STABLE GEOMETRIC DIMENSION OF VECTOR BUNDLES OVER ODD-DIMENSIONAL REAL PROJECTIVE SPACES

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Abstract. In [6], the geometric dimension of all stable vector bundles over real projective space P^n was determined if n is even and sufficiently large with respect to the order 2^e of the bundle in $\widetilde{KO}(P^n)$. Here we perform a similar determination when n is odd and $e > 6$. The work is more delicate since P^n does not admit a v_1-map when n is odd. There are a few extreme cases which we are unable to settle precisely.

1. Statement of results

The geometric dimension $\mathrm{gd}(\theta)$ of a stable vector bundle θ over a space X is the smallest integer m such that θ is stably equivalent to an m-plane bundle. Equivalently, $\mathrm{gd}(\theta)$ is the smallest m such that the classifying map $X \xrightarrow{\theta} BO$ factors through $BO(m)$. The group $\widetilde{KO}(P^n)$ of equivalence classes of stable vector bundles over real projective space is a finite cyclic 2-group generated by the Hopf line bundle ξ_n.

In [6], it was shown that, for sufficiently large even n, the geometric dimension of a stable vector bundle over P^n depends only on its order in $\widetilde{KO}(P^n)$ and the mod 8 value of n. For bundles of order 2^e, this value, called $\mathrm{sgd}(n, e)$ or $\mathrm{sgd}(\overline{n}, e)$, where $\overline{n}$ is the mod 8 residue of n, was completely determined; its approximate value is $2e$. A key role in this analysis was played by KO-equivalences $P^{n+8}_{k+8} \to P^n_k$, defined if n is even, k is odd, and $n + 8 < 2k - 1$. Such maps do not exist when n is odd, and so the methods and results are somewhat more complicated. The term "stable" geometric dimension (sgd) refers to the fact that the geometric dimension achieves a stable value as n gets large within its congruence class.

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An important role in [6] was played by the v_1-periodic spectrum functor Φ described in [7, 7.2]. We are interested in the stable portion of $[P^n, \Phi BSO(m)]$, i.e. the portion which persists under $j_m : BSO(m) \to BSO$. To achieve this, we define the **stable** portion

$$\mathbf{s}[P^n, \Phi BSO(m)] = [P^n, \Phi BSO(m)]/\ker(j_{m*}),$$

and similarly for spectral sequence groups that approximate these groups. The group $\mathbf{s}[P^n, \Phi BSO(m)]$ is cyclic since it maps injectively to the cyclic group $[P^n, \Phi BSO]$.

In [6], we proved that, if n is even,

$$\text{sgd}(n, e) \le m \quad \text{iff} \quad \nu(\mathbf{s}[P^n, \Phi BSO(m)]) \ge e. \tag{1.1}$$

Here and throughout, $\nu(-)$ denotes the exponent of 2 in an integer, and if C is a cyclic group, then $\nu(C)$ denotes $\nu(|C|)$. The backwards implication has a simple and natural proof ([6, 1.5]), while the forward implication was proved by noting that all the requisite nonlifting results were already in the literature.

For odd n, we determine $\nu(\mathbf{s}[P^n, \Phi BSO(m)])$ completely in Theorem 1.2, provided $m \ge 12$. We prove in 2.1 that the backwards implication of (1.1) holds when n is odd, except that here this sgd refers to stable bundles of order 2^e over projective spaces of sufficiently large dimension $\equiv n \bmod 2^L$, with L usually, but perhaps not always, equal to 3. We will observe in Theorem 1.3 that, in almost all cases, known nonlifting results of Section 3 imply the converse; i.e. (1.1) holds in almost all cases when n is odd. However, there are some rare cases in which our computation of $\nu(\mathbf{s}[P^n, \Phi BSO(m)])$ suggests there should be an extra nonlifting result which we have been unable to establish.

Most of our work is devoted to proving the following theorem.

Theorem 1.2. *If $m = 8i + d \ge 12$, then $\nu(\mathbf{s}[P^n, \Phi BSO(m)])) = 4i + t$, where t is given by the following table. The two entries indicated by asterisks must be decreased by 1 if $\nu(n + 1 - m) \ge \frac{1}{2}m - 2$.*

		d							
		0	1	2	3	4	5	6	7
	1	0	0	1^*	1	2	2	3	3
n *mod* 8	3	0	0	1	2	3	3	3	3
	5	0	0	1	1	2	2	3^*	3
	7	0	0	0	0	1	1	2	3

Combining this with 2.1 for liftings, and using 3.1 and 3.2 for nonliftings, yields the following result, which is our main theorem.

Theorem 1.3. *Define* $\delta(\overline{n}, e)$ *by the table*

		$e \mod 4$			
		0	1	2	3
	1	0	0^*	0	0
$\overline{n}$	3	0	0	-1	-2
	5	0	0	0	0^*
	7	0	2	2	1

Let $e \geq 7$. *For sufficiently large* $n \equiv \overline{n} \mod 8$,[1] *the geometric dimension of stable vector bundles of order* 2^e *over* P^n *equals* $2e + \delta(\overline{n}, e)$, *except that entries indicated with an asterisk might be 1 greater than indicated if* $\nu(n + 1 - 2e) \geq e - 2$.

The idea of stable geometric dimension was first proposed in [10]. It was claimed there that if $e \geq 75$, then $\text{sgd}(n, e) \leq 2e + \delta(\overline{n}, e)$ with $\delta(\overline{n}, e)$ as in Theorem 1.3, ignoring the asterisks. We do not contradict those results here. However, if the exotic nonlifting results mentioned above can be proved, they would contradict this lifting result of [10], for certain extreme cases with n odd. This does not seem to be out of the question, for the sentence near the bottom of [10, p.60] which includes a commutative diagram seems to lack justification, which could render that proof invalid.

For even-dimensional projective spaces, we also obtained, in [6], results about stable geometric dimension for bundles of order 2^e when $e < 7$. We could do that here for odd-dimensional projective spaces, but the arguments are extremely delicate. Consequently, we will defer these cases of small m and e to the future.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We begin with a general result similar to [6, 1.6].

Proposition 2.1. *Let* n *be odd and* e *a fixed positive integer. For each* m, *there exists an integer* L *such that if* $\nu(\mathbf{s}[P^n, \Phi BSO(m)]) \geq e$ *then, for sufficiently large* N *satisfying* $N \equiv n \mod 2^L$, *the geometric dimension of any stable vector bundle of order* 2^e *over* P^N *is less than or equal to* m.

[1]If the asterisked entries are increased to 1, then $n \equiv \overline{n} \bmod 8$ must be modified to $n \equiv \overline{n} \bmod 2^{e-2}$ in these cases.

Proof. From the definition of ΦX in [11][2] as a periodic spectrum whose spaces are telescopes of

$$\Omega^{L_1}X \to \Omega^{L_1+2^L}X \to \cdots \to \Omega^{L_1+k2^L}X \to \cdots,$$

with $L_1 \equiv 0 \mod 2^L$ for the 0^{th} space, it follows, using James periodicity, that

$$[P^n, \Phi BSO(m)] \approx \operatorname*{colim}_k [P^{n+k2^L}_{1+k2^L}, BSO(m)].$$

Thus the hypothesis implies that the stable bundle of order 2^e over P^{n+k2^L} lifts to $BSO(m)$ if k is sufficiently large. ■

The informal claim that we made in Section 1 that L can usually be chosen to be 3 can be seen either from the fact that $\nu(\mathbf{s}[P^n, BSO(m)])$ determined in 1.2 usually only depends on n mod 8, or by restricting to P^{n-1} and using the result from [6] that geometric dimension over these even-dimensional projective spaces eventually only depends on the mod 8 value of $n-1$. The way in which Proposition 2.1 will be used in the proof of Theorem 1.2 is to use known nonlifting results (3.1 and 3.2) to assert that $\nu(\mathbf{s}[P^n, \Phi BSO(m)]) < e$ for various values of the parameters.

The proof of the following result occupies most of the rest of this section.

Theorem 2.2. *Let n be odd, $m \geq 12$, and $\phi_{n,m}$ denote the restriction homomorphism*

$$\mathbf{s}[P^n, \Phi BSO(m)] \to \mathbf{s}[P^{n-1}, \Phi BSO(m)]$$

between cyclic 2-groups. Then

$$|\ker(\phi_{n,m})| = \begin{cases} 2 & \text{if } n \equiv 1 \mod 8 \\ 1 & \text{otherwise} \end{cases}$$

$$|\operatorname{coker}(\phi_{n,m})| = \begin{cases} 2 & \text{if } n \equiv 1 \mod 4 \text{ and } n-m \equiv 0,1,2 \mod 8 \\ 2 & \text{if } n \equiv 1 \mod 4 \text{ and } \nu(n+1-m) \geq m/2-2 \\ 1 & \text{otherwise} \end{cases}$$

Theorem 1.2 follows directly from 2.2 and the following recapitulation of results of [6].

[2] called $\mathbf{Tel}_1 X$ there

Theorem 2.3. ([6, 1.7,1.8,1.10]) *If* $n \equiv 6, 8 \mod 8$ *and* $8i + d \geq 9$*, then*

$$\nu(\mathbf{s}[P^n, \Phi BSO(8i+d)]) = 4i + \begin{cases} -1 & d = -1 \\ 0 & d = 0,1,2,3 \\ 1 & d = 4,5 \\ 2 & d = 6. \end{cases}$$

If $n \equiv 2, 4 \mod 8$ *and* $8i + d \geq 9$*, then*

$$\nu(\mathbf{s}[P^n, \Phi BSO(8i+d)]) = 4i + \begin{cases} 0 & d = 0,1 \\ 1 & d = 2 \\ 2 & d = 3 \\ 3 & d = 4,5,6,7. \end{cases}$$

The lengthy proof of Theorem 2.2 will occupy the remainder of this section. We let $n = 2k+1$. Viewing $\mathbf{s}[P, \Phi BSO(m)]$ as

$$\operatorname{im}([P, \Phi BSO(m)] \xrightarrow{j_{m*}} [P, \Phi BSO],$$

it is clear that the kernel of $\phi_{2k+1,m}$ in 2.2 equals the kernel of

$$[P^{2k+1}, \Phi BSO] \xrightarrow{i^*} [P^{2k}, \Phi BSO].$$

The proof of 2.1 implies that this kernel equals that of

$$\operatorname{colim}[P^{2k+1+c2^L}, BSO] \xrightarrow{i^*} \operatorname{colim}[P^{2k+c2^L}, BSO],$$

which, by the calculation of $\widetilde{KO}(P^n)$ in [1], has order 2 if $k \equiv 0 \bmod 4$, and is trivial otherwise. This establishes the kernel part of 2.2.

The cokernel of $\phi_{2k+1,m}(= \mathbf{s}i^*)$ is much more delicate. It involves the exact sequence

$$[P^{2k+1}, \Phi BSO(m)] \xrightarrow{i^*} [P^{2k}, \Phi BSO(m)] \xrightarrow{\alpha^*} v_1^{-1}\pi_{2k}(BSO(m)), \tag{2.4}$$

where α denotes the attaching map. The following proposition is elementary.

Proposition 2.5. *Let* $x \in [P^{2k}, \Phi BSO(m)]$ *satisfy* $j_{m*}(x) \neq 0$*, so its equivalence class* $[x]$ *is a nonzero element in* $\mathbf{s}[P^{2k}, \Phi BSO(m)]$*.*

- *If* $\alpha^*(x) = 0$*, then* $[x] \in \operatorname{im}(\phi_{2k+1,m})$*.*
- *If* $\alpha^*(x) \neq 0$ *and there is no* $y \in \ker(j_{m*})$ *such that* $\alpha^*(y) = \alpha^*(x)$*, then* $[x]$ *is a nonzero element of* $\operatorname{coker}(\phi_{2k+1,m})$*.*

The main point here is the necessity of checking for y.

The proof of the cokernel part of 2.2 varies depending on the mod 4 value of k and mod 8 value of m in (2.4).

Case 1: $k \equiv 2 \mod 4$, $m \equiv -1, 0, 1 \mod 8$. Here $v_1^{-1}\pi_{2k}(BSO(m)) = 0$ by [3, 1.2,3.4,3.6] and so by Proposition 2.5 $\phi_{2k+1,m}$ is surjective in 2.2 in this case.

Case 2: $k \equiv 2 \mod 4$, $m \equiv 3, 4, 5 \mod 8$. By §3[3],

$$\nu(\mathbf{s}[P^{8\ell+5}, \Phi BSO(8i+d)]) \le 4i + \begin{cases} 1 & d = 3 \\ 2 & d = 4, 5. \end{cases}$$

By Theorem 2.3,

$$\nu(\mathbf{s}[P^{8\ell+4}, \Phi BSO(8i+d)]) = 4i + \begin{cases} 2 & d = 3 \\ 3 & d = 4, 5. \end{cases}$$

Thus $\phi_{2k+1,m}$ in 2.2 must have nontrivial cokernel when $m \equiv 3, 4, 5 \mod 8$ (and still $k \equiv 2 \mod 4$). This cokernel can have order at most 2 because $v_1^{-1}\pi_{2k}(BSO(m)) = \mathbf{Z}/2$ if $m \equiv 3, 5 \mod 8$ by [3, 3.10], while $v_1^{-1}\pi_{2k}(BSO(8i+4)) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Case 3: $k \equiv 0 \mod 4$, $m \equiv -1, 0, 1 \mod 8$. By §3,

$$\nu(\mathbf{s}[P^{8\ell+1}, \Phi BSO(8i+d)]) \le 4i + \begin{cases} -1 & d = -1 \\ 0 & d = 0, 1. \end{cases}$$

By 2.3

$$\nu(\mathbf{s}[P^{8\ell}, \Phi BSO(8i+d)]) = 4i + \begin{cases} -1 & d = -1 \\ 0 & d = 0, 1. \end{cases}$$

We have already proved $\ker(\phi_{8\ell+1,m}) = \mathbf{Z}/2$, and hence $\text{coker}(\phi_{8\ell+1,m}) \ne 0$. We must prove the order of this cokernel is only 2.

By [3, 1.2,1.3,1.4], $v_1^{-1}\pi_{8\ell-1}(SO(m))$ is an extension of two $\mathbf{Z}/2$-vector spaces[4], one in filtration 2 and the other in filtration 4. We will show that the filtration-4 elements are in the image of α^* in (2.4); they are hit not by the stable summand but rather by elements of order 2. This implies that the desired cokernel has order only 2.

[3]As was remarked prior to Theorem 1.3, all the lower bounds of that theorem are immediate from 3.1 and 3.2, and by 2.1, all the non-asterisked "$\le$" parts of 1.2 follow from this. When we invoke one of these ($\text{sgd}(-,-) \le -$)-results, we will just say "By §3."

[4]This is the first time of many that we will utilize the isomorphism $v_1^{-1}\pi_i(SO(m)) \approx v_1^{-1}\pi_{i+1}(BSO(m))$.

The attaching map for the top cell of $P^{8\ell+1}$ is η on the $(8\ell-1)$-cell. By [6, (2.4)],

$$[P^{8\ell}, \Phi BSO(m)] \approx [P_{1-8\ell}^{0}, \Phi BSO(m)] \approx [M^0(2^{4\ell}), \Phi BSO(m)].$$

Since, by [6, (2.6)], the stable summand of $[M^0(2^{4\ell}), \Phi BSO(m)]$ comes from the bottom cell of the Moore space, α^* in (2.4) is equivalent to

$$\mu_\ell^* : v_1^{-1}\pi_{-1}(BSO(m)) \to v_1^{-1}\pi_{8\ell}(BSO(m)), \tag{2.6}$$

where μ_ℓ is the element of highest Adams filtration in the $(8\ell+1)$-stem, detected by $P^\ell h_1$ in the Adams spectral sequence. This is seen by observing that

$$S^{8\ell} \xrightarrow{\alpha} P^{8\ell} \xrightarrow{\phi^\ell} P_{1-8\ell}^{0}$$

and

$$S^{8\ell} \xrightarrow{\mu_\ell} S^{-1} \xrightarrow{\deg 1} P_{1-8\ell}^{0}$$

become equal in $\pi_{8\ell}(P_{1-8\ell}^0 \wedge J) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$, where each equals the element of highest filtration. Thus, since $v_1^{-1}\pi_*(P) \approx v_1^{-1}\pi_*(P \wedge J)$ for spectra P by [12], the two composites become equal in $v_1^{-1}\pi_{8\ell}(P_{1-8\ell}^0)$. Thus they are equal in $v_1^{-1}\pi_{8\ell}(BSO(m))$. Here we have used the 2-local J-spectrum which is the fiber of $\psi^3 - 1 : bo \to \Sigma^4 bsp$. This spectrum played a key role in the early days of v_1-periodic homotopy theory, especially in [12].

In the spectral sequence of [3] converging to $v_1^{-1}\pi_*(SO(m))$, elements in filtration ≥ 2 occur in eta-towers, with their Pontryagin duals described by elements in $QK^1(\mathrm{Spin}(m))/\operatorname{im}(\psi^2)$, occurring with period 4. Dual to the composition (2.6) is

$$E_2^{s+1,t+2+8\ell}(\mathrm{Spin}(m))^\# \xrightarrow{v_1^{4\ell}} E_2^{s+1,t+2}(\mathrm{Spin}(m))^\# \xrightarrow{h_1^\#} E_2^{s,t}(\mathrm{Spin}(m))^\#, \tag{2.7}$$

where v_1^4 is the isomorphism which shifts eta towers to elements with the same name, and $h_1^\#$ stays in the same eta tower. To see this, note that, with $Y = \mathrm{Spin}(m)$, if $g \in \pi_n(Y)$, then $g \circ \mu_\ell (= \mu_\ell^*(g)$ in (2.6)) can be obtained as the composite

$$S^{8\ell+n+1} \hookrightarrow M^{8\ell+n+2}(2) \xrightarrow{A^\ell} M^{n+2}(2) \xrightarrow{\widetilde{\eta}} S^n \xrightarrow{g} Y, \tag{2.8}$$

where A is an Adams map and $\widetilde{\eta}$ an extension over the mod-2 Moore spectrum of $S^{n+1} \xrightarrow{\eta} S^n$. Then (2.7) is dual to the horizontal composition in Diagram 2.9, while (2.8) induces the composition around the top. The vertical maps ∂ are Bockstein homomorphisms for $\cdot 2$.

Diagram 2.9. *Diagram involving Bockstein and h_1*

$$\begin{array}{ccccc} & & E_2^{s,s+n+2}(Y;\mathbf{Z}_2) & \xrightarrow{v_1^{4\ell}} & E_2^{s,s+n+2+8\ell}(Y;\mathbf{Z}_2) \\ & \nearrow^{\widetilde{\eta}^*} & \downarrow \partial & & \downarrow \partial \\ E_2^{s,s+n}(Y) & \xrightarrow{h_1} & E_2^{s+1,s+n+2}(Y) & \xrightarrow{v_1^{4\ell}} & E_2^{s+1,s+n+2+8\ell}(Y) \end{array}$$

Now the claim about filtration-4 elements y being $\alpha^*(x)$ with x an element of filtration 3 follows from (2.7), since x is the element in an earlier eta-tower with the same name as y. This completes the proof of Case 3.

For the remaining cases, we will need the following result, where $Q(-)$ denotes the indecomposables.

Theorem 2.10. *For any positive integers n and m, there is a spectral sequence $E_r(n,m)$ converging to $[P^n, \Phi SO(m)]_*$ with*

$$E_2^{s,t}(n,m) = \operatorname{Ext}^s_{\mathcal{A}}(K^*(\Phi\operatorname{Spin}(m)), K^*(\Sigma^t P^n)). \qquad (2.11)$$

If n is even, then $E_2^{s,2r}(n,m) = 0$, and if n is also sufficiently large, there is a short exact sequence

$$\begin{aligned} 0 \quad &\to \quad \operatorname{Ext}^s_{\mathcal{A}}(QK^1\operatorname{Spin}(m)/\operatorname{im}(\psi^2), K^1S^{2r+1}) \to E_2^{s,2r+1}(n,m) \\ &\xrightarrow{\delta} \quad \operatorname{Ext}^{s+1}_{\mathcal{A}}(QK^1\operatorname{Spin}(m)/\operatorname{im}(\psi^2), K^1S^{2r+1}) \to 0. \qquad (2.12) \end{aligned}$$

If n is odd and sufficiently large, there is a split short exact sequence

$$0 \to \operatorname{Ext}^{s,n+t}_{\mathcal{A}}(QK^*(\operatorname{Spin}(m))/\operatorname{im}(\psi^2)) \xrightarrow{q^*} E_2^{s,t}(n,m) \xrightarrow{i^*} E_2^{s,t}(n-1,m) \to 0. \qquad (2.13)$$

Several remarks are in order here. (i) We omit 2-adic coefficients from all $K^*(-)$-groups, and will continue to do so. (ii) $\mathcal{A}$ is the category of 2-adic stable Adams modules.([7]) (iii) We have replaced $SO(m)$ by its double cover $\operatorname{Spin}(m)$. This does not change $v_1^{-1}\pi_*(-)$, and indeed $\Phi SO(m) = \Phi\operatorname{Spin}(m)$. But for calculations such as (2.14), it is essential that the underlying space be simply-connected. (iv) Beginning with (2.13), we will often abbreviate $\operatorname{Ext}^s_{\mathcal{A}}(M, K^*S^t)$ as $\operatorname{Ext}^{s,t}_{\mathcal{A}}(M)$. (v). The splitting of (2.13) is just claimed for E_2, not necessarily for the entire spectral sequence.

Proof. By [7, 7.2], the spectrum $\Phi SO(m)$ is $K/2_*$-local, and so the existence of the spectral sequence follows from [7, 10.4].[5] By [8, 9.1], there is an isomorphism in A

$$K^i(\Phi\,\mathrm{Spin}(m)) \approx \begin{cases} 0 & i=0 \\ QK^1(\mathrm{Spin}(m))/\,\mathrm{im}(\psi^2) & i=1. \end{cases} \tag{2.14}$$

By [1], if n is even, then

$$K^i(P^n) \approx \begin{cases} \mathbf{Z}/2^{n/2} & i=0 \\ 0 & i=1 \end{cases}$$

with $\psi^k = 1$ on $K^0(P^n)$.

Let $M_r = K^*(S^{2r+1}) = \begin{cases} \mathbf{Z}_2^\wedge & *=1 \\ 0 & *=0 \end{cases}$ with $\psi^k = k^r$. With n still even, there is a short exact sequence in $\mathcal{A}$

$$0 \to M_r \xrightarrow{2^{n/2}} M_r \to K^*(\Sigma^{2r+1}P^n) \to 0. \tag{2.15}$$

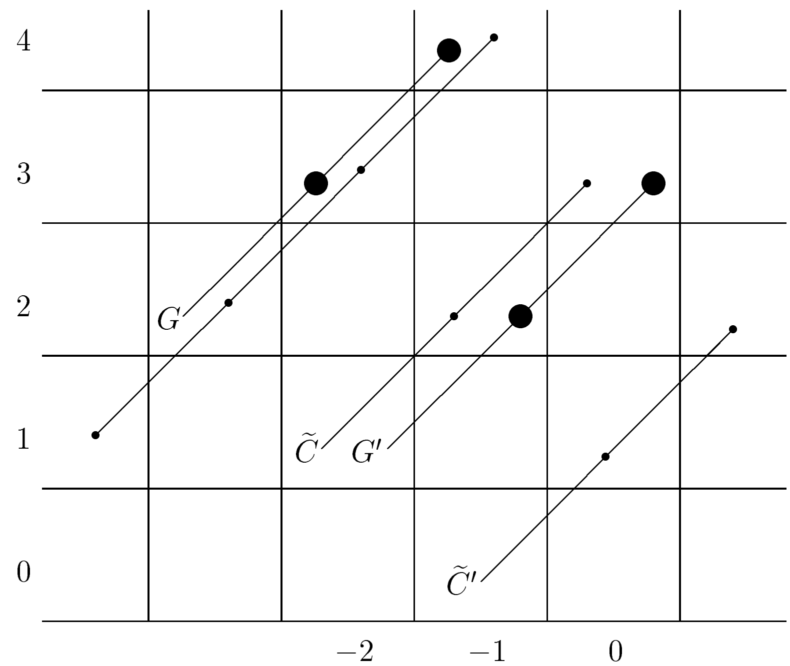
We choose n larger than any of the exponents of Ext groups that occur (roughly $m/2$). Then the long exact sequence with (2.15) in the second variable of $\mathrm{Ext}_{\mathcal{A}}(K^*(\Phi\,\mathrm{Spin}(m)), -)$ breaks up into short exact sequences (2.12).

If n is odd, the cofibration $P^{n-1} \to P^n \to S^n$ induces a split short exact sequence in $K^*(-)$. In fact, $K^*(S^n)$ and $K^*(P^{n-1})$ are nonzero only in distinct gradings. The split short exact sequence (2.13) is immediate from this. ■

By (2.12), if n is even and sufficiently large, the E_2-chart is independent of n, and, using results of [3] about the general form of $\mathrm{Ext}_{\mathcal{A}}^{**}(QK^1\,\mathrm{Spin}(m)/\,\mathrm{im}(\psi^2))$, the chart, in the vicinity of $t-s=-1$, has the form pictured in Diagram 2.16.

[5] Although [7] just deals with odd primes, this result is also valid for the prime 2.

Diagram 2.16. General form of $E_2^{*,*}(n,m)$ when n is even and large



The notation here is as follows. As is customary with Adams spectral sequence charts, the group in position $(t-s,s)$ is $E_2^{s,t}$. In [3, esp. 1.3,3.7,3.12], charts for $\mathrm{Ext}_{\mathcal{A}}^{*,*}(QK^1\,\mathrm{Spin}(m))$ are presented for various mod 8 congruences of m. The group $\widetilde{C}$ of Diagram 2.16 is usually[6] a sum of two cyclic groups usually denoted $C_1 \oplus C_2$ in [3]. Our group $\widetilde{C}'$ is a group isomorphic to $\widetilde{C}$ coming from the second half of (2.12). The summand C_1 in $\widetilde{C}'$ is our stable summand $\mathbf{s}E_2^{0,-1}(n,m)$. The groups G and G' have the same order as $\widetilde{C}$, but usually have many more summands; they are also denoted by G in the charts of [3]. The big $\bullet$'s in 2.16 are sums of $\mathbf{Z}_2$'s.

By the proof of [6, 1.7 and 1.10.1], (2.12) splits as spectral sequences, and the stable summand in which we are interested occurs in the summand which comes from δ^{-1}. We may ignore the other summand and, if $n \equiv 6$ or 8 mod 8, think of the spectral sequence for $[P^n, \Phi SO(m)]_*$ as being the spectral sequence for $v_1^{-1}\pi_*(SO(m))$ shifted one unit down and one unit to the right. If $n \equiv 2$ or 4 mod 8, we may think of the spectral sequence for $[P^n, \Phi SO(m)]_*$ as a similar shift of the spectral sequence of [6, 2.16] converging to $v_1^{-1}\pi'_*(SO(m))$. We will review these $v_1^{-1}\pi'_*(-)$-groups later.

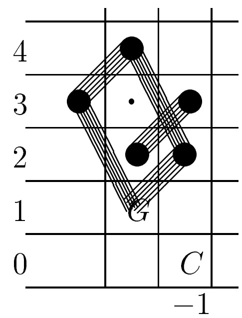
[6] If $m \equiv 0 \mod 4$, there are three summands.

When n is odd, the Ext groups from the two parts of (2.13) occur in distinct bigradings. The group $\text{Ext}_{\mathcal{A}}^{s,n+t}(QK^1(\text{Spin}(m))/\operatorname{im}(\psi^2))$ is nonzero if t is even and $s \geq 1$, while, as depicted in Diagram 2.16, $E_2^{s,t}(n-1,m)$ is nonzero if t is odd and $s \geq 0$. For odd n, appended to Diagram 2.16 should be a chart such as those of [3] shifted left by n gradings. The issue for α^* in (2.4) is whether the group $\widetilde{C}'$ in 2.16 supports a d_2- or d_4-differential in this new spectral sequence.

Now we return to the consideration of the various cases in the proof of Theorem 2.2.

Case 4: $k \equiv 0 \mod 4$, $m \equiv 3, 4, 5 \mod 8$. Let $k = 4\ell$. We first consider the cases when $m \equiv 3$ or $5 \bmod 8$. In this case, the relevant elements of $E_2^{*,*}(8\ell+1, m)$ are depicted in Diagram 2.17.

Diagram 2.17. A portion of $E_2^{*,*}(8\ell+1,m)$ when $m \equiv 3$ or $5 \mod 8$



In (2.13), the part in i^{*-1} (resp. $\operatorname{im}(q^*)$) is that in positions (x, y) with $x + y$ odd (resp. even). The indicated d_2-differentials are a consequence of the argument of Case 3; see especially the last paragraph of the proof. We consider the morphism of spectral sequences

$$E_r^{*,*}(8\ell+1, m) \xrightarrow{i^*} E_r^{*,*}(8\ell, m). \qquad (2.18)$$

The result for $\mathbf{s}[P^{8\ell}, \Phi BSO(m)]$ in [6, 1.7,1.8] was obtained from a nonzero d_3-differential from $E_3^{1,-1}$ in the spectral sequence for $v_1^{-1}\pi_*(\text{Spin}(m))$ as established in [3, 3.8], which implies that $d_3 \neq 0$ on $\mathbf{s}E_3^{0,-1}(8\ell, m)$. Hence either $d_2 \neq 0$ or $d_3 \neq 0$ on the generator of C in Diagram 2.17. To know that $\operatorname{coker}(\phi_{8\ell+1,m}) = 0$, we need to know that it is not the case that d_2 is nonzero on the generator of C, and also d_3 nonzero on twice the generator; this follows by naturality using (2.18), since i^* is injective on C and the $\mathbf{Z}_2$ in filtration 3.

If $m \equiv 4 \mod 8$, the same situation applies. There are more target classes for differentials, but those in filtration 4 are killed by d_2-differentials, as indicated in Diagram 2.17, because the relevant new classes from $E_2(S^{m-1})$ occur in the same sort of eta-towers as did those in $E_2(\mathrm{Spin}(m-1))$. (See, e.g., [3, 3.16].) The filtration-3 targets map isomorphically to those in $E_2(8\ell, m)$, and $d_3 \neq 0$ on $\mathbf{s}E_3^{0,-1}(8\ell, m)$, this time by [3, 3.14]. Thus the same naturality argument implies that it is impossible that both d_2 and d_3 are nonzero from $E_2^{0,-1}$. Hence $\mathrm{coker}(\phi_{8\ell+1,m}) = 0$. This completes the proof of Case 4.

Case 5: $k \equiv 2 \mod 4$, $m \equiv 6 \mod 8$. Let $k = 4\ell + 2$ and $m = 8i + 6$. We use the commutative diagram of exact sequences

$$\begin{array}{ccccc}
[P^{8\ell+5}, \Phi BSO(8i+5)] & \xrightarrow{i^*} & [P^{8\ell+4}, \Phi BSO(8i+5)] & \xrightarrow{\alpha_1^*} & v_1^{-1}\pi_{8\ell+3}(SO(8i+5)) \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_3} \\
[P^{8\ell+5}, \Phi BSO(8i+6)] & \xrightarrow{i'^*} & [P^{8\ell+4}, \Phi BSO(8i+6)] & \xrightarrow{\alpha_2^*} & v_1^{-1}\pi_{8\ell+3}(SO(8i+6))
\end{array}$$

By [6, 1.10], j_2 on stable summands is an isomorphism of $\mathbf{Z}/2^{4i+3}$. By §3,

$$\nu(\mathbf{s}[P^{8\ell+5}, \Phi BSO(8i+5)]) < 4i+3,$$

and hence $\phi_{8\ell+5,8i+5}(= \mathbf{s}i^*)$ is not surjective. By [3, 3.7,3.8,3.10],

$$v_1^{-1}\pi_{8\ell+3}(SO(8i+5)) \approx \mathbf{Z}/2,$$

with generator D. By [3, 3.11,3.12,3.13], $v_1^{-1}\pi_{8\ell+3}(SO(8i+6)) \approx \mathbf{Z}/2^{\min(4i+2,\nu(\ell-i)+4)}$. (The 2-line group has exponent 1 larger than this, but it supports a nonzero differential.) Thus, with gen denoting a generator of the stable summand, $\alpha_2^*(\mathrm{gen}) = j_3(D)$ and $\alpha_2^*(2\cdot\mathrm{gen}) = 0$. Hence $|\,\mathrm{coker}(\phi_{8\ell+5,8i+6})| \leq 2$ and it equals 2 if and only if $j_3^\#$ sends the generator of $E_2^{2,8\ell+5}(\mathrm{Spin}(8i+6))^\#$ to $D \in E_2^{2,8\ell+5}(\mathrm{Spin}(8i+5))^\#$.

In the proof of [3, 3.11], which appears near the end of [3, §7], it is proved that the relevant summand of $E_2^{2,8\ell+5}(\mathrm{Spin}(8i+6))^\#$ is $\mathbf{Z}/2^{4i+3}$ generated by D_+ if $\nu(\ell-i) > 4i-2$, while if $\nu(\ell-i) \leq 4i-2$, it is $\mathbf{Z}/2^{5+\nu(\ell-i)}$ generated by $2^{4i-2-\nu(\ell-i)}D_+ - x_{4i-1}$. Since restriction $j_3^\#$ to $\mathrm{Spin}(8i+5)$ sends D_+ to D and x_{4i-1} to x_{4i-1}, we deduce that $j_3^\#$ maps onto D if and only if $\nu(\ell-i) \geq 4i-2$, establishing the claim in 2.2 about $\mathrm{coker}(\phi_{8\ell+5,8i+6})$, one of the asterisk cases in 1.2 and 1.3.

Case 6: $k \equiv 0 \mod 4$, $m \equiv 2 \mod 8$. The argument is similar to that of Case 5, although it has one additional complication. We use a diagram of exact sequences analogous to that of Case 5, with dimensions of projective spaces and indices of $\Phi BSO(-)$ decreased by 4. By [6, 1.7,1.8], $\mathbf{s}j_2$ is an isomorphism of $\mathbf{Z}/2^{4i}$. Using §3, $\nu(\mathbf{s}[P^{8\ell+1}, BSO(8i+1)]) < 4i+1$. As we showed at the beginning of the proof of 2.2, $\ker(\phi_{8\ell+1,8i+1}) = \mathbf{Z}/2$, and hence $\phi_{8\ell+1,8i+1}$ cannot be surjective.

What complicates the argument as compared to Case 5 is that $v_1^{-1}\pi_{8\ell-1}(SO(8i+1))$ and $v_1^{-1}\pi_{8\ell-1}(SO(8i+2))$ are larger than the corresponding groups that appeared in Case 5. These groups are taken from [3, 1.3,3.12]. Both of these groups have a large $\mathbf{Z}_2$-vector space in filtration 4, which maps isomorphically under j_3. It is not an issue as possible image of α_1^* on the stable summand because, as in Case 3, it is in the image under α_1^* from a similar sum of $\mathbf{Z}_2$'s. From the point of view of the spectral sequence of 2.10, they are already hit by d_2-differentials, and so we don't have to worry about whether they are hit by d_4's.

What is more of a worry is that $E_\infty^{2,8\ell+1}(\mathrm{Spin}(8i+1))$ and $E_\infty^{2,8\ell+1}(\mathrm{Spin}(8i+2))$ have, in addition to, respectively, the $\mathbf{Z}_2$-class D and the larger cyclic summand C' that they had in Case 5, also a summand L, which is the sum of many $\mathbf{Z}_2$'s and maps isomorphically under j_3, while the first group also has an additional $\mathbf{Z}_2$-class labeled x_{4i-3}. The summand L is depicted by the big dots in [3, 1.3,3.12] and has dimension $[\log_2(\frac{4}{3}(4i-1))]$. We will show that α_1^* sends the generator of the stable summand to just the class D. The analysis of whether D hits the element of order 2 in C' proceeds exactly as in Case 5. We obtain that j_3 sends D nontrivially, and hence $\mathrm{coker}(\phi_{8\ell+1,8i+2}) = \mathbf{Z}/2$, if and only if $\nu(\ell-i) \geq 4i-4$, which translates to the claim of the theorem in this case, the other asterisk case in 1.2 and 1.3.

It remains to verify the claim about α_1^*, which is done by applying Pontryagin duality. By (2.6) and (2.7), $\alpha_1^\#$ is determined by

$$E_2^{2,1}(\mathrm{Spin}(8i+1))^\# \xrightarrow{h_1^\#} E_2^{1,-1}(\mathrm{Spin}(8i+1))^\#.$$

That this sends only the class D nontrivially to the stable summand is proved exactly as in the two paragraphs of [6] which appear shortly after Diagram 2.24 of that paper. The first of the two paragraphs begins "In order to show that $d_3(g_1) = 0$." In summary, a presentation of $E_2^{1,-1}(\mathrm{Spin}(8i+1))^\#$ is given, and, for each basis element b of $E_2^{2,1}(\mathrm{Spin}(8i+1))^\#$, $(h_1)^\#(b)$ is interpreted as an element in that presented group, and it is observed that only $(h_1)^\#(D)$ is nonzero.

Case 7: $k \equiv 0 \mod 4$, $m \equiv 6 \mod 8$. Let $k = 4\ell$ and $m = 8i + 6$. This time the diagram of the sort used in Case 5 does not quite work because j_2 is not surjective, due to a d_3-differential in $[P^{8\ell}, \Phi BSO(8i+5)]$ not present in $[P^{8\ell}, \Phi BSO(8i+6)]$. We can, however, consider an E_2-version of the diagram, where α_1^* and α_2^* are, after dualizing, given by (2.7). The diagram below addresses what amounts to the d_2-differential on $\mathbf{s}E_2^{0,-1}(8\ell+1, 8i+6)$. The d_4-differential on this summand is then eliminated similarly to Cases 3, 4, and 6.

$$\begin{array}{ccccc} \mathbf{s}E_2^{0,-1}(8\ell, 8i+5)^{\#} & \xrightarrow{\approx} & \mathbf{s}E_2^{1,-1}(\mathrm{Spin}(8i+5))^{\#} & \xleftarrow{v_1^{4\ell}h_1^{\#}} & E_2^{2,8\ell+1}(\mathrm{Spin}(8i+5))^{\#} \\ \approx \uparrow & & j_2^{\#} \uparrow \approx & & j_3^{\#} \uparrow \\ \mathbf{s}E_2^{0,-1}(8\ell, 8i+6)^{\#} & \xrightarrow{\approx} & \mathbf{s}E_2^{1,-1}(\mathrm{Spin}(8i+6))^{\#} & \xleftarrow{v_1^{4\ell}h_1^{\#}} & E_2^{2,8\ell+1}(\mathrm{Spin}(8i+6))^{\#} \end{array}$$

As in Case 6, the $v_1^{4\ell}h_1^{\#}$ on $\mathrm{Spin}(8i+5)$ sends only D nontrivially, and $j_3^{\#}$ sends the generator of the C'-summand to x_{4i-1}, since $\nu((8\ell+1)-(8i+5)) = 2$. Thus $v_1^{4\ell}h_1^{\#}$ on $\mathrm{Spin}(8i+6)$ is 0, and hence $\phi_{8\ell+1,8i+6}$ is surjective.

Case 8: $k \equiv 2 \mod 4$, $m \equiv 2 \mod 8$. Let $k = 4\ell + 2$. The argument is similar to that of Case 7, but is complicated by $P^{8\ell+4}$ not being K-equivalent to a Moore spectrum. Let, as in [6, 2.14],

$$T^n = S^n \cup_\eta e^{n+2} \cup_2 e^{n+3}.$$

From [6, (2.11),(2.13)], we have

$$\mathbf{s}[P^{8\ell+4}, \Phi BSO(m)] \approx \mathbf{s}v_1^{-1}\pi'_{-2}(SO(m)), \tag{2.19}$$

where, by [6, (2.17)],

$$v_1^{-1}\pi'_n(X) \approx [T^n, \Phi(X)]. \tag{2.20}$$

The analogue of (2.6) is that the morphism α^* in (2.4) is equivalent to

$$\zeta_\ell^* : v_1^{-1}\pi'_{-1}(BSO(m)) \to v_1^{-1}\pi_{8\ell+4}(BSO(m)),$$

where $\zeta_\ell : S^{8\ell+5} \to T^0$ is the element of highest filtration $(4\ell+2)$ in its stem in the Adams spectral sequence of T^0. It is $\eta\mu_\ell$ on the top cell. The reason for this is similar to the discussion between (2.6) and (2.7). In this case, both

$$S^{8\ell+4} \xrightarrow{\alpha} P^{8\ell+4} \xrightarrow{\phi^\ell} P^4_{1-8\ell}$$

and

$$S^{8\ell+4} \xrightarrow{\zeta_\ell} T^{-1} \xrightarrow{f} P^4_{1-8\ell},$$

where f is, up to periodicity, a restriction of the map in [6, 2.8], become equal in $\pi_{8\ell+4}(P^4_{1-8\ell} \wedge J) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$, where each is the element of highest filtration. Note that f has Adams filtration -1. Thus the two composites are equal in $v_1^{-1}\pi_{8\ell+4}(P^4_{1-8\ell})$, and hence, following by any element g of $[P^4_{1-8\ell}, \Phi BSO(m)] \approx [P^{8\ell+4}, \Phi BSO(m)]$, $\alpha^*(g) = \zeta_\ell^*(g \circ f)$ in $\pi_{8\ell+4}(\Phi BSO(m))$. Note that f induces the isomorphism obtained from (2.19) and (2.20).

Let $M^6 \xrightarrow{\widetilde{\zeta}} T^0$ be an extension of ζ. Here M^n is the mod-2 Moore spectrum with top cell in dimension n. We claim that

$$\widetilde{\zeta}^* : K^0(T^0) \to K^0(M^6) \tag{2.21}$$

is the nontrivial morphism from $\mathbf{Z}_2^\wedge$ to $\mathbf{Z}/2$. One way to see this is to obtain $ku_*(D(\widetilde{\zeta}))$ from $ko_*(D(\widetilde{\zeta}))$ by using $bu = bo \cup_\eta \Sigma^2 bo$. Here D denotes the S-dual. There is a cofiber sequence

$$M^{-6} \to D(MC(\widetilde{\zeta})) \to D(T^0).$$

In the chart below, the solid dots are from the M^{-6} and the circles from $D(T^0)$. The differential in the ko_*-chart is due to the η^2 connection. It implies the differential in the ku_*-chart, which is the asserted homomorphism (2.21).

Diagram 2.22. *$ko_*(D(MC(\widetilde{\zeta})))$ and $ku_*(D(MC(\widetilde{\zeta})))$*

From e.g. [4, p.488] or [3, 3.6,3.16], $\mathrm{Ext}_{\mathcal{A}}^{1,n+6}(PK^1(S^n)) \approx \mathbf{Z}/2$. We will name the nonzero class $v_1^2 h_1$. In the spectral sequence converging to $v_1^{-1}\pi_*(S^n)$, this element supports a d_3-differential, but in that converging to $v_1^{-1}\pi'_*(S^n)$, it survives to a homotopy class, which is the class ζ discussed above. (See [6, 2.18].) We obtain the following analogue of Diagram 2.9.

Diagram 2.23. *Diagram involving Bockstein and* $v_1^2 h_1$

$$\begin{array}{ccccc} & \nearrow^{\widetilde{\zeta}^*} & E_2^{s,s+n+6}(Y;\mathbf{Z}_2) & \xrightarrow{v_1^{4\ell}} & E_2^{s,s+n+6+8\ell}(Y;\mathbf{Z}_2) \\ & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ E_2^{s,s+n}(Y) & \xrightarrow{v_1^2 h_1} & E_2^{s+1,s+n+6}(Y) & \xrightarrow{v_1^{4\ell}} & E_2^{s+1,s+n+6+8\ell}(Y) \end{array}$$

Here Y could be any space, but we use $Y = \mathrm{Spin}(m)$. The point of the diagram is that the composition around the top is α^*, while the composition on the bottom sends an eta-tower to one with the same name. The claim about (2.21) was needed to establish commutativity of the triangle.

Now that we have related α^* to $v_1^{4\ell+2}h_1$, we obtain the following analogue of the diagram in Case 7.

$$\begin{array}{ccccc} \mathbf{s}E_2^{0,-1}(8\ell+4,8i+1)^{\#} & \xrightarrow{\approx} & \mathbf{s}E_2^{1,-1}(\mathrm{Spin}(8i+1))^{\#} & \xleftarrow{v_1^{4\ell+2}h_1^{\#}} & E_2^{2,8\ell+5}(\mathrm{Spin}(8i+1))^{\#} \\ \approx\uparrow & & j_2^{\#}\uparrow\approx & & j_3^{\#}\uparrow \\ \mathbf{s}E_2^{0,-1}(8\ell+4,8i+2)^{\#} & \xrightarrow{\approx} & \mathbf{s}E_2^{1,-1}(\mathrm{Spin}(8i+2))^{\#} & \xleftarrow{v_1^{4\ell+2}h_1^{\#}} & E_2^{2,8\ell+5}(\mathrm{Spin}(8i+2))^{\#} \end{array}$$

The same argument as in Case 7 now implies

$$d_2 = 0 : \mathbf{s}E_2^{0,-1}(8\ell+5,8i+2) \to E_2^{2,0}(8\ell+5,8i+2).$$

The d_3-differential on $\mathbf{s}E_3^{0,-1}(8\ell+5,8i+2)$ is as it was on $\mathbf{s}E_3^{0,-1}(8\ell+4,8i+2)$, which was shown to be 0 in [6].[7] That $d_4 = 0$ on $\mathbf{s}E_4^{0,-1}(8\ell+5,8i+2)$ is seen as in most of the previous cases, using Diagram 2.23 to assert that the target was already hit by d_2 applied to eta-towers with the same name.

Case 9: $k \equiv 3 \mod 4$, $m \not\equiv 2 \mod 4$, and $m \geq 12$. We decompose α^* in (2.4) as

$$[P^{2k}, \Phi BSO(m)] \xrightarrow{\widetilde{\alpha}^*} [M^{2k+1}, \Phi BSO(m)] \xrightarrow{i^*} v_1^{-1}\pi_{2k-1}(SO(m)), \tag{2.24}$$

where $M^n = M^n(2)$, and $\widetilde{\alpha}$ is the attaching map for the top two cells of P^{2k+2}. Let $k = 4\ell - 1$. There is a commutative diagram in which rows are cofiber sequences and columns are K-equivalences

[7] It was done in the paragraph of [6] near the end of Section 2, which begins "We prove now that $d_3 = 0$ on $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+2))$."

$$\begin{array}{ccccccc}
M^{8\ell-1} & \xrightarrow{\widetilde{\alpha}} & P^{8\ell-2} & \longrightarrow & P^{8\ell} & \longrightarrow & M^{8\ell} \\
{\scriptstyle A^\ell}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M^{-1} & \xrightarrow{\alpha'} & P_{1-8\ell}^{-2} & \longrightarrow & P_{1-8\ell}^{0} & \longrightarrow & M^0 \\
{\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & {\scriptstyle =}\uparrow \\
M^{-1} & \xrightarrow{q} & M^0(2^{4\ell-1}) & \xrightarrow{2} & M^0(2^{4\ell}) & \longrightarrow & M^0
\end{array} \tag{2.25}$$

The top vertical maps are just the v_1-maps. The middle square on the bottom is from [6, 2.2], which was originally from [11]. The construction in [11] implies commutativity of the lower right square. If this cofiber sequence is pushed one space farther, a commutative square is obtained which is the suspension of the lower left square. Hence the lower left square commutes.

Thus we obtain a commutative diagram

$$\begin{array}{ccc}
\mathbf{s}[P^{8\ell-2}, \Phi BSO(m)] & \xrightarrow{\widetilde{\alpha}^*} & [M^{8\ell-1}, \Phi BSO(m)] \\
{\scriptstyle \approx}\uparrow & & {\scriptstyle \approx}\uparrow \\
\mathbf{s}[P_{1-8\ell}^{-2}, \Phi BSO(m)] & \xrightarrow{\alpha'^*} & [M^{-1}, \Phi BSO(m)] \\
{\scriptstyle \approx}\downarrow & & {\scriptstyle =}\downarrow \\
\mathbf{s}v_1^{-1}\pi_{-2}(SO(m)) & \xrightarrow{q^*} & [M^{-1}, \Phi BSO(m)],
\end{array} \tag{2.26}$$

where q is the collapse map. In the bottom row, $\mathbf{s}[M^0(2^{4\ell-1}), \Phi BSO(m)]$ has been replaced by $\mathbf{s}v_1^{-1}\pi_{-1}(BSO(m)) \approx \mathbf{s}v_1^{-1}\pi_{-2}(SO(m))$ because ℓ can be taken to be arbitrarily large, and so the maps from the top cell of the Moore space are ephemeral. When the $\widetilde{\alpha}^*$ in the top row is followed by i^* into $v_1^{-1}\pi_{8\ell-3}(SO(m))$ to yield (2.24), we obtain from the diagram something agreeing up to isomorphisms with that obtained by applying $\mathbf{s}[-, \Phi BSO(m)]$ to the composite

$$S^{8\ell-2} \hookrightarrow M^{8\ell-1} \xrightarrow{A^\ell} M^{-1} \xrightarrow{q} S^{-1}. \tag{2.27}$$

By [2], this composite is the element of order 2 in the stable image of J in the $(8\ell-1)$-stem; however, we will compute it using (2.27) rather than this imJ description.

We will show that the composite

$$\begin{aligned}\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(m)) \underset{q^*}{\xrightarrow{\rho_2}} \ & E_2^{1,-1}(\mathrm{Spin}(m);\mathbf{Z}_2) \xrightarrow{A^\ell} E_2^{1,8\ell-1}(\mathrm{Spin}(m);\mathbf{Z}_2)\\ \underset{i^*}{\xrightarrow{\partial}} \ & E_2^{2,8\ell-1}(\mathrm{Spin}(m)) \qquad (2.28)\end{aligned}$$

is 0.[8] Noting that

$$E_\infty^{4,8\ell+1}(\mathrm{Spin}(m)) = 0 \qquad (2.29)$$

by [3, 1.3,3.6,3.7], Theorem 2.2 follows in this case.

We show that the Pontryagin dual of (2.28) is 0. Let

$$C_0 \xrightarrow{d_1} C_1 \xrightarrow{d_2} C_2$$

be the sequence of free $\mathbf{Z}_{(2)}$-modules associated to the sequence of free $\mathbf{Z}_2^\wedge$-modules in [3, 11.9]. Thus $C_0 = F$, $C_1 = F \oplus F \oplus F$, and $C_2 = F \oplus F \oplus F \oplus F$, where F is a free $\mathbf{Z}_{(2)}$-module on $[m/2]$ generators. The transpose of the matrix of d_1 is

$$(0 \quad \Psi^2 \quad \Theta_{4\ell-1}), \qquad (2.30)$$

and the transpose of the matrix of d_2 is

$$\begin{pmatrix} -2 & \Psi^2 & \Theta_{4\ell-1} & 0\\ 0 & 0 & 0 & \Theta_{4\ell-1}\\ 0 & 0 & 0 & -\Psi^2 \end{pmatrix}, \qquad (2.31)$$

and then the homology at C_s is $\mathrm{Ext}_{\mathcal{A}}^{s,8\ell-1}(PK^1(\mathrm{Spin}(m)/\mathrm{im}(\psi^2))$. Here Ψ^2 (resp. Θ_j) is the matrix of ψ^2 (resp. $\psi^3 - 3^j$) on $PK^1(\mathrm{Spin}(m))$. We are using here that for a rationally acyclic complex of finitely generated free $\mathbf{Z}_{(2)}$-modules, the inclusion induces an isomorphism $H_*(-;\mathbf{Z}_{(2)}) \to H_*(-;\mathbf{Z}_2^\wedge)$. In the remainder of this proof, we will write $\mathbf{Z}$ when we really mean $\mathbf{Z}_{(2)}$.

As observed in [3, proof of 11.3], $E_2^{s,8\ell-1}(\mathrm{Spin}(m))^\#$ is the homology at C_{s-1}^* of the chain complex C^* given by

$$C_0^* \xleftarrow{d_1^*} C_1^* \xleftarrow{d_2^*} C_2^*, \qquad (2.32)$$

where $C_s^* = \mathrm{Hom}(C_s, \mathbf{Z})$ and the matrices of d_1^* and d_2^* are those of (2.30) and (2.31). The shift from s to $s-1$ is due to the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

[8]Note that ρ_2 and ∂ are parts of different Bockstein exact sequences, and so it is not automatic that the composite is 0.

Note that $E_2^{s,4\ell-1}(\mathrm{Spin}(m);\mathbf{Z}/2)^\#$ is the homology at $C_s^*/2$ of the mod 2 reduction of (2.32), and

$$\rho_2^\# : E_2^{1,8\ell-1}(\mathrm{Spin}(m);\mathbf{Z}/2)^\# \to E_2^{1,8\ell-1}(\mathrm{Spin}(m))^\#$$

is the boundary homomorphism δ in the exact sequence of homology groups induced by the short exact sequence of chain complexes

$$0 \to C^* \xrightarrow{2} C^* \to C^*/2 \to 0. \tag{2.33}$$

To see this, note that the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \frac{1}{2}} & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Q} & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \longrightarrow & 0 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} H_1(C^*/2) & \xrightarrow{\delta} & H_0(C^*) \\ \downarrow{\scriptstyle \rho_2^*} & & \downarrow{\scriptstyle =} \\ H_1(C^* \otimes \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\approx} & H_0(C^*), \end{array}$$

from which the agreement of δ and ρ_2^* is immediate.

The composite which we wish to show is 0 (dual to (2.28)) may now be identified as

$$H_1(C^*_{(4\ell-1)}) \xrightarrow{\rho_{2*}} H_1(C^*_{(4\ell-1)}/2) \xrightarrow{=} H_1(C^*_{(-1)}/2) \xrightarrow{\delta} \mathbf{s}H_0(C^*_{(-1)}). \tag{2.34}$$

Here the parenthesized subscript of C^* is the subscript of Θ, and $C^*/2$ means the mod 2 reduction of C^*. The identity map in the middle is due to the subscript not mattering mod 2, and the fact that A^* is the identity homomorphism of $K^*(M)$. Since, for the same parenthesized subscript, $\mathrm{im}(\rho_2^*) = \ker(\delta)$, we are reduced to proving

$$\ker(H_1(C^*/2) \xrightarrow{\delta_\ell} H_0(C^*_{(4\ell-1)})) \subset \ker(H_1(C^*/2) \xrightarrow{\delta_0} \mathbf{s}H_0(C^*_{(-1)})). \tag{2.35}$$

We will need the following result, culled from [3].

Theorem 2.36. *Suppose $m \geq 12$.*

- *If* $m = 2n+1$, *then*

$$H_0(C^*_{(4\ell-1)}) \approx \begin{cases} \mathbf{Z}/2^n \oplus \mathbf{Z}/2^n & n \le \nu(\ell)+4 \\ \mathbf{Z}/2^e \oplus \mathbf{Z}/2^{\nu(\ell)+4} & n > \nu(\ell)+4 \end{cases} \tag{2.37}$$

with $e > n$. *The group is presented by a matrix*

$$\begin{pmatrix} 2^{A_1} & 0 \\ u_2 2^{A_2} & 2^n \\ u_3 2^n & 2^v \end{pmatrix}, \tag{2.38}$$

where u_i *is odd,* $A_i > n$, *and* $v = \min(\nu(\ell)+4, 2n+1)$. *The columns of this matrix correspond to generators* ξ_1 *and* D *of* $PK^1(\mathrm{Spin}(m))$ *under the isomorphism*

$$H_0(C^*_{(4\ell-1)}) \approx E_2^{1,8\ell-1}(\mathrm{Spin}(m))^{\#} \approx PK^1(\mathrm{Spin}(m))/(\psi^2, \theta_{4\ell-1}), \tag{2.39}$$

where $\theta_j = \psi^3 - 3^j$. *The first row of (2.38) is due to a combination of relations of the form* $\psi^2(\xi_i)$ *and* $\theta_{4\ell-1}(\xi_i)$, *while the second row is a combination of such relations together with* $\psi^2(D)$ *(with coefficient 1), and the third row is a combination of such relations together with* $1\cdot\theta_{4\ell-1}(D)$. *The first summand of (2.37) is the stable summand; it corresponds to the first* (ξ_1) *column of (2.38).*

- *If* $m = 4a$, *then*

$$H_0(C^*_{(4\ell-1)}) \approx \begin{cases} \mathbf{Z}/2^{2a} \oplus \mathbf{Z}/2^{2a-1} \oplus \mathbf{Z}/2^{\nu(a)+2} & 2a \le \nu(\ell)+5 \\ \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{e_2} \oplus \mathbf{Z}/2^{e_3} & \textit{otherwise,} \end{cases}$$

with $e_1 > 2a$ *and* $e_3 \le e_2 < 2a$. *The group is presented by a matrix*

$$\begin{pmatrix} 2^{A_1} & 0 & 0 \\ 0 & 2^M & -2^M \\ u_2 2^{A_2} & 2^{2a-1} & 0 \\ 2^{2a-1} & u_3 2^{v_1} & u_4 2^{v_2} \end{pmatrix} \tag{2.40}$$

with u_i *odd,* $A_i > 2a$, $M = \min(2a-1, \nu(2\ell-a)+3)$, $v_1 = \min'(\nu(a)+2, \nu(\ell)+4)$, *and* $v_2 = \nu(a)+2$. *Here* $\min'(A,B) = \min(A,B)$ *unless* $A = B$, *in which case it is greater than either.*

Under the isomorphisms of (2.39), the columns of (2.40) correspond to generators ξ_1, D_+, and D_-, and of the rows (relations) only the last one involves an odd multiple of $\theta_{4\ell-1}(D)$.

Proof. For the first part, we use [3, 3.1,3.2] and [5, 3.18]. The proof of [5, 3.15] explains how the rows of the presentation matrix are obtained, while [5, §4] derives the inequalities for the exponents in those relations. Actually, [5, 3.18] only proves $A_i \geq n$. The stronger result needed here follows by a more careful analysis of the proof of [5, §4]. It follows from [5, 3.18], refined to say that $eSp(4\ell+1, n) > n+1$ and the coefficients of ξ_1 in [5, (3.19)] and [5, (3.20)] are divisible by 2^{n+1}.

By [3, 8.1], $eSp(-, n)$ is divisible by $(2n+1)!$, which is divisible by 2^{n+1} for $n \geq 2$. The divisibility of [5, (3.20)] is proved using its representation as

$$(n-1)2^{2n-4} + \sum_{j=2}^{n/2} \binom{n-j}{j} 2^{2n-4j} \sum_{i \geq j-1} 8^i \binom{2\ell-1}{i} S_{i,j}$$

with

$$S_{i,j} = \sum_{t=0}^{j-2} (-1)^t \binom{2j-1}{t} (2j-2t-1) \binom{j-t}{2}^i$$

given in [5, (4.20)]. The term $(n-1)2^{2n-4}$ is divisible by 2^{n+1} for $n \geq 5$. The other terms are divisible by 2^{2n-j-3} with $2 \leq j \leq n/2$, which will be sufficiently divisible except when (n, j) is (6,3). In this case, the additional divisibility is provided by $S_{2,3} = 30$.

The divisibility of [5, (3.19)] is proved similarly using its representation as

$$(n+1)2^{2n-3} \sum_{j \geq 2} 2^{2n+1-4j} \left(\binom{n+2-j}{j} - \binom{n-j}{j-2} \right) \sum_{i \geq j-1} 8^i \binom{2\ell}{i} S_{i,j},$$

with $S_{i,j}$ as above, from [5, p.54]. The lead term $(n+1)2^{2n-3}$ is divisible by 2^{n+1} for $n \geq 3$. Other terms are divisible by 2^{2n-j-2} with $2 \leq j \leq n/2$, which is divisible by 2^{n+1}.

For the second part, we use [3, 3.3] and its proof in [3, §4]. The classes ξ_i, D, D_+, and D_- in $PK^1(\mathrm{Spin}(m))$ are as in [5, 3.10] and [3, 4.1], but do not play a major role in this paper. ■

We remark that the condition $m \geq 12$ is necessary for the divisibilities of the entries of the matrices to hold.

By the definition of δ using (2.33), if $\mathbf{x} = (x_1, x_2, x_3) \in C_1^*/2$ is a cycle representing an element of $H_1(C^*_{(4\ell-1)}/2)$, then

$$\delta(\mathbf{x}) = \tfrac{1}{2}\psi^2(x_2) + \tfrac{1}{2}\theta_{4\ell-1}(x_3), \tag{2.41}$$

viewed as an element in the group presented by one of the matrices of 2.36. Here $x_i \in F^*$ or $F/2^*$. We write δ_0 and δ_ℓ for the boundaries δ associated to $C^*_{(-1)}$ and $C^*_{(4\ell-1)}$, respectively. Note that the relations $\xi_j = j^{4\ell-1}\xi_1$ are used to bring these elements into the 2- or 3-generator form of 2.36. This relation is a consequence of [5, 3.9], which says that modding out by $\psi^j - j^{4\ell-1}$ for $j = 3$ and -1 also accomplishes modding out by $\psi^j - j^{4\ell-1}$ for other odd j.

The matrix (2.38) implies that when $m = 2n+1$, $\mathbf{s}H_0(C^*_{(-1)})$ is isomorphic to $\mathbf{Z}/2^n$ generated by ξ_1, since $v = 2n+1$ in this case, and that in (2.41) with $\ell = 0$, $\delta_0(x_1, x_2, x_3) \neq 0 \in \mathbf{s}H_0(C^*_{(-1)})$ if and only if the D-component of x_3 is odd. This key point may warrant some explanation. The interpretation of the rows of (2.38) given after (2.39) implies that when $\psi^2(x_2)$ or $\theta_{-1}(x_3)$ are written in terms of ξ_1 and D, using $\xi_j = j^{-1}\xi_1$, the ξ_1-component of each will be divisible by 2^{n+1} unless the D-component of x_3 is odd, and when these are multiplied by $1/2$, as they are in (2.41), the only way to obtain a nonzero component in the ξ_1-component of the $\mathbf{Z}/2^n$-group presented by (2.38) is then to have this D-component of x_3 be odd.

If the D-component of x_3 is odd, then

$$\delta_\ell(x_1, x_2, x_3) \neq 0 \in H_0(C^*_{(4\ell-1)}), \tag{2.42}$$

since it is $\frac{1}{2}$ times the last row of (2.38) plus perhaps $\frac{1}{2}$ times the other rows. Such a vector is easily seen to be nonzero in the group presented by (2.38), regardless of the value of v. This establishes the contrapositive of (2.35).

The same argument applies when $m = 4a$, using the matrix (2.40). The previous paragraph carries through verbatim, with n replaced by $2a-1$.

Case 10: $k \equiv 3 \mod 4$, $m \equiv 2 \mod 4$. The method of Case 9 does not apply here, since $\psi^{-1} \neq -1$ in $PK^1(\mathrm{Spin}(m))$ when $m \equiv 2 \mod 4$. However the result here follows by naturality from Case 9.

Let $k = 4\ell+3$ and $m = 4j+2$. The morphism $\mathbf{s}E_2^{0,-1}(8\ell+7, 4j+1) \to \mathbf{s}E_2^{0,-1}(8\ell+7, 4j+2)$ is bijective by [3, 3.3]. As we have just seen that $d_2 = 0$ on the former, it must also be 0 on the latter. Note that d_3 on $\mathbf{s}E_3^{0,-1}(8\ell+7, 4j+2)$ equals d_3 on

$\mathbf{s}E_3^{0,-1}(8\ell+6, 4j+2)$, by the general form of the spectral sequence, and this equals d_3 on $E_3^{1,-1}(\mathrm{Spin}(4j+2))$ by the paragraph after Diagram 2.16 beginning "By the proof." By [3, 3.12], this is zero. As there is nothing for d_4 to hit by (2.29)[9], we deduce that the generator of $E_2^{0,-1}(2k+1, m)$ is an infinite cycle in this case, establishing Theorem 2.2 in this case.

Case 11: $k \equiv 1 \mod 4$, $m \not\equiv 2 \mod 4$, $m \geq 12$. Let $k = 4\ell + 1$. Similarly to (2.25), we have, using [6, 2.8], a commutative diagram in which rows are cofibrations and columns are K-equivalences.

$$\begin{array}{ccccc}
M^{8\ell+3} & \xrightarrow{\widetilde{\alpha}} & P^{8\ell+2} & \longrightarrow & P^{8\ell+4} \\
\downarrow & & \downarrow & & \downarrow \\
M^3 & \longrightarrow & P^2_{1-8\ell} & \longrightarrow & P^4_{1-8\ell} \\
\uparrow & & \uparrow & & \uparrow \\
\Sigma^{2^{4\ell+1}L}F & \longrightarrow & N^{2^{4\ell+1}L}(2^{4\ell}) & \xrightarrow{2} & N^{2^{4\ell+1}L}(2^{4\ell+1})
\end{array}$$

where $N^n(k) = M^n(k) \cup_\eta e^{n+1} \cup_2 e^{n+2}$, the map labeled 2 has degree 2 on the bottom cell, and $\Sigma^{2^{4\ell+1}L}F$ is the stable fiber of this map. Thus

$$F = M^{-1} \cup_\eta M^1 \cup_2 M^2,$$

and, with $T^n = S^n \cup_\eta e^{n+2} \cup_2 e^{n+3}$ as in Case 8, there is a cofiber sequence

$$T^{-2} \to F \to T^{-1} \xrightarrow{2} T^{-1}. \qquad (2.43)$$

Similarly to (2.26), we obtain a commutative diagram, using [6, (2.13)]

$$\begin{array}{ccc}
\mathbf{s}[P^{8\ell+2}, \Phi BSO(m)] & \xrightarrow{\widetilde{\alpha}^*} & [M^{8\ell+3}, \Phi BSO(m)] \\
\approx\uparrow & & \approx\uparrow \\
\mathbf{s}[P^2_{1-8\ell}, \Phi BSO(m)] & \longrightarrow & [M^3, \Phi BSO(m)] \\
\approx\downarrow & & \approx\downarrow \\
\mathbf{s}v_1^{-1}\pi'_{2^{4\ell+1}L-2}(SO(m)) & \longrightarrow & [\Sigma^{2^{4\ell+1}L}F, \Phi BSO(m)].
\end{array}$$

[9] which also holds when $m \equiv 2 \mod 4$

Since ℓ is large, the $\Sigma^{2^{4\ell+1}L}$ may be omitted by periodicity, and so α^* in (2.4) is obtained as the composite

$$\mathbf{s}v_1^{-1}\pi'_{-2}(SO(m)) \to [M^3, \Phi BSO(m)] \xrightarrow{\approx} [M^{8\ell+3}, \Phi BSO(m)] \xrightarrow{i^*} v_1^{-1}\pi_{8\ell+1}(SO(m)). \tag{2.44}$$

This can be considered as the d_2- and d_4-differentials in the spectral sequence described prior to Case 4. Recall from [6, 2.16] that the E_2-term for $v_1^{-1}\pi'_*(-)$ equals that for $v_1^{-1}\pi_*(-)$.

The cofibration (2.43) yields a short exact sequence

$$0 \to K^{-1}(T^{-1}) \xrightarrow{2} K^{-1}(T^{-1}) \to K^{-1}(F) \to 0$$

which is

$$0 \to \mathbf{Z}_2^\wedge \xrightarrow{2} \mathbf{Z}_2^\wedge \to \mathbf{Z}/2 \to 0.$$

Thus (2.44) is, at the E_2-level, given by

$$\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(m)) \xrightarrow{\rho_2} E_2^{1,3}(\mathrm{Spin}(m);\mathbf{Z}/2) \xrightarrow{\approx} E_2^{1,8\ell+3}(\mathrm{Spin}(m);\mathbf{Z}/2) \xrightarrow{\partial} E_2^{2,8\ell+3}(\mathrm{Spin}(m)), \tag{2.45}$$

similarly to (2.28). We can justify the ρ_2 between distinct bigradings in two ways. (a) $\mathrm{Ext}_{\mathcal{A}}^{s,t}(-;\mathbf{Z}/2)$ has period 4 in t; (b) The morphism is induced by $F \to T^{-1}$, and there is a K-equivalence $F \to M^3$.

Hence, by the same argument used in Case 9 to go from (2.28) to (2.35), showing that $d_2 = 0$ on $\mathbf{s}E_2^{0,-1}(8\ell+3, m)$ is equivalent to proving

$$\ker(H_1(C^*/2) \xrightarrow{\delta'_\ell} H_0(C^*_{(4\ell+1)})) \subset \ker(H_1(C^*/2) \xrightarrow{\delta_0} \mathbf{s}H_0(C^*_{(-1)})). \tag{2.46}$$

Here $\delta'_\ell(x_1, x_2, x_3) = \frac{1}{2}\psi^2(x_2) + \frac{1}{2}\theta_{4\ell+1}(x_3)$.

The proof that (2.46) holds is similar to that of Case 9, except that the matrix, using $\psi^3 - 3^{4\ell+1}$ instead of $\psi^3 - 3^{4\ell-1}$ has a slightly different form. The matrix is described in Lemma 2.50 when m is odd. One must prove, analogous to (2.42), that if the D-component of x_3 is odd, then $\delta'_\ell(x_1, x_2, x_3) \neq 0 \in H_0(C^*_{(4\ell+1)})$. This is easier than in Case 9 because of the 2^3 in the last row of (2.51). As before, the last row is characterized by being the relation due to $\theta_{4\ell+1}(D)$ plus other terms. Hence $\delta'_\ell(x_1, x_2, x_3)$ will involve $1/2$ times the last row of (2.51), which, because of the 2^3 is certainly nonzero in the group presented by (2.51).

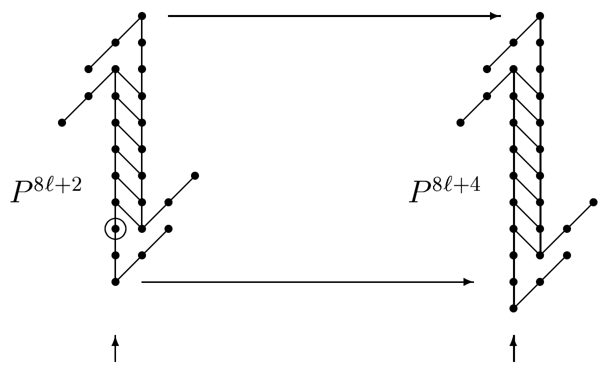
Finally, we must show $d_4 = 0$ on $\mathbf{s}E_4^{0,-1}(8\ell+3, m)$. The composite (2.44) may be viewed as applying $[-, \Phi BSO(m)]$ to

$$S^{8\ell+2} \xrightarrow{\alpha} P^{8\ell+2} \to P^2_{1-8\ell} \to v_1^{-1}P^2_{1-8\ell} \simeq v_1^{-1}N^0(2^{4\ell}). \tag{2.47}$$

The class of this composite is divisible by 4 in $v_1^{-1}\pi_{4\ell+2}(N^0(2^{4\ell})) \approx v_1^{-1}\pi_{4\ell+2}(P^{8\ell+2})$. Call it 4γ.

To see this divisibility, we use that α goes to 0 in $v_1^{-1}\pi_{8\ell+2}(P^{8\ell+4})$, since it is an attaching map. Diagram 2.48, which is similar to those of [12, pp 94-5], depicts $v_1^{-1}\pi_*(P^{8\ell+2}) \to v_1^{-1}\pi_*(P^{8\ell+4})$ near $* = 8\ell+2$. The group where $* = 8\ell+2$ is indicated with an arrow, and the nonzero element in the kernel of this homomorphism is circled.

Diagram 2.48. $v_1^{-1}\pi_*(P^{8\ell+2}) \to v_1^{-1}\pi_*(P^{8\ell+4})$ **near** $* = 8\ell + 2$



This chart also depicts $v_1^{-1}\pi_*(N^0(2^{4\ell}))$, and the circled element equals the composite (2.47) (since the α is nontrivial, because Sq^4 is nonzero in its mapping cone). The inclusion $v_1^{-1}T^{-1} \xrightarrow{i_T} v_1^{-1}N^0(2^{4\ell})$ induces in $\pi_{8\ell+2}(-)$ an injection $\mathbf{Z}/8 \to \mathbf{Z}/8 \oplus \mathbf{Z}/2$.[10]

Let g denote the generator of $v_1^{-1}\pi_{-2}(T^{-1})$, and let $2^e g$ denote an extension of $2^e g$ over an appropriate Moore spectrum. Then (2.47) equals the top row of the commutative diagram (2.49) followed by i_T.

$$\begin{array}{ccccccc}
S^{8\ell+3} & \xrightarrow{i} & M^{8\ell+3} & \xrightarrow{A^\ell} & M^3 & \xrightarrow{4g} & v_1^{-1}T^{-1} \\
{\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & {\scriptstyle =}\downarrow \\
S^{8\ell+3} & \xrightarrow{i} & M^{8\ell+3}(4) & \xrightarrow{A^\ell} & M^3(4) & \xrightarrow{2g} & v_1^{-1}T^{-1}
\end{array} \tag{2.49}$$

[10] $v_1^{-1}T^{-1}$ can be defined to be $T^{-1} \wedge v_1^{-1}J$.

Here $2: M^{8\ell+3} \to M^{8\ell+3}(4)$ from the mod 2 Moore spectrum to the mod 4 Moore spectrum has degree 2 on the bottom cell and degree 1 on the top cell.

Since $E_2^{3,8\ell+4}(\mathrm{Spin}(m))$ and $E_2^{4,8\ell+5}(\mathrm{Spin}(m))$ are $\mathbf{Z}_2$-vector spaces, and there can be no extension from filtration 2 to filtration 3 by naturality, the only way that α^* in (2.44) could hit an element in filtration 4 is if γ^* hits an element of order 4 in filtration 2, and there is a nontrivial extension. We will show that $(2\gamma)^*$ cannot be nonzero in filtration 2.

Since $\alpha^*(= (4\gamma)^*)$ is given by applying $[-, \Phi BSO(m)]$ to the top composite in (2.49), then $(2\gamma)^*$ is given by applying $[-, \Phi BSO(m)]$ to the bottom composite. The E_2-version of this bottom composite is just like (2.45) with $\mathbf{Z}/2$ replaced by $\mathbf{Z}/4$. Thus showing that $(2\gamma)^*$ is 0 in filtration 2 is equivalent to proving the analogue of (2.46) with $C^*/2$ replaced by $C^*/4$.

We need the following lemma.

Lemma 2.50. *The matrix, analogous to (2.38) in the interpretations of its rows and columns, which presents $H_0(C^*_{(4\ell+1)})$ for $\mathrm{Spin}(2n+1)$ with $n > 5$ is*

$$\begin{pmatrix} 2^{A_1} & 0 \\ u_2 2^{A_2} & 2^n \\ u_3 2^n & 2^3 \end{pmatrix} \tag{2.51}$$

with u_i odd and $A_i \geq n+1$.

This is proved similarly to 2.36. It differs in that it involves $4\ell+1$ rather than $4\ell-1$. It is just [5, 3.18] with a lower bound for some exponents being 1 larger than was proved in [5]. As we don't need this refinement here, we will not present the details of the proof, which are extremely similar to those of 2.36.

Now the analogue of (2.46) with 4 instead of 2 is proved by the same method used for 2. Now we have that $\delta_0(x_1, x_2, x_3) \neq 0 \in \mathbf{s}H_0(C^*_{(-1)})$ if and only if the D-component of x_3 is not divisible by 4. Here we need that $A_i \geq n+1$ in (2.38) when $\ell = 0$, which was proved in 2.36. In this case, $\delta'_\ell(x_1, x_2, x_3)$ is nonzero in $H_0(C^*_{(4\ell+1)})$ because it is $\frac{1}{4}$ or $\frac{1}{2}$ times the last row of (2.51) plus $\frac{1}{4}$ times multiples of the other rows. This will be nonzero because of the 2^3 in the second column.

This completes the argument (for Case 11) when m is odd. If $m = 4a$ a similar argument works. A matrix of the same general form as (2.40) presents $H_0(C^*_{(4\ell+1)})$. Its rows and columns have analogous interpretations. As in the case m odd, the key point is a 2^3 which occurs in the last row, second column. This is due to the $(3^{m+1}-1)$-factor in [3, (4.27)]. The m of that paper is our $4\ell+1$. This 2^3 will cause (2.46) to hold, and with the 2 replaced by a 4, just as it did when m is odd.

Case 12: $k \equiv 1 \mod 4$, $m \equiv 2 \mod 4$. Similarly to Case 10, the method of Case 11 does not apply because the chain complex used there required $\psi^{-1}=-1$. Again, we can make the required deductions by naturality. The morphism $\mathbf{s}E_2^{0,-1}(8\ell+3,4j+1) \to \mathbf{s}E_2^{0,-1}(8\ell+3,4j+2)$ is bijective by [3, 3.3]. If j is odd, the generator of $E_2^{0,-1}(8\ell+3,4j+1)$ is a permanent cycle by Case 11, and hence so is its image. Now let j be even. The same naturality argument shows that $d_2=0$ on $\mathbf{s}E_2^{0,-1}(8\ell+3,4j+2)$. That $d_3=0$ is proved by the method of Case 10, using that $d_3=0$ on $\widetilde{E}_3^{1,-1}(\mathrm{Spin}(4j+2))$ by [6, 2.23]. Finally we consider d_4. We cannot use naturality from $E_4(8\ell+3,4j+1)$ because it had a nonzero d_3 by [6, 2.23]. Instead we use the argument in Case 11, that the attaching map α equals 4γ. We use naturality from $E_2(8\ell+3,4j+1)$ to see that $(2\gamma)^*$ must be zero in filtration 2, and deduce as in Case 11 that α^* is 0 in filtration 4.

3. Nonlifting results

In [9], the following result was proven.

Theorem 3.1. *If u is odd and $2^{4b+\epsilon} > 4k+t$, then*

$$\mathrm{gd}(u2^{4b+\epsilon}\xi_{4k+t}) \geq 4k-8b+d,$$

where d is given in the following table.

		ϵ			
		0	1	2	3
	1	0	-2	-2	-4
t	2	2	2	0	-4
	3	2	2	0	-4
	4	4	2	2	0

Several more nonlifting results could have been obtained by the same method. The author of [9] did not give careful enough consideration to P_b^t with $t \equiv 1 \mod 4$ or $b \equiv 2 \mod 4$. We sketch a proof of the following result. Theorems 3.1 and 3.2 together provide all the nonlifting results in Theorem 1.3, and those of [6, 1.1(2)].

Theorem 3.2. *If u is odd and $2^{4b+\epsilon} > 4k+t$, then*

$$\mathrm{gd}(u2^{4b+\epsilon}\xi_{4k+t}) \geq 4k-8b+\delta$$

if $(\epsilon,t,\delta)=(0,2,3)$, $(0,3,3)$, $(1,4,3)$, $(1,1,0)$, *or* $(0,1,2)$.

Proof. We must show there does not exist an axial map

$$P^{4k+t}\times P^{u2^{4b+\epsilon}-4k+8b-\delta} \to P^{u2^{4b+\epsilon}-1}.$$

This is done by showing that ψ^3-1 applied to the dual class in

$$ko_{-2}(P^{-2}_{-4k-t-1}\wedge P_{-u2^{4b+\epsilon}+4k-8b+\delta-1}\wedge P^{u2^{4b+\epsilon}-1}) \tag{3.3}$$

is nonzero. This class is called the axial class.

Lemma 3.4. *Let $X=P^{-2}_{-4k-t-1}\wedge P_{-u2^{4b+\epsilon}+4k-8b+\delta-1}$. Then $ko_*(X\wedge P^{u2^{4b+\epsilon}-1})$ contains summands*

$$ko_*(X\wedge S^{u2^{4b+\epsilon}-1})\oplus ko_*(X\wedge P^{u2^{4b+\epsilon}-2}).$$

The upper edge of the second of these summands extends one filtration higher than that of the first.

Proof. Let A_1 denote the subalgebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2. We use that the Adams spectral sequence converging to $ko_*(X)$ has $E_2=\mathrm{Ext}_{A_1}(H^*X)$. (We omit writing $\mathbf{Z}_2$ in the second variable.) Let N denote the A_1-module with classes in grading 0, 2, 3, and 5 with $\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\neq 0$, and let N_0 be defined by the short exact sequence of A_1-modules

$$0\to\Sigma^5\mathbf{Z}_2\to N\to N_0\to 0.$$

If M is an A_1-module which is free as a module over the subalgebra A_0 generated by Sq^1, then $\mathrm{Ext}_{A_1}(M\otimes N)=0$ in filtration >0, and hence, for $s>0$, we have

$$\mathrm{Ext}^{s,t}_{A_1}(M\otimes\Sigma^4\mathbf{Z}_2)\approx\mathrm{Ext}^{s,t+1}_{A_1}(M\otimes\Sigma^5\mathbf{Z}_2)\xrightarrow{\approx}\mathrm{Ext}^{s+1,t+1}_{A_1}(M\otimes N_0). \tag{3.5}$$

The first of these groups can correspond roughly to the first summand of the lemma, and the last to the other summand, after adjoining many copies of $\mathrm{Ext}_{A_1}(M\otimes N)$. The filtration shift in (3.5) yields the conclusion of the lemma.

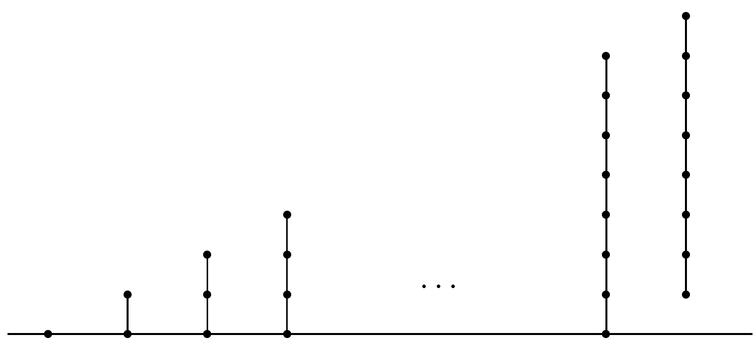
Here we have used that, except in its bottom few cells, the A_1-module $H^*P^{u2^{4b+\epsilon}-2}$ is built by short exact sequences from many copies of Σ^iN and one of $\Sigma^{u2^{4b+\epsilon}-5}N_0$. A deviation due to the bottom few cells of $P^{u2^{4b+\epsilon}-2}$ will not alter the Ext groups in the region of interest. Note that H^*X is A_0-free except in the case where $t = 3 = \delta$, in which case it is a direct sum of an A_0-free summand and one that is inconsequential here. ■

Using some suspension isomorphisms, the part of (3.3) corresponding to the first summand in 3.4 is

$$ko_{-1}(P_{-4k-t-1}^{-2} \wedge P_{4k-8b+\delta-1}).$$

The subscript of one P is odd[11] and the other $\equiv 2 \bmod 4$. The $P_{4\ell+2}$ is built from copies of N, which, after tensoring with the other P, give no Ext in positive filtration, together with $\langle g_{4\ell+2}, \mathrm{Sq}^2(g_{4\ell+2})\rangle$, which changes bo to bu. Thus the chart for the portion of 3.4 due to the top cell is given by the diagram below, with the bottom class in dimension $-8b+\delta-t-2$.

Diagram 3.6.



All of our cases[12] have $\delta - t = 1 - 2\epsilon$. Thus the chart starts in $-8b-2\epsilon-1$, and its top element in dimension -1 is in filtration $4b+\epsilon$. The summand of (3.3) corresponding to the second summand of 3.4 has top element in filtration $4b+\epsilon+1$.

According to the third case of Table 12 of [9], the axial class has a component $2 \cdot u2^{4b+\epsilon}$ in this second summand, i.e. at height $4b+\epsilon+1$, and so is nonzero. ■

[11]except for the case $(0,3,3)$, which is equivalent to $(0,2,3)$ plus an additional split summand

[12]with the exception noted in the previous footnote

References

[1] J. F. Adams, *Vector fields on spheres*, Annals of Math **75** (1962) 603-632.
[2] ———, *On the groups $J(X)$, IV*, Topology **5** (1966) 21-71.
[3] M. Bendersky and D. M. Davis, *The v_1-periodic homotopy groups of $SO(n)$*, Memoirs Amer Math Soc **815** (2004).
[4] ———, *2-primary v_1-periodic homotopy groups of $SU(n)$*, Amer Jour Math **114** (1991) 529-544.
[5] ———, *The 1-line of the K-theory Bousfield-Kan spectral sequence for* $\mathrm{Spin}(2n+1)$, Contemp Math AMS **279** (2001) 37-56.
[6] M. Bendersky, D. M. Davis, and M. Mahowald, *Stable geometric dimension of vector bundles over even-dimensional real projective spaces*, to appear in Trans Amer Math Soc. http://www.lehigh.edu/~dmd1/sgd2.html
[7] A. K. Bousfield, *The K-theory localization and v_1-periodic homotopy groups of finite H-spaces*, Topology **38** (1999) 1239-1264.
[8] ———, *On the 2-primary v_1-periodic homotopy groups of spaces*, Topology **44** (2005) 381-413.
[9] D. M. Davis, *Generalized homology and the generalized vector field problem*, Quar Jour Math Oxford **25** (1974) 169-193.
[10] D. M. Davis, S. Gitler, and M. Mahowald, *The stable geometric dimension of vector bundles over real projective spaces*, Trans Amer Math Soc **268** (1981) 39-61.
[11] D. M. Davis and M. Mahowald, *Homotopy groups of some mapping telescopes*, Annals of Math Studies **113** (1987) 126-151.
[12] M. Mahowald, *The image of J in the EHP sequence*, Annals of Math **116** (1982) 65-112.

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