On vanishing Tate cohomology and decompositions in Goodwillie calculus

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Abstract

Our main result is that if F is a functor from a pointed category $\mathcal{C}$ to spectra, the Goodwillie tower of F evaluated at X splits rationally when X is a co-H-object of $\mathcal{C}$. We show that the layers of $F(X)$ in this case are easy to identify. The splitting of the Goodwillie tower gives a decomposition of $F(X)$ into a product of its layers. We use this to recover the rational decompositions of Hochschild and higher Hochschild homology [P00], [L98], [GS87]. Finally, we extend the main theorem to include dual calculus to recover the Poincaré-Birkhoff-Witt theorem, and improve the theorem in the special case in which the comultiplication map is cocommutative.

1 Introduction

Let F be a functor from any pointed category $\mathcal{C}$ to any abelian category. As an application of [G90], [G92] and [G02], Johnson-McCarthy provide a

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Goodwillie calculus theory for such functors ([JM4]). There is a tower

$$\begin{tikzcd}
 & \vdots \arrow[d, "q_{n+1}"] \\
F(X) \arrow[ur] \arrow[r, "p_n"] \arrow[dr, "p_{n-1}"] & P_nF(X) \arrow[d, "q_n"] \\
 & P_{n-1}F(X) \arrow[d] \\
 & \vdots
\end{tikzcd}$$

of universal degree n approximations $P_nF(X)$ to $F(X)$. The layers $D_nF(X)$ of $F(X)$ are the fibers of the maps q_n. This model of calculus can be extended to functors F from $\mathcal{C}$ to spectra, as in [McC02] (by dualizing). In this paper, we provide a criterion on X for the Goodwille tower of F to split rationally when evaluated at X. In particular, we show that if X is an object with a "comultiplication" map ∇ from X to the coproduct $X \vee X$ – in other words, if X is a co-H-object in $\mathcal{C}$ – then rationally the first map of the fiber sequence

$$D_nF(X) \longrightarrow P_nF(X) \xrightarrow{q_n} P_{n-1}F(X)$$

has a splitting map. This results in the main theorem of the paper:

Theorem 1.1. *If F is a homotopy functor and X is a co-H-object of $\mathcal{C}$ then rationally*

$$P_nF(X) \simeq \prod_{i=1}^{n} D_iF(X)$$

and consequently $P_\infty F(X) \simeq \prod_{n\geq 1} D_nF(X)$.

The proof of the main theorem is constructive. We construct maps

$$D_nF(X) \longrightarrow P_nF(X) \xrightarrow{P_nF(\nabla)} P_nF(\bigvee_n X) \longrightarrow D_nF(X)$$

such that the composite is multiplication by $n!$. To obtain the splitting, we simply invert $n!$ (thus the result is rational). The bulk of the proof involves showing that this composite map is induced by the norm map (§3), which in turn induces the Tate map and shows that the composite is multiplication by $n!$.

Since X is a co-H-object, it is equipped with "covering maps" Φ^r defined by

$$X \xrightarrow{\nabla^r} \bigvee_r X \xrightarrow{+} X$$

where $+$ is the fold map. An important consequence of the proof of the main theorem is that the maps Φ^r induce multiplication by r^n on $D_nF(X)$, and hence can be used to identify the layers of $F(X)$. We use this idea to show that the known rational decompositions of (higher) Hochschild homology ([GS87], [L98], [P00], [B]) are actually splittings of the Goodwillie tower of the forgetful functor from augmented commutative k-algebras to k-modules. We then compute the layers of the tower associated to higher Hochschild homology, and show that they are suspensions of the layers of the tower associated to Hochschild homology.

We also prove a version of the main theorem for dual calculus ([McC02]). In this case, we examine the H-objects of $\mathcal{C}$. Using the dual version of our main theorem, we are able to show that rationally, the Poincaré-Birkhoff-Witt theorem is a decomposition of the universal enveloping algebra of a free Lie algebra into the dual layers of a dual Goodwillie tower associated to it.

Finally, we show that sometimes the Goodwillie tower splits more often than "rationally". Recall that the Tate map provides a map from the homotopy orbits to the homotopy fixed points of a spectrum. We are particularly interested in the Tate map applied to the spectrum $D_1^{(n)}\mathrm{cr}_nF(X)$, the multi-additivitation of the n-th cross effects of F (see section 2 for the definition and close relationship to D_nF). Define the n-th Tate cohomology of F evaluated at X to be the cofiber of this map applied to $\mathrm{cr}_nF(X)$. In the final section of the paper, we show that if the comultiplication map is cocommutative, then the Goodwillie tower actually splits whenever the Tate cohomology vanishes. Kuhn has recently observed that work of Greenlees, Hovey and Sadofsky shows that the Tate cohomology of a $K(n)_*$-local spectrum vanishes - hence the Goodwillie calculus towers of functors from cocommutative co-H-objects to spectra split upon $K(n)_*$-localization.

This paper is organized as follows. In section 2 we define and examine the cross effects of a functor. It is important to note that in this section we do not construct the universal degree n approximations to F. However, we do provide all of the results and constructions that we will need in the context of this paper. A more detailed account can be had from [JM4] and [M] (for the case where the target category is spectra) or [McC02]. In section 3 we describe the norm map and its relationship to the Tate map, which will be essential in constructing the splitting map to $D_nF(X) \to P_nF(X)$. Here we only provide statements of the results we will need regarding these maps; a more detailed account of the Tate map is given in the appendix of the preprint version [BMc]. In section 4 we make explicit our definition of co-H-objects. We also check that the properties we require of F are preserved by cr_n, D_n and P_n. Section 5 is the statement and proof of the main theorem. We conclude by giving two extensions of the main theorem; first the extension to dual calculus (section 6) and finally to the case of vanishing Tate cohomology (section 7).

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2 Preliminaries in Goodwillie Calculus

Let $\mathcal{S}$ be any category of rational spectra, that is, modules over the Eilenberg-MacLane spectrum $H\mathbb{Q}$. Let F be a functor from a pointed category $\mathcal{C}$ to $\mathcal{S}$. The building blocks used to construct the universal degree n approximation to F are called the cross effects. In this paper, we will usually phrase all important information about the Goodwillie tower of F in terms of cross effects. For this reason, it is important for us to spend some time introducing them.

It is easiest to think of the cross effects as the total fibers of certain cubical diagrams. Let $\mathbf{n} = \{1, \ldots, n\}$ be a finite set of n elements. Let $P(\mathbf{n})$ be the category whose objects are the subsets of $\mathbf{n}$ and whose morphisms are ordered inclusions. Note that the subset $\emptyset$ is an initial object of $P(\mathbf{n})$ while $\mathbf{n}$ is the terminal object. One can visualize this category as a cubical diagram with an object of $P(\mathbf{n})$ at each corner and morphisms as edges. The objects $\emptyset$ and $\mathbf{n}$ occupy opposite corners of the cube.

An $\mathbf{n}$-cube in a pointed category $\mathcal{C}$ (i.e. a category whose initial object is also its terminal object) with coproducts is a contravariant functor from $P(\mathbf{n})$ to $\mathcal{C}$. To construct the cross effects, we will use a particular $\mathbf{n}$-cube. Let $\vee$ denote the coproduct in $\mathcal{C}$. Let $g : \mathbf{n} \to \mathrm{obj}(\mathcal{C})$ be a function which generates a list of objects $X_1, \ldots, X_n$ of $\mathcal{C}$. Define an $\mathbf{n}$-cube χ_g to be the contravariant functor which is given on a subset S of $\mathbf{n}$ by

$$\chi_g(S) = \bigvee_{c \notin S} g(c)$$

and which takes an inclusion $S \hookrightarrow S'$ to the projection

$$\bigvee_{d \notin S} g(d) \to \bigvee_{c \notin S'} g(c).$$

By convention, $\chi_g(\mathbf{n}) = *$ (the base point of $\mathcal{C}$). For $n = 2$, χ_g is the square diagram:

$$\begin{tikzcd} X_1 \vee X_2 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & * \end{tikzcd}$$

Let F be a covariant functor from $\mathcal{C}$ to $\mathcal{S}$. Let $P_0(\mathbf{n})$ be the subcategory of $P(\mathbf{n})$ consisting of non-empty subsets and let $X_i = g(i)$.

Definition 2.1. The n-th cross effect, $\mathrm{cr}_n F(X_1, \ldots, X_n)$, is the fiber of the map

$$\mathrm{holim}_{S \in P(\mathbf{n})} F(\chi_g(S)) \to \mathrm{holim}_{S \in P_0(\mathbf{n})} F(\chi_g(S)).$$

Note that $\mathrm{holim}_{S \in P(\mathbf{n})} F(\chi_g(S)) = F(\vee_{i=1}^n X_i)$.

The cross effect $\mathrm{cr}_n F(X_1, \ldots, X_n)$ is an n multi-variable functor from $C^{\times n}$ to $\mathcal{S}$. Let $\mathrm{cr}_n F(X) = \mathrm{cr}_n F(X, \ldots, X)$. Notice that there is a convenient map $\rho : \mathrm{cr}_n F(X) \to F(\vee_n X)$ arising from the definition of the cross effect.

Definition 2.2. A functor $F : \mathcal{C} \to \mathcal{S}$ is a degree n functor if $\mathrm{cr}_k F \simeq *$ for every $k > n$.

If F is degree n, then the Goodwillie tower of F is truncated - i.e. $D_n F$ is the largest non-trivial layer of F and $P_k F$ is equivalent to F for all $k \geq n$. This is sometimes taken as the definition of degree n. Furthermore, if F is degree n, then $\mathrm{cr}_n F$ is linear in each variable. That is, e.g.

$$\begin{aligned} &\mathrm{cr}_n F(X \vee Y, X_2, \ldots, X_n) \\ &\qquad \simeq \mathrm{cr}_n F(X, X_2, \ldots, X_n) \times \mathrm{cr}_n F(Y, X_2, \ldots, X_n). \quad (1) \end{aligned}$$

We enumerate here some properties of the cross effects and Goodwillie towers of degree n functors which we will frequently use.

Lemma 2.3. *If F is a degree n functor, then there exists a natural equivalence*

$$\mathrm{cr}_n F(\bigvee_{i=1}^{m} X_i) \;\simeq\; \prod_{\alpha\in \mathrm{Hom}(\mathbf{n},\mathbf{m})} \mathrm{cr}_n F(X_{\alpha(1)},\dots,X_{\alpha(n)}) \tag{2}$$

Proof. This is a consequence of the fact that for degree n functors, $\mathrm{cr}_n F(X)$ is linear in each variable. □

Remark 2.4. Let $u_k : \vee_{i=1}^m X_i \to X_k$ $(1 \le k \le m)$ be the map which is the identity on the X_k component of $\vee_{i=1}^m X_i$ and which sends X_j to the basepoint for $j \neq k$. The weak equivalence of Lemma 2.3 can be realized by the map $\omega : \mathrm{cr}_n F(\vee_{i=1}^m X_i) \to \prod_{\alpha\in\mathrm{Hom}(\mathbf{n},\mathbf{m})} \mathrm{cr}_n F(X_{\alpha(1)},\dots,X_{\alpha(n)})$ defined by

$$\omega = \Big(\prod_{\alpha\in\mathrm{Hom}(\mathbf{n},\mathbf{m})} \mathrm{cr}_n F(u_{\alpha(1)},\dots,u_{\alpha(n)})\Big)\circ\Delta$$

where Δ is the diagonal map $\mathrm{cr}_n F(\vee_{i=1}^m X_i) \to \prod_{\mathrm{Hom}(n,m)} \mathrm{cr}_n F(\vee_{i=1}^m X_i)$.

Let $c_k : X_k \to \vee_{i=1}^m X_i$ be the dual map to u_k which includes the k-th summand. Keeping in mind that coproducts and products in $\mathcal{S}$ are weakly equivalent, we construct a homotopy inverse to ω given by the composition of

$$\coprod_{f\in\mathrm{Hom}(\mathbf{n},\mathbf{m})} \mathrm{cr}_n F(X) \xrightarrow{\coprod_{f\in\mathrm{Hom}(\mathbf{n},\mathbf{m})} \mathrm{cr}_n F(c_{f(1)},\dots,c_{f(n)})} \coprod_{f\in\mathrm{Hom}(\mathbf{n},\mathbf{m})} \mathrm{cr}_n F(\bigvee_m X)$$

with the fold map.

Lemma 2.5. *If F is a degree n functor, then there exists a natural equivalence*

$$\mathrm{cr}_n \mathrm{cr}_n F(X_1,\dots,X_n) \;\simeq\; \prod_{\sigma\in\Sigma_n} \mathrm{cr}_n F(X_{\sigma(1)},\dots,X_{\sigma(n)}). \tag{3}$$

Proof. Let $g(i) = X_i$ and let cS denote the complement in $\mathbf{n}$ of a subset S of $\mathbf{n}$. Using the definition of the cross effect and lemma 2.3, we compute that

$$\mathrm{cr}_n\mathrm{cr}_n F(X_1,\dots,X_n) = \mathrm{fiber}\{\operatorname*{holim}_{S\in P(\mathbf{n})} \mathrm{cr}_n F(\bigvee_{c\in cS} X_c) \to \operatorname*{holim}_{S\in P_0(\mathbf{n})} \mathrm{cr}_n F(\bigvee_{c\in cS} X_c)\}$$

is equivalent to the fiber of

$$\operatorname*{holim}_{S\in P(\mathbf{n})}\prod_{\substack{\sigma\in\\ \mathrm{Hom}(\mathbf{n},cS)}}\mathrm{cr}_nF(X_{\sigma(1)},\dots,X_{\sigma(n)})\to\operatorname*{holim}_{S\in P_0(\mathbf{n})}\prod_{\substack{\sigma\in\\ \mathrm{Hom}(\mathbf{n},cS)}}\mathrm{cr}_nF(X_{\sigma(1)},\dots,X_{\sigma(n)}).$$

The first homotopy limit is just $\prod_{\sigma\in\mathrm{Hom}(\mathbf{n},\mathbf{n})}\mathrm{cr}_nF(X_{\sigma(1)},\dots,X_{\sigma(n)})$ (since this limit diagram has an initial object) and the second homotopy limit is $\prod_{\sigma\in\mathrm{Hom}_0(\mathbf{n},\mathbf{n})}\mathrm{cr}_nF(X_{\sigma(1)},\dots,X_{\sigma(n)})$ where $\mathrm{Hom}_0(\mathbf{n},\mathbf{n})$ are the non-bijective set maps from $\mathbf{n}$ to itself. So this is equivalent to the fiber of

$$\prod_{\sigma\in\mathrm{Hom}(\mathbf{n},\mathbf{n})}\mathrm{cr}_nF(X_{\sigma(1)},\dots,X_{\sigma(n)})\to\prod_{\sigma\in\mathrm{Hom}_0(\mathbf{n},\mathbf{n})}\mathrm{cr}_nF(X_{\sigma(1)},\dots,X_{\sigma(n)})$$

which is

$$\prod_{\sigma\in\mathrm{Bij}(\mathbf{n},\mathbf{n})}\mathrm{cr}_nF(X_{\sigma(1)},\dots,X_{\sigma(n)})$$

where $\mathrm{Bij}(\mathbf{n},\mathbf{n}):=\Sigma_n$ is the group of bijective maps. □

Remark 2.6. Using remark 2.4, one can realize the weak equivalence of 2.5 by a map $\omega':\mathrm{cr}_n\mathrm{cr}_nF(X)\to\prod_{\mathrm{Bij}(n,n)}\mathrm{cr}_nF(X)$ defined by

$$\omega'=\Big(\prod_{\alpha\in\mathrm{Bij}(n,n)}\mathrm{cr}_nF(u_{\alpha(1)},\dots,u_{\alpha(n)})\Big)\circ\Delta\circ\rho$$

where $\rho:\mathrm{cr}_n\mathrm{cr}_nF(X)\to\mathrm{cr}_nF(\vee_nX)$ is the natural map. This is accomplished by applying the map ω to $\mathrm{cr}_nF(\vee_{c\in cS}X_c)$ for each $S\in P(\mathbf{n})$ (or $P_0(\mathbf{n})$).

Proposition 2.7 (Lemma 3.9 [JM4]). *If F is a degree n functor from $\mathcal{C}$ to $\mathcal{S}$ then $D_nF\simeq(\mathrm{cr}_nF)_{h\Sigma_n}$.*

Proposition 2.8 (Remark 2.8 [JM4]). *If F is a degree n functor from $\mathcal{C}$ to $\mathcal{S}$ then $F\simeq P_nF$.*

By using fibers instead of cofibers in the definition of the cross effect, we obtain another cross effect, called the co-cross effect, $\widetilde{\mathrm{cr}}_n$. Let $P_1(\mathbf{n})$ denote the full subcategory of $P(\mathbf{n})$ consisting of proper subsets of $\mathbf{n}$. Let $\widetilde{\chi}$ be the covariant functor from $P(\mathbf{n})$ to $\mathcal{C}$ with $\widetilde{\chi}(S)=\bigvee_{c\in S}g(c)$ and which takes inclusions in $P(\mathbf{n})$ to inclusions in $\mathcal{C}$.

Definition 2.9. We define the co-cross effect $\widetilde{\mathrm{cr}}_n F(X_1, \ldots, X_n)$, to be the **cofiber** of the map

$$\mathrm{holim}_{S \in P_1(\mathbf{n})} F(\widetilde{\chi}_g(S)) \to \mathrm{holim}_{S \in P(\mathbf{n})} F(\widetilde{\chi}_g(S)).$$

In the category of spectra, since cofibration and fibration sequences are equivalent, we have that $\mathrm{cr}_n F \simeq \widetilde{\mathrm{cr}}_n F$. There is a convenient map

$$F(\vee_n X) \to \widetilde{\mathrm{cr}}_n F(X)$$

(dual to the map ρ) which we will need.

Lemma 2.10. *If F is a degree n functor, then there exist natural equivalences*

$$\widetilde{\mathrm{cr}}_n F(\bigvee_n X) \simeq \coprod_{\mathrm{Hom}(n,n)} \widetilde{\mathrm{cr}}_n F(X) \tag{4}$$

$$\widetilde{\mathrm{cr}}_n \widetilde{\mathrm{cr}}_n F(X) \simeq \coprod_{\Sigma_n} \widetilde{\mathrm{cr}}_n F(X). \tag{5}$$

Proof. This is the analogue of lemmas 2.3 and 2.5, and the proof is straightforward. $\square$

3 The Tate Map

If a finite group G acts on a k-module M, there is a natural map $t : M_G \to M^G$ induced by the norm map. The norm map is constructed using the diagonal Δ, the action of g for each $g \in G$, and the addition map $+$ as follows:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[rrr, bend left, "t'"] & \oplus_{|G|} M \oplus_{g \in G} g \arrow[r] & \oplus_{|G|} M \arrow[r, "+"] & M \end{tikzcd}$$

We can extend this to a map $t : M_G \to M^G$ by making two observations. First, t' extends in the following diagram (where p is the projection map onto the orbits)

$$\begin{tikzcd} M \arrow[r, "t'"] \arrow[d, "p"'] & M \\ M_G \arrow[ur, dashed] & \end{tikzcd}$$

since for each $g \in G$, $t'(m) = t'(gm)$. Second, note that the image of t' actually lands in M^G since $t'(m) = \sum_{g \in G} gm$ so that for any $h \in G$,

$$ht'(m) = h \sum_{g \in G} gm = \sum_{g \in G} hgm = \sum_{g \in G} gm = t'(m)$$

since G is finite. Thus we have a map t which factors t' as:

$$\begin{tikzcd} M \arrow[r, "t'"] \arrow[d, "p"'] & M \\ M_G \arrow[r, "t"] & M^G \arrow[u, hook, "i"'] \end{tikzcd}$$

where i is the inclusion of the fixed points. The map t is an equivalence whenever the order of G is invertible. In fact, the inverse is given by $p \circ i$ and we have $p \circ i \circ t[m] = |G|[m]$.

We wish to extend this map to the category $\mathcal{S}$ of $H\mathbb{Q}$-modules. Suppose that E is an $H\mathbb{Q}$-module with an action of a finite group G. We can extend the norm map T'. Again, we do this by using the diagonal map and the fold map. This time, we must use the weak equivalence between the product and the coproduct. The following diagram defines T':

$$E \xrightarrow{\Delta} \prod_{g \in G} E \xrightarrow{\prod_{g \in G} g \times (-)} \prod_{g \in G} E \xhookleftarrow{\simeq} \coprod_{g \in G} E \xrightarrow{+} E$$

Motivated by this diagram, we make the following definition

Definition 3.1. Let I be an indexing set and $\{X_i\}_{i \in I}$ be a collection of objects of $\mathcal{S}$. Let $f_{i,j} : X_i \to X_j$ be maps in $\mathcal{S}$ for each $i, j \in I$ and $g_{i,j} : X_i \xleftarrow{\simeq} X_j$ be weak equivalences in $\mathcal{S}$ for each $i, j \in I$. A *weak map* in $\mathcal{S}$ from $X_{i_m} \to X_{i_n}$ is a collection of objects $X_{i_m}, X_{i_{m+1}}, \ldots, X_{i_{m+k}} = X_{i_n}$ and arrows $f_{i,j}$ and $g_{i,j}$ such that for any adjacent pair $X_{i_{m+j}}$, $X_{i_{m+j+1}}$ of objects in the collection, there is either a map $f_{i_{m+j}, i_{m+j+1}}$ or a weak equivalence $g_{i_{m+j}, i_{m+j+1}}$ between them.

A typical weak map might look like a zig-zag

$$X_i \xrightarrow{f_{i,j}} X_j \xleftarrow[\simeq]{g_{j,k}} X_k.$$

This weak map is denoted $X_i \overset{g_{j,k}^{-1}\circ f_{i,j}}{\dashrightarrow} X_j$, where the dashed line denotes that it's a weak map.

Note that each weak map has associated to it a map in the homotopy category. We say that a diagram of weak maps commutes if the corresponding diagram of maps in the homotopy category commutes. There is a weak map T which extends the weak map T' to homotopy orbits and homotopy fixed points just as t extended t' in the case of modules. This weak map is called the Tate (weak) map. A construction of t can be found in e. g. [McC].

We will need the following properties of the Tate map. The proofs of these can be found in [McC] or [BMc].

Lemma 3.2. *The Tate map is the restriction of the Norm map to homotopy orbits. That is, the following diagram of weak maps commutes:*

$$\begin{tikzcd} E \arrow[r, "\Delta"] \arrow[d] & \prod_G E \arrow[r, "\prod_{g\in G} g"] & \prod_G E \arrow[r, dashed, "\simeq"] & \coprod_G E \arrow[r, "+"] & E \\ E_{hG} \arrow[rrrr, dashed, "T"] & & & & E^{hG} \arrow[u, hook] \end{tikzcd}$$

where T refers to the composite that defines the Tate map and the composite of the top (weak) maps is the norm map.

Proposition 3.3. *Let Σ_n be the n-th symmetric group. The composition*

$$T_{h\Sigma_n} \xrightarrow{t} T^{h\Sigma_n} \longrightarrow T \longrightarrow T_{h\Sigma_n}$$

induces multiplication by $n!$ on homotopy groups.

4 The category of homotopy comonoids

Let $\mathcal{C}$ be a pointed model category.

Definition 4.1. Let X be a cofibrant object of $\mathcal{C}$. We say that X is a *co-H-object* of $\mathcal{C}$ if there exists a map $\nabla : X \to X \vee X$ which is coassociative up to homotopy, and counital up to homotopy with counit $c : X \to *$.

The co-H-objects are the comonoids of $\mathcal{C}$ (up to homotopy). They form a subcategory of $\mathcal{C}$ whose morphisms from X to Y are the maps of $f \in \mathcal{C}$ which make the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\nabla_X"] \arrow[d, "f"'] & X \vee X \arrow[d, "f \vee f"] \\ Y \arrow[r, "\nabla_Y"'] & Y \vee Y \end{tikzcd} .$$

Since the fold map is unital with respect to the unit $u : * \to X$, we can think of the fold map as a unital multiplication with which $\mathcal{C}$ is already equipped. The co-H-objects of $\mathcal{C}$ are exactly those whose comultiplication maps ∇ act like algebra maps with respect to the multiplication map defined by the fold map. In other words, for a co-H-object X the following diagram commutes up to homotopy:

$$\begin{tikzcd} \vee_n X \arrow[r, "\vee_n \nabla^n"] \arrow[d, "+^n"'] & \vee_n(\vee_n) \arrow[r, "\tau"] & \vee_n(\vee_n X) \arrow[d, "\vee_n +^n"] \\ X \arrow[rr, "\nabla^n"] & & \vee_n X \end{tikzcd} . \tag{6}$$

If one arranges $\vee_{\mathrm{n}} \vee_{\mathrm{n}} X$ in an $n \times n$-array, one can think of the map τ as the transpose map. It is a reordering of the copies of X.

The first examples of co-H-objects are co-H-spaces in the category of pointed topological spaces. In particular, the basepointed circle S^1 is a co-H-space. The map ∇ in this case is the pinch map which identifies the basepoint with its antipodal point. Since the circle is a co-H-space, so are all suspensions $\Sigma X = S^1 \wedge X$ using the map $\nabla \wedge 1$.

For the main theorem, we will be considering a co-H-object X and a functor $F : \mathcal{C} \to \mathcal{S}$ which preserves weak equivalences. If F preserves weak equivalences, then when F is applied to diagram 6 the resulting diagram still commutes up to homotopy. The following lemma will allow us to use F, $\mathrm{cr}_n F$, P_n and $D_n F$ interchangeably with respect to diagrams which commute up to homotopy.

Lemma 4.2. *Let $F : \mathcal{C} \to \mathcal{S}$ be a functor which preserves weak equivalences. Then $\mathrm{cr}_{n+1} F$, $P_n F$ and $D_n F$ preserve weak equivalences between cofibrant objects.*

Proof. Recall (Definition 2.1) that $\mathrm{cr}_{n+1}F(X)$ is the fiber of

$$\mathrm{holim}_{P(\mathbf{n+1})}F(\chi_g) \to \mathrm{holim}_{P_0(\mathbf{n+1})}F(\chi_g).$$

We claim that if $\alpha : A \to B$ is a weak equivalence then $\mathrm{cr}_{n+1}F(\alpha) : \mathrm{cr}_{n+1}F(A) \to \mathrm{cr}_{n+1}F(B)$ is also a weak equivalence, as long as A and B are both cofibrant. This is true because if A and B are both cofibrant, then

$$\vee_k \alpha : \vee_k A \to \vee_k B$$

is again a weak equivalence for all $k \leq n+1$. Thus α induces a weak equivalence on each term $F(\chi_g)$ of the cube defining $\mathrm{cr}_{n+1}F$, and hence induces weak equivalences on $\mathrm{holim}_{P(\mathbf{n+1})}F(\chi_g)$ and $\mathrm{holim}_{P_0(\mathbf{n+1})}F(\chi_g)$. Then the map induced on the fiber is also a weak equivalence.

Each of the remaining constructions defining P_nF and D_nF is a homotopy construction involving only $\mathrm{cr}_{n+1}F$ and F.

□

Remark 4.3. We only require $\mathcal{C}$ to be a model category in order to provide us with the correct notion of "weak equivalences" in $\mathcal{C}$, and co-H-objects are only cofibrant to insure that diagrams involving co-H-objects which commute up to homotopy will still commute up to homotopy after P_nF or other related functors have been applied. We can also define co-H-objects in categories $\mathcal{C}$ which are not model categories by requiring that all of the maps involved in the definition commute up to isomorphism. Since all functors preserve isomorphisms, lemma 4.2 is then unnecessary. For example, in the category of commutative algebras over k, the co-H-objects are almost the Hopf algebras. Notice that the co-H-objects only differ from Hopf algebras because they lack the antipodal map with which Hopf algebras are equipped.

5 The Splitting Theorem

We can now state the main Theorem.

Theorem 5.1. *If F is a functor from $\mathcal{C}$ to $\mathcal{S}$ and X is a co-H-object of $\mathcal{C}$, then rationally the fiber sequence*

$$D_nF(X) \xrightarrow{j} P_nF(X) \xrightarrow{q_n} P_{n-1}F(X)$$

splits on the homotopy category. Consequently,

$$P_\infty F(X) \simeq \prod_{n\geq 0} D_n F(X)$$

is a rational equivalence.

Without loss of generality, we may assume F is a degree n functor by replacing F with P_nF. We remark that this replacement is why Lemma 4.2 is needed.

The map j exists because D_nF is defined to be the fiber of the map q_n. We can reformulate j ([JM4]) in terms of cross effects. Since F is degree n, there are equivalences $D_nF(X) \simeq (\mathrm{cr}_n F(X))_{h\Sigma_n}$ and $P_nF(X) \simeq F(X)$. Thus, the above fiber sequence becomes

$$(\mathrm{cr}_n F(X))_{h\Sigma_n} \xrightarrow{j} F(X) \longrightarrow P_{n-1}F(X)$$

for each n. Recall that there is a map $\rho : \mathrm{cr}_n F(X) \to F(\bigvee_n X)$. The map j is induced by ρ and the fold map as follows:

$$\begin{tikzcd}
\mathrm{cr}_n F(X) \arrow[rr, "\rho"] \arrow[dd, two heads] & & F(\bigvee_n X) \arrow[dl, two heads] \arrow[dd, "F(+)"] \\
& F(\bigvee_n X)_{h\Sigma_n} \arrow[dr] & \\
(\mathrm{cr}_n F(X))_{h\Sigma_n} \arrow[ur, "\rho_{h\Sigma_n}"] \arrow[rr, "j"] & & F(X)
\end{tikzcd} \tag{7}$$

The Σ_n action on $F(\vee_n X)$ permutes copies of X. However, the fold map $F(+) : F(\vee_n X) \longrightarrow F(X)$ has $F(+)(F(\sigma \cdot \vee_n X)) = F(+)(F(\vee_n X))$. Hence $F(+)$ factors through homotopy orbits. Since the map ρ is Σ_n-equivariant, ρ extends to a map $\rho_{h\Sigma_n}$ on orbits. The resulting map j is the map for which we seek to provide a splitting.

When X is a co-H-object, we may extend this diagram to:

$$\begin{tikzcd}
\mathrm{cr}_n F(X) \arrow[r, hook, "\rho"] \arrow[d] & F(\bigvee_n X) \arrow[r, "\tau\circ F(\bigvee_n \nabla^n)"] \arrow[d, "F(+^n)"] & F(\bigvee_n(\bigvee_n X)) \arrow[d, "F(\bigvee_n +^n)"] \\
\mathrm{cr}_n F(X)_{h\Sigma_n} \arrow[r, hook, "j"] & F(X) \arrow[r, "F(\nabla^n)"] & F(\bigvee_n X) \arrow[d, "\delta"] \\
 & & \widetilde{\mathrm{cr}}_n F(X)
\end{tikzcd} \tag{8}$$

where δ is the map dual to ρ (see definition 2.9).

Recall that for functors whose target category is $\mathcal{S}$, $\mathrm{cr}_n F(X) \simeq \widetilde{\mathrm{cr}}_n F(X)$. We wish to show that the composite $\delta \circ F(\nabla^n) \circ F(+^n) \circ \rho$ is the norm map T' of section 3. If this is the case, then the following diagram commutes (as a diagram of weak maps)

$$\begin{tikzcd}
\mathrm{cr}_n F(X) \arrow[r, "\rho"] \arrow[d, "q"'] & F(\bigvee_n X) \arrow[d, "F(+)"] & \\
\mathrm{cr}_n F(X)_{h\Sigma_n} \arrow[r, "j"] \arrow[dr, "T"] & F(X) \arrow[r, "F(\nabla^n)"] & F(\bigvee_n X) \arrow[d, "\delta"'] \\
 & \widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n} \arrow[r, "i"] & \widetilde{\mathrm{cr}}_n F(X)
\end{tikzcd} \tag{9}$$

where T is the Tate map. We note that the lower trapezoid of this diagram also commutes. The key point is that the Tate map, extended to a map from homotopy orbits to homotopy orbits (by including the fixed points into the orbits) induces multiplication by $n!$ on π_*, which is rationally invertible. Thus, if f and g are two maps from $\mathrm{cr}_n F(X)_{h\Sigma_n}$ to $\widetilde{\mathrm{cr}}_n F(X)$ such that $f \circ q \simeq g \circ q$ then since $q \circ i \circ T$ is rationally invertible, we actually have that $f \simeq g$. Furthermore, the fact that $q \circ i \circ T \simeq n!$ provides the splitting.

Using the square diagram 6 from section 4 as a central square, we can further extend the diagram to

$$\begin{tikzcd}
 & & \mathrm{cr}_n\mathrm{cr}_n F(X) \arrow[d, "\mathrm{cr}_n(p)"] \arrow[ddrr, "\eta_\tau"] & & \\
 & \mathrm{cr}_n F(X) \arrow[ur, "\eta_\Delta"] \arrow[r] \arrow[d, "\rho"] \arrow[dr, phantom, "A"] & \mathrm{cr}_n F(\bigvee_n X) \arrow[d, "\rho"] \arrow[dr, phantom, "C"] & & \\
\mathrm{cr}_n F(X) \arrow[ur, "="] \arrow[r, "\rho"] \arrow[d] & F(\bigvee_n X) \arrow[r, "\tau\circ F(\bigvee_n \nabla^n)"'] \arrow[d, "F(+^n)"'] & F(\bigvee_n \bigvee_n X) \arrow[r, "\delta"] \arrow[d, dashed, "F(\bigvee_n +^n)"'] \arrow[dr, phantom, "B"] & \widetilde{\mathrm{cr}}_n F(\bigvee_n X) \arrow[r, "\mathrm{cr}_n(\delta)"] \arrow[d] & \widetilde{\mathrm{cr}}_n\widetilde{\mathrm{cr}}_n F(X) \arrow[dl, "\eta_+"] \\
\mathrm{cr}_n F(X)_{h\Sigma_n} \arrow[r, "j"] \arrow[dr, "T"] \arrow[ddrr, bend right, "\times n!"'] & F(X) \arrow[r, "F(\nabla^n)"] & F(\bigvee_n X) \arrow[r, "\delta"] \arrow[d, "\delta"'] & \widetilde{\mathrm{cr}}_n F(X) \arrow[dl, "="] & \\
 & \widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n} \arrow[r] & \widetilde{\mathrm{cr}}_n F(X) \arrow[d, two heads] & & \\
 & & \widetilde{\mathrm{cr}}_n F(X)_{h\Sigma_n} & &
\end{tikzcd} \tag{10}$$

The squares A and B commute by the functoriality of cr_n (resp. $\widetilde{\mathrm{cr}}_n$). Let η_τ be the composition of maps which makes the triangle C commute. We will show that the lifts η_Δ and η_+ exist. Then, by the commutativity of the diagram, it will suffice to show that $\eta = \eta_+ \circ \eta_\tau \circ \eta_\Delta$ is the norm map.

Lemma 5.2. *There exists a lift η_Δ such that the diagram*

$$\begin{tikzcd}
 & \mathrm{cr}_n\mathrm{cr}_n F(X) \arrow[d, hook] \\
\mathrm{cr}_n F(X) \arrow[ur, "\eta_\Delta"] \arrow[r, "\mathrm{cr}_n F(\nabla^n)"] \arrow[d, hook] & \mathrm{cr}_n F(\bigvee_n X) \arrow[d, hook] \\
F(\bigvee_n X) \arrow[r, "F(\bigvee_n \nabla^n)"] & F(\bigvee_n \bigvee_n X)
\end{tikzcd}$$

commutes. Moreover, η_Δ is homotopic to a diagonal map.

Proof. First, notice that by Remark 2.6 the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathrm{cr}_n\mathrm{cr}_n F(X) \arrow[r, "\omega'", "\simeq"'] \arrow[d, "\rho"'] & \prod_{\mathrm{Bij}(n,n)} \mathrm{cr}_n F(X) \\
\mathrm{cr}_n F(\bigvee_n X) \arrow[r, "\Delta"] & \prod_{\mathrm{Bij}(n,n)} \mathrm{cr}_n F(\bigvee_n X) \arrow[u, "u"']
\end{tikzcd}$$

where Δ is the diagonal map and $u = \prod_{\alpha\in\mathrm{Bij}(n,n)} \mathrm{cr}_n F(u_{\alpha(1)}, \ldots, u_{\alpha(n)})$ is built of the maps u_i of Remark 2.4.

If we rearrange this diagram, one sees that this provides a lift:

$$\begin{tikzcd}
& \prod_{\mathrm{Bij}(n,n)} \mathrm{cr}_n F(X) & \mathrm{cr}_n\mathrm{cr}_n F(X) \arrow[l, "\omega'"', "\simeq"] \arrow[dl, "\rho"] \\
\mathrm{cr}_n F(X) \arrow[ur, "\eta_\Delta"] \arrow[r, "\mathrm{cr}_n F(\nabla)"] & \mathrm{cr}_n F(\bigvee_n X) \arrow[u, "u\circ\Delta"'] &
\end{tikzcd}$$

In fact, since X is a co-H-object, the map ∇ is counital. Therefor $u_i \circ \nabla$ is homotopic to the identity on X for each $1 \le i \le n$. It follows that η_Δ is homotopic to the diagonal map. We will not distinguish between the lift η_Δ and the associated map $(\omega')^{-1} \circ \eta_\Delta$.

□

Lemma 5.3. *There exists a lift η_+ such that the diagram*

$$\begin{tikzcd}
F(\bigvee_n \bigvee_n X) \arrow[r] \arrow[d, "F(+)"'] & \widetilde{\mathrm{cr}}_n F(\bigvee_n X) \arrow[r] \arrow[d, "\mathrm{cr}_n F(+)"'] & \widetilde{\mathrm{cr}}_n\widetilde{\mathrm{cr}}_n F(X) \arrow[dl, "\eta_+"] \\
F(\bigvee_n X) \arrow[r] & \widetilde{\mathrm{cr}}_n F(X) &
\end{tikzcd}$$

commutes. Moreover, η_+ is homotopic to a fold map.

Proof. This is formally dual to Lemma 5.2, using Lemma 2.10. The details of the proof are left as an exercise to the reader. □

The map τ with

$$\tau : \prod_{f\in\mathrm{Bij}(n,n)} \mathrm{cr}_n F(X) \to \prod_{f\in\mathrm{Bij}(n,n)} \mathrm{cr}_n F(X)$$

is given on the factor indexed by $f \in \mathrm{Bij}(n, n)$ by shuffling each of the n variables of $\mathrm{cr}_n F(X)$ by f. Call τ the twist map.

Lemma 5.4. *The map η_τ, which is the composite*

$$\begin{tikzcd}
\mathrm{cr}_n\mathrm{cr}_n F(X) \arrow[r, "\omega'", "\simeq"'] \arrow[d] & \prod\limits_{\mathrm{Bij}(n,n)} \mathrm{cr}_n F(X) \arrow[d, hook] \arrow[ddrr, "\eta_\tau"] & & \\
\mathrm{cr}_n F(\bigvee\limits_n X) \arrow[r, "\omega", "\simeq"'] & \prod\limits_{\mathrm{Hom}(n,n)} \mathrm{cr}_n F(X) \arrow[d, "\bar{\rho}"'] \arrow[dr, "\eta'_\tau"] & & \\
& F(\bigvee\limits_n \bigvee\limits_n X) \arrow[r, "\bar{d}"'] & \prod\limits_{\mathrm{Hom}(n,n)} \widetilde{\mathrm{cr}}_n F(X) \arrow[r, two heads] \arrow[d, "\simeq"'] & \prod\limits_{\mathrm{Bij}(n,n)} \widetilde{\mathrm{cr}}_n F(X) \arrow[d, "\simeq"] \\
& & \widetilde{\mathrm{cr}}_n F(\bigvee\limits_n X) \arrow[r] & \widetilde{\mathrm{cr}}_n \widetilde{\mathrm{cr}}_n F(X)
\end{tikzcd}$$

is the twist map.

Proof. Note that here we have used Lemma 2.10 and the fact that coproducts are weakly equivalent to products in the lower corner of the diagram.

We will begin by examining the map η'_τ. Recall that $\rho : \mathrm{cr}_n F(X) \to F(\vee_n X)$ and $\delta : F(\vee_n X) \to \widetilde{\mathrm{cr}}_n F(X)$ are the usual maps. On the factor of $\prod_{\mathrm{Hom}(n,n)} \mathrm{cr}_n F(X)$ indexed by f, notice that $\omega^{-1} : \mathrm{cr}_n F(X) \to \mathrm{cr}_n F(\vee_n X)$ is given by $\mathrm{cr}_n F(c_{f(1)}, \ldots, c_{f(n)})$ where $c_i : X \to \vee_n X$ is the inclusion into the i-th summand. Let $c_f := c_{f(1)} \vee \cdots \vee c_{f(n)}$. Then the map $\bar{\rho}$ is given on the f-th factor by either of the composites

$$\begin{tikzcd}
\mathrm{cr}_n F(X) \arrow[r, "\omega^{-1}", "\mathrm{cr}_n F(c_{f(1)}, \ldots, c_{f(n)})"'] \arrow[d, "\rho"'] & \mathrm{cr}_n F(\bigvee\limits_n X) \arrow[d, "\rho"] \\
F(\bigvee\limits_n X) \arrow[r, "F(c_f)"] & F(\bigvee\limits_n \bigvee\limits_n X)
\end{tikzcd}$$

Similar computations provide a factorization of $\bar{d}$, projected onto the g-th factor, using the map $u_i : \vee_n X \to X$ which is the identity on the i-th summand and 0 elsewhere (this is dual to c_i). Let $u_g := u_{g(1)} \vee \cdots \vee u_{g(n)}$. We obtain a commuting diagram

$$\begin{tikzcd}
\mathrm{cr}_n F(X) \arrow[r, "\omega^{-1}", "\mathrm{cr}_n F(c_{f(1)},\dots,c_{f(n)})"'] \arrow[d, "\rho"'] & \mathrm{cr}_n F(\bigvee_n X) \arrow[d, "\rho"] & \\
F(\bigvee_n X) \arrow[r, "F(c_f)"] \arrow[dr, dashed] & F(\bigvee_n \bigvee_n X) \arrow[r, "d"] \arrow[d, "F(u_g)"] & \widetilde{\mathrm{cr}}_n F(\bigvee_n X) \arrow[d, "\widetilde{\mathrm{cr}}_n F(u_{g(1)},\dots,u_{g(n)})"] \\
 & F(\bigvee_n X) \arrow[r, "d"] & \widetilde{\mathrm{cr}}_n F(X)
\end{tikzcd} \tag{11}$$

The key point now is to understand the composition represented by the dashed arrow. By functoriality, we need only understand the composition $(u_g) \circ (c_{f(1)} \vee \cdots \vee c_f)$. To understand this composition, it is easiest to introduce the labels

$$\begin{tikzcd}
\begin{pmatrix} X_{1,*} \\ \vee \\ \vdots \\ \vee \\ X_{n,*} \end{pmatrix} \arrow[r, "(c_f)"] & \begin{pmatrix} X_{1,1} & \vee & \dots & \vee & X_{1,n} \\ \vee & & & & \vee \\ \vdots & & & & \vdots \\ \vee & & & & \vee \\ X_{n,1} & \vee & \dots & \vee & X_{n,n} \end{pmatrix} \arrow[d, "(u_g)"] \\
 & \begin{pmatrix} X_{*,1} & \vee & \dots & \vee & X_{*,n} \end{pmatrix}
\end{tikzcd}$$

keeping in mind that $X_{*,*} = X$ for all choices of $*$'s. On any summand, $X_{i,*}$, we have

$$X_{i,*} \xrightarrow{c_{f(i)}} X_{i,1} \vee \cdots \vee X_{i,n} \xrightarrow{u_{g(f(i))}} X_{g(f(i)),f(i)}$$

which is non-zero if and only if $g(f(i)) = i$. Since this is true for all $1 \leq i \leq n$, this implies that f has an inverse and that $g = f^{-1}$. That is, $f \in \mathrm{Bij}(n, n)$.

Furthermore, the composition of $u_{f^{-1}}$ with c_f yields

$$\begin{pmatrix} X_{1,*} \\ \vee \\ \vdots \\ \vee \\ X_{n,*} \end{pmatrix} \longrightarrow \left(X_{1,f(1)} \quad \vee \quad \ldots \quad \vee \quad X_{n,f(n)}\right)$$

which is exactly the twist by the action of f.

We have shown the composition is exactly non-zero when $f \in \mathrm{Bij}$ so that diagram 11 actually defines

$$\eta_\tau : \prod_{\mathrm{Bij}(n,n)} \mathrm{cr}_n F(X) \to \prod_{\mathrm{Bij}(n,n)} \widetilde{\mathrm{cr}}_n F(X).$$

By the equivariance of ρ and d, we see that this extends to show that η_τ is the twist map. $\square$

Now, looking at the composites of the maps of lemmas 5.2, 5.3 and 5.4 we find by lemma 3.2 that we have constructed the norm map. Thus the induced map T is indeed the Tate map which gives us the rational equivalence we sought. This concludes the proof of the Theorem 5.1.

Remark 5.5. In fact, the proof of the theorem only requires the condition that there exists a maps

$$F(X) \to F(\vee_n X)$$

which are compatible with $F(+) : F(\vee_n X) \to F(X)$ in the sense of definition 4.1. Since the structure maps ∇ are actually induced as maps in $\mathcal{C}$ in most of our examples, we choose to state the theorem in terms of co-H-objects. However, one should note that the theorem could be stated more generally by adjusting definition 4.1 to accommodate this situation.

Example 5.1. The following example is due to Tom Goodwillie.

Let $F : Top \to S$ be the functor $F(X) = C_*(X \wedge X/\Delta)$ where Δ is the diagonal map $\Delta : X \to X \wedge X$. We show that rationally the Goodwillie tower of F splits when X is a co-H-space, but not in general.

First note that the cofiber sequence

$$X \xrightarrow{\Delta} X \wedge X \longrightarrow \frac{X \wedge X}{\Delta}$$

induces a short exact sequence

$$C_*(X) \longrightarrow C_*(X \wedge X) \longrightarrow C_*(\tfrac{X \wedge X}{\Delta})\ .$$

The functor $X \mapsto C_*(X)$ is a homogeneous functor of degree one and the functor $X \mapsto C_*(X \wedge X)$ is homogeneous of degree two. Since D_n (as well as P_n) preserve short exact sequences, we have that $F(X)$ must also be a functor of degree at most two.

We want to examine the fiber sequence

$$D_2F(X) \to P_2F(X) \to P_1F(X)$$

The following 3 by 3 diagram with exact rows and columns captures all of the essential information for the Goodwillie tower of $F(X)$:

$$\begin{tikzcd}
D_2C_*(X) \arrow[r] \arrow[d] & D_2C_*(X \wedge X) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & D_2F(X) \arrow[d] \\
P_2C_*(X) \arrow[r] \arrow[d] & P_2C_*(X \wedge X) \arrow[r] \arrow[d] & P_2F(X) \arrow[d] \\
D_1C_*(X) \arrow[r] & D_1C_*(X \wedge X) \arrow[r] & D_1F(X)
\end{tikzcd}$$

The columns are exact since we have a fiber sequence $D_2 \to P_2 \to P_1$ and, since all of the functors involved are reduced, $P_1 = D_1$. Since $C_*(X)$ is a homogeneous functor of degree 1, $D_2C_*(X) = 0$ and similarly, since $C_*(X \wedge X)$ is a homogeneous functor of degree 2, $D_1C_*(X \wedge X) = 0$. That means we have equivalences $D_2F(X) \simeq D_2C_*(X \wedge X) = C_*(X \wedge X)$ and $D_1F(X) \simeq \Sigma D_1C_*(X) = C_*(\Sigma X)$. Since F is of degree two, we also have that $P_2F(X) \simeq C_*(X \wedge X/\Delta)$. That is, we have an exact sequence

$$C_*(X \wedge X) \longrightarrow C_*(X \wedge X/\Delta) \longrightarrow C_*(\Sigma X).$$

inducing the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_{*+1}(\Sigma X) \arrow[d, "\partial"] & & \\
& H_*(X \wedge X) \arrow[r] & H_*(X \wedge X/\Delta) \arrow[r] & H_*(\Sigma X) \arrow[d, "\partial"] & \\
& & & H_{*-1}(X \wedge X) \arrow[r] & \cdots
\end{tikzcd}$$

on homology. If this map splits, then the map ∂ is zero. It is easy to find spaces for which this can not be the case. One simple example is given by S^0. Also, notice that up to the suspension isomorphism the map ∂ can be expressed as

$$H_*(X) \xrightarrow{\Delta_*} H_*(X \wedge X)\,.$$

On (rational) cohomology, the map $\Delta^* : H^*(X \wedge X) \to H^*(X)$ is the cup product. Therefore, if this map is zero then X has trivial cup products. Since spaces don't generally have trivial cup products, this generally doesn't split. However, when X is a co-H-space, the Hopf algebra $H^*(X)$ is exterior on the indecomposables, the cup products are trivial.

We end this section by showing that the layers $D_nF(X)$ can be identified by the image of a certain map. Let $\Phi^r : F(X) \to F(X)$ be the composite $\Phi^r = F(+) \circ F(\nabla^r)$. Note that in the case $X = S^1$, this induces an r-fold covering map of S^1.

Remark 5.6. The map $D_n(\Phi^r) := P_\infty(\Phi^r)|_{D_nF(X)}$ from $D_nF(X) \to D_nF(X)$ induces multiplication by r^n.

To see this, recall that $D_nF(X) = (D_1^{(n)}\mathrm{cr}_nF(X))_{h\Sigma_n}$ by [JM4]. We have $D_1^{(n)}\mathrm{cr}_nF(\vee_k X) \cong \prod_{\mathrm{Hom}(n,r)=r^n} D_1^{(n)}\mathrm{cr}_nF(X)$ since $D_1^{(n)}\mathrm{cr}_nF$ is a multilinear functor. The following diagram describes Φ^r:

$$\begin{tikzcd}
D_1^{(n)}\mathrm{cr}_nF(X) \arrow[d, "F_*(\nabla^r)"] \arrow[dr, "\Delta"] & \\
D_1^{(n)}\mathrm{cr}_nF(\bigvee_r X) \arrow[r, "\simeq"] \arrow[d, "F_*(+)"] & \prod_{\mathrm{Hom}(n,r)=r^n} D_1^{(n)}\mathrm{cr}_nF(X) \arrow[dl, "+"] \\
D_1^{(n)}\mathrm{cr}_nF(X) &
\end{tikzcd}$$

Now, by arguments analogous to Lemmas 5.2 and 5.3, we have that the maps Δ and $+$ of the diagram are the diagonal and fold map (recall that coproducts and products are weakly equivalent). Hence, Φ^r is multiplications by r^n.

The result follows from the fact that Φ^r is a Σ_n equivariant map, so passes to homotopy orbits.

5.1 Higher Hochschild Homology

Let A be a commutative algebra over a field k of characteristic zero. Let $X.$ be any finite pointed simplicial set. We can form a simplicial k-algebra by composing the functor $X.$ with the functor $- \otimes A$ from the category of finite pointed sets to A-algebras which takes the set $[[n]] = \{0, 1, \dots n\}$ to $[[n]] \otimes A = A^{\otimes_k n}$. The chain complex associated to the resulting simplicial algebra computes the Hochschild homology of A when $X \simeq S^1$ and computes the n-th higher Hochschild homology when $X \simeq S^n$. Denote the homology of the chain complex associated to $X. \otimes A$ by $HH_*^X(A)$. Rational decompositions of Hochschild homology have been studied extensively [L98], [GS87], [R93]. A rational decomposition of higher Hochschild homology which recovers the known decomposition of Hochschild homology has been discovered by Pirashvili [P00] and a rational decomposition for $HH_*^{S^1 \wedge X}(A)$ was found by the first author, recovering the decompositions of the other cases [B].

The goal is to compute the layers of the Goodwillie tower computing higher Hochschild homology in terms of the layers of Hochschild homology. The chain complex $(S^1 \wedge X) \otimes A$ is a commutative differential graded Hopf algebra [B]. Using the comultiplication map, consider $(S^1 \wedge X) \otimes A$ as a co-H-object in the category of commutative augmented A-algebras, ${}_A Comm_A$.

Let U be the forgetful functor from the category ${}_A Comm_A$ to the category $A - mod$ of A-modules. By Theorem 5.1, $HH_*^{S^1 \wedge X} = \bigoplus_n D_n U((S^1 \wedge X) \otimes A)$. The Goodwillie tower of the functor U has been computed [K-M]. Let M be a cofibrant object of ${}_A Comm_A$. Then the linear part, $D_1 U(M)$, is $I/I^2(M)$ where $I(M)$ is the augmentation ideal of M over A. The higher layers are given by $D_n U(M) = (I/I^2(M))^{\otimes n}_{\Sigma_n} = S^n(I/I^2(M))$ where S^n denotes the n-th part of the symmetric algebra. Since A is a cofibrant object in ${}_A Comm_A$, so is $S^d \otimes A$ and we will use this to compute the Goodwille tower.

By Remark 5.6 the "r-fold cover map" Φ^r induces multiplication by r^n on each layer $D_n U((S^1 \wedge X) \otimes A)$ and one can use this to show that the decomposition using Theorem 5.1 agrees with the decomposition of [B] and hence also those of [P00], [L98] and [GS87].

Since $D_1 U$ is a linear functor, it commutes with suspensions. Note that the suspension in ${}_A Comm_A$ is given by $S^1 \otimes -$ and the suspension in $A - mod$ is given by $\tilde{\mathbb{Z}}[S^1] \otimes_A -$ where $\tilde{\mathbb{Z}}[S^1]$ is the free module generated by S^1. Therefore $D_1 U(S^d \otimes A) = \widetilde{\mathbb{Z}}[S^d] \otimes_A D_1 U(S^0 \otimes A)$. Since $D_n U(S^d \otimes A) =$

$(D_1U(S^d \otimes A))^{\otimes n}_{\Sigma_n}$ we have

$$D_nU(S^d \otimes A) = (\widetilde{\mathbb{Z}}[S^d] \otimes D_1U(S^0 \otimes A))^{\otimes n}_{\Sigma_n} \tag{12}$$
$$= [\widetilde{\mathbb{Z}}[S^{dn}] \otimes (D_1U(S^0 \otimes A))^{\otimes n}]_{\Sigma_n} \tag{13}$$
$$\tag{14}$$

where the action of Σ_n on the last line is given diagonally. On the first factor, Σ_n acts by permuting the copies of S^d. Each flip of factors S^d induces multiplication by $(-1)^d$ on homology. On the second factor, Σ_n acts by permuting the factors $D_1U(S^0 \otimes A)$. Taking the orbits of this action produces the homogeneous degree n part of the symmetric algebra. Taking both of these actions together, we have

$$D_nU(S^d \otimes A) = \begin{cases} \Sigma^{dn} S^n D_1U(S^0 \otimes A) & d \text{ is even;} \\ \Sigma^{dn} \Lambda^n D_1U(S^0 \otimes A) & d \text{ isodd,} \end{cases}$$

where Λ^n is the homogeneous degree n part of the exterior algebra. Finally, if one then computes that $D_nU(S^1 \otimes A) = \Sigma^n \Lambda^n D_1U(S^0 \otimes A)$ then we can express the layers of higher Hochschild homology in terms of Hochschild homology by realizing that (up to a sign) $D_nU(S^d \otimes A)$ is an d-fold suspension of $D_nU(S^1 \otimes A)$. That is, the layers of higher Hochschild homology depend only on the layers of Hochschild homology.

6 Dual Calculus

There is a version of calculus which is strictly dual to the Goodwillie calculus tower which we have been using so far. To obtain this theory, one simply replaces homotopy limits with homotopy colimits, coproducts with products, fibers with cofibers, etc. For details, see [McC02].

If one replaces coproducts by products and fibers by cofibers in 2.1, one obtains the dual cross effects, $\mathrm{cr}^n F(X)$. The dual cross effects can be thought of as the total cofibers of cubical diagrams. There is a natural map $\rho : F(\prod_n X) \to \mathrm{cr}^n F(X)$.

One can use the dual cross effects to define a codegree n approximation to F, and these assemble into a tower

$$\begin{tikzcd}
\vdots \\
P^nF(X) \arrow[u] \arrow[r, "p^n"] & F(X) \\
P^{n-1}F(X) \arrow[u, "q^n"] \arrow[ur, "p^{n-1}"'] \\
\vdots \arrow[u]
\end{tikzcd}$$

which is universal with respect to maps to F from codegree n functors. The n-th dual layer of F, D^nF, is the cofiber of the map $q^n : P^{n-1}F \to P^nF$.

There is also a notion of dual co-cross effects analogous to the co-cross effects and obtained by replacing coproducts with products and limits with colimits. There is a natural map $\tilde{\rho} : \widetilde{\mathrm{cr}}^n F(X) \to F(\prod_n X)$.

To state the dual version of the theorem, we must describe the appropriate counterpart for co-H-objects. The following is the expected definition:

Definition 6.1. A fibrant object X of $\mathcal{C}$ is an H-object if X is equipped with a map $\mu : X \times X \to X$ which is unital and associative up to homotopy.

The unit map for μ is given by the inclusion of the basepoint. Let Δ be the diagonal map $\Delta : X \to X \times X$. The following diagram commutes up to homotopy:

$$\begin{tikzcd}
\prod\limits_n X \arrow[r, "\tau\circ\prod\Delta^n"] \arrow[d, "\mu^n"'] & \prod\limits_n\prod\limits_n X \arrow[d, "\prod\mu^n"] \\
X \arrow[r, "\Delta^n"] & \prod\limits_n X
\end{tikzcd}$$

where τ is the map which transposes the entries of $\prod_n \prod_n X$. The H-objects in the category of pointed topological spaces are precisely the H-spaces.

Theorem 6.2. *If F is a functor from $\mathcal{C}$ to $\mathcal{S}$ which preserves weak equivalences and if X is an H-object of $\mathcal{C}$, then rationally the cofiber sequence*

$$P^{n-1}F(X) \xrightarrow{q^n} P^nF(X) \xrightarrow{j} D^nF(X)$$

splits in the homotopy category. Consequently,

$$P^\infty F(X) \simeq \prod_{n\geq 0} D^n F(X)$$

is a rational equivalence.

The proof of this dual version of our main theorem proceeds in essentially the same manner as the proof of the main theorem. Here is a sketch:

We first reduce to the case where F is a codegree n functor, since if F is not codegree n we may replace F by $P^n F$. When F is codegree n, we have equivalences $P^n F(X) \simeq F(X)$ and $D^n F(X) \simeq \mathrm{cr}^n F(X)^{h\Sigma_n}$. We can then express the map j as in the following diagram:

$$\begin{tikzcd}
F(X) \arrow[rr, "j"] \arrow[dr] \arrow[dd, "F(\Delta)"'] & & \mathrm{cr}^n F(X)^{h\Sigma_n} \arrow[dd, "i"] \\
& F(\prod_n X)^{h\Sigma_n} \arrow[ur, "\rho^*"] \arrow[dl] & \\
F(\prod_n X) \arrow[rr, "\rho"] & & \mathrm{cr}^n F(X)
\end{tikzcd}$$

where $F(\Delta)$ factors through the fixed points since it is Σ_n-fixed, and ρ^* extends because ρ is Σ_n-equivariant.

Now, using the fact that X is an H-object and using the map $\widetilde{\rho} : \widetilde{\mathrm{cr}^n} F(X) \to F(\prod_n X)$, we can expand this diagram to

$$\begin{tikzcd}
\widetilde{\mathrm{cr}}^n F(X) \arrow[d, "\tilde{\rho}"] & & \\
F(\prod\limits_n X) \arrow[r, "F(\mu)"] \arrow[d, "F(\Delta)"] & F(X) \arrow[r, "j"] \arrow[d, "F(\Delta)"] & \mathrm{cr}^n F(X)^{h\Sigma_n} \arrow[d, "i"] \\
F(\prod\limits_n \prod\limits_n X) \arrow[r, "F(\mu\circ\tau)"] & F(\prod\limits_n X) \arrow[r, "\rho"] & \mathrm{cr}^n F(X)
\end{tikzcd}$$

which commutes. We claim that the "outside" map - that is, the composition $\rho \circ F(\mu\circ\tau)\circ F(\Delta)\circ\tilde{\rho}$ - is the norm map. Just as before, one can show this by using the equivalences $\mathrm{cr}^n F(\prod_n X) \simeq \prod_{Hom(n,n)} \mathrm{cr}^n F(X)$ and $\mathrm{cr}^n \mathrm{cr}^n F(X) \simeq \prod_{\Sigma_n} \mathrm{cr}^n F(X)$ and by showing that the relevant maps are the diagonal, twist and fold maps. Once this is done, we can further expand our diagram to

$$
\begin{tikzcd}
\widetilde{\mathrm{cr}}^n F(X) \arrow[r, "q"] \arrow[d, "\tilde{\rho}"] & \mathrm{cr}^n F(X)_{h\Sigma_n} \arrow[dr, "T"] & \\
F(\prod\limits_n X) \arrow[r, "F(\mu)"] \arrow[d, "F(\Delta)"] & F(X) \arrow[r, "j"] \arrow[d, "F(\Delta)"] & \mathrm{cr}^n F(X)^{h\Sigma_n} \arrow[d, "i"] \arrow[ull, "i"'] \\
F(\prod\limits_n \prod\limits_n X) \arrow[r, "F(\mu \circ \tau)"] & F(\prod\limits_n X) \arrow[r, "\rho"] & \mathrm{cr}^n F(X)
\end{tikzcd}
$$

where q and i are the relevant quotient and inclusion maps, respectively. The map T is the Tate map, and this commutes with the rest of the diagram as before because the outside composition is the norm map. In fact, the entire diagram commutes except possibly for the map i. We wish to show that i provides a splitting map for j. In other words, we'll show that $j \circ F(\mu) \circ \widetilde{\rho} \circ i$ is rationally homotopic to the identity map on $\mathrm{cr}^n F(X)^{h\Sigma_n}$. This is the same as showing that $T \circ q \circ i$ is rationally homotopic to the identity map. However, recalling the definition of the map T, we have that $T \circ q \circ i \sim N \circ i$ where N is the norm map since N actually lands in the fixed points. One can easily check that $N \circ i$ is multiplication by $n!$ which is rationally equivalent to the identity map on $\mathrm{cr}^n F(X)^{h\Sigma_n}$. Thus i provides the splitting.

Remark 6.3. Define a map $\Phi^r := F(\mu) \circ F(\Delta)$. Then $D^n(\Phi^r)$ induced multiplication by r^n. The argument for this is exactly dual to Remark 5.6.

6.1 The Poincaré-Birkhoff-Witt Theorem.

We seek to recover the rational version of the Poincaré-Birkhoff-Witt Theorem as an application of Theorem 6.2. First, we recall the definitions required to state this theorem. This background can be found in e.g. [W94].

Let Lie_k be the category of Lie algebras over a field k of characteristic 0. If A is any algebra over k, A can be thought of as a Lie algebra by giving it the bracket $[x, y] = xy - yx$, where $x, y \in A$. Denote A as a Lie algebra by $\mathcal{A}$. Let $\mathcal{G}$ be a Lie algebra over k with bracket $[\ ,\ ]$.

Definition 6.4. The universal enveloping algebra of $\mathcal{G}$ is an algebra over k, $U(\mathcal{G})$ with associated Lie algebra $\mathcal{U}(\mathcal{G})$ together with a morphism $i : \mathcal{G} \to \mathcal{U}(\mathcal{G})$ which is universal with respect to algebras over k. That is, if $f : \mathcal{G} \to \mathcal{A}$ is a map of Lie algebras, then there is a map $g : U(\mathcal{G}) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{U}(\mathcal{G}) \arrow[d, "g_*"] \\ & \mathcal{A} \end{tikzcd}$$

where g_* is the map induced by g.

One can construct $U(\mathcal{G})$ as follows: Let $i : \mathcal{G} \to T(\mathcal{G})$ be the inclusion of $\mathcal{G}$ into its tensor algebra induced by including $\mathcal{G}$ into the first graded piece of $T(\mathcal{G})$. Let I be the ideal of $T(\mathcal{G})$ generated by the relations

$$i([x, y]) = i(x)i(y) - i(y)i(x)$$

where $x, y \in \mathcal{G}$. Then $U(\mathcal{G}) = T(\mathcal{G})/I$.

Let $\pi : T(\mathcal{G}) \to U(\mathcal{G})$ be the quotient map and let $T_m(\mathcal{G}) = \oplus_{j=0}^{m} \mathcal{G}^{\otimes j}$. One can see that $U(\mathcal{G})$ inherits a grading from $T(\mathcal{G})$ by setting $U_m(\mathcal{G}) = \pi(T_m(\mathcal{G}))$. If $\mathcal{G}$ is free as a module over k, then the Poincaré-Birkhoff-Witt theorem shows that $U_m(\mathcal{G})/U_{m-1}(\mathcal{G}) \cong S_m$ as k-modules, where $S_m := \mathcal{G}^{\otimes m}/\Sigma_m$ is the m-th homogeneous graded piece of the symmetric algebra, and that $U(\mathcal{G}) \cong S$ as k-modules, where S is the whole symmetric algebra. In fact, if $\{e_i\}_{i \in \mathcal{I}}$ is a basis for $\mathcal{G}$ as a k-module, then

$$\{e_{i_1} \otimes \cdots \otimes e_{i_m} | i_j \in \mathcal{I}; i_1 \leq \cdots \leq i_m; m \geq 1\}$$

is a basis for $U(\mathcal{G})$ (where $\mathcal{I}$ is some indexing set).

From [MM65] we know that $U(\mathcal{G} \oplus \mathcal{G}) \cong U(\mathcal{G}) \otimes U(\mathcal{G})$ and that $U(\mathcal{G})$ is a cocommutative Hopf algebra. The multiplication structure map for the Hopf algebra is induced by concatenation on $T(\mathcal{G})$, and the comultiplication is induced by the diagonal map $\Delta : \mathcal{G} \to \mathcal{G} \oplus \mathcal{G}$. Let $\mathcal{C}$ be the category of cocommutative, coaugmented coalgebras over k. In $\mathcal{C}$, the product is given by $\otimes$ and the diagonal map for any object is given by the comultipication. The Hopf algebra structure makes $U(\mathcal{G})$ an H-object in the category $\mathcal{C}$.

Let F be the forgetful functor from $\mathcal{C}$ to the category of k-modules. We can now apply theorem 6.2 to $F(U(\mathcal{G}))$ to obtain a splitting of $U(\mathcal{G})$ as a k-module. We want to show that this splitting recovers the Poincaré-Birkhoff-Witt theorem. In other words, we want to show that $D^n F(U(\mathcal{G})) \cong S_n$.

Note that for $g \in U_1(\mathcal{G})/U_0(\mathcal{G})$, the image of the comultiplication map is $\Delta(g) = g \otimes 1 + 1 \otimes g$. Denote an element in $U_n(\mathcal{G})/U_{n-1}(\mathcal{G})$ by $(g_1, \ldots, g_n)$. By induction, we have

$$\Delta(g_1, \ldots, g_n) = \sum_{\substack{p+q=n \\ \sigma \in (p,q)-\text{shuffles}}} (g_{\sigma(1)}, \ldots, g_{\sigma(p)}) \otimes (g_{\sigma(p+1)}, \ldots, g_{\sigma(p+q)})$$

where σ, a (p,q)-shuffle means that σ is a permutation of $\{1, \ldots, n\}$ with $\sigma(1) \leq \cdots \leq \sigma(p)$ and $\sigma(p+1) \leq \cdots \leq \sigma(p+q)$ Since multiplication is induced by concatenation, the map $\Phi^2 = \mu \circ \Delta$ is

$$\mu \circ \Delta(g_1, \ldots, g_n) = \sum_{\substack{p+q=n \\ \sigma \in (p,q)-\text{shuffles}}} (g_{\sigma(1)}, \ldots, g_{\sigma(p)}, g_{\sigma(p+1)}, \ldots, g_{\sigma(p+q)}).$$

However, in $U(\mathcal{G}) = T(\mathcal{G})/I$, for any permutation $\sigma \in \Sigma_n$ we have

$$(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = (g_1, \ldots, g_n).$$

Therefore $\mu \circ \Delta$ is simply multiplication by the number of ways of shuffling $(g_1, \ldots, g_n)$ into two factors. An easy inductive argument shows that the number of (p,q)-shuffles with $p + q = n$ is 2^n.

We now know that the map $\Phi^2 = \mu \circ \Delta$ induces multiplication by 2^n on $U_n(\mathcal{G})/U_{n-1}(\mathcal{G}) \cong S_n$. However we also know that Φ^2 induces multiplication by 2^n on $D_n F(U(\mathcal{G}))$. This shows that the two decompositions are the same - the map Φ^2 plays the role of a linear operator on the k-vector space $U(\mathcal{G})$ whose image determines the decomposition associated to it. Hence theorem 6.2 recovers the Poincaré-Birkhoff-Witt theorem.

7 Splittings and Tate Cohomology

We want to consider another extension to Theorem 5.1.

Definition 7.1. [McC02] Let F be a homotopy functor from $\mathcal{C} \to \mathcal{S}$. Define the *$n$-th Tate cohomology of F at X* to be

$$Tate^n(F;X) := cofiber((D_1^{(n)}\mathrm{cr}_n F(X))_{h\Sigma_n} \to (D_1^{(n)}\mathrm{cr}_n F(X))^{h\Sigma_n})$$

where the map from the homotopy orbits to the homotopy fixed points is the Tate map.

Theorem 7.2. *Let F be a homotopy functor from $\mathcal{C} \to \mathcal{S}$ and let X be a co-H-object of $\mathcal{C}$. If the map ∇ is cocommutative, then the fiber sequence*

$$D_n F(X) \to P_n F(X) \to P_{n-1} F(X)$$

splits whenever the Tate cohomology vanishes for all n.

Proof. The proof only requires a small adjustment to the proof of theorem 5.1. If ∇ is cocommutative (hence ∇^n is cocommutative), then it is a Σ_n-fixed map and since δ is Σ_n-equivariant we have a factorization

$$\begin{tikzcd}
F(X) \arrow[rr, "F(\nabla^n)"] \arrow[dr] \arrow[dd, "\sigma"'] & & F(\bigvee_n X) \arrow[dd, "\delta"] \\
& F(\bigvee_n X)^{h\Sigma_n} \arrow[ur] \arrow[dl, "\delta^{h\Sigma_n}"] & \\
\widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n} \arrow[rr] & & \widetilde{\mathrm{cr}}_n F(X)
\end{tikzcd}$$

This extends Diagram 5 to:

$$\begin{tikzcd}
\mathrm{cr}_n F(X) \arrow[r, "\rho"] \arrow[d] & F(\bigvee_n X) \arrow[d, "F(+)"] & \\
\mathrm{cr}_n F(X)_{h\Sigma_n} \arrow[r, "j"] \arrow[dr, "\simeq"] & F(X) \arrow[r, "F(\nabla^n)"] \arrow[d, "\sigma"'] & F(\bigvee_n X) \arrow[d, "\delta"'] \\
& \widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n} \arrow[r] & \widetilde{\mathrm{cr}}_n F(X)
\end{tikzcd} \tag{15}$$

We wish to show that $\sigma \circ j$ is a weak equivalence. When the n-th Tate cohomology vanishes, the map T is an equivalence so that $\mathrm{cr}_n F(X)_{h\Sigma_n} \simeq \mathrm{cr}_n F(X)^{h\Sigma_n}$ which in turn is equivalent to $\widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n}$.

Let F and G be functors to spectra. From the construction of $P_n F$, we know that if $\mathrm{cr}_n F(X) \simeq \mathrm{cr}_n G(X)$ via a natural transformation $\omega : F \to G$, then there is a pull back diagram:

$$\begin{tikzcd} P_n F(X) \arrow[r, "P_n(\omega)"] \arrow[d] & P_n G(X) \arrow[d] \\ P_{n-1} F(X) \arrow[r, "P_{n-1}(\omega)"] & P_{n-1} G(X) \end{tikzcd} \tag{16}$$

In this case, we wish to use $G(X) = \widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n}$ and $\omega = \sigma$. By using Lemma 5.2, we know that $\mathrm{cr}_n(\delta) \circ \mathrm{cr}_n F(\nabla^n)$ is the diagonal map into $\widetilde{\mathrm{cr}}_n \widetilde{\mathrm{cr}}_n F(X) \simeq \prod_{\Sigma_n} \widetilde{\mathrm{cr}}_n F(X)$. Here, we are making use of the equivalence $\widetilde{\mathrm{cr}}_n F(X) \simeq \mathrm{cr}_n F(X)$ repeatedly. From there, one sees that

$$\Big(\prod_{\Sigma_n} \widetilde{\mathrm{cr}}_n F(X)\Big)^{h\Sigma_n} \simeq \widetilde{\mathrm{cr}}_n F(X)$$

so that the map $\widetilde{\mathrm{cr}}_n(\sigma)$ is an equivalence. Thus, we have a pullback diagram

$$\begin{tikzcd} P_n F(X) \arrow[r, "P_n(\sigma)"] \arrow[d] & P_n(\widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n}) \arrow[d] \\ P_{n-1} F(X) \arrow[r, "P_{n-1}(\sigma)"] & P_{n-1}(\widetilde{\mathrm{cr}}_n F(X)^{h\Sigma_n}) \end{tikzcd}$$

Now, since the Tate map is an equivalance, and since $cr_n F(X)_{h\Sigma_n}$ is a homogeneous degree n functor, we have that $cr_n F(X)^{h\Sigma_n}$ is also a homogeneous degree n functor. So, this pullback diagram is actually

$$\begin{tikzcd} F(X) \arrow[r, "\sigma"] \arrow[d] & \mathrm{cr}_n F(X)^{h\Sigma_n} \arrow[d] \\ P_{n-1} F(X) \arrow[r] & * \end{tikzcd}$$

and the result follows by taking parallel fibers. □

Remark 7.3. Note that the theorem only actually requires that the map $F(X) \to F(\vee_n X)$ be "cocommutative", i.e. that this map is fixed under the action of Σ_n. In [McC02], McCarthy proves a similar theorem for "stable functors", that is, for functors to spectra with the property that the inclusion map

$$X \vee Y \to X \times Y$$

induces an equivalence $F(X \vee Y) \to F(X \times Y)$ for all X and Y in $\mathcal{C}$. If F is a stable functor, then every object $X \in \mathcal{C}$ becomes a co-H-object via the diagonal map

$$F(X) \to F(X \times X) \simeq F(X \vee X).$$

Since the diagonal map is cocommutative, this recovers McCarthy's theorem.

In particular, functors from the category $\mathcal{C}$ of left (resp. right) k-modules are stable functors since it is already the case in $\mathcal{C}$ that products are equivalent to coproducts. Note also in this case that every object is both a co-H-object and an H-object, so that both the dual towers and the regular towers split when the Tate groups vanish. McCarthy's result shows that in fact the layers of the dual tower and the regular tower, and hence the two splittings, agree.

8 Appendix

Recall from section 3 that if a finite group G acts on a k-module M, there is a natural map $t : M_G \to M^G$ factoring the norm map as:

$$\begin{tikzcd} M \arrow[r, "t'"] \arrow[d, "p"'] & M \\ M_G \arrow[r, "t"] & M^G \arrow[u, hook, "i"'] \end{tikzcd}$$

which is an equivalence whenever the order of G is invertible.

We wish to extend this map to the category $\mathcal{S}$ of $H\mathbb{Q}$-modules. Let T be an FSP. Let $\underline{E}$ be it's associated spectrum

$$\underline{E}_k = \operatorname{hocolim}_n \Omega^n E(S^{n+k})$$

and suppose $\underline{E}$ is an $H\mathbb{Q}$-module with an action of a finite group G. We wish to explicitly construct a weak map $\underline{E}_{hG} \to \underline{E}^{hG}$ which will be the correct analogue for the Tate map for modules. Such constructions can be found in e.g. [WW], [GM], [DGM], [McC]. The construction we provide here is due to Tom Goodwillie and closely follows the recollection provided in the appendix of [McC]. We begin with some background.

8.1 Group Actions

Let Top be the category of base pointed topological spaces. In the following, G is a finite group. The definitions and constructions which follow hold for all groups G, but since we are only interested in the case $G = \Sigma_n$, it won't hurt for us to assume that G is a finite group throughout.

An action of a group G on a space X is a map $G_+ \wedge X \to X$. We denote the image of $g \wedge x$ by gx. Equivalently, a group action is a functor $X : \mathcal{G} \to \mathcal{T}\wr\surd$ where $\mathcal{G}$ is the category with one object and $\mathrm{Hom}_{\mathcal{G}}(*, *) = G$. In this case, X denotes both the functor and the topological space X which is the image of the unique object of $\mathcal{G}$. We say that G acts freely on X if for all non-basepoint elements $x \in X$, $gx \neq x$ unless g is the identity element of G.

The following is a small collection of constructions involving G actions.

The orbits of the G action on X are defined to be

$$X_G = \operatorname{colim}_{\mathcal{G}} X = X/\{x \sim gx\}.$$

If X is a G-space and Y is a G^{op}-space (the action of G is on the right), then we may define a smash product $Y \wedge X$ with G-action $g(y \wedge x) = yg^{-1} \wedge gx$. Define $Y \wedge_G X = \{yg \wedge x \sim y \wedge gx\}$ and notice that

$$(Y \wedge X)_G = Y \wedge_G X.$$

The homotopy orbits of the G action on X are

$$X_{hG} = \operatorname{hocolim}_{\mathcal{G}} X = (EG_+ \wedge X)_G$$

where EG is any contractibe free G space. Later, we will use a specific model of EG given by the simplicial construction

$$EG = |[q] \mapsto \bigwedge^{q+1} G_+|$$

where $+$ denotes a disjoint basepoint.

The fixed points and homotopy fixed points of the G action on X are respectively

$$X^G = \lim_{\mathcal{G}} X = \{x | gx = x \text{ for all } g \in G\}$$

and

$$X^{hG} = \operatorname{holim}_{\mathcal{G}} X = \operatorname{Map}(EG_+, X)^G.$$

If X is a G-space, there are two important free G spaces associated to X. The first is $G_+ \wedge X$ with G-action given by $h(g \wedge x) = gh^{-1} \wedge hx$. The second is $\operatorname{Map}(G_+, X)$ with G-action given by $h(f)(g) = hf(gh)$. There are equivalences

$$\begin{aligned} (G_+ \wedge X)_G &\to X \\ g \wedge x &\mapsto gx \end{aligned}$$

and

$$\begin{aligned} X &\to \operatorname{Map}(G_+, X)^G \\ x &\mapsto f(g) = gx. \end{aligned}$$

Since the action of G on $G_+ \wedge X$ and $Map(G_+, X)$ is free, there are G-equivariant equivalences

$$(G_+ \wedge X)_{hG} := (EG_+ \wedge G_+ \wedge X) \to (G_+ \wedge X)_G$$

and

$$\operatorname{Map}(G_+, X)^G \to \operatorname{Map}(EG_+, \operatorname{Map}(G_+, X))^G =: \operatorname{Map}(G_+, X)^{hG}.$$

We conclude this section by defining an important relationship between these two free G-spaces. Using the inclusion $\vee_G X \to \prod_G X$ we can produce a map

$$G_+ \wedge X \cong \bigvee_G X \to \prod_G X \cong \operatorname{Map}(G_+, X).$$

Let this composition be named γ and notice that γ is given by

$$\gamma(g \wedge x)(u) = \begin{cases} x & u = g \\ * & \text{otherwise.} \end{cases}$$

By the Blakers-Massey theorem, if X is k-connected then γ is $(2k-1)$-connected. We will take advantage of this in the future to produce equivalences using γ and stabilization.

8.2 G-actions on Functors with Smash Product

In this section, we assume a familiarity with the definition of functors with smash products and we describe the group actions on these. Readers who are not familiar with these definitions should consult e.g. [McC].

If E is an FSP, then we denote by $\underline{E}$ the spectrum associated to E with $\underline{E}_k = E(S^k)$. Since every $\underline{E}$ is naturally equivalent to an Ω-spectrum (see [McC] for a construction), we may assume that $\underline{E}$ is an Ω-spectrum.

Define the homotopy orbits of E to be the FSP E_{hG} with $E_{hG}(X) = \Omega^\infty(E(X)_{hG})$. (Technically, we have to indicate why this is again an FSP.) Note that since

$$\begin{aligned} E_{hG}(X) &= \Omega^\infty(E(X)_{hG}) \\ &= \mathrm{hocolim}_n\, \Omega^n(E(S^n \wedge X)_{hG}) \\ &= \mathrm{hocolim}_n\, \Omega^n(\mathrm{hocolim}_G\, E(S^n \wedge X)), \end{aligned}$$

E_{hG} is a homotopy colimit construction. Therefore, if E_* is a simplicial FSP, the natural map $|E_*|_{hG} \to |(E_*)_{hG}|$ is an equivalence since the geometric realization is also a colimit construction and "colimits commute". Let $\underline{E}_{hG}$ be the spectrum associated to E_{hG}.

The homotopy fixed points of E is the FSP with $E^{hG}(X) = (\Omega^\infty E(X))^{hG}$. Since homotopy fixed points are a limit construction we cannot make a similar statement relating $|(E_*)^{hG}|$ with $|E_*|^{hG}$. However, there is a natural map

$$|(E_*)^{hG}| \to |E_*|^{hG}.$$

Let $\underline{E}^{hG}$ be the spectrum associated to E^{hG}.

We have FSP's $(G_+ \wedge E)(X) = G_+ \wedge E(X)$ and $\mathrm{Map}(G_+, E)(X) = \mathrm{Map}(G_+, E(X))$. Note that the relevant homotopy orbit and homotopy fixed point constructions are given by

$$\begin{aligned} (G_+ \wedge E)_{hG}(X) &= \Omega^\infty[(G_+ \wedge E(X))_{hG}] \\ &= \mathrm{hocolim}_n\, \Omega^n[(G_+ \wedge E(\Sigma^n X))_{hG}] \\ Map(G_+, E)^{hG}(X) &= [\Omega^\infty \mathrm{Map}(G_+, E(X))]^{hG} \\ &= [\mathrm{hocolim}_n\, \Omega^n \mathrm{Map}(G_+, E(\Sigma^n X))]^{hG} \end{aligned}$$

8.3 The Tate Map

We would like to produce a weak map $\underline{E}_{hG} \xrightarrow{T} \underline{E}^{hG}$ which factors the (weak) norm map for spectra. That is, we'd like T to satisfy the following diagram of weak maps

$$\begin{tikzcd}
\underline{E} \arrow[r, "\text{diagonal}"] \arrow[d] & \prod_{g\in G} \underline{E} \arrow[r, "\sigma"] & \prod_{g\in G} \underline{E} \arrow[r, dashed] & \bigvee_{g\in G} \underline{E} \arrow[r, "\text{fold}"] & \underline{E} \\
\underline{E}_{hG} \arrow[rrrr, dashed, "T"] & & & & \underline{E}^{hG} \arrow[u]
\end{tikzcd}$$

where the vertical maps are the natural maps and the dotted lines are the weak maps (the dotted arrow in the top row is given by the inclusion of the coproduct into the product and the Blakers-Massey theorem). The map σ is the "shuffle map" $\prod_{g\in G} g$ which acts on the factor of $\prod_{g\in G} X$ indexed by g by g.

Before we construct the map T for a general FSP, we first consider the special case where the FSP is of the form $G_+ \wedge E$. In this case, we'll produce an equivalence $(G_+ \wedge E)_{hG} \to (G_+ \wedge E)^{hG}$.

If X is a space, we almost have an equivalence $(G_+ \wedge X)_{hG} \simeq (G_+ \wedge X)^{hG}$ given by assembling the maps from subsection 8.1:

$$\begin{tikzcd}
(G_+ \wedge X)_{hG} \arrow[d, "\simeq"'] & & (G_+ \wedge X)^{hG} \arrow[d, "\gamma^{hG}"] \\
(G_+ \wedge X)_G \arrow[d, "\cong"'] & & \\
X \arrow[r, "\cong"] & \mathrm{Map}(G_+, X)^G \arrow[r, "\simeq"] & \mathrm{Map}(G_+, X)^{hG}
\end{tikzcd} \quad . \tag{17}$$

Of course, if X is just a space, we do not know that γ^{hG} is a weak equivalence. Before venturing into the stable world to correct this, we make the following observation.

Remark 8.1. The weak equivalence we have constructed between $(G_+ \wedge X)_{hG}$ and $\mathrm{Map}(G_+, X)^{hG}$ has a very encouraging property. Notice that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Map}(G_+, X) \arrow[rrr, "\sigma"] & & & \mathrm{Map}(G_+, X) \\
G_+ \wedge X \arrow[u, "\Delta"] \arrow[r] & (G_+ \wedge X)_{hG} \arrow[r, "\simeq"] & X \arrow[r, "\simeq"] & \mathrm{Map}(G_+, X)^{hG} \arrow[u]
\end{tikzcd}$$

where the vertical arrow on the left hand side is the diagonal map sending $g \wedge x$ to the constant map $f(u) = gx$ (this map is closely related to γ). So far, we have considered the map $\sigma := \prod_{g \in G} g$ to be a self map of $\prod_G X$. Here, we are using the identification of $\prod_G X$ with $\mathrm{Map}(G_+, X)$. The image of the constant map $f(u) = gx$ under $\prod_{u \in G} u$ is the map $h(u) = ugx$, which is exactly the image of $g \wedge x$ under the bottom row (off by a negative sign...? change the shuffle map to twist by g^{-1}). In other words, we have shown that this forms a piece of the Tate map.

Now, if E is an FSP then since the product and the coproduct in the category of spectra agree, we have that the G-equivariant map γ induces a G equivariant equivalence

$$\mathrm{hocolim}_n \, \Omega^n (G_+ \wedge E(\Sigma^n X)) \to \mathrm{hocolim}_n \, \Omega^n \mathrm{Map}(G_+, E(\Sigma^n X)).$$

We would like to use this to produce the analogue of diagram 8.3 for FSP's. Note that since the homotopy orbits are a homotopy colimit construction, we have a (G-equivariant) equivalence

$$(G_+ \wedge E)_{hG}(X) := \mathrm{hocolim}_n \, \Omega^n [(G_+ \wedge E(\Sigma^n X)_{hG}] \simeq [\mathrm{hocolim}_n \, \Omega^n (G_+ \wedge E(\Sigma^n X))]_{hG}.$$

Using this, the equivalences

$$\begin{tikzcd}
(G_+ \wedge E)_{hG}(X) \arrow[r, "\simeq"] \arrow[d, ":="'] & \Omega^\infty E(X) \arrow[d, ":="] \\
\mathrm{hocolim}_n \, \Omega^n [(G_+ \wedge E(\Sigma^n X)_{hG}] \arrow[r, "\simeq"] & \mathrm{hocolim}_n \, \Omega^n E(\Sigma^n X)
\end{tikzcd}$$

and

$$\mathrm{hocolim}_n \, \Omega^n E(\Sigma^n X) \xrightarrow{\simeq} \mathrm{Map}(G_+, \mathrm{hocolim}_n \, \Omega^n E(\Sigma^n X))^{hG}$$

are induced by diagram 8.3, with $\Omega^\infty E(X)$ playing the role of X. Note that since G is finite, we also have a G-equivariant equivalence

$$\mathrm{hocolim}_n \,\mathrm{Map}(G_+, \Omega^n E(\Sigma^n X)) \longrightarrow \mathrm{Map}(G_+, \mathrm{hocolim}_n \,\Omega^n E(\Sigma^n X))\ .$$

Since G acts trivially on Ω^n, the adjunction

$$\mathrm{Map}(G_+, \Omega^n E(\Sigma^n X) \simeq \Omega^n \mathrm{Map}(G_+, E(\Sigma^n X)$$

is also G-equivariant. Assembling these, we have a G-equivariant equivalence

$$\mathrm{Map}(G_+, \Omega^\infty E(X))^{hG} \longleftarrow [\underset{n}{\mathrm{ho}\overset{\simeq}{\mathrm{co}}\mathrm{lim}}\,\Omega^n \mathrm{Map}(G_+, E(\Sigma^n X))]^{hG} := \mathrm{Map}(G_+, E)^{hG}(X)\ .$$

Now we can apply the equivalence induced by γ^{hG} to obtain

$$\begin{tikzcd}
(G_+ \wedge E)_{hG}(X) \arrow[rr, dashed] \arrow[d, "\simeq"'] & & (G_+ \wedge E)^{hG}(X) \arrow[d, "\simeq"', "\gamma^{hG}"] \\
\Omega^\infty E(X) \arrow[r, "\simeq"] & \mathrm{Map}(G_+, \Omega^\infty E(X))^{hG} & \mathrm{Map}(G_+, E(X))^{hG}. \arrow[l, "\simeq"']
\end{tikzcd}$$

These assemble into a natural equivalence of functors with stablilization with G-action $(G_+ \wedge E)_{hG} \simeq (G_+ \wedge E)^{hG}$. Call the weak map representing the composite of weak equivalence we've just described $\Gamma : (G_+ \wedge E)_{hG} \dashrightarrow (G_+ \wedge E)^{hG}$.

Lemma 8.2. *The equivalence $(G_+ \wedge E)_{hG} \simeq (G_+ \wedge E)^{hG}$ factors the shuffle map $\sigma = \prod_G g$.*

Proof. This follows from the fact that the diagram

$$\begin{tikzcd}
G_+ \wedge E \arrow[r, "\gamma", "\simeq"'] \arrow[d] & \mathrm{Map}(G_+, E) \arrow[r, "\sigma"] & \mathrm{Map}(G_+, E) & G_+ \wedge X \arrow[l, "\gamma"', "\simeq"] \\
(G_+ \wedge E)_{hG} \arrow[r, "\simeq"'] & \Omega^\infty E \arrow[r, "\simeq"'] & \mathrm{Map}(G_+, E)^{hG} \arrow[u] & (G_+ \wedge X)^{hG} \arrow[l, "\gamma"', "\simeq"] \arrow[u]
\end{tikzcd}$$

commutes. The rectangle on the left hand side commutes by remark 8.1 and the fact that the weak equivalences indicated are induced by weak equivalences of spaces. The square on the right commutes because γ is a G-equivariant map. ☐

We now have enough information to construct the Tate (weak) map. Let π be the G-equivariant equivalence $EG_+ \wedge E \to E$. The Tate map is the composite

$$\begin{tikzcd}
E_{hG} & & \\
(EG_+ \wedge E)_{hG} \arrow[u, dashed, "\simeq"', "\pi"] \arrow[r, "{:=}"'] & {|[q] \mapsto (\bigwedge^{q+1} G_+ \wedge E)|_{hG}} & \\
& {|[q] \mapsto (\bigwedge^{q+1} G_+ \wedge E)_{hG}|} \arrow[u, dashed, "\simeq"', "f"] \arrow[d, dashed, "\simeq"', "|\Gamma|"] & \\
& {|[q] \mapsto (\bigwedge^{q+1} G_+ \wedge E)^{hG}|} \arrow[d, "g"] & \\
& {|[q] \mapsto (\bigwedge^{q+1} G_+ \wedge E)|^{hG}} & {|EG_+ \wedge E|^{hG}} \arrow[l, "{:=}"'] \arrow[d, "\simeq"', "\pi"] \\
& & E^{hG}
\end{tikzcd}$$

where f and g are induced by the universal properties of the homotopy orbits and homotopy fixed points. Note that g is not neccessarily an equivalence.

Remark 8.3. Technically, we have only defined the weak map Γ on simplicial level 0. We can extend Γ to a simplicial map by using the extra degeneracy, d_0, with which EG is equipped (since it is the path space of BG). See for example ([W94], section 8.3) for details.

Lemma 8.4. *The Tate map is the restriction of the Norm map to homotopy orbits. That is, the following diagram of weak maps commutes:*

$$\begin{tikzcd}
E \arrow[r, "\Delta"] \arrow[d] & \prod_G E \arrow[r, "\prod_{g \in G} g"] & \prod_G E & \coprod_G E \arrow[l, dashed, "\cong"'] \arrow[r, "+"] & E \\
E_{hG} \arrow[rrrr, dashed, "t"] & & & & E^{hG} \arrow[u, hook]
\end{tikzcd} \tag{18}$$

where T refers to the composite that defines the Tate map.

Proof. Using the equivalences $G_+ \wedge E \simeq \vee_G E \simeq \prod_G E \simeq Map(G_+, E)$, we can rewrite the Norm map as

$$E \xrightarrow{\Delta} G_+ \wedge E \xrightarrow{\sigma} G_+ \wedge E \xrightarrow{+} E$$

where all of the maps are homotopic to the maps in diagram 18 above, but composed with the appropriate weak equivalences.

Under the identification $G_+ \wedge E \simeq \vee_G E$, the fold map is the same as the projection map $G_+ \wedge E \to E$. Since the map $\pi : EG_+ \wedge E \to E$ is given simplicially by the projections

$$\bigwedge^{q+1} G_+ \wedge E \to E$$

π is the fold map in simplicial degree 0. If one identifies $G_+ \wedge E$ with $\mathrm{Map}(G_+, E)$ instead, the diagonal map given (on the space level) by taking $x \in E(X)$ to the constant map with value x is a homotopy inverse to the fold map $G_+ \wedge E \to E$. Therefore, π^{-1} is the diagonal map in simplicial degree 0.

For any space or FSP X, let $X_.$ denote the constant simplicial set whose value in every simplicial dimension is X and whose face and degeneracy maps are id_X. We wish to show first that the weak map π^{-1} factors the diagonal map $\Delta : |E_.| \to |(G_+ \wedge E)_.|$ which makes up the first part of the norm map (recall that $G_+ \wedge E \simeq \prod_G E$). To do this, we want to show that

$$\begin{tikzcd}
{|E_.|} \arrow[rr, "\Delta"] \arrow[dr] \arrow[ddr] & & {|(G_+ \wedge E)_.|} \arrow[d, "\epsilon"] \\
& {|E_.|_{hG}} \arrow[r, dashed, "\pi^{-1}"] \arrow[d, "\simeq"] & {|[q] \mapsto \bigwedge^{q+1} G_+ \wedge E|_{hG}} \arrow[d, "\simeq"'] \\
& {|(E_{hG})_.|} \arrow[r, dashed, "|\pi^{-1}|"] & {|[q] \mapsto (\bigwedge^{q+1} G_+ \wedge E)_{hG}|}
\end{tikzcd}$$

commutes. By remark 8.3, we need only worry about simplicial map involving $EG_+ := |[q] \mapsto \bigwedge^{q+1} G_+|$ in simplicial degree 0, since such maps naturally extend to all degrees by using d_0. The map ϵ is the inclusion of simplicial degree 0 of $|(G_+ \wedge E)_.|$ into $|[q] \mapsto \bigwedge^{q+1} G_+ \wedge E|$, extended in this way, and followed by the map to homotopy orbits. The square which comprises the lower right hand corner of the diagram commutes by the equivariance of Δ, and of π. The triangle on the left hand side of the diagram commutes by the universal properties of the colimits involved. In simplicial degree 0, the outer diagram is

$$\begin{tikzcd}
E \arrow[r, "\Delta"] \arrow[d] & G_+ \wedge E \arrow[d] \\
E_{hG} \arrow[r, "\pi^{-1}"] & (G_+ \wedge E)_{hG}
\end{tikzcd}$$

which commutes because the diagonal map is a homotopy inverse to π, and by the equivariance of π. Once again, we extend this to a diagram of simplicial maps by using d_0 and this whole diagram commutes.

We already know that

$$\begin{tikzcd}
G_+ \wedge E \arrow[r, "\sigma"] \arrow[d] & G_+ \wedge E \\
(G_+ \wedge E)_{hG} \arrow[r, dashed, "\Gamma"] & (G_+ \wedge E)^{hG} \arrow[u]
\end{tikzcd}$$

commutes by Lemma 8.2. We can again extend this to a commuting diagram of simplicial sets by remark 8.3.

Finally, dualize the argument used with the diagonal map to show that

$$\begin{tikzcd}
\lvert (G_+ \wedge E)_. \rvert \arrow[rr, "+"] & & \lvert E_. \rvert \\
\lvert [q] \mapsto (\bigwedge^{q+1} G_+ \wedge E)^{hG} \rvert \arrow[u] \arrow[d] \arrow[r, "\lvert\pi\rvert"] & \lvert (E^{hG})_. \rvert \arrow[ur] \arrow[d] & \\
\lvert [q \mapsto \bigwedge^{q+1} G_+ \wedge E \rvert^{hG} \arrow[r, "\pi"] & \lvert E_. \rvert^{hG} \arrow[uur] &
\end{tikzcd}$$

commutes.

Assembling these diagrams completes the proof. $\square$

Proposition 8.5. *The Tate map is an equivalence whenever the order of G is invertible.*

Proof. Evaluating the Norm map on the homotopy orbits results in a commuting diagram:

$$
\begin{tikzcd}
E_{hG} \arrow[r, "\Delta"] \arrow[d, "="] & \prod\limits_G E_{hG} \arrow[r, "\sigma", "\simeq"'] & \prod\limits_G E_{hG} \arrow[r, "\simeq"'] & \coprod\limits_G E_{hG} \arrow[r, "+"] & E_{hG} \\
E_hG \arrow[rrrr, dashed, "T"] & & & & E^{hG} \arrow[u, "j"']
\end{tikzcd}
$$

where j is the compositions of the natural maps $E^{hG} \to E$ and $E \to E_{hG}$. Since σ is an equivalence on homotopy orbits, the norm map (the top map) is just the $|G|$-fold covering map. This induces multiplication by $|G|$ on homotopy groups. On the other hand, evaluating the norm map on the homotopy fixed points yields

$$
\begin{tikzcd}
E^{hG} \arrow[r, "\Delta"] \arrow[d, "j"] & \prod\limits_G E^{hG} \arrow[r, "\sigma", "\simeq"'] & \prod\limits_G E^{hG} \arrow[r, "\simeq"'] & \coprod\limits_G E^{hG} \arrow[r, "+"] & E_{hG} \\
E_hG \arrow[rrrr, dashed, "T"] & & & & E^{hG}. \arrow[u, "="']
\end{tikzcd}
$$

Again, the map σ is an equivalence and the norm map is the $|G|$-fold covering map. So, if $j \circ T$ and $T \circ J$ both induce multiplication by $|G|$ on homotopy. Thus T is an equivalence if $|G|$ is invertible.

□

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