pushout

$$\underline{Y} = \underline{X} \amalg_{(\amalg \underline{S}_{n_\alpha; f_\alpha})} \left( \amalg \underline{T}_{n_\alpha; f_\alpha} \right)$$

over a coproduct of maps  $\lambda_{n_\alpha; f_\alpha}$  is a stable equivalence. This is clear for a finite coproduct (since then for  $n$  large, the map on  $n$ -th objects is a deformation retract). The proof for the general case follows from the finite case, provided we can identify the homotopy groups  $\pi_* \underline{Y}$  as the filtered colimit of the homotopy groups  $\pi_* \underline{Y}_A$  where  $A$  ranges over the finite subsets of the index sets. For this it is sufficient to identify the homotopy groups of the  $n$ -th object of  $\underline{Y}$  as the filtered colimit of the homotopy groups of the  $n$ -th objects. As observed above, for each  $\alpha$ ,  $\lambda_{n_\alpha; f_\alpha}$  is on  $n$ -th objects either an isomorphism (if  $n < n_\alpha$ ), the map  $f_\alpha$  from  $I_{\mathcal{C}}$  (when  $n = n_\alpha$ ), or the inclusion of a deformation retraction (if  $n > n_\alpha$ ). The argument is therefore completed by the following lemma.

**Lemma 7.11.** *Let  $\{f_\alpha: S_\alpha \rightarrow T_\alpha\}$  be a set of maps in  $I_{\mathcal{C}}$ , let  $X$  be an object in  $\mathfrak{C}$  and let*

$$Y = X \amalg_{(\amalg S_\alpha)} \left( \amalg T_\alpha \right).$$

*Any element of  $\tilde{\pi}_q Y$  is represented in  $\tilde{\pi}_q Y_{\alpha_1, \dots, \alpha_m}$ ,*

$$Y_{\alpha_1, \dots, \alpha_m} = X \amalg_{(S_{\alpha_1} \amalg \dots \amalg S_{\alpha_m})} (T_{\alpha_1} \amalg \dots \amalg T_{\alpha_m}),$$

*for some finite subset of indexes  $\alpha_1, \dots, \alpha_m$ . If some element of  $\tilde{\pi}_q Y_{\alpha_1, \dots, \alpha_m}$  is zero in  $\tilde{\pi}_q Y$ , then it is zero in  $\tilde{\pi}_q Y_{\alpha_1, \dots, \alpha_p}$  for some subset of indexes  $\alpha_1, \dots, \alpha_p$ .*

*Proof.* In order to fix notation, we treat the case  $\mathfrak{C} = \mathfrak{C}_B/B$ , but the arguments for the other cases are similar. (We continue to write  $\amalg$  instead of  $\wedge$  for the coproduct, however, to avoid confusion for the other cases.) Since  $\tilde{\pi}_q$  is a subset of  $\pi_q$ , it suffices to prove the analogous lemma for  $\pi_q$ , and for this, it suffices to work in  $\mathfrak{C}_B$ , the category of commutative  $B$ -algebras. In this context, each map  $f_\alpha$  is just a map  $\mathbb{P}S_B^{m_\alpha} \rightarrow \mathbb{P}CS_B^{m_\alpha}$  for some integer  $m_\alpha$ . For a finite subset  $A = \{\alpha_1, \dots, \alpha_n\}$ , we set  $Y_A = Y_{\alpha_1, \dots, \alpha_n}$  (or  $Y_A = X$  when  $A$  is empty), and we set

$$M_A = \bigvee_{\alpha \notin A} S_B^{m_\alpha},$$

so that  $Y \cong Y_A \amalg_{\mathbb{P}M_A} \mathbb{P}(CM_A)$ . We construct various filtrations (of  $B$ -modules) on  $Y$  using the bar construction, which we denote as “ $\beta_\bullet$ ”. Let  $\beta_\bullet(Y_A, \mathbb{P}M_A, \mathbb{P}^*)$  be

the simplicial object in  $\mathfrak{C}_B$ , with

$$\beta_n(Y_A, \mathbb{P}M_A, \mathbb{P}*) = Y_A \amalg \underbrace{\mathbb{P}M_A \amalg \cdots \amalg \mathbb{P}M_A}_n \amalg \mathbb{P}*,$$

with faces induced by the maps  $M_A \rightarrow Y$ ,  $M_A \rightarrow *$ , and the codiagonal maps, and with degeneracy maps induced by inserting an extra summand of  $\mathbb{P}M_A$ . We write  $\beta[A]$  for the geometric realization and we use the notation

$$Y_A = \beta[A]_0 \longrightarrow \beta[A]_1 \longrightarrow \cdots \longrightarrow \beta[A]_n \longrightarrow \cdots, \quad \beta[A] = \operatorname{Colim} \beta[A]_n,$$

for the filtration arising from the geometric realization. Since the degeneracy maps are inclusions of wedge summands of  $B$ -modules, this filtration is a filtration by  $h$ -cofibrations of  $B$ -modules. The geometric realization of a simplicial object in  $\mathfrak{C}_B$  is an object in  $\mathfrak{C}_B$ , and we have isomorphisms

$$\begin{aligned} \beta[A] &= |\beta_\bullet(Y_A, \mathbb{P}M_A, \mathbb{P}*)| \cong Y_A \amalg_{\mathbb{P}M_A} |\beta_\bullet(\mathbb{P}M_A, \mathbb{P}M_A, \mathbb{P}M_A)| \amalg_{\mathbb{P}M_A} \mathbb{P}^* \\ &\cong Y_A \amalg_{\mathbb{P}M_A} \mathbb{P}(M_A \wedge I) \amalg_{\mathbb{P}M_A} \mathbb{P}^* \cong Y_A \amalg_{\mathbb{P}M_A} \mathbb{P}(CM_A) \cong Y \end{aligned}$$

(see [5, VII.3.2]). For  $A \subset A'$ , the map  $\beta[A] \rightarrow \beta[A']$  covering the identity map of  $Y$  is not induced by a simplicial map but does preserve the filtrations above.

Now given an element  $x$  of  $\pi_q Y$ , we show that  $x$  is the image of an element of  $\pi_q Y_A$  for some finite subset of indexes  $A$ . Let  $A_0$  be the empty set. Using the isomorphisms

$$\pi_q Y \cong \pi_q(\beta[A_0]) \cong \operatorname{Colim} \pi_q(\beta[A_0]_n),$$

we can represent  $x$  as the image of an element  $x_0$  of  $\pi_q(\beta[A_0]_n)$  for some  $n$ . Now suppose by induction, we have constructed a finite set  $A_k$  and found an element  $x_k$  of  $\pi_q(\beta[A_k]_{n-k})$  whose image in  $\pi_q Y$  is  $x$ . The quotient  $\beta[A_k]_{n-k}/\beta[A_k]_{n-k-1}$  is the quotient of  $\Sigma^{n-k}\beta_{n-k}(Y_{A_k}, \mathbb{P}M_{A_k}, \mathbb{P}^*)$  by the degeneracies. Since the degeneracies are inclusions of wedge summands, this quotient is still a wedge sum of  $B$ -modules, each summand of which involves only finitely many of the indexes  $\alpha$ . The image of  $x_k$  factors through  $\pi_q$  of a finite wedge sum of these summands; let  $A_{k+1}$  be the union of  $A_k$  and the finite set of indexes involved in these summands. Since the map  $\beta[A_k] \rightarrow \beta[A_{k+1}]$  is compatible with the filtration, and by construction, the image of  $x_k$  in  $\pi_q(\beta[A_{k+1}]_{n-k}/\beta[A_{k+1}]_{n-k-1})$  is zero, we can find an element  $x_{k+1}$  in  $\pi_q(\beta[A_{k+1}]_{n-k-1})$  whose image in  $\pi_q(\beta[A_{k+1}]_{n-k})$  is the image of  $x_k$ . It follows that the image of  $x_{k+1}$  in  $\pi_q Y$  is  $x$ . Continuing in this way, we get a finite subset of indexes  $A_n$  and an element  $x_n$  of  $\pi_q(\beta[A_n]_0) = \pi_q Y_{A_n}$  whose image in  $\pi_q Y$  is  $x$ .

The argument for a relation is similar, using a relative class in  $\pi_{q+1}$  and starting with  $A_0 = \{\alpha_1, \dots, \alpha_m\}$ .  $\square$

## 8. COHOMOLOGY THEORIES FOR OPERADIC ALGEBRAS

We have stated and proved the results in this paper in terms of the special case of particular interest, the category of algebras over the operad  $Com$ . Most of these results hold quite generally for the categories of algebras over other operads. The purpose of this section is to give precise statements of these general results and to indicate how to adapt the arguments in the earlier sections to the more general case. We prove the following theorem.

**Theorem 8.1.** *Let  $\mathcal{G}$  be an operad of (unbased) spaces with each  $\mathcal{G}(n)$  of the homotopy type of a  $\Sigma_n$ -CW complex. Let  $B$  be a cofibrant  $\mathcal{G}$ -algebra in EKMM  $R$ -modules, and let  $UB$  be its universal enveloping algebra. Let  $\mathfrak{G}_{R/B}$  be the category of  $\mathcal{G}$ -algebras of EKMM  $R$ -modules lying over  $B$ .*

1. *Topological Quillen Cohomology with coefficients in a left  $UB$ -module induces an equivalence from the homotopy category of left  $UB$ -modules to the category of cohomology theories on  $\mathfrak{G}_{R/B}$ .*
2. *The category of homology theories on  $\mathfrak{G}_{R/B}$  is equivalent to the category of homology theories on the category of left  $UB$ -modules.*
3. *The stable category of  $\mathcal{G}$ -algebras over and under  $B$  is equivalent to the category of left  $UB$ -modules.*
4. *The equivalence in the previous statement takes the suspension spectrum of an  $\mathcal{G}$ -algebra  $A$  over  $B$  to the  $UB$ -module of infinitesimal  $UB$ -deformations.*

We review the general definition of the universal enveloping algebra  $UB$  and of Topological Quillen Cohomology below; the proof of the theorem essentially amounts to formulating the definitions in a framework parallel to the case for  $\mathcal{G} = \text{Com}$ . In the case  $\mathcal{G} = \text{Com}$ ,  $UB$  is just  $B$ , the category of left  $UB$ -modules is the category of  $B$ -modules, and Topological Quillen Cohomology is Topological André–Quillen Cohomology, as in the theorems in the introduction.

In general, unlike the case of the operad  $\text{Com}$ , a weak equivalence of  $\mathcal{G}$ -algebras does not necessarily induce a weak equivalence of enveloping algebras, and this is why we need to assume that  $B$  is cofibrant from the outset in the theorem above. The theorem combined with Proposition 1.2 implies that for general  $B$ , the category of cohomology theories on  $\mathfrak{G}_{R/B}$  is equivalent to the homotopy category of left  $UB'$ -modules and the category of homology theories on  $\mathfrak{G}_{R/B}$  is equivalent to the category of homology theories on the category of left  $UB'$ -modules, where  $B' \rightarrow B$  is a cofibrant approximation.

For  $\mathcal{G} = \text{Ass}$ , the operad for associative algebras,  $UA$  is  $A \wedge_R A^{\text{op}}$ , and so the category of left  $UA$ -modules is the category of  $A$ -bimodules. When  $A$  is cofibrant, one typically writes  $A^e$  for  $A \wedge_R A^{\text{op}}$ . More generally,  $A^e$  denotes  $A' \wedge_R A^{\text{op}}$ , for some fixed choice of cofibrant approximation  $A' \rightarrow A$ . Lazarev [7] identifies Topological Quillen Cohomology in terms of Topological Hochschild Cohomology, and identifies the module of infinitesimal deformations of an associative algebra  $A$  as the homotopy fiber of the multiplication map  $A^e \rightarrow A$ . Part 1 of the previous theorem then has the following corollary.

**Corollary 8.2.** *Let  $B$  be an associative  $R$ -algebra. Every cohomology theory on the category of associative  $R$ -algebras lying over  $B$  is of the form*

$$h^*(X, A) = \pi_{-*} \text{Fib}(\text{THH}_R(X, M) \longrightarrow \text{THH}_R(A, M))$$

for some left  $B^e$ -module  $M$ .

We now return to the general case of Theorem 8.1. We begin by describing the universal enveloping algebra  $UB$ . At the same time, we describe the “universal enveloping operad”  $UB$ .

**Definition 8.3.** The universal enveloping operad  $\mathcal{U}B$  is the operad in  $R$ -modules that has  $n$ -th object  $\mathcal{U}B(n)$  defined by the coequalizer

$$\bigvee_k \mathcal{G}(n+k) \wedge_{\Sigma_k} (\mathbb{G}B)^{(k)} \rightrightarrows \bigvee_k \mathcal{G}(n+k) \wedge_{\Sigma_k} B^{(k)} \longrightarrow \mathcal{U}B(n)$$

where  $\mathbb{G}$  denotes the free  $\mathcal{G}$ -algebra functor, one map is the  $\mathcal{G}$ -action map of  $B$  and the other is induced by the operadic multiplication of  $\mathcal{G}$ . The operadic multiplication on  $\mathcal{U}B$  is induced by the operadic multiplication of  $\mathcal{G}$ . The universal enveloping algebra  $UB$  is the  $R$ -algebra  $\mathcal{U}B(1)$ .

The fundamental property of the universal enveloping operad is given by the following proposition, which is an easy consequence of the definitions.

**Proposition 8.4.** *The category of  $\mathcal{G}$ -algebras lying under  $B$  is equivalent to the category of  $\mathcal{U}B$ -algebras.*

In particular, we have that  $\mathcal{U}B(0) = B$ . Let  $\tilde{\mathcal{U}}B$  be the operad with

$$\tilde{\mathcal{U}}B(n) = \begin{cases} * & n = 0 \\ \mathcal{U}B(n) & n > 0 \end{cases}$$

The category of  $\tilde{\mathcal{U}}B$ -algebras plays the role for  $\mathcal{G}$ -algebras under and over  $B$  that the category of nucas plays for commutative algebras.

**Definition 8.5.** Let  $\mathfrak{G}_{B/B}$  denote the category of  $\mathcal{U}B$ -algebras lying over  $B$ , and let  $\mathfrak{N}_{\tilde{\mathcal{U}}B}$  denote the category of  $\tilde{\mathcal{U}}B$ -algebras. Let  $K: \mathfrak{N}_{\tilde{\mathcal{U}}B} \rightarrow \mathfrak{G}_{B/B}$  denote the functor that takes a  $\tilde{\mathcal{U}}B$ -algebra  $N$  to  $B \vee N$ . Let  $I: \mathfrak{G}_{B/B} \rightarrow \mathfrak{N}_{\tilde{\mathcal{U}}B}$  denote the functor that takes  $A$  to the (point-set) fiber of the augmentation  $A \rightarrow B$ .

The functors  $K$  and  $I$  are adjoint, and the following proposition holds just as in the commutative algebra case.

**Theorem 8.6.** *The categories  $\mathfrak{G}_{B/B}$  and  $\mathfrak{N}_{\tilde{\mathcal{U}}B}$  are topological closed model categories with weak equivalences the weak equivalences of the underlying  $R$ -modules. The adjunction  $(K, I)$  is a Quillen equivalence.*

*Proof.* The topological closed model structures and easy consequences of the general theory in EKMM [5, VII§4]. When we take  $\mathbb{T}$  to be the monad associated to the operad  $\mathcal{U}B$  or  $\tilde{\mathcal{U}}B$ , or indeed any operad in the category of  $R$ -modules, the proof of the ‘‘Cofibration Hypothesis’’ of [5, VII§4] for  $\mathbb{T}$  follows just like the proof for associative and commutative algebras in [5, VII§3]. The key observation is that since colimits and smash products of  $R$ -modules commute with geometric realization of simplicial  $R$ -modules, the monad  $\mathbb{T}$  commutes with geometric realization. This gives the geometric realization of a simplicial  $\mathbb{T}$ -algebra a  $\mathbb{T}$ -algebra structure and also proves that geometric realization commutes with colimits of  $\mathbb{T}$ -algebras. As a consequence, we obtain the analogue of [5, VII.3.7]: For any maps of  $\mathbb{T}$ -algebras  $A \rightarrow A'$  and  $A \rightarrow A''$ , we can identify the geometric realization of the ‘‘bar construction’’

$$\beta_n(A', A, A'') = (A, \mathbb{T}M, \mathbb{T}*) = A' \amalg \underbrace{A \amalg \cdots \amalg A}_n \amalg A'',$$

as the double pushout in  $\mathbb{T}$ -algebras

$$A' \amalg_A (A \otimes I) \amalg_A A'',$$

where  $A \otimes I$  is the tensor with the (unbased) interval (see for example the argument for Lemma 6.7). In the special case when  $A'' = \mathbb{T}^*$  and  $A = \mathbb{T}M$  for some  $R$ -module  $M$ , the degeneracy maps are the inclusion of wedge summands, and so the filtration on the geometric realization is a filtration by  $h$ -cofibrations. In particular, in this case, the inclusion of the lowest filtration level  $A' = A' \amalg \mathbb{T}^*$  in the geometric realization,  $A' \amalg_{\mathbb{T}M} \mathbb{T}(CM)$ , is an  $h$ -cofibration; this is the Cofibration Hypothesis.

Since  $I$  preserves fibrations and acyclic fibrations,  $(K, I)$  is a Quillen adjunction. Finally, given any  $\tilde{U}B$ -algebra  $N$ , and any fibrant  $UB$ -algebra  $X$  over  $B$ , it is clear from the effect of  $K$  and  $I$  on homotopy groups that a map  $N \rightarrow IX$  is a weak equivalence if and only if the adjoint map  $KN \rightarrow X$  is a weak equivalence, and so  $(K, I)$  is a Quillen equivalence.  $\square$

The proof of the previous theorem used the free functor from  $R$ -modules to  $\tilde{U}B$ -algebras, but there is in addition a free functor  $\mathbb{N}_{\tilde{U}B}$  from left  $UB$ -modules to  $\tilde{U}B$ -algebras, left adjoint to the forgetful functor from  $\tilde{U}B$ -algebras to left  $UB$ -modules, defined by

$$\mathbb{N}_{\tilde{U}B}M = \bigvee_{n \geq 1} (UB(n) \wedge_{UB(n)} M^{(n)}) / \Sigma_n,$$

for a left  $UB$ -module  $M$ . We also have a zero multiplication functor  $Z$  that gives  $M$  a  $\tilde{U}B$ -action where

$$\tilde{U}B(n) \wedge_{UB(n)} M^{(n)} \longrightarrow M$$

is the left  $UB$ -action map for  $n = 1$  and the trivial map for  $n > 1$ . This functor has a left adjoint  $Q$  defined by the coequalizer

$$\mathbb{N}_{\tilde{U}B}X \rightrightarrows X \longrightarrow QX,$$

where one map is the  $\tilde{U}B$ -algebra action map on  $X$ , and the other map is the unit on the  $\tilde{U}B(1) \wedge_{UB} X$  summand and the trivial map on the summands  $(UB(n) \wedge_{UB(n)} M^{(n)}) / \Sigma_n$  for  $n > 1$ . Since the zero multiplication functor  $Z$  preserves fibrations and weak equivalences, we obtain the following proposition.

**Proposition 8.7.** *The  $(Q, Z)$  adjunction is a Quillen adjunction.*

As a consequence of the previous two propositions, for a cofibrant  $\mathcal{G}$ -algebra  $A$  over  $B$  and any left  $UB$ -module  $M$ , we obtain bijections of sets (in fact, isomorphisms of abelian groups)

$$\begin{aligned} \mathrm{Ho} \mathfrak{G}_{R/B}(A, B \vee ZM) &\cong \mathrm{Ho} \mathfrak{G}_{B/B}(B \amalg A, B \vee ZM) \\ &\cong \mathrm{Ho} \mathfrak{G}_{B/B}(I^{\mathbf{R}}(B \amalg A), ZM) \cong \mathrm{Ho} \mathfrak{M}_{UB}(Q^{\mathbf{L}}I^{\mathbf{R}}(B \amalg A), M), \end{aligned}$$

where  $\mathbf{L}$  and  $\mathbf{R}$  denote left and right derived functors. This leads to the following definition.

**Definition 8.8.** For a  $\tilde{U}B$ -algebra  $N$ , the module of infinitesimal  $UB$ -deformations is the left  $UB$ -module  $Q^{\mathbf{L}}N$ . For a cofibrant  $\mathcal{G}$ -algebra  $A$  over  $B$ , the  $UB$ -module of infinitesimal  $UB$ -deformations of  $A$  is the left  $UB$ -module  $Q^{\mathbf{L}}I^{\mathbf{R}}(B \amalg A)$ , and for a cofibration of  $\mathcal{G}$ -algebras  $A \rightarrow X$  over  $B$ , the  $UB$ -module of infinitesimal  $UB$ -deformations of  $X$  relative to  $A$  is the left  $UB$ -module  $Q^{\mathbf{L}}I^{\mathbf{R}}(B \amalg_A X)$ . For a left  $UB$ -module  $M$ , we define the Topological Quillen Cohomology of  $X$  relative to  $A$  with coefficients in  $M$  by

$$D_{\mathcal{G}}^*(X, A; M) = \mathrm{Ext}_{UB}^*(Q^{\mathbf{L}}I^{\mathbf{R}}(B \amalg_A X), M).$$

For general  $A$  and a general map  $A \rightarrow X$  of  $\mathcal{G}$ -algebras over  $B$ , the Topological Quillen Cohomology is defined using a cofibration  $A' \rightarrow X'$  covering  $A \rightarrow X$  for cofibrant approximations  $A' \rightarrow A$  and  $X' \rightarrow X$ .

Since  $(I, K)$  is a Quillen equivalence,  $I^{\mathbf{R}}$  preserves coproducts and cofibration sequences. Since  $Q$  is a Quillen left adjoint,  $Q^{\mathbf{L}}$  also preserves coproducts and cofibration sequences. From this, it is easy to see that for any left  $UB$ -module  $M$ , Topological Quillen Cohomology with coefficients in  $M$  forms a cohomology theory, with the connecting maps  $\delta$  induced by  $\text{Ext}$ .

We now have the underlying theory for  $\mathcal{G}$ -algebras parallel to the theory for commutative algebras, and the argument for Theorem 8.1 parallels the proof of Theorems 1–4. The Brown’s Representability argument in Section 1 applies generally to any category to which the arguments of EKMM [5, VII§4] apply. The only argument in Sections 1–5 and 7 that does not immediately generalize in this framework is the appeal to [5, III.5.1] in Lemma 2.9. (For the proof of Lemma 7.11, note that the coproduct of  $UB$ -algebras  $X \amalg \mathbb{P}_{UB}M$  has as its underlying module  $\bigvee (\mathcal{U}X(n) \wedge_R M^{(n)})/\Sigma_n$  by the analogue of Proposition 8.4 for  $X$ .) The following lemma fills this gap.

**Lemma 8.9.** *Let  $M$  be a cofibrant left  $UB$ -module. The natural map*

$$E\Sigma_{m+} \wedge_{\Sigma_m} (\mathcal{U}B(m) \wedge_{UB(m)} M^{(m)}) \longrightarrow (\mathcal{U}B(m) \wedge_{UB(m)} M^{(m)})/\Sigma_m$$

*is a weak equivalence.*

The proof depends strongly on the hypothesis that the spaces of  $\mathcal{G}$  have equivariant CW homotopy types and that  $B$  is a cofibrant  $\mathcal{G}$ -algebra. This latter hypothesis implies that  $B$  is a retract of a “cell  $\mathcal{G}$ -algebra” [5, VII.4.11]. If  $B \rightarrow B' \rightarrow B$  is such a retraction (with  $B \rightarrow B$  the identity), then we get a retraction of operads  $UB \rightarrow UB' \rightarrow UB$ . The analogue of Lemma 8.9 for  $B'$ , then implies the lemma as stated for  $B$ . Thus, it suffices to consider the case when  $B$  is a cell  $\mathcal{G}$ -algebra. Specifically, this means that we can write  $B$  as  $\text{Colim } B_n$  where  $B_0 = \mathcal{G}(0)_+ \wedge R$ , and  $B_{n+1} = B_n \amalg_{\mathbb{G}W_n} \mathbb{G}(CW_n)$ , where  $W$  is a wedge of sphere  $R$ -modules  $S_R^m$ .

The filtration of  $B$  allows us to get a better hold on the  $\Sigma_m$ -equivariant  $R$ -modules  $UB(m)$ . For example,  $UB_0(m) = \mathcal{G}(m)_+ \wedge R$ , and

$$UB_1(m) = \bigvee_{k \geq 0} \mathcal{G}(m+k) \wedge_{\Sigma_k} (\Sigma W_0)^{(k)}.$$

More generally, we have the following lemma.

**Lemma 8.10.** *For each  $n \geq 0$ ,  $UB_{n+1}(m)$  has a filtration by  $\Sigma_m$ -equivariant h-cofibrations  $UB_{n+1} = \text{Colim}_k UB_{n+1}(m)_k$ , with  $UB_{n+1}(m)_0 = UB_n(m)$ , and the filtration quotients*

$$UB_{n+1}(m)_k/UB_{n+1}(m)_{k-1} \cong UB_n(m+k) \wedge_{\Sigma_k} (\Sigma W_n)^{(k)}.$$

*Proof.* We set  $UB_{n+1}(m)_0 = UB_n(m)$  as required. The idea is that  $UB_{n+1}(m)_k$  is the image in  $UB_{n+1}(m)$  of  $UB_n(m+k) \wedge_{\Sigma_k} (CW_n)^{(k)}$ . Precisely, consider the  $\Sigma_k$ -equivariant filtration

$$W_n^{(k)} = F^0 CW_n \longrightarrow \cdots \longrightarrow F^{k-1} CW_n \longrightarrow F^k CW_n = (CW_n)^{(k)}$$

obtained as the smash power of the filtration on  $CW_n$  that has  $W_n$  in level zero and  $CW_n$  in level one. For  $k \geq 1$ , define  $\mathcal{U}B_{n+1}(m)_k$  as the pushout

$$\begin{array}{ccc} \mathcal{U}B_n(m+k) \wedge_{\Sigma_k} F^{k-1}CW_n & \longrightarrow & \mathcal{U}B_n(m+k) \wedge_{\Sigma_k} (CW_n)^{(k)} \\ \downarrow & & \downarrow \\ \mathcal{U}B_{n+1}(m)_{k-1} & \longrightarrow & \mathcal{U}B_{n+1}(m)_k. \end{array}$$

Then it is clear that the map  $\mathcal{U}B_{n+1}(m)_{k-1} \rightarrow \mathcal{U}B_{n+1}(m)_k$  is an  $h$ -cofibration, and the quotient is as indicated in the statement. The identification of  $\mathcal{U}B_{n+1}$  with  $\text{Colim}_k \mathcal{U}B_{n+1}(m)_k$  follows from the universal properties and the fact that the map

$$\mathcal{U}B_n(m+k) \wedge_{1 \times \Sigma_{k-1}} (W_n \wedge (CW_n)^{(k-1)}) \longrightarrow \mathcal{U}B_n(m+k) \wedge_{\Sigma_k} F^{k-1}CW_n$$

is a categorical epimorphism.  $\square$

*Proof of Lemma 8.9.* By the usual retract argument, it suffices to prove the lemma when  $M$  is a cell left  $UB$ -module, and the usual filtration argument then reduces to the case when  $M = UB \wedge_R S_R^q$ , that is, to proving that the map

$$E\Sigma_{m+} \wedge_{\Sigma_m} (\mathcal{U}B(m) \wedge_R (S_R^q)^{(m)}) \longrightarrow (\mathcal{U}B(m) \wedge_R (S_R^q)^{(m)})/\Sigma_m$$

is a weak equivalence. This now follows from previous lemma using the argument of [1, §9] (which generalizes the argument of [5, III.5.1]).  $\square$

## 9. WEAK EQUIVALENCES AND EXCISION IN MODEL CATEGORIES

The axioms listed in the introduction for a homology or cohomology theory on a closed model category  $\mathfrak{C}$  patently depend on the cofibrations. The reduced version of these axioms in Section 1 implicitly depends on the cofibrations in terms of the definition of cofibration sequences. The purpose of this section is to prove the following theorem.

**Theorem 9.1.** *Let  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  be closed model structures on the same category with the same weak equivalences. Let  $h^*$  be a contravariant functor from the category of pairs to the category of graded abelian groups and let  $\delta^n: h^n(A) \rightarrow h^{n+1}(X, A)$  be natural transformations of abelian groups. Then  $(h^*, \delta)$  is a cohomology theory on  $\mathfrak{C}_1$  if and only if it is a cohomology theory on  $\mathfrak{C}_2$ . Likewise, for a covariant functor and natural transformations,  $(h_*, \partial)$  is a homology theory on  $\mathfrak{C}_1$  if and only if it is a homology theory on  $\mathfrak{C}_2$ .*

As a technical point, with the Product Axiom and Direct Sum Axiom as stated in the introduction, we need the standard assumption that the closed model category has all small colimits. See Remark 9.5 below for further discussion of the case when this assumption does not hold.

The theorem above does not appear to be well known, and we offer it here for its intrinsic interest; it has not been used in the previous sections. The basic idea is that although cofibrations determine the notion of excision in a category, an appropriate notion of “weak excision” defined in terms of “homotopy cocartesian” diagrams depends only on the weak equivalences. The following definition is standard.

**Definition 9.2.** A commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in a closed model category is homotopy cocartesian means that there exists a commutative diagram

$$\begin{array}{ccccc} X' & \longleftarrow & A' & \longrightarrow & B' \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ X & \longleftarrow & A & \longrightarrow & B \end{array}$$

with  $A'$  cofibrant, the top horizontal arrows cofibrations, and vertical arrows weak equivalences, such that the induced map  $Y' = X' \cup_{A'} B' \rightarrow Y$  is a weak equivalence.

More concisely, the diagram is homotopy cocartesian if the canonical map in the homotopy category from the homotopy pushout to  $Y$  is an isomorphism. One class of examples of homotopy cocartesian squares is given by the squares where  $A$  is cofibrant,  $A \rightarrow X$  and  $A \rightarrow B$  are cofibrations, and  $Y$  is the pushout  $X \cup_A B$ . Another class of examples is given by the squares where  $A \rightarrow B$  and  $X \rightarrow Y$  are weak equivalences. With these examples in mind, the following proposition is clear from the definition.

**Proposition 9.3.** *Let  $\mathfrak{C}$  be a closed model category and let  $h$  be a contravariant functor from the category of pairs to the category of abelian groups. The following are equivalent:*

- (a)  $h$  satisfies the Homotopy Axiom and the Excision Axiom:
  - (i) (Homotopy) If  $(X, A) \rightarrow (Y, B)$  is a weak equivalence of pairs, then  $h(Y, B) \rightarrow h(X, A)$  is an isomorphism.
  - (iii) (Excision) If  $A$  is cofibrant,  $A \rightarrow B$  and  $A \rightarrow X$  are cofibrations, and  $Y$  is the pushout  $X \cup_A B$ , then the map of pairs  $(X, A) \rightarrow (Y, B)$  induces an isomorphism  $h(Y, B) \rightarrow h(X, A)$ .
- (b)  $h$  satisfies the following Weak Excision Axiom: Whenever

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

is homotopy cocartesian, the map of pairs  $(X, A) \rightarrow (Y, B)$  induces an isomorphism  $h(Y, B) \rightarrow h(X, A)$ .

The Weak Excision Axiom does not depend on the cofibrations but only on the weak equivalences:

**Lemma 9.4.** *Let  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  be closed model structures on the same category  $\mathfrak{C}$  with the same weak equivalences. Then a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

*is homotopy cocartesian in  $\mathfrak{C}_1$  if and only if it is homotopy cocartesian in  $\mathfrak{C}_2$ .*

*Proof.* Assume the diagram is homotopy cocartesian in  $\mathfrak{C}_1$ . Using the factorization properties of  $\mathfrak{C}_2$  (starting with  $A$ ), we can find a commutative diagram

$$\begin{array}{ccccc} X_2 & \xleftarrow{2} & A_2 & \xrightarrow{2} & B_2 \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ X & \xleftarrow{\quad} & A & \xrightarrow{\quad} & B \end{array}$$

with  $A_2$  cofibrant in  $\mathfrak{C}_2$ , the top horizontal arrows cofibrations in  $\mathfrak{C}_2$ , and the vertical arrows weak equivalences. Likewise, using the factorization properties of  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ , we can extend this to a commutative diagram

$$\begin{array}{ccccc} X' & \xleftarrow{2} & A' & \xrightarrow{2} & B' \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ X_1 & \xleftarrow{1} & A_1 & \xrightarrow{1} & B_1 \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ X_2 & \xleftarrow{2} & A_2 & \xrightarrow{2} & B_2 \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ X & \xleftarrow{\quad} & A & \xrightarrow{\quad} & B \end{array}$$

where  $A_1$  is cofibrant in  $\mathfrak{C}_1$ ,  $A'$  is cofibrant in  $\mathfrak{C}_2$ , the horizontal arrows labeled with the number  $i$  are cofibrations in  $\mathfrak{C}_i$ , and all the vertical arrows are weak equivalences. Taking  $Y_i = X_i \cup_{A_i} B_i$  and  $Y' = X' \cup_{A'} B'$ , the previous diagram induces maps

$$Y' \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Y.$$

Clearly the map  $Y' \rightarrow Y_2$  is a weak equivalence (see, for example, the characterization of homotopy pushouts by Dwyer and Spalinski [4, 10.7]). The hypothesis that the original diagram is homotopy cocartesian in  $\mathfrak{C}_1$  implies that the map  $Y_1 \rightarrow Y$  is a weak equivalence. It follows that the map  $Y' \rightarrow Y$  is a weak equivalence, and so the original diagram is homotopy cocartesian in  $\mathfrak{C}_2$ .  $\square$

Since coproducts of cofibrant objects represent the coproduct in the homotopy category, it is clear that the Product Axiom of the introduction is equivalent to the following axiom.

- (iv)<sub>w</sub> If  $\{X_\alpha\}$  is a set of objects and  $X$  is the coproduct in the homotopy category, then the natural map  $h^*(X) \rightarrow \prod h^*(X_\alpha)$  is an isomorphism.

The analogous observation holds for the Direct Sum Axiom. Since the homotopy category depends only on the weak equivalences in the model structure and not the cofibrations, this completes the proof of Theorem 9.1.

*Remark 9.5.* It is sometimes useful to consider model categories that do not have all small colimits but (as in the original definition in [14]) are only assumed to have finite colimits. For these categories, it appears unlikely that the version of the Product Axiom above is equivalent to the one in the introduction, and it depends on the application which axiom, if either, is the “right” one. Typically, the most useful version of the Product Axiom in this case is one where we assume the isomorphism only for index sets of certain fixed cardinalities; when coproducts of the given cardinalities always exist in the point-set category, then the two versions of the axioms are again equivalent. The version of the Product Axiom for finite cardinalities follows from the other axioms.

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