COMPOSITIONS IN THE v_1-PERIODIC HOMOTOPY GROUPS OF SPHERES

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Abstract. Let $\rho_i \in \pi_{n+8i-1}(S^n)$ denote an element which suspends to a generator of the image of the stable J- homomorphism. We determine the image of the composite $\rho_j \circ \rho_k$ in v_1-periodic homotopy $v_1^{-1}\pi_{n+8i+8j-2}(S^n)$. The method is to use Adams operations to compute the 1-line of an unstable homotopy spectral sequence constructed by Bendersky and Thompson.

1. Main theorem

The p-primary v_1-periodic homotopy groups of a space X, denoted $v_1^{-1}\pi_*(X;p)$, are a localization of the portion of $\pi_*X_{(p)}$ detected by K-theory.([14]) The v_1-periodic homotopy groups of spheres contain the image of the J-homomorphism.([18]) Until the last two sections of this paper, we will deal with 2-primary homotopy theory, let $\nu(-)$ denote the exponent of 2 in an integer, and let $v_1^{-1}\pi_*(X) = v_1^{-1}\pi_*(X;2)$.

We need the following known result.

Proposition 1.1. *i.)* $v_1^{-1}\pi_{2n+8i-1}(S^{2n+1}) \approx \mathbf{Z}/2^{\min(n,\nu(i)+4)}$;

ii.) there are morphisms $v_1^{-1}: \pi_*(S^{2n+1}) \to v_1^{-1}\pi_*(S^{2n+1})$, $* > 2n+1$, *which are split surjections for* $* = 2n+8i-1$ *with* $4i - \nu(i) \geq n+8$.

Proof. i.) See Theorem 2.1 and the accompanying diagrams, 2.2 and 2.3. (ii.) In [14, 1.7] and [12, 2.4] it is shown that if $p^e : \Omega^n X \to \Omega^n X$ is null homotopic, then there is a natural morphism $v_1^{-1} : \pi_j(X) \to v_1^{-1}\pi_j(X)$ for $j > n$. In [22], it is noted that $2^{3n/2+1}$ is null homotopic on $\Omega^{2n}S^{2n+1}\langle 2n+1\rangle$. Thus v_1^{-1} is defined on $\pi_*(S^{2n+1})$ for $* > 2n+1$. The split surjection follows from [18, 1.3,1.5]. ■

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In this paper, we determine the v_1-periodic homotopy class of the composition

$$S^{N+8j+8k-2} \xrightarrow{\rho_k} S^{N+8j-1} \xrightarrow{\rho_j} S^N, \tag{1.2}$$

where ρ_i denotes a map (or its homotopy class) which suspends to a generator of the image of the stable J-homomorphism in the 2-primary $(8i-1)$-stem. That is, we will determine $v_1^{-1}(\rho_j \circ \rho_k)$ in $v_1^{-1}\pi_{N+8j+8k-2}(S^N)$, and will sometimes say that $\rho_j \circ \rho_k$ is a certain multiple of the generator in the v_1-periodic summand to mean that $v_1^{-1}(\rho_j \circ \rho_k)$ equals this multiple.

These compositions were first considered in [19] and were determined in [20] when $\nu(k) \le 4j-5$. Although our main theorem, 1.3, will be stated when N in (1.2) is an arbitrary integer, most of our discussion will, for simplicity, center around odd values of N; the minor interpolation required for even values of N can be obtained by the same methods, using results from [18].

Let $\phi(n)$ denote the number of positive integers i satisfying $i \le n$ and $i \equiv 0,1,2,4$ mod 8, and let $\delta = \begin{cases} 3 & \text{if } \nu(j)+4j > \nu(k) \\ 2 & \text{if } \nu(j)+4j \le \nu(k) \end{cases}$. Our main theorem is

Theorem 1.3. *Let* $m_{j,k} = \max(\nu(j)+4, \nu(k)+4-4j)$, *and let* N *satisfy*

$$\phi(N+\delta) \ge m_{j,k}+3. \tag{1.4}$$

Then[1] $v_1^{-1}(\rho_j \circ \rho_k)$ *is* $2^{\phi(N+3)-(m_{j,k}+3)+\nu(e(j,k))}$ *times a generator of* $v_1^{-1}\pi_{N+8j+8k-2}(S^N)$, *where*

$$e(j,k) = \begin{cases} \frac{1}{k}\Big(\sum_{i\ge1} 80^{i-1}\Big(\binom{j}{i} + (2^{4j}-1)\binom{j+k}{i}\Big)\Big) & \text{if } \nu(k) = 4j+\nu(j) \\ 0 & \text{if } \nu(k) \ne 4j+\nu(j). \end{cases}$$

An integer N satisfies (1.4) if and only if the composite (1.2) is defined; an explicit version is given by:

$$N \ge 2m_{j,k} + 4 - \delta + \begin{cases} 0 & \text{if } m_{j,k} \equiv 0,3 \mod 4 \\ 1 & \text{if } m_{j,k} \equiv 2 \mod 4 \\ 2 & \text{if } m_{j,k} \equiv 1 \mod 4. \end{cases} \tag{1.5}$$

Our theorem may be summarized by saying that

[1]except in the (rare) cases specified in Proposition 2.14, in which cases we can only assert that $v_1^{-1}(\rho_j \circ \rho_k)$ is a multiple of the specified element

Remark 1.6. *$\rho_j \circ \rho_k$ generates the v_1-periodic summand the first time it is defined, with two exceptions:*

(i) If $\nu(k) > \nu(j) + 4j$ and $\nu(k) \equiv 1, 2 \bmod 4$, then $\rho_j \circ \rho_k$ is 2 times a generator. (ii) If $\nu(k) = \nu(j) + 4j$, then $\rho_j \circ \rho_k$ is $e(j,k)$ or $2e(j,k)$ times a generator, and $e(j,k)$ is even, since, when $\nu(k) = 4j + \nu(j)$, the mod 8 value of $e(j,k)$ is that of $(2^{4j}j - k)/k$, which is even. We hope that Remark 1.6 illuminates the technical content of Theorem 1.3. Once we know $\rho_j \circ \rho_k$ on the smallest sphere on which it is defined, its value on larger spheres is easily determined from the well-known effect of the suspension homomorphism on the unstable v_1-periodic homotopy groups of spheres.

In [20], the noncommutativity of these compositions when $\nu(j) \neq \nu(k)$ is emphasized; here this is seen from the fact that $m_{j,k} \neq m_{k,j}$ in these cases. The results of [20] are a subset of our cases in which $\nu(k) < 4j + \nu(j)$; [20] does not see the change that occurs when $\nu(k) = \nu(j) + 4j$.

It is immediate to read off from Theorem 1.3 that (provided $\nu(k) \neq 4j + \nu(j)$) $v_1^{-1}(\rho_j \circ \rho_k) = 0$ in $v_1^{-1}\pi_{N+8j+8k-2}(S^N)$ if and only if

$$\phi(N + \delta) \geq m_{j,k} + \nu(j + k) + 7. \qquad (1.7)$$

However, we cannot conclude that $\rho_j \circ \rho_k$ is 0 when (1.7) is satisfied, for it might have a nontrivial v_1-torsion component. The composite $\rho_1 \circ \rho_1$ is an extreme example of this, which will be discussed at the end of Section 2.

In [8] the first author and Thompson constructed a spectral sequence $E_r^{s,t}(X)$ converging to the p-primary homotopy groups of the K-completion $X_K^\wedge$ of a space X. The second author wishes to call this the BTSS. Our theorems are proved by using Adams operations in K-theory to compute what would be the 1-line of the BTSS of $B(\rho_j)$, which is defined to be the total space of a fibration

$$S^{2n+1} \to B(\rho_j) \to S^{2n+8j+1} \qquad (1.8)$$

with attaching map ρ_j, if such a fibration existed. Our algebraic calculation has the desired homotopy-theoretic implication, even if the fibration does not exist.

The odd-primary analogue of the composition (1.2) was considered in [4] and [17], and results obtained in a large family of cases. In Section 6 we give complete results at the odd primes, following exactly the same methods that we employ at the prime

2. The odd-primary case is slightly simpler because the d_3-differentials in the BTSS which make the 2-primary case complicated to state are not present at the odd primes.

In Section 7, we determine compositions of elements which suspend to elements of stable ImJ which are not necessarily generators. This generalizes our main theorem, 1.3, and is not implied by it, since the multiples of generators are defined on smaller spheres than are the generators.

In [12], homotopy-theoretic methods were used to determine the 2-primary v_1-periodic homotopy groups of F_4/G_2, which, by [15], is an S^{15}-bundle over S^{23} with attaching map ρ_1. The results obtained there are not consistent with those obtained in this paper for such a sphere bundle. The reason for the discrepancy is apparently a mistake in an aspect of the proof in [12] which was also used in computing the v_1-periodic homotopy groups of G_2 in [15]. The results for G_2 in [15] are correct, but the proof requires some modification. These matters will be discussed in Section 5.

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2. Outline of proof

In this section, we outline the proof of Theorem 1.3. Indeed, in this section we reduce the proof of this theorem to a certain calculation in the BTSS. This calculation is then performed in Section 4. The required relationship of the BTSS with compositions is established in Section 3.

We begin by discussing the relevant features of the spectral sequence for odd-dimensional spheres.

Theorem 2.1. *Let $s = 1$ or 2, $j \geq 1$, and $n > 2$.*

1. $E_2^{s,2n+8j+1}(S^{2n+1}) \approx \mathbf{Z}/2^{\min(n,\nu(j)+4)}$.
2. *The double suspension*
$$E_2^{s,2n+8j-1}(S^{2n-1}) \to E_2^{s,2n+8j+1}(S^{2n+1})$$
is multiplication by 2 of isomorphic groups if $s = 2$ and $n > \nu(j) + 4$, and is injective otherwise.

3. *There is an isomorphism*

$$v_1^{-1}\pi_*(S^{2n+1}) \approx v_1^{-1}\pi_*((S^{2n+1})^\wedge_K)$$

and an exact sequence

$$0 \to K_1 \oplus K_2 \to v_1^{-1}\pi_{2n+8j+1-s}(S^{2n+1}) \to E_2^{s,2n+8j+1}(S^{2n+1}) \to C \to 0,$$

with

$$C = \begin{cases} \mathbf{Z}/2 & \text{if } n \equiv 1,2 \mod 4 \text{ and } (s=2 \text{ or } n \le \nu(j)+4) \\ 0 & \text{otherwise} \end{cases}$$

$$K_1 = \begin{cases} \mathbf{Z}/2 & \text{if } n \equiv 1,2 \mod 4 \text{ and } s=2 \text{ and } n > \nu(j)+4 \\ 0 & \text{otherwise} \end{cases}$$

$$K_2 = \begin{cases} \mathbf{Z}/2 & \text{if } n \equiv 0,3 \mod 4 \text{ and } s=1 \\ 0 & \text{otherwise.} \end{cases}$$

The extension is nontrivial into K_1*, and* K_2 *is split.*

4. *The localization* $\pi_i((S^{2n+1})^\wedge_K) \to v_1^{-1}\pi_i((S^{2n+1})^\wedge_K)$ *is bijective if* $i > 2n+1$.

The reader may find the following charts of the BTSS of S^{2n+1} helpful in understanding this theorem, and also, perhaps, the proof of Theorem 1.3 presented later in this section. These charts are similar to those presented in [7, p.488] and [3, p.58]. The primary difference is that those presented E_2, while these present E_∞ when $n \equiv 0,3$ mod 4, and a subquotient of E_3 after most of the d_3-differentials have been taken into account when $n \equiv 1,2$ mod 4. That the charts of [7] and [3] were of the v_1-periodic UNSS while the ones here are of the BTSS is inconsequential, since the two spectral sequences are isomorphic in dimension $> 2n+1$.

The horizontal coordinate $t-s$ corresponds to the homotopy group, dots are $\mathbf{Z}/2$, integers are cyclic groups of the indicated order, diagonal lines of slope 1 are multiplication by η, and dotted vertical lines are extensions (multiplication by 2). In these charts $\nu = \min(n, \nu(j)+4)$, and the dotted d_3-differential occurs iff $\nu = n$.

Diagram 2.2.

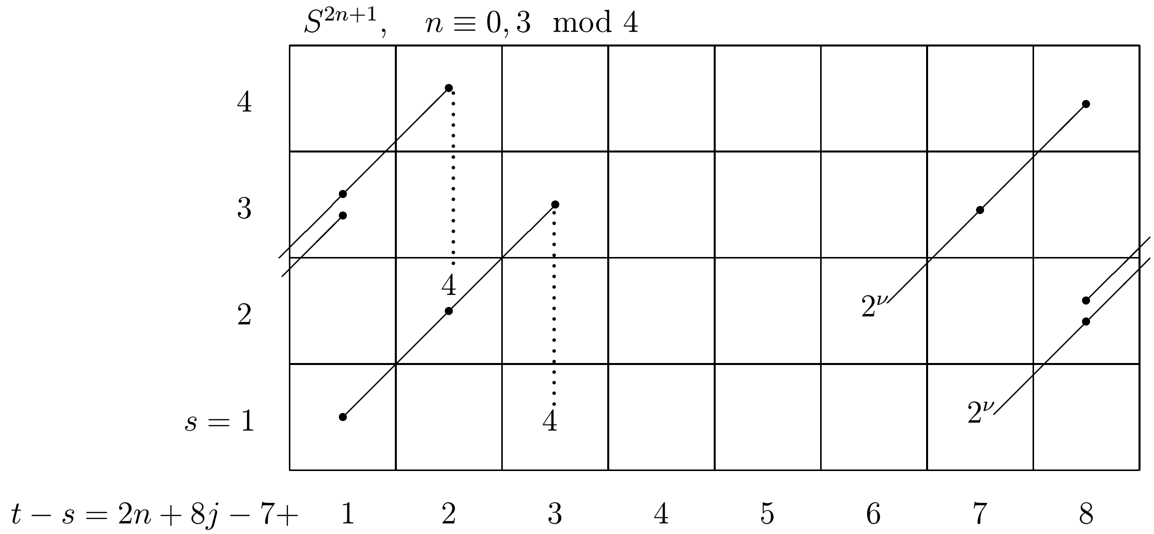
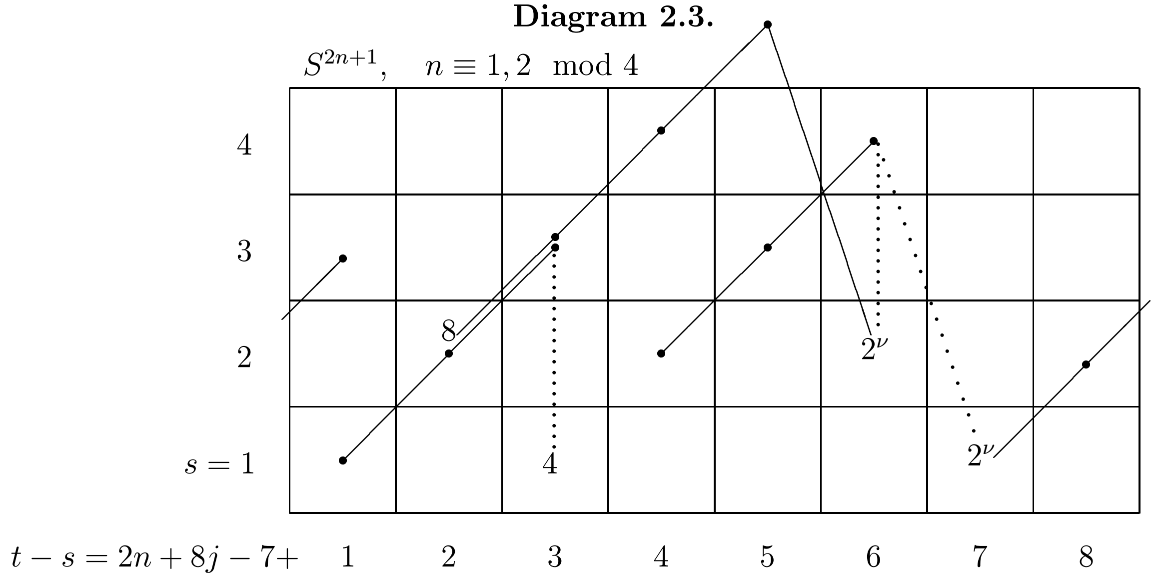


Diagram 2.3.



Proof of Theorem 2.1. There is also a v_1-periodic BTSS, denoted $v_1^{-1}E_r^{s,t}(X)$, with the localization performed as in [3]; it is isomorphic to the (unlocalized) BTSS in $t - s > \dim(X)$. In [8, 5.2] it is proved that, for any prime p, the mod p v_1-periodic BTSS of S^{2n+1} is isomorphic to the mod p v_1-periodic unstable Novikov spectral sequence

(UNSS). (The BTSS is an unstable Adams spectral sequence based on periodic K-theory, while the UNSS is an unstable Adams spectral sequence based on BP-theory.) A Bockstein spectral sequence argument implies that the two v_1-periodic spectral sequences are isomorphic (integrally). Note that [8, 5.2] just deals with E_2, but since there is a morphism of spectral sequences which is an isomorphism of E_2-terms, it is an isomorphism of spectral sequences. Thus parts 1 and 2 follow from computations made in [3] of the 2-primary v_1-periodic UNSS.

The first part of part 3 follows since the periodic UNSS converges to $v_1^{-1}\pi_*(S^{2n+1})$, while the periodic BTSS converges to $v_1^{-1}\pi_*((S^{2n+1})^\wedge_K)$, and the spectral sequences are isomorphic. The second part of part 3 follows from results about the periodic UNSS in [3], as depicted in our charts here. Cases of $C = \mathbf{Z}/2$ correspond to a nonzero d_3-differential on E_2^s, and cases of $K_1 = \mathbf{Z}/2$ correspond to a nontrivial extension into E_2^{s+2}. The $K_2 = \mathbf{Z}/2$ when $n \equiv 0, 3$ is the element in filtration 3. Part 4 follows from the isomorphism of the BTSS and periodic BTSS above $\dim(X)$. ■

If K_*X is a free commutative algebra, then $E_2^{s,t}(X) \approx \mathrm{Ext}^s_{\mathcal{U}}(PK_*S^t, PK_*X)$, where $\mathcal{U}$ is an abelian category of unstable K_*K-comodules.([8, 4.9]) We abbreviate $\mathrm{Ext}_{\mathcal{U}}(PK_*S^t, M)$ as $\mathrm{Ext}^{s,t}_{\mathcal{U}}(M)$. Let $\alpha_{i/e} = -d(v_1^i)/2^e \in K_{2i}(K)$ be as in [8, §5].

Definition 2.4. *Assume $n \geq \nu(j) + 4$ and $e \geq 0$. Let $\phi_j = \alpha_{4j/\nu(j)+4}$. Let $M_{n,j,e}$ be the $\mathcal{U}$-object which is a free K_*-module on generators g_n and g_{n+4j} with $|g_i| = 2i + 1$ and coaction*

$$\psi(g_n) = 1 \otimes g_n, \qquad \psi(g_{n+4j}) = 1 \otimes g_{n+4j} + 2^e\phi_j \otimes g_n.$$

Proposition 2.5. *With the notation and hypotheses of Definition 2.4, then*

1. $\phi_j\iota_{2n+1}$ *generates* $E_2^{1,2n+1+8j}(S^{2n+1})$, *and*
2. *If the fibration (1.8) exists, then* $PK_1(B(\rho_j)) \approx M_{n,j,0}$.

Proof. 1. In the BP-based UNSS, the generator of $E_2^{1,2n+1+8j}(S^{2n+1})$ described in [5, 9.10][2] is $\pm d(v_1^{4j} + 2^{\nu(j)+3}v_1^{4j-3}v_2)/2^{\nu(j)+4}$. In the K-based UNSS, the correction term

[2] Actually, [5, 9.10] contains a misprint in the exponent; the result here is what it should have said.

$2^{\nu(j)+3}v_1^{4j-3}v_2$ is not needed to obtain maximal divisibility. To see this, we note that

$$K_*K \approx v_1^{-1}BP_*BP/(v_i, \eta_R v_i : i > 1).$$

The relation $\eta_R v_2$ implies that $v_1^2 h_1 + v_1 h_1^2$ is divisible by 2. On the other hand,

$$d(v_1^{4j})/2^{\nu(j)+3} = ((v_1-2h_1)^{4j} - v_1^{4j})/2^{\nu(j)+3} \equiv v_1^{4j-3}(v_1^2h_1+v_1h_1^2) \mod 2,$$

and hence this is divisible by 2 in K_*K.

2. If e is as in (2.8), then $2^e\phi_j$ survives the BTSS to an element $2^e\rho_j \in v_1^{-1}\pi_{2n+8j}(S^{2n+1})$. To see this, note that $e = 1$ iff $C = \mathbf{Z}/2$ in Theorem 2.1 when $s = 1$ iff there is a nonzero differential in the BTSS emanating from $E_3^{1,2n+1+8j}(S^{2n+1})$. It is immediate that if (1.8) exists, then $PK_1(B(\rho_j)) \approx M_{n,j,0}$ because the K_*K-coaction detects the attaching map. ■

Regardless of whether or not the fibration exists, algebraic analysis yields information about topological compositions by the following result, whose proof appears in Section 3.

Theorem 2.6. *If $n \geq \nu(j) + 4$ and $e \geq 0$, there are exact sequences*

$$0 \to E_2^{1,t}(S^{2n+1}) \to \mathrm{Ext}_{\mathcal{U}}^{1,t}(M_{n,j,e}) \to E_2^{1,t}(S^{2n+1+8j}) \xrightarrow{\partial_e} E_2^{2,t}(S^{2n+1}), \tag{2.7}$$

where $\partial_e(x\iota_{2n+1+8j}) = x \otimes 2^e\phi_j\iota_{2n+1}$. Let $i : \pi_*(S^N) \to \pi_*((S^N)^\wedge_K)$ denote the completion. If $x\iota_{2n+1+8j}$ represents $i(\Sigma^2\tau)$ with $\tau \in \pi_{t-3}(S^{2n+8j-1})$, and if

$$e = \begin{cases} 1 & \text{if } n \equiv 1, 2 \text{ mod } 4, \text{ and } n = \nu(j) + 4 \\ 0 & \text{otherwise,} \end{cases} \tag{2.8}$$

then $\partial_e(x\iota_{2n+1+8j})$ represents the composite $i(2^e\rho_j \circ \Sigma\tau) \in \pi_{t-2}(S^{2n+1})^\wedge_K \approx v_1^{-1}\pi_{t-2}(S^{2n+1})$.

The compositions we have to compute lie in filtration 2 in the BTSS. Theorem 2.6 implies that the order of $\mathrm{Ext}_{\mathcal{U}}^{1,t}(M_{n,j,e})$ determines the order of the composition class. The following result is an algebraic analogue of [6, 1.1] and [6, 2.2] which reduces the calculation of the 1-line of the BTSS to a calculation involving Adams operations.

Theorem 2.9. *For any positive integers n, j, and k, there is an isomorphism*

$$\mathrm{Ext}_{\mathcal{U}}^{1,2n+8j+8k+1}(M_{n,j,0}) \approx \langle G, G' : R_2, R_2', R_3, R_3' \rangle^\#, \tag{2.10}$$

where

$$\begin{aligned} R_t &= (t^{n+4j+4k} - t^n)G + t^n(t^{4j}-1)/2^{-\nu(j)-4}G' \\ R'_t &= (t^{n+4j+4k} - t^{n+4j})G', \end{aligned}$$

and $(-)^{\#}$ *denotes the Pontryagin dual.*

Proof. Because the argument is so similar to that of [6, 1.1], we merely sketch. We begin with a result similar to [6, 2.2] for a K_*K-module M which is free as a K_*-module. The proof is essentially the same as that of [6, 2.2], and is omitted.

We say that $s\otimes y \in K_*K\otimes M$ is *unstable* if s is in the K_*-span of $\{h^I : 2\sum i_j < |y|\}$.

Proposition 2.11. *An element* $s\otimes y \in K_*K\otimes M_{2d+1}\otimes \mathbf{Q}$ *is* (*integral and*) *unstable if and only if* $t^d\langle \psi^t, s\rangle \in \mathbf{Z}_{(2)}$ *for all integers* t.

We define $U(M) \subset K_*K\otimes M$ to be the span of unstable elements. As in [6], we have

$$\mathrm{Ext}^{1,2t+1}_{\mathcal{U}}(M) \approx \ker(\bar d : M_{2t+1}\otimes \mathbf{Q}/\mathbf{Z} \to K_*K\otimes M\otimes \mathbf{Q}/U(M)),$$

where $\bar d$ is induced by the boundary d. We apply 2.11 to determine $\ker(\bar d)$ when $M = M_{n,j,0}$ and $t = n+4j+4k$. A basis for M in this grading is $\{v_1^{4j+4k}g_n, v_1^{4k}g_{n+4j}\}$ with

$$\begin{aligned} \bar d(v_1^{4j+4k}g_n) &= ((\eta_R v_1)^{4j+4k} - v_1^{4j+4k})\otimes g_n \\ \bar d(v_1^{4k}g_{n+4j}) &= ((\eta_R v_1)^{4k} - v_1^{4k})\otimes g_{n+4j} + v_1^{4k}((\eta_R v_1)^{4j} - v_1^{4j})2^{-\nu(j)-4}\otimes g_n. \end{aligned}$$

By [9, p.676] $\langle \psi^t, \eta_R(v_1^i)\rangle = t^i v_1^i$ and $\langle \psi^t, v_1^i\rangle = v_1^i$, and hence, by 2.11, $\ker(\bar d)$ is the intersection over all integers t of the kernel of the morphism of free $\mathbf{Q}/\mathbf{Z}_{(2)}$-modules with matrix

$$A_t = \begin{pmatrix} t^n(t^{4j+4k}-1) & t^n(t^{4j}-1)2^{-\nu(j)-4} \\ 0 & t^{n+4j}(t^{4k}-1) \end{pmatrix}$$

By [6, 2.5], this intersection is Pontryagin dual to the abelian group presented by all matrices A_t stacked. By an argument, presented in the next paragraph, similar to that of [6, 3.9], the relations from A_{-1}, A_2 and A_3 imply all other relations, but $A_{-1} = 0$, completing the proof of Theorem 2.9.

Although the rows of A_t may not be $t^{n+4j+4k} - \psi^t$ acting on $PK^1(X)$ for a topological space X, they behave as if they were:

i: Analogous to [2, 5.1], we verify directly that $A_t \equiv A_{t+m^e}$ mod $m^e/2^{\nu(j)+4}$;

ii: Transformations $\widetilde{\psi}^t$ defined by $\widetilde{\psi}^t(x_1) = t^n x_1 - t^n(t^{4j}-1)2^{-\nu(j)-4}x_2$, $\widetilde{\psi}^t(x_2) = t^{n+4j}x_2$ satisfy $\widetilde{\psi}^s\widetilde{\psi}^t = \widetilde{\psi}^{st}$.

Let s be odd. Use [1, 2.9] to write $s = (-1)^\epsilon 3^\ell + c2^N$ with $\epsilon = 0$ or 1, $c \in \mathbf{Z}$, and N sufficiently large. By (i), mod $2^{N-\nu(j)-4}$, $A_s \equiv A_{(-1)^\epsilon 3^\ell} = (-1)^{\epsilon n}A_{3^\ell}$. The relations in A_3 say that $\widetilde{\psi}^3 = 3^{n+4j+4k}$ on the group presented by A_3. Thus by (ii), $\widetilde{\psi}^{3^\ell} = (3^{n+4j+4k})^\ell$ on this group. Thus the rows of A_{3^ℓ}, and hence of A_s, are consequences of those of A_3. Similarly, using (ii), the relations in A_{2s} are a consequence of those in A_2 and those in A_s. ■

In Section 4, we shall calculate the right hand side of (2.10), obtaining the following result.

Theorem 2.12. *Let* $m = m_{j,k} = \max(\nu(j)+4, \nu(k)+4-4j)$. *Then*

$$\operatorname{Ext}_{\mathcal{U}}^{1,2m+8j+8k+1}(M_{m,j,0}) \approx \mathbf{Z}/2^{\nu(k)+4+f(j,k)},$$

where

$$f(j,k) = \begin{cases} 0 & \text{if } \nu(k) \neq 4j+\nu(j) \\ \min(\nu(e(j,k)), \nu(j)+4) & \text{if } \nu(k) = 4j+\nu(j), \end{cases}$$

with $e(j,k)$ *as in Theorem 1.3.*

Proof of Theorem 1.3. We first assume $\nu(k) < 4j + \nu(j)$. We will first prove that, under this assumption, $v_1^{-1}(\rho_j \circ \rho_k)$ generates $v_1^{-1}\pi_{N+8j+8k-2}(S^N)$ when

$$N = 2m_{j,k} + \begin{cases} 1 & \text{if } m_{j,k} \equiv 0,3 \mod 4 \\ 2 & \text{if } m_{j,k} \equiv 2 \mod 4 \\ 3 & \text{if } m_{j,k} \equiv 1 \mod 4 \end{cases} \tag{2.13}$$

is the smallest integer such that $\phi(N+3) \geq m_{j,k}+3$.

The exact sequence (2.7) with $n = m_{j,k}$, $e = 0$, and $t = 2n+8j+8k+1$ is

$$0 \to \mathbf{Z}/2^{\nu(j+k)+4} \to \mathbf{Z}/2^{\nu(k)+4} \to \mathbf{Z}/2^{\nu(k)+4} \xrightarrow{\partial_0} \mathbf{Z}/2^{\nu(j+k)+4}$$

unless $\nu(k) = \nu(j)$, in which case $\nu(j+k)$ should be replaced by $\nu(j)$ twice in the exact sequence. This easily implies that ∂_0 is surjective (e.g., by Euler characteristic). We emphasize that it is here that the calculation of $\operatorname{Ext}_{\mathcal{U}}(M_{n,j,0})$ is being used.

Now let e be as in (2.8), with n and t as in the previous paragraph. By the previous paragraph, ∂_e in (2.7) is multiplication by 2^e. We apply Theorem 2.6 with $x\iota_{2n+1+8j}$ representing 2^a times a generator of $E_2^{1,t}(S^{2n+1+8j})$, subject to the condition that this element is the double suspension of a homotopy class which suspends to $2^a\rho_k$. This is possible since $n+4j \geq \nu(k)+4$, which implies that $E_2^{1,2n+8j+8k+1}(S^{2n+1+8j})$ surjects under stabilization. Using Theorem 2.1, we find that this can be done (i.e., 2^a times a generator of $E_2^{1,2n+8j+8k+1}(S^{2n+1+8j})$ is the double suspension of a permanent cycle) if

$$a = \begin{cases} 2 & \text{if } \nu(j)+4j \leq \nu(k) \text{ and } m_{j,k} \equiv 2,3 \mod 4 \\ 1 & \text{if } \nu(j)+4j \leq \nu(k) \text{ and } m_{j,k} \equiv 0,1 \mod 4 \\ 1 & \text{if } \nu(j)+4j = \nu(k)+1 \text{ and } m_{j,k} \equiv 2,3 \mod 4 \\ 0 & \text{otherwise.} \end{cases}$$

We deduce from 2.6 that $v_1^{-1}(2^e\rho_j \circ 2^a\rho_k)$ is represented by 2^{e+a} times a generator of $v_1^{-1}E_2^{2,2m_{j,k}+8j+8k+1}(S^{2m_{j,k}+1})$.

If $m_{j,k} \equiv 0,3 \mod 4$, then (by 2.1.3) the E_2^2-generator represents the generator of $v_1^{-1}\pi_{2m_{j,k}+8j+8k-1}(S^{2m_{j,k}+1})$, and so, since we can divide by 2^{e+a} in this group of order divisible by 2^4, $v_1^{-1}(\rho_j \circ \rho_k)$ generates the homotopy group in S^N with $N = 2m_{j,k}+1$ in this case, as desired. If $m_{j,k} \equiv 1 \mod 4$, then the E_2-generator does not yield a homotopy class, but by 2.1 (or see [7, p.483]) its double suspension generates $v_1^{-1}\pi_{N+8j+8k-2}(S^N)$ with $N = 2m_{j,k}+3$, and so we deduce that $v_1^{-1}(\rho_j \circ \rho_k)$ generates this group in this case, as desired. Similar considerations apply when $m_{j,k} \equiv 2 \mod 4$, except that only a single suspension is required in order that the E_2 generator yield a homotopy generator (This can be deduced from [16]), and so $\rho_j \circ \rho_k$ generates in S^N with $N = 2m_{j,k}+2$ in this case.

The case of an arbitrary value of N in Theorem 1.3 now follows from the fact that $v_1^{-1}\pi_{N+8j+8k-3}(S^{N-1}) \xrightarrow{\Sigma} v_1^{-1}\pi_{N+8j+8k-2}(S^N)$ is surjective if $N+3 \not\equiv 0,1,2,4 \mod 8$, and hits the multiples of 2 if $N+3 \equiv 0,1,2,4 \mod 8$. This is probably best seen in the table at the top of page 483 of [7], with an obvious intermediate tower interpolated when $N \equiv 6 \mod 8$.

Now assume $\nu(k) > 4j+\nu(j)$. The entire argument above goes through, except that $S^{N+8j+8k-2} \xrightarrow{\rho_k} S^{N+8j-1}$ is not defined. One more suspension is required. That is why, in this case, N in (1.5) must be 1 greater than its value in (2.13). The statement in the previous paragraph about the effect on one suspension on the v_1-periodic $(8i-2)$-stem implies the minor modifications required in the proof in this case.

The situation when $\nu(k) = \nu(j) + 4j$ is exactly the same except that ∂_0 in (2.7) is multiplied by $e(j,k)$ because of Theorem 2.12 and the Euler characteristic argument. There is, however, a problem that if $e + a$ is as in the preceding paragraphs and if $2^{e+a}e(j,k)$ is 0 in $\mathbf{Z}/2^{\nu(j)+4}$, we cannot divide by 2^{e+a} to make an assertion about the image of $\rho_j \circ \rho_k$. This causes the exceptional case in Theorem 1.3.

Proposition 2.14. *Let*

$$E = \begin{cases} 1 & \text{if } \nu(j) \equiv 0 \mod 4 \\ 2 & \text{if } \nu(j) \equiv 1,3 \mod 4 \\ 3 & \text{if } \nu(j) \equiv 2 \mod 4. \end{cases}$$

If $\nu(k) = \nu(j) + 4j$ and $\nu(e(j,k)) \geq \nu(j) + 4 - E$, we can only assert in Theorem 1.3 that $v_1^{-1}(\rho_j \circ \rho_k)$ is a multiple of $2^{\phi(N+3)-(m_{j,k}+3)+\nu(e(j,k))}$ (and not that it equals this 2-power times a generator).

We close this section by discussing the special case $\rho_1 \circ \rho_1$. Theorem 1.3 and (1.7) say $v_1^{-1}(\rho_1 \circ \rho_1)$ is a generator of $v_1^{-1}\pi_{23}(S^9)$ and is 0 in $v_1^{-1}\pi_{N+14}(S^N)$ for $N \geq 21$. Yet $\rho_1 \circ \rho_1$ is nonzero in $\pi_{N+14}(S^N)$ for all $N \geq 9$.

An explanation for this is given by noting that the v_1-localization for S^{2n+1} can be obtained as the composite (see [18] or [12])

$$\pi_{2n+8i-1}(S^{2n+1}) \xrightarrow{s_*} J_{2n+8i-1}(\Sigma^{2n+1}P^{2n}) \xrightarrow{v} v_1^{-1}J_{2n+8i-1}(\Sigma^{2n+1}P^{2n}) \approx v_1^{-1}\pi_{2n+8i-1}(S^{2n+1}).$$

By [18, 1.5], s_* is surjective for $4i - \nu(i) \geq n + 8$. The morphism v is bijective if $4i-2 \geq n$; for smaller values of i, $v_1^{-1}J_{2n+8i-1}(\Sigma^{2n+1}P^{2n})$ involves elements of negative filtration, which are not present in $J_{2n+8i-1}(\Sigma^{2n+1}P^{2n})$. The class $\rho_1 \circ \rho_1$ is present and nonzero in filtration 2 in $J_{2n+15}(\Sigma^{2n+1}P^{2n})$ for $n \geq 4$, but for $n \geq 10$ it is killed by a d_3-differential from filtration -1 in $v_1^{-1}J_{2n+8i}(\Sigma^{2n+1}P^{2n})$.

This is the only case of $\rho_j \circ \rho_k$ for which this happens—that $\rho_j \circ \rho_k$ is nonzero in $J_{2n+8j+8k-1}(\Sigma^{2n+1}P^{2n})$ while 0 in $v_1^{-1}J_{2n+8j+8k-1}(\Sigma^{2n+1}P^{2n})$. To avoid such a situation, we need (roughly)

$$4(j+k) - 2 \geq m_{j,k} + \nu(j+k) + 5,$$

which is satisfied unless $j = k = 1$.

3. Compositions in the BTSS

In this section, we prove Theorem 2.6. We begin with some background on pairings in the BTSS.

Let $\{E_r^{s,t}\mathcal{X}\}$ be the homotopy spectral sequence of an augmented cosimplicial space $\mathcal{X}$. ([11]) For a space X, a K-based cosimplicial space $\mathcal{K}\mathcal{X}$ is constructed in [8]. The BTSS of X, $\{E_r^{s,t}(X)\}$, is the homotopy spectral sequence of $\mathcal{K}\mathcal{X}$.

We wish to show that the Yoneda pairing of E_2-terms of the BTSS induces a pairing of spectral sequences which survives to the composition pairing of homotopy groups. One problem is that the BTSS converges to the homotopy groups of the K-completion of X rather than those of X itself. More importantly, the cosimplicial space used to construct the spectral sequence in [8] is not termwise abelian. The required generalization of the construction in [11] will appear in [10]. In order to understand why one needs this generalization, we first review the constructions and results of [11].

If X is a simply-connected space and R is a commutative ring, then $\mathbf{R}X$ is the cosimplicial space obtained by resolving X with respect to R. The R-completion of X, $X_R^\wedge$, is defined to be $\mathrm{Tot}(\mathbf{R}X)$. The Bousfield-Kan spectral sequence, $\{E_r\mathbf{R}X\}$, converges to the homotopy groups of $X_R^\wedge$.

In [11, §10], an action

$$E_r^{s,t+m}\mathbf{R}X \otimes E_r^{s',t'}\mathbf{R}S^m \xrightarrow{\ *\ } E_r^{s+s',t+t'}\mathbf{R}X$$

is constructed for $t - s \geq 1$. This action survives to a pairing

$$\pi_{t+m-s}X_R^\wedge \otimes \pi_{t'-s'}(S^m)_R^\wedge \xrightarrow{\ *\ } \pi_{t+t'-s-s'}X_R^\wedge$$

for $t - s \geq 1$. In [11, 10.2], it is shown that this pairing is compatible with the composition pairing in homotopy, in the sense that, with $i : X \to X_R^\wedge$ the natural map, there is a commutative diagram

$$\begin{array}{ccc} \pi_{t+m}X \otimes \pi_{t'}S^m & \xrightarrow{\ \circ\ } & \pi_{t+t'}X \\ \downarrow{\scriptstyle i_*\otimes i_*} & & \downarrow{\scriptstyle i_*} \\ \pi_{t+m}X_R^\wedge \otimes \pi_{t'}(S^m)_R^\wedge & \xrightarrow{\ *\ } & \pi_{t+t'}X_R^\wedge \end{array}$$

In [11, §18], this pairing is shown to correspond to the Yoneda preoduct.

The results of [11] do not immediately apply to our situation. The pairing $*$ depends on a map c, constructed in [11, 9.1], between cosimplicial spaces. That c is a map of cosimplicial spaces uses the fact that $\mathbf{R}X$ is termwise abelian. Since the cosimplicial space in [8] is not termwise abelian, we cannot use the results of [11]. However, Bousfield has shown in [10] that there is a pairing

$$E_r^{s,t+m}X \otimes E_r^{s',t'}S^m \xrightarrow{*} E_r^{s+s',t+t'}X$$

(with $t - s \geq 1$ and $r \geq 2$) which generalizes the pairing in [11] to the K-theory situation. Specializing to odd spheres, we have the following result.

Theorem 3.1. *The $*$-pairing is compatible with the $\circ$-pairing in the sense that the following diagram commutes.*

$$\begin{array}{ccccc}
\pi_{t+2m+1}S^{2n+1} \otimes \pi_{t'}S^{2m+1} & \xrightarrow{\circ} & \pi_{t+t'}S^{2n+1} & \xrightarrow{v_1^{-1}} & v_1^{-1}\pi_{t+t'}S^{2n+1} \\
\downarrow{\scriptstyle i\otimes i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
\pi_{t+2m+1}(S^{2n+1})^\wedge_K \otimes \pi_{t'}(S^{2m+1})^\wedge_K & \xrightarrow{*} & \pi_{t+t'}(S^{2n+1})^\wedge_K & \xrightarrow{\approx} & v_1^{-1}\pi_{t+t'}(S^{2n+1})^\wedge_K
\end{array}$$

We apply this theorem to the element $2^e\rho_j \otimes \Sigma\tau$ of Theorem 2.6, and deduce that the element $i(2^e\rho_j \circ \Sigma\tau)$ in the conclusion of that theorem equals

$$i(2^e\rho_j) * i(\Sigma\tau). \tag{3.2}$$

Now we interpret the pairing in the BTSS in terms of the usual pairing on Ext. This follows since K_*S^{2n+1} is isomorphic to a free commutative algebra, so that, as noted prior to 2.4, $E_2^{s,t}(S^{2n+1}) \approx \mathrm{Ext}^s_{\mathcal{U}}(K_*S^t, K_*S^{2n+1})$.

Lemma 3.3. *The $*$ pairing corresponds to the Yoneda product*

$$\begin{aligned}
&\mathrm{Ext}^s_{\mathcal{U}}(K_*S^{t+2m+1}, K_*S^{2n+1}) \otimes \mathrm{Ext}^{s'}_{\mathcal{U}}(K_*S^{t'}, K_*S^{2m+1}) \\
\to\ &\mathrm{Ext}^s_{\mathcal{U}}(K_*S^{t+2m+1}, K_*S^{2n+1}) \otimes \mathrm{Ext}^{s'}_{\mathcal{U}}(K_*S^{t+t'}, K_*S^{t+2m+1}) \\
\to\ &\mathrm{Ext}^{s+s'}_{\mathcal{U}}(K_*S^{t+t'}, K_*S^{2n+1}).
\end{aligned}$$

Proof. This follows as in [11, §18], using the cosimplicial pairings constructed in [10]. ■

In the notation of Theorem 2.6, the homotopy pair (3.2) comes from the E_2 pair $2^e\phi_j\iota * x\iota$. The value of e given in (2.8) is required in order that $2^e\phi_j$ is a permanent cycle in the BTSS. By Lemma 3.3, this $*$ product of E_2 classes is the Yoneda product of Ext elements. The relationship of this Yoneda product with the boundary in the exact sequence (2.7) is standard, and is formalized and proved in [21, 2.3.4].

Indeed the sequence (2.7) is induced from the short exact sequence in $\mathcal{U}$

$$0 \to K_*S^{2n+1} \to M_{n,j,e} \to K_*S^{2n+1+8j} \to 0, \tag{3.4}$$

and [21, 2.3.4] asserts that the boundary morphism in (2.7) is given by Yoneda product with the element of Ext^1 corresponding to the extension (3.4). Our expression of this boundary as $x \otimes 2^e\phi_j\iota$ is easily seen by the usual way of seeing such a boundary morphism by diagram-chasing.

4. The calculation

In this section we prove Theorem 2.12 by computing the right hand side of (2.10) with $n = m_{j,k}$. We begin by letting n be arbitrary. We remove terms which are divisible by $2^{n+4j+4k}$, for, as we shall see, the groups will be of order smaller than this. The four relations, after multiplying by unit factors, are

$$\begin{gathered}2^nG - 2^{n-4-\nu(j)}(2^{4j}-1)G' \\ 2^{n+4j}G' \\ (3^{4(j+k)}-1)G + (3^{4j}-1)2^{-4-\nu(j)}G' \\ (3^{4k}-1)G'\end{gathered}$$

We use that $\nu(3^i-1) = \nu(i)+2$ if i is even, and note that the coefficient of G' in the third relation is a unit. We use the third relation to eliminate G'. With the second and fourth relations combined, we have two relations (on G)

$$2^n + 2^n(2^{4j}-1)(3^{4j+4k}-1)/(3^{4j}-1) \tag{4.1}$$

$$2^{\nu(j+k)+4+\min(n+4j,\nu(k)+4)}. \tag{4.2}$$

Multiplying (4.1) by the unit $(3^{4j}-1)/2^{\nu(4j)+4}$ yields

$$2^{n-4-\nu(j)}(3^{4j}-1+(2^{4j}-1)(3^{4j+4k}-1)).$$

Expanding $3^{4i}-1=(80+1)^i-1$ brings this to the form

$$5\cdot 2^{n-\nu(j)}\sum_{i\ge1}80^{i-1}\left(\binom{j}{i}+(2^{4j}-1)\binom{j+k}{i}\right). \tag{4.3}$$

The 2-exponent of our desired group is the smaller of the exponents of (4.2) and (4.3) when $n=m_{j,k}$. Considering several cases, one verifies that if $\nu(k)\ne 4j+\nu(j)$, then the exponent of (4.3) is $\nu(k)+4$, while that of (4.2) is larger. If $\nu(k)=4j+\nu(j)$, then the exponent of (4.2) is $\nu(j)+\nu(k)+8$, while that of (4.3) is $\nu(e(j,k))+\nu(k)+4$, as claimed. ■

5. Discussion of previous work on G_2 and F_4/G_2

In an attempt to determine $v_1^{-1}\pi_*(F_4/G_2)$, the second author proposed a proof in [12, 8.15] that $v_1^{-1}(\rho_1\circ\rho_k)$ is 8 times a generator of $v_1^{-1}\pi_{8k+21}(S^{15})\approx \mathbf{Z}/16$. According to our Theorem 1.3, its actual coefficient is

$$\begin{cases} 8 & \text{if } \nu(k)<4\\ 0 & \text{if } \nu(k)=4\\ 2^{7-\nu(k)} & \text{if } 5\le\nu(k)\le 6.\end{cases}$$

The composite is not defined on S^{15} if $\nu(k)>6$.

The proof of [12, 8.15] relied on the existence of a certain map

$$\Omega^\infty(\Sigma^{2n+1}P^{2n}\wedge J)\to(S^{2n+1})_K \tag{5.1}$$

when $n=7$. Such a map had been produced when $n=2$ in [15, 4.6]. It appears that the maps (5.1) must not exist, because of the contradiction that they imply for $\rho_1\circ\rho_k$ when $\nu(k)\ge 4$, and many other similar composites. The method of [12] was similar to that employed by Mahowald and Thompson in [20], except that [20] did not use maps (5.1). If maps (5.1) exist, this method applied to many composites $\rho_j\circ\rho_k$ disagrees with Theorem 1.3. The results of [20] about $\rho_j\circ\rho_k$ did not deal with cases where existence of (5.1) yields results that contradict our Theorem 1.3.

One aspect of the proof of [15, 4.6] seems incomplete. This is apparently the cause of the problem. A diagram

$$\begin{array}{ccc} \Omega^\infty(\Sigma^{2n+1}P\wedge J) & \longrightarrow & \Omega^\infty(\Sigma^{2n+1}P_{2n+1}\wedge J)\\ \downarrow & & \downarrow\\ \Omega^\infty(\Sigma^\infty S^{2n+1})_K & \longrightarrow & \Omega^\infty(\Sigma^{2n+1}P_{2n+1})_K\end{array}$$

is constructed. Its commutativity was tacitly assumed. The bottom map was induced from the K-localization of the Snaith map $QS^{2n+1} \to Q\Sigma^{2n+1}P_{2n+1}$. It is probably not an infinite loop map. So there seems no clear way to establish commutativity of this diagram.

The (apparently false) result [15, 4.6] was used in computing a family of higher differentials in an Adams-type spectral sequence converging to $v_1^{-1}\pi_*(G_2)$. There were two possibilities for this family of differentials, yielding two possible results for $v_1^{-1}\pi_*(G_2)$. Another approach to $v_1^{-1}\pi_*(G_2)$ is via the BTSS. Our expertise with the BTSS is not yet at the level where we can determine, strictly within BTSS technology, the entire BTSS for G_2. However, we can use [6, 1.1] to calculate the 1-line, and the result of this calculation is consistent with the picture of $v_1^{-1}\pi_*(G_2)$ claimed in [15], and is not consistent with the alternate picture. Thus we assert that [15, 1.3], the explicit listing of the groups $v_1^{-1}\pi_i(G_2)$, is correct, even though the argument there apparently had a gap.

Indeed, we have that

$$E_2^{1,2m+1}(G_2) \approx (PK^1(G_2)/\operatorname{im}(\psi^2, \psi^3 - 3^m))^{\#}.$$

Using software `LiE` as in [13], we find that $PK^1(G_2)_{(2)}$ has basis $\{w_1, w_2\}$ satisfying $\psi^k(w_1) = kw_1 + \frac{1}{2}(k - k^5)w_2$ and $\psi^k(w_2) = k^5 w_2$. This agrees with results of [23, 2.5]. From this, we obtain

$$E_2^{1,4i+3}(G_2) \approx \mathbf{Z}/2^{3+\min(\nu(i-2),3)}.$$

These groups are isomorphic to the groups $v_1^{-1}\pi_{4i+2}(G_2)$ asserted in [15, 1.3]. A picture of $v_1^{-1}\pi_*(G_2)$ is presented on [15, p.666]. The two possibilities there are whether a differential d is nonzero for even values of k or for odd values of k. In either case, the fact that the element in the top of the tower is divisible by η^2 implies that in the BTSS there must be an extension from the 1-line group to a $\mathbf{Z}/2$ on the 3-line. In order to make the E_2-group and the homotopy group have the same order, there must be a compensating d_3-differential on the generator of the 1-line group. On the other hand, there is no way to create a BTSS picture with 1-line as computed which yields the picture of [15, p.666] with the opposite pattern for d.

6. The odd-primary analogue

The exact same methods which were applied at the prime 2 yield similar results when applied at an odd prime p. In this section, p is an odd prime, $q = 2(p-1)$, and $\nu(-)$ is the exponent of p in an integer. Let ρ_i denote a map (or its homotopy class) which suspends to a generator of the image of the stable J-homomorphism in the p-primary $(qi-1)$-stem.

It was shown in [4] that $v_1^{-1}\pi_{2n+qi-1}(S^{2n+1}) \approx \mathbf{Z}/p^{\min(n,\nu(i)+1)}$, and

$$\pi_{2n+qi-1}(S^{2n+1}) \xrightarrow{v_1^{-1}} v_1^{-1}\pi_{2n+qi-1}(S^{2n+1})$$

is split surjective if $i - \nu(i) \geq n+1$. Our theorem at the odd prime p, analogous to Theorem 1.3 is

Theorem 6.1. *Let $n_{j,k} = \max(\nu(j)+1, \nu(k)+1-(p-1)j)$. Then, by [5], $S^{2n_{j,k}+1}$ is the smallest odd sphere on which $\rho_j \circ \rho_k$ is defined. If $n \geq n_{j,k}$, then $v_1^{-1}(\rho_j \circ \rho_k)$ is $2^{n-n_{j,k}+\nu(e(j,k))}$ times a generator of $v_1^{-1}\pi_{2n+qj+qk-1}(S^{2n+1})$, where*

$$e(j,k) = \begin{cases} 1 & \text{if } \nu(k) \neq (p-1)j + \nu(j) \\ \frac{1}{k}\sum_{i\geq 1}(\alpha p)^{i-1}\left(\binom{j}{i} + (p^{(p-1)j}-1)\binom{j+k}{i}\right) & \text{if } \nu(k) = (p-1)j + \nu(j), \end{cases}$$

where $\alpha \not\equiv 0$ mod p satisfies $\alpha p + 1 = r^{p-1}$ with r a generator of $(\mathbf{Z}/p^2)^\times$.

Again it is true that $\rho_j \circ \rho_k$ generates the v_1-periodic summand the first time it is defined, provided $\nu(k) \neq (p-1)j + \nu(j)$. Also note that one can easily read off when the v_1-periodic component of $\rho_j \circ \rho_k$ is 0, but cannot usually deduce that $\rho_j \circ \rho_k$ is in fact 0. The proof of Theorem 6.1 is a direct adaptation of our proof in the 2-primary case, and is omitted.

In [4, 4.1], the first author used the UNSS to prove Theorem 6.1 when $\nu(k) < (p-1)j + \nu(j)$.

7. Compositions of non-generators

In this section, we show how our main theorem generalizes to compositions of any elements which suspend to elements in the image of J, not necessarily the generators. Results for multiples of the generators are not usually implied by results for the generators because the multiples are defined on smaller spheres than are the generators, and the double suspension homomorphism is often not injective on the unstable homotopy groups where these compositions lie.

For simplicity, we just treat the odd-primary case here, and we do not deal with borderline cases which cause coefficients such as $e(j,k)$ of Theorem 6.1 to arise. We just write what the composite is on the smallest sphere on which it is defined. Results for these composites on larger spheres are, of course, then determined since the double suspension induces multiplication by 2.

We use the notation that $\alpha_{j/e}$ is an element of order p^e in the $(qj-1)$-stem. This is defined provided $e \le \nu(j)+1$. Then ρ_j of Section 6 is $\alpha_{j/\nu(j)+1}$. The following result generalizes Theorem 6.1. Composites of this sort were considered in a limited range of cases in [4] and [17].

Theorem 7.1. *The smallest odd sphere S^{2n_0+1} on which $\alpha_{j/e}\circ\alpha_{k/f}$ is defined has $n_0 = \min(e, f-(p-1)j+1)$. The composite $v_1^{-1}(\alpha_{j/e}\circ\alpha_{k/f})$ equals $p^{c_1+c_2}$ times a generator of $v_1^{-1}\pi_{qj+qk+2n_0-1}(S^{2n_0+1})$, where*

$$\begin{aligned} c_1 &= \max(0, \min(\nu(k)-(p-1)j, \nu(j))+1-e) \\ c_2 &= \max(0, \min(\nu(k)+1, e+(p-1)j-1)-f), \end{aligned}$$

provided $\nu(k)\ne\nu(j)+(p-1)j$.

Note that if $\alpha_{j/e}=\rho_j$, then $c_1=0$, and if $\alpha_{k/f}=\rho_k$, then $c_2=0$, and so the result agrees with Theorem 6.1, which states that the composite of elements which suspend to generators of ImJ is a generator the first time it is defined.

The proof is similar to that of Theorem 6.1, which was similar to that of Theorem 1.3. We define an unstable K_*K-comodule M which depends on n_0, j, and e, and fits into an exact sequence

$$0\to E_2^{1,t}(S^{2n_0+1})\to \operatorname{Ext}_{\mathcal{U}}^{1,t}(M)\to E_2^{1,t}(S^{2n_0+1+qj})\xrightarrow{\partial} E_2^{2,t}(S^{2n_0+1}).$$

We work with $t_0=2n_0+1+qj+qk$. Then $\alpha_{k/f}$ is represented by p^{c_2} times a generator of $E_2^{1,t_0}(S^{2n_0+1+qj})$. Similarly to Theorem 2.12, we compute the order of $\operatorname{Ext}_{\mathcal{U}}^{1,t_0}(M)$. The group is not always cyclic. It turns out that

$$|\operatorname{Ext}_{\mathcal{U}}^{1,t_0}(M)|/|E_2^{1,t_0}(S^{2n_0+1+qj})| = p^{c_1}.$$

It is elementary to conclude that ∂ sends a generator to p^{c_1} times a generator. By an analogue of Theorem 2.6, ∂ sends $\alpha_{k/f}$ to $\alpha_{j/e} \circ \alpha_{k/f}$, yielding Theorem 7.1.

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