On the coalgebraic ring and Bousfield-Kan spectral sequence for a Landweber exact spectrum

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5 December 2001

Abstract

We construct a Bousfield-Kan (unstable Adams) spectral sequence based on an arbitrary (and not necessarily connective) ring spectrum E with unit and which is related to the homotopy groups of a certain unstable E completion $X^{\wedge}_{E}$ of a space X. For E an $\mathbb{S}$-algebra this completion agrees with that of the first author and R. Thompson [7]. We also establish in detail the Hopf algebra structure of the unstable cooperations

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[1]MSC: 55P60, 55Q51, 55S25, 55T15. Secondary: 55P47.

[2]Keywords: Bousfield-Kan spectral sequence, unstable Adams spectral sequence, Hopf rings, coalgebraic algebra, Landweber exact, unstable cooperations, unstable completions

(the coalgebraic module) $E_*(\underline{E}_*)$ for an arbitrary Landweber exact spectrum E, extending work of the second author with M. Hopkins [15] and with P. Turner [20] and giving basis-free descriptions of the modules of primitives and indecomposables. Taken together, these results enable us to give a simple description of the E_2-page of the E-theory Bousfield-Kan spectral sequence when E is any Landweber exact ring spectrum with unit. This extends work of the first author and others and gives a tractable unstable Adams spectral sequence based on a v_n-periodic theory for all n.

Introduction

An unstable Adams spectral sequence computes homotopy theoretic information for a space X from homological information. More specifically, such a spectral sequence based on a homology theory $E_*(-)$ seeks, under certain hypotheses, to compute the homotopy of an appropriate E-completion of X from an Ext group (in a suitable category) involving $E_*(X)$. This paper identifies, for E a general ring spectrum with unit, an unstable E-completion $X^\wedge_E$ of X and an associated E-theory Bousfield-Kan spectral sequence with E_2-term the homology of a certain unstable cobar complex. When E is an arbitrary Landweber exact spectrum [22] we obtain a more tractable description of the E_2-term, and, when E additionally has the structure of an $\mathbb{S}$-algebra in the sense of [13], our completion $X^\wedge_E$ and spectral sequence agree with those of [7]. In order to obtain the description of the E_2-term we prove a number of results on the generalised homology of the spaces in the Ω-spectrum for a Landweber exact theory E, that is, on the coalgebraic ring $F_*(\underline{E}_*)$. These results are of independent interest; see, for example, [14].

The first example of an unstable Adams spectral sequence based on a theory E other than ordinary homology was that of the first author with E. Curtis and H. Miller [5] which considered the case of a connective theory E and concentrated in particular on the case of BP-theory. This provided a sequence that converged to the p-localisation of the unstable homotopy of an odd dimensional sphere and identified the E_2-term as an Ext group in a non-abelian category of unstable $BP_*(BP)$-coalgebras. Using results of Wilson [25], the E_2-term was given a simpler, and more computationally practical, interpretation as the homology of an unstable cobar complex and which could be further considered [4] as the homology of a certain subcomplex of the *stable* cobar complex. This spectral sequence and subsequent variations were generalisations of that of Bousfield and Kan [9] and we refer throughout to all these models as Bousfield-Kan spectral sequences.

A more mysterious gadget however has been that of a Bousfield-Kan spectral sequence based on a *periodic* theory E. Theories E one would naturally wish to consider include complex K-theory, the Johnson-Wilson theories $E(n)$ and the Morava E- and K-theories; for technical reasons one is probably going to make easiest headway with those theories E which are also Landweber exact, as in the connective example BP successfully dealt with by [5] and subsequent papers. With R. Thompson, the first author has developed a framework [7] to define and study sequences based on periodic theories. The requirements on E to set such a sequence up, to identify the E_2-term in a practical and computable manner, and to prove convergence to an identifiable object are however significant. In brief, convergence is proved, in appropriate cases, to a certain 'unstable E-completion' of the underlying space X, where this completion is defined as Tot of a certain cosimplicial space, and is defined only when E is represented by an $\mathbb{S}$-algebra in the sense of [13]. Of the example theories E listed above, to present knowledge this rules out all but complex K-theory and the Morava E-theories.

The understanding of the E_2-page of any of these Bousfield-Kan spectral sequences involves in large part having good and well understood structure in the un stable E-theory cooperation algebras, that is, in the *coalgebraic ring* [19] or *Hopf ring* [24] $E_*(\underline{E}_*)$ where the $\underline{E}_r$ denote the spaces in the Ω-spectrum for E-theory. In [7] the E_2-term was identified in a practical manner under the hypotheses that each $E_*(\underline{E}_r)$ was free as an E_*-module and that the submodule of primitives $PE_*(\underline{E}_r)$ inject under infinite stabilisation in the stable cooperation ring $E_*(E)$. Work of the second author and M. Hopkins [15] showed that these hypotheses were satisfied for a Landweber exact theory E whose coefficients were 'not too large': this incuded the cases of K-theory and the Johnson-Wilson theories $E(n)$, but not the $\mathbb{S}$-algebra examples of the Morava E-theories. Between these two sets of requirements on E – for convergence and for the computation of the E_2-term – fully satisfactory results in [7] were obtainable only when E was taken as complex K-theory.

The main results of this paper fall into three sections. In section 1 we study in depth the homology and generalised homology of the spaces $\underline{E}_r$ in the Ω-spectrum for an arbitrary Landweber exact theory E, assuming only that the coefficients E_* are concentrated in even dimensions (an assumption which fails only in rather artifical examples). These results extend those of [15], removing the size restrictions on the coefficients, proving for example (Theorem 1.4) that the algebras $F_*(\underline{E}_r)$ for a wide class of homology theories $F_*(-)$ are polynomial or exterior for r even or odd respectively. However, they go further than the type of results in [15], giving also basis-free descriptions (Theorem 1.10) of the modules of primitives and indecomposables associated to the $F_*(\underline{E}_r)$. Unusually for results on the coalgebraic ring $F_*(\underline{E}_*)$ for theories F and E, these results give explicit descriptions of the individual Hopf algebras $F_*(\underline{E}_r)$, rather than just implicit descriptions in terms of the global object $F_*(\underline{E}_*)$. We also relate (Corollary 1.11) the modules $PF_*(\underline{E}_r)$ and $QF_*(\underline{E}_r)$ to the primitives and indecomposables of the universal example $MU_*(\underline{MU}_*)$. These results are of independent interest in the study of the homology of Ω-spectra, having applications, for example, to group cohomology [14] and [18]. In the case of certain completed spectra, such as the Morava E-theories and the Baker-Würgler completions $\widehat{E(n)}$ [3], these results have parallels with those of [16] where the homological effects of completion on Ω-spectra are examined using rather different methods.

In section 2 we suppose E merely to be a ring spectrum with a unit. For a space X we define a notion (Definition 2.2) of E-completion of X, denoted $X^{\wedge}_E$. If E is an $\mathbb{S}$-algebra then the space $X^{\wedge}_E$ turns out to be homotopy equivalent to the E-completion of X as defined in [7]. The E-theory Bousfield-Kan spectral sequence related to the homotopy groups of this space $X^{\wedge}_E$ is introduced and we identify (Theorem 2.8) the E_2-page as the homology of an unstable cobar complex.

The results of section 2 are very general but as they stand offer small hope for specific computation. In section 3 we build on them in the special case of a Landweber exact ring spectrum (with unit), using the work of section 1 on the coalgebraic ring for such a spectrum. The main result here is a 'change of rings' theorem that identifies (Theorem 3.1) the E_2-page of the E-theory Bousfield-Kan spectral sequence of section 2 as an Ext group in a convenient, moreover abelian, category. This applies to spaces X such as torsion free H-spaces and odd dimensional spheres. We note also (Remark 3.11) that a similar result holds for spaces such as ΩS^{2n+1}, though care is needed for such examples as, by the work of section 1, the relevant Hopf algebras $E_*(\underline{E}_{2r})$ in the computation are not primitively generated. Taken together, the results of this article allow for the construction and description of an unstable Adams spectral sequence based on a v_n-periodic theory for any positive integer n, extending the framework of [7] which established the v_1-periodic case.

Notation The convention we use for denoting spaces, spectra, *etc.* related to a theory E is as follows. For a theory E we write $E_*(-)$ and $E^*(-)$ for the generalised E-homology and cohomology, $\mathbf{E}$ for the associated spectrum when we wish to consider it as an explicit object in the stable category, and $\underline{E}_r$ and $\underline{E}_*$ for the spaces in the Ω-spectrum and for the Ω-spectrum itself. Thus the space $\underline{E}_r$ represents the cohomological functor $E^r(-)$ in the sense that $E^r(X) = [X, \underline{E}_r]$ for any space X. The $\underline{E}_r$ are related by equivalences $\Omega\underline{E}_{r+1} \simeq \underline{E}_r$.

1 The coalgebraic module $F_*(\underline{E}_*)$

Throughout this section we shall assume that $\underline{E}_*$ is an Ω-spectrum representing a Landweber exact cohomology theory [22]. Such theories include the examples of complex cobordism MU and the Brown-Peterson theories BP [1], the Johnson-Wilson theories $E(n)$ [21] and their I_n-adic completions $\widehat{E(n)}$ [3] as well as Morava E-theory, complex K-theory, various forms of elliptic cohomology [23] and their completions. For simplicity in the statement of our results, we assume the coefficients E_* are concentrated in even degrees; this is satisfied by all standard examples including all those just mentioned. As $\mathbf{E}$ is necessarily a module spectrum over $\mathbf{MU}$, the mod p homology $H_*(\underline{E}_*;\mathbb{F}_p)$ will be a *coalgebraic module* over both $H_*(\underline{MU}_*;\mathbb{F}_p)$ and $\mathbb{F}_p[MU^*]$ in the sense of [19]; if, as will in fact generally be the case, $\mathbf{E}$ is a ring spectrum, $H_*(\underline{E}_*;\mathbb{F}_p)$ will be a coalgebraic ring (Hopf ring) and a coalgebraic algebra over these objects as well. We assume the notation and results on $H_*(\underline{MU}_*;\mathbb{F}_p)$ to be found in [24] and the notions of coalgebraic algebra as in [19, 24]. If $\mathbf{E}$ is a p-local spectrum then similar statements hold on replacement of MU by BP.

The work of [19] establishes in particular a tensor product $\overline{\otimes}$ in the category of $\mathbb{F}_p[MU^*]$ coalgebraic modules; note that this is quite distinct from the tensor product of the underlying $\mathbb{F}_p$ coalgebras. The main theorem of [20] tells us

Theorem 1.1 $H_*(\underline{E}_*;\mathbb{F}_p) \cong H_*(\underline{MU}_*;\mathbb{F}_p) \underset{\mathbb{F}_p[MU^*]}{\overline{\otimes}} \mathbb{F}_p[E^*]$. □

Corollary 1.2 *$H_*(\underline{E}_{2r+1};\mathbb{F}_p)$ is an exterior algebra.*

Proof Consider the indecomposable quotient $QH_*(\underline{E}_{2r+1};\mathbb{F}_p)$. Unwinding the definition of $\overline{\otimes}$, elements in this quotient are represented by sums of $\circ$ products of elements of the form $q\overline{\otimes}x$ where q represents an indecomposable in an odd MU space and $x \in \mathbb{F}_p[E^*] = H_0(\underline{E}_*;\mathbb{F}_p)$ (this follows from the fact that E^* is concentrated in even dimensions). As $QH_*(\underline{MU}_s;\mathbb{F}_p)$ lies in odd homological dimensions if s is odd, [24], we conclude that $QH_*(\underline{E}_{2r+1};\mathbb{F}_p)$ lies in odd homological dimension.

Thus any finite dimensional sub-Hopf algebra of $H_*(\underline{E}_{2r+1};\mathbb{F}_p)$ lies in a finite dimensional sub-Hopf algebra generated by odd dimensional elements, and so is an exterior algebra. As $H_*(\underline{E}_{2r+1};\mathbb{F}_p)$ is the colimit of its finite dimensional subalgebras, the result follows. □

Corollary 1.3 *$H_*(\underline{E}_{2r};\mathbb{F}_p)$ is a polynomial algebra and homology suspension induces an isomorphism $QH_*(\underline{E}_{2r};\mathbb{F}_p) \cong QH_*(\underline{E}_{2r+1};\mathbb{F}_p)$.*

Proof This is an immediate consequence of Corollary 1.2 and the homology Eilenberg-Moore spectral sequence [12]

$$\mathrm{Cotor}^{H_*(\underline{E}_{2r+1};\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \Longrightarrow H_*(\underline{E}_{2r};\mathbb{F}_p).$$

As $H_*(\underline{E}_{2r+1};\mathbb{F}_p)$ is exterior, the E^2-page is already polynomial and is concentrated in even degrees. The sequence thus collapses and the result follows. □

We require knowledge of $F_*(\underline{E}_*)$ for more general theories F than just mod p homology; for example we need results with $F = E$ for the unstable homotopy spectral sequences later, but other examples are of importance too. For the remainder of this article we shall assume that F is a p-local ring spectrum, with coefficients torsion free and concentrated in even dimensions. We shall have occasion also to consider a version of a theory F with coefficients reduced mod p; such homology of a space X will be denoted $F_*(X;\mathbb{F}_p)$.

Theorem 1.4 *Suppose $\underline{E}_*$ and F are as above. Then $F_*(\underline{E}_s)$ is a free F_* module, with algebra structure polynomial for s even and exterior for s odd.*

Proof Begin with the case $F = H\mathbb{Z}_{(p)}$. As $H_*(\underline{E}_{2r};\mathbb{F}_p)$ is polynomial, and in even dimensions, its generators lift to polynomial generators of the torsion free algebra $H_*(\underline{E}_{2r};\mathbb{Z}_{(p)})$. From this we can deduce that the homology of the odd spaces $H_*(\underline{E}_{2r+1};\mathbb{Z}_{(p)})$ are torsion free, exterior algebras, generated by the suspensions of generators of $H_*(\underline{E}_{2r};\mathbb{Z}_{(p)})$. The result for general F follows by a collapsing Atiyah-Hirzebruch spectral sequence argument. □

Corollary 1.5 *For $\underline{E}_*$ and F as above, $F_*(\underline{E}_*)$ is a $F_*(\underline{MU}_*)$ coalgebraic module; if $\mathbf{E}$ is a ring spectrum, $F_*(\underline{E}_*)$ is also a coalgebraic ring.*

Proof This result is a standard and formal argument (see, for example, [24]) and follows as soon as a Künneth theorem

$$F_*(\underline{E}_r \times \underline{E}_s) \cong F_*(\underline{E}_r) \underset{F_*}{\otimes} F_*(\underline{E}_s)$$

is established. This holds by the freeness result of (1.4). □

For $\mathbf{E}$ a ring spectrum, recall the algebraic model coalgebraic rings $F_*^R(\underline{E}_*)$ and $F_*^Q(\underline{E}_*)$ constructed in [24] and [17] respectively. The former, $F_*^R(\underline{E}_*)$, is the free $F_*[E^*]$ coalgebraic algebra generated by certain classes arising from the complex orientation on E, modulo specific relations arising from the interaction of the E and F formal group laws; the latter, $F_*^Q(\underline{E}_*)$, can be constructed as a certain sub-coalgebraic ring of the rational object $F\mathbb{Q}_*(\underline{E\mathbb{Q}}_*)$. There are natural maps

$$F_*(\underline{E}_*) \xleftarrow{\tau} F_*^R(\underline{E}_*) \longrightarrow F_*^Q(\underline{E}_*).$$

Corollary 1.6 *For E and F as above, there are isomorphisms of coalgebraic rings*

$$F_*(\underline{E}_*) \cong F_*^R(\underline{E}_*) \cong F_*^Q(\underline{E}_*).$$

Proof That τ is an isomorphism follows from [20] and the corresponding result for MU, [24]. This, together with the fact that $F_*(\underline{E}_*)$ is torsion free by Theorem 1.4, gives the second isomorphism using [17] corollary 6.3. □

We seek now to describe the modules of primitives $PF_*(\underline{E}_s)$ and indecomposables $QF_*(\underline{E}_s)$ for the Hopf algebras $F_*(\underline{E}_s)$. One of the main ideas of [17] is that when $F_*(\underline{E}_*)$ is torsion free (as here), simple descriptions of its algebra structure can be obtained by identifying its image in the rational coalgebraic ring $F\mathbb{Q}_*(\underline{E\mathbb{Q}}_*)$. The following result allows analagous descriptions of $PF_*(\underline{E}_s)$ and $QF_*(\underline{E}_s)$ by embedding these modules in the *stable* object $F_*(E)$.

Proposition 1.7 *For E and F as above, homology suspension induces monomorphisms*

$$\begin{aligned} PF_*(\underline{E}_s) &\longrightarrow F_{*-s}(E) \\ QF_*(\underline{E}_s) &\longrightarrow F_{*-s}(E). \end{aligned}$$

Proof The proof is essentially given by the following commutative diagram

$$\begin{array}{ccccc} PF_*(\underline{E}_s) & \xrightarrow{\iota} & QF_*(\underline{E}_s) & \xrightarrow{\sigma_s} & F_{*-s}(E) \\ & & \Big\downarrow Q\rho_* & & \Big\downarrow \rho_* \\ & & QF\mathbb{Q}_*(\underline{E}_s) & \xrightarrow{\sigma_s} & F\mathbb{Q}_{*-s}(E). \end{array}$$

The map ι is the natural map from primitives to indecomposables; as $F_*(\underline{E}_*)$ is torsion free, by (1.4), this is an inclusion. Infinite homology suspension from the s^{th} space is denoted by σ_s and ρ indicates the rationalisation map $F \to F\mathbb{Q}$. Then Theorem 1.4 tells us that the left hand vertical map $Q\rho_*$ is a monomorphism, and the analysis of rational coalgebraic rings in [17] shows that the suspension $\sigma_s\colon QF\mathbb{Q}_*(\underline{E}_s) \to F\mathbb{Q}_{*-s}(E)$ is also monic. □

This result allows us to give a basis free description of $PF_*(\underline{E}_s)$ and $QF_*(\underline{E}_s)$ as a submodule of the stable module $F_*(E)$. This generalizes the construction of [4], definition 2.13. First though we need to recall some standard notation for elements in coalgebraic rings; see [24] for further details.

Recall there are classes $b_t \in H_{2t}(\underline{MU}_2)$. When localized at a prime p (as here) it is customary to denote the class b_{p^s} by $b_{(s)} \in H_{2p^s}(\underline{MU}_2)$. We shall use this notation throughout, reserving the names b_t (without brackets) for elements of the *stable* module $F_*(E)$, as below. There is also the suspension element $e \in H_1(\underline{MU}_1)$ with the relation $e \circ e = -b_{(0)}$. For v a homogeneous element of MU_*, say $v \in MU_{|v|} = \pi_0(\underline{MU}_{-|v|})$, we have the element $[v] \in H_0(\underline{MU}_{-|v|})$, its Hurewitz image; note that $v \in MU_{|v|} = MU^{-|v|}$ and $|v| \geqslant 0$. By [24], $H_*(\underline{MU}_*; \mathbb{Z}_{(p)})$ is generated as a coalgebraic ring by the classes $[v]$, e and $b_{(s)}$. The algebraic models

$F_*^R(\underline{E}_*)$ are by definition generated by the analogous elements $[v]$, e and $b_{(s)}$ with the v homogenous elements of E_*, and by Corollary 1.6 we know the corresponding classes also generate $F_*(\underline{E}_*)$.

The suspension homomorphisms, $\sigma_s\colon F_*(\underline{E}_s) \to F_{*-s}(E)$ send $b_{(i)}$ to $b_i \in F_{2p^i-2}(E)$, kill $*$ products and take $\circ$ products to multiplication in $F_*(E)$; note in particular that $\sigma_2\colon b_{(0)} \mapsto 1$. For v a homogeneous element of E_*, we denote also by v the image of $[v]$ under suspension in $F_{|v|}(E)$; this is additionally the image of v in $F_*(E)$ under the right unit $F_* \to F_*(E)$.

We shall denote the free E_* module generated by a class ι_n in dimension n by M_n, or, when we need to indicate the Ω spectrum considered, by M_n^E. This is useful for keeping track of the domain of the stabilization map, $\sigma_s\colon F_*(\underline{E}_s) \to F_{*-s}(E)$ and is accomplished by defining the range of σ_s to be $F_{*-s}(E) \underset{E_*}{\otimes} M_s$. In this notation $b_{(i)} \in F_{2p^i}(\underline{E}_2)$ maps to $b_i \otimes \iota_2$ while a class such as $b_{(i)} \circ [v] \in F_{2p^i}(\underline{E}_{2-|v|})$ maps to $b_i \otimes v\iota_{2-|v|}$. Notice that the suspension homomorphisms now preserve dimension.

For each finite sequence of non-negative integers $I = (i_1, i_2, \ldots, i_n)$ we write b^I for the stable element

$$b^I = b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} \in F_*(E).$$

The *length* of I is the integer $l(I) = i_1 + \cdots + i_n$. Write $b^{\circ I}$ for the *unstable* element $b_{(1)}^{\circ i_1} \circ \cdots \circ b_{(n)}^{\circ i_n} \in F_*(\underline{E}_{2l(I)})$. Of course $\sigma_{2l(I)}\colon b^{\circ I} \mapsto b^I$. More generally, any element of the form $(b_{(0)}^{\circ r} \circ b^{\circ I} + \text{decomposables})$ suspends to b^I.

Definition 1.8 *Let M be a free, graded E_*-module. Write $U_F(M)$ for the sub-F_*-module of $F_*(E) \underset{E_*}{\otimes} M$ spanned by all elements of the form*

$b^I \otimes m$ *where* $2l(I) < |m|$.

Definition 1.9 *Let* M *be a free, graded* E_**-module. Write* $V_F(M)$ *for the sub-*F_**-module of* $F_*(E) \underset{E_*}{\otimes} M$ *spanned by all elements of the form* $b^I \otimes m$ *where* $2l(I) \leqslant |m|$.

The special case of the next result for $E = F = BP$ was proved in [4, 5].

Theorem 1.10 *The image of the suspension homomorphism,* $\sigma_s\colon\ QF_*(\underline{E}_s) \to F_{*-s}(E) \underset{E_*}{\otimes} M_s \cong F_*(E)$ *lies in* $V(M_s)$ *and*

$$\sigma_s\colon QF_*(\underline{E}_s) \to V(M_s)$$

is an isomorphism.

Furthermore the image of $\sigma_s|_{PF_*(\underline{E}_s)}$ *lies in* $U(M_s)$ *and*

$$\sigma_s\colon PF_*(\underline{E}_s) \to U(M_s)$$

is an isomorphism.

Proof We start with the identification of the image of the indecomposables $QF_*(\underline{E}_s)$ with $V_F(M_s)$. As σ_s on $QF_*(\underline{E}_s)$ is monomorphic this will prove the first statement. We begin with the case where s is even.

By Corollary 1.6 we know that any element of $QF_*(\underline{E}_s)$ can be written as a F_*-linear sum of elements of the form $b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v]$ with $r \geqslant 0$. Such an element suspends to $b^I \otimes v\iota_s$. The condition that an element $b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v]$ lies in the F-homology of the s^{th} space $\underline{E}_s$ is that $2r + 2l(I) - |v| = s$, thus $2l(I) \leqslant |v| + s = |v\iota_s|$ and so the image of σ_s lies in $V_F(M_s)$.

Conversely, if $b^I \otimes v\iota_s$ lies in $V_F(M_s)$ then $2l(I) - |v| \leqslant s$ and so $b^I \otimes v\iota_s = \sigma_s(b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v])$ where $2r = s + |v| - 2l(I) \geqslant 0$. Hence σ_s is onto $V_F(M_s)$ and the isomorphism for even spaces is shown.

The result for odd spaces is very similar; note that circle multiplication by e induces a one to one correspondence between $QF_*(\underline{E}_{2t})$ and $QF_*(\underline{E}_{2t+1})$.

The result for primitives is again similar and follows immediately after making the observation that $PF_*(\underline{E}_s)$ for even s is the F_*-linear span of elements of the form $b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v]$ with $r > 0$. Also, for odd s there is an isomorphism $PF_*(\underline{E}_s) \cong QF_*(\underline{E}_s)$. □

The F_*-module $QF_*(\underline{E}_s)$ is not as it stands an E_* module, but may be modified to be so. Looking at all the spaces together, the bigraded object $QF_*(\underline{E}_*)$ is an E_* module under the action $x \otimes v \mapsto x \circ [v]$ for $x \in QF_*(\underline{E}_*)$ and $v \in E_*$. (Verification that $x \circ [v+w] = x \circ [v] + x \circ [w]$ in $QF_*(\underline{E}_*)$ is left as an exercise in coalgebraic modules: see the axioms listed in [24].)

We may modify the construction of $V_F(M_s)$ so as also to carry the action of E_* by considering the corresponding bigraded object $V_F(M_*)$ equipped with the action $(y \otimes v\iota_s) \otimes w \mapsto y \otimes vw\iota_{s-|w|}$. Define a global suspension map $\sigma\colon QF_*(\underline{E}_*) \longrightarrow V_F(M_*)$ as σ_s on the component $QF_*(\underline{E}_s)$. With these definitions and the previous result it may easily be checked that σ is a F_*-E_* bimodule isomorphism. Similar constructions may be made and results established for the objects of primitives $PF_*(\underline{E}_*)$ and $U_F(M_*)$.

Our second description of the modules of primitives and indecomposables for $F_*(\underline{E}_s)$ can now be given in terms of a simple relation to those of the universal theories. As in the underlying philosophy of [24], *etc.*, this requires us to consider all spaces $\underline{E}_s$ together.

Corollary 1.11 *Let E and F be as above. Then there are isomorphisms*

$$QF_*(\underline{E}_*) \;=\; F_* \underset{MU_*}{\otimes} QMU_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_* \;=\; QF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*$$

$$PF_*(\underline{E}_*) \;=\; F_* \underset{MU_*}{\otimes} PMU_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_* \;=\; PF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*$$

where $\otimes$ means tensor product of modules in the standard sense. Assuming E is p-local, analgous results hold on replacing MU by BP.

Proof We prove the first line, concerning the indecomposable functor: the proof of the version involving the primitives is essentially identical. Note also that the equality

$$F_* \underset{MU_*}{\otimes} QMU_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_* = QF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*$$

follows immediately since each $MU_*(\underline{MU}_s)$ is a free (left) MU_* algebra and hence $F_* \underset{MU_*}{\otimes} QMU_*(\underline{MU}_*) = QF_*(\underline{MU}_*)$.

We show that $QF_*(\underline{E}_*) = QF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*$. By Theorem 1.10 it suffices to show that $V_F(M_*^E) = V_F(M_*^{MU}) \underset{MU_*}{\otimes} E_*$. Of these, the left hand side is the (bigraded) sub-F_*-module of $F_*(E) \underset{E_*}{\otimes} M_*^E$ spanned in grading s by elements $b^I \otimes m\iota_s$ satisfying $2l(I) \leq |m\iota_s|$. As E is Landweber exact, and, by definition, M_*^E and M_*^{MU} are free over E_* and MU_* respectively in each grading,

$$F_*(E) \underset{E_*}{\otimes} M_*^E = F_*(MU) \underset{MU_*}{\otimes} E_* \underset{E_*}{\otimes} M_*^E = F_*(MU) \underset{MU_*}{\otimes} M_*^{MU} \underset{MU_*}{\otimes} E_*\,.$$

Under this equivalence, the element $b^I \otimes m\iota_s \in V_F(M^E_*) \subset F_*(E) \underset{E_*}{\otimes} M^E_*$ is then identified with

$$b^I \otimes \iota_s \otimes m \in V_F(M^{MU}_*) \underset{MU_*}{\otimes} E_* \subset F_*(MU) \underset{MU_*}{\otimes} M^{MU}_* \underset{MU_*}{\otimes} E_* .$$

This is also onto $V_F(M^{MU}_*) \underset{MU_*}{\otimes} E_*$ as the map $MU \to E$ induces a left inverse. $\square$

The constructions U_F and V_F may be extended to other E_* modules M. For an arbitrary non-negatively graded left E_*-module M let

$$F_1 \xrightarrow{f} F_0 \to M \to 0$$

be exact with F_0 and F_1 free over E_*. Then U_F may be extended to M by defining

$$U_F(M) = \mathrm{coker}(U_F(f) \colon U_F(F_1) \to U_F(F_0)).$$

V_F is similarly extended to such E_*-modules.

Proposition 1.12

$$\begin{aligned} V_F(M_s \otimes \mathbb{Z}/p) &\cong QF_*(\underline{E}_s; \mathbb{F}_p) \\ U_F(M_s \otimes \mathbb{Z}/p) &\cong \mathrm{Im}(PF_*(\underline{E}_s; \mathbb{F}_p) \to QF_*(\underline{E}_s; \mathbb{F}_p)). \end{aligned}$$

Proof Since $F_*(\underline{E}_s)$ is a free algebra, there is a diagram with rows short exact

$$\begin{array}{ccccccccc} 0 & \to & QF_*(\underline{E}_s) & \xrightarrow{\times p} & QF_*(\underline{E}_s) & \to & QF_*(\underline{E}_s; \mathbb{Z}/p) & \to & 0 \\ & & \| & & \| & & & & \\ 0 & \to & V_F(M_s) & \xrightarrow{\times p} & V_F(M_s) & \to & V_F(M_s \otimes \mathbb{Z}/p) & \to & 0. \end{array}$$

Hence there is an induced isomorphism $QF_*(\underline{E}_s; \mathbb{F}_p) \to V_F(M_s \otimes \mathbb{Z}/p)$. A similar proof gives the second isomorphism. $\square$

Remark 1.13 Since in practice our cohomology theories tend to be $\mathbb{Z}_{(p)}$ local it can be advantageous to use BP generators. The generators $h_i = c(t_i)$, where the t_i are the standard generators for $BP_*(BP)$ and c denotes the canonical anti-isomorphism, have proven to be useful for unstable calculations. Following ([5], 8.5) we may replace the generator b_i with h_i in Theorem 1.10.

We conclude this section with an example to help clarify these definitions.

Example 1.14 Let $F = E = BP$. We claim that $ph_1 \otimes \iota_1$ defines a non-zero element in $V_{BP}(M_1 \otimes \mathbb{Z}/p) = QBP_*(\underline{BP}_1; \mathbb{F}_p)$, but which suspends to zero in $QBP_*(\underline{BP}_2; \mathbb{F}_p)$. To see that $ph_1 \otimes \iota_1$ is indeed in $V_{BP}(M_1 \otimes \mathbb{Z}/p)$, note that the right action formula tells us that $ph_1 = v_1 \cdot 1 - 1 \cdot v_1$. Thus

$$ph_1 \otimes \iota_1 = v_1 \otimes \iota_1 - 1 \otimes v_1 \cdot \iota_1 \,,$$

an element of $V_{BP}(M_1)$. This element is not divisible by p in $V_{BP}(M_1)$, and so is not zero in $V_{BP}(M_1 \otimes \mathbb{Z}/p)$. On the other hand, $h_1 \otimes \iota_2$ is an element of $V_{BP}(M_1)$ and so $ph_1 \otimes \iota_2$ is p-divisible in $V_{BP}(M_1)$ and thus is zero in $V_{BP}(M_1 \otimes \mathbb{Z}/p)$. In general, $V_F(M_s \otimes \mathbb{Z}/p)$ is not a submodule of $F_*(E)$: when working mod p the unstable classes do not necessarily inject into the stable module.

In a similar fashion the right action formula for ph_1 can be used to show that ph_1^n is a non-zero element in $V_{BP}(M_{2n-1} \otimes \mathbb{Z}/p)$ but which suspends to zero in $V_{BP}(M_{2n} \otimes \mathbb{Z}/p)$.

Example 1.15 Consider the Araki generators $w_i \in BP_{2p^i-2}$, as in [2],

$$pm_n = \sum_{0 \leqslant j \leqslant n} m_i(w_{n-i}^{p^i}); \quad w_0 = p.$$

We prefer the Araki generators to the Hazewinkel generators because of the integral form of Ravenel's formulæ

$$\sum^{F^*} h_j^{p^i} \cdot w_i = \sum^{F^*} w_j^{p^i} \cdot h_i \,.$$

Here $\Sigma^F c(\gamma_i)$ is the formal group sum, c is the canonical anti-isomorphism and $\Sigma^{F^*} \gamma_i = c(\Sigma^F c(\gamma_i))$. (*i.e.*, $\sum^{F^*}$ looks like the usual formal group law, but the formal group coefficients act on the right). It is easy to check that the Ravenel formulæ imply that $\Sigma^{F^*} h_j^{p^i} \cdot w_i = \Sigma^{F^*} w_j^{p^i} \cdot h_i$ hold also in $V_F(M_s^E)$ and $V_F(M_s^E \otimes \mathbb{Z}/p)$ with $s \geqslant 2$. There are similar formulæ involving the Hazewinkel generators, but they are only true stably mod p. We do not know if the Hazewinkel generators satisfy similar, mod p formulæ, unstably.

If $E = E(1)$, Adams' summand of p-local K-theory, or equivalently the first Johnson-Wilson theory and we take F to be H, integral homology, these formulæ reduce to

$$\left(\sum^{F^*} h_j^p \cdot w_1 \right) \otimes \iota_s = 0 \text{ in } V_{H_*}(M_s \otimes \mathbb{Z}/p) \text{ if } s \geqslant 2.$$

Using the grading, this implies that $h_j^p \cdot w_1 \otimes \iota_s = 0$ here. (Notice that $h_j^p \cdot w_1 \otimes \iota_2 = h_j^p \otimes w_1 \cdot \iota_2$ and $w_1 \cdot \iota_2$ has degree $2p$ so this class is defined.)

2 The BKSS for E-theory

Let $\mathcal{S}$ be the category of pointed CW complexes and suppose $\mathbf{E}$ is a ring spectrum with unit. Associated to $\mathbf{E}$ is a functor $T_E \colon \mathcal{S} \to \mathcal{S}$ given by sending X to $\Omega^\infty(\mathbf{E} \wedge \Sigma^\infty X)$. There are natural transformations $\phi \colon 1_{\mathcal{S}} \to T_E$ and $\mu \colon T_E^2 = T_E \circ T_E \to T_E$ induced by the unit and the multiplication in E respectively and these make (T_E, ϕ, μ) a *triple* up to homotopy. See, for example, [6, §2], [7, §4] and [8] for details of the notions of triple, cotriple, their associated categories and derived functors, as used in this and the next section.

If $\mathbf{E}$ is an $\mathbb{S}$-algebra in the sense of [13] (for example, K-theory), it is shown in [7] that (T_E, ϕ, μ) is in fact a triple on the category $\mathcal{S}$. Following [10] there is then a cosimplicial space, $\mathbf{T}_E X$, with coface maps and codegeneracies denoted d^i and s^j respectively. The *completion of X with respect to* $\mathbf{E}$ is taken as

$$X_E^\wedge = \mathrm{Tot}(\mathbf{T}_E X).$$

The E_2-page of the Bousfield-Kan spectral sequence associated to $X_E^\wedge$ is identified [7] with the homology of the *unstable cobar complex*,

$$E_2^s(X) = \pi^s \pi_* \mathbf{T}_E X = H^s(\pi_* \mathbf{T}_E X, \partial),$$

where $\pi_* \mathbf{T}_E X$ is considered as a cochain complex with coboundary map $\partial = \Sigma(-1)^i \pi_* d^i$.

We wish however to be able to consider an 'E-completion' of a space X and a corresponding E-theory Bousfield-Kan spectral sequence whenever E is an arbitrary ring spectrum with a unit. In this section we use the results of [11] to construct (2.2) a space $X_E^\wedge$ for any such E, and prove it to be homotopic to the construction in [7] if $\mathbf{E}$ is an $\mathbb{S}$-algebra. In Theorem 2.8 we identify the E_2-term of the E-theory Bousfield-Kan spectral sequence as an unstable cobar complex.

We recall the notion [11] of a *restricted cosimplicial space*, *i.e.*, a "cosimplicial space" without the codegeneracies.

Definition 2.1 *Suppose* (T, ϕ) *is an augmented functor on* $\mathcal{S}$*,* i.e.*, a functor* $T\colon \mathcal{S} \to \mathcal{S}$ *equipped with a natural transformation* $\phi\colon 1_{\mathcal{S}} \to T$*. Let* X *be a space in* $\mathcal{S}$*. Define the* restricted cosimplicial space $\widehat{\mathbf{T}}X$ *to be the restricted cosimplicial resolution with respect to* T *given by*

$$(\widehat{\mathbf{T}}X)^k = T^{k+1}X$$

in codimension k*, and coface maps given by*

$$((\widehat{\mathbf{T}}X)^{k-1} \xrightarrow{d^i} (\widehat{\mathbf{T}}X)^k) = (T^k X \xrightarrow{T^i \phi T^{k-i}} T^{k+1}X).$$

We may describe a restricted cosimplicial space as a diagram in $\mathcal{S}$ as follows. Let Δ_{rest} denote the *restricted simplicial category*, that is the category whose objects are finite ordered sets $[n] = \{0, 1, \cdots, n\}$ $(n \geqslant 0)$ and whose morphisms are strictly monotone maps. A restricted, unaugmented, cosimplicial space, $\mathbf{C}_{\text{rest}}$ is equivalent to a functor

$$\mathbf{C}_{\text{rest}}\colon \Delta_{\text{rest}} \to \mathcal{S}.$$

In particular $\widehat{\mathbf{T}}_E X \in \mathcal{S}^{\Delta_{\text{rest}}}$.

The full simplicial category, Δ, is the category whose objects are the sets $[n]$ and whose morphisms are all weakly monotone maps. Then a cosimplicial space is a functor

$$\mathbf{C}\colon \Delta \to \mathcal{S}.$$

So $\mathbf{C} \in \mathcal{S}^{\Delta}$.

Let $J\colon \Delta_{\text{rest}} \to \Delta$ be the inclusion functor. Then there is a natural transformation

$$J^*\colon \mathcal{S}^{\Delta} \to \mathcal{S}^{\Delta_{\text{rest}}},$$

essentially the forgetful functor from cosimplical spaces to restricted cosimplical spaces.

Definition 2.2 *For a general ring spectrum with unit* $\mathbf{E}$*, define* $X^{\wedge}_{E}$*, the* E*-completion of* X*, to be* $\underleftarrow{\operatorname{holim}}\, \widehat{\mathbf{T}}_E X$.

Strictly speaking, this definition only requires $\mathbf{E}$ to have a unit. However, we shall need $\mathbf{E}$ to have a ring structure directly after the next definition, which introduces an object lying between a cosimplicial space and a restricted cosimplicial space.

Definition 2.3 *A* modified cosimplical space *is a restricted cosimplicial space with codegeneracies that satisfy cosimplicial-like identities*

$$\begin{array}{rclc} d^j d^i & = & d^i d^{j-1} & i < j \\ s^j d^i & \simeq & d^i s^{j-1} & i < j \\ & \simeq & \text{id} & i = j, j+1 \\ & \simeq & d^{i-1} s^j & i > j+1 \\ s^j s^i & \simeq & s^{i-1} s^j & i > j \end{array}$$

where the first identity is the usual cosimplicial identity, but the rest are required to hold only up to homotopy.

Remark 2.4 If $\mathbf{E}$ is a ring spectrum with unit, then, for $X \in \mathcal{S}$, the triple (T_E, ϕ, μ) induces a modified cosimplicial space which we also denote by $\mathbf{T}_E X$. Clearly any cosimplicial space $\mathbf{C}$ is also a modified cosimplicial space and so if X is an $\mathbb{S}$-algebra the two objects denoted $\mathbf{T}_E X$ agree.

Remark 2.5 Corollary 3.9 of [11] proves that $\mathrm{Tot}(\mathbf{C}) = \underleftarrow{\mathrm{holim}}(\mathbf{C}_{\mathrm{rest}})$ when $\mathbf{C} = J^* \mathbf{C}_{\mathrm{rest}}$. In particular, if $\mathbf{E}$ is an $\mathbb{S}$-algebra, the completion $X^\wedge_E$ defined in [7] agrees with that of Definition 2.2.

Remark 2.6 It is not possible to apply Tot to modified cosimplicial spaces. However, after applying π_* we obtain a cosimplicial group $\pi_* \mathbf{T}_E X$ which we view as a diagram $\pi_* \mathbf{T}_E X \in \mathcal{A}^\Delta$, where $\mathcal{A}$ is the category of abelian groups. Applying π_* to $\widehat{\mathbf{T}}_E X$ gives an object in $\mathcal{A}^{\Delta_{\mathrm{rest}}}$ which is $J^*(\pi_* \mathbf{T}_E X)$.

For a wide class of diagrams $\underline{X} \in \mathcal{S}^I$ Bousfield and Kan [10], XI 7.1, define a spectral sequence related to the groups $\pi_* \underleftarrow{\mathrm{holim}}\, \underline{X}$.

Definition 2.7 *For $X \in \mathcal{S}$ and $\mathbf{E}$ a ring spectrum with unit, define $E_r^{*,*}(X)$, the E-*theory Bousfield-Kan spectral sequence of X, *as the Bousfield-Kan spectral sequence for $\widehat{\mathbf{T}}_E X \in \mathcal{S}^{\Delta_{\mathrm{rest}}}$.*

Theorem 2.8 *$E_2^{s,*}(X)$ is isomorphic to the homology of the unstable cobar complex. That is to say $E_2^{s,*}(X) = \pi^s \pi_* \mathbf{T}_E X$*

Remark 2.9 Recall the cohomotopy $\pi^s \underline{A}$ of a cosimplicial abelian group $\underline{A}$ is defined [10], X 7.1, as the cohomology $H^s(ch(\underline{A}), \partial)$ where $(ch(\underline{A}), \partial)$ is the cochain complex given by $ch(\underline{A})^n = \underline{A}^n$ and $\partial = \sum (-1)^i d^i$.

Proof of (2.8) Let I be either Δ or Δ_{rest}. For $\underline{X} \in \mathcal{S}^I$ the E_2-page is given by

$$E_2^{s,t} = \underleftarrow{\lim}^s \pi_t \underline{X}$$

([10] page 309). Since $\pi_*\mathbf{T}_E X$ is a cosimplicial group, $\varprojlim^s \pi_*\mathbf{T}_E X = \pi^s\pi_*\mathbf{T}_E X$ ([10], XI 7.3 (i)) and it suffices to show that

$$\varprojlim^s \pi_*\widehat{\mathbf{T}}_E X = \varprojlim^s \pi_*\mathbf{T}_E X.$$

For any fixed n, denote by $\underline{K}^I(n) \in \mathcal{S}^I$ the diagrams of Eilenberg-Mac Lane spaces $K(A,n)$ which correspond to $\pi_*\mathbf{T}_E X \in \mathcal{A}^\Delta$ and $\pi_*\widehat{\mathbf{T}}_E X \in \mathcal{A}^{\Delta_{\text{rest}}}$ for the respective I (see [10], XI 7.2). Then for $s \leqslant n$ (again from [10], XI 7.2)

$$\begin{aligned}\varprojlim^s \pi_*\mathbf{T}_E X &= \pi_{n-s} \operatorname{holim}_{\leftarrow} \underline{K}^\Delta(n)\\ \varprojlim^s \pi_*\widehat{\mathbf{T}}_E X &= \pi_{n-s} \operatorname{holim}_{\leftarrow} \underline{K}^{\Delta_{\text{rest}}}(n)\end{aligned}$$

However, $J\colon \Delta_{\text{rest}} \to \Delta$ is left cofinal ([11] page 193). Thus

$$J^*\colon\ \operatorname{holim}_{\leftarrow} \underline{K}^\Delta(n) \to \operatorname{holim}_{\leftarrow} \underline{K}^{\Delta_{\text{rest}}}(n)$$

is a homotopy equivalence. Since n was arbitrary, it follows that $\varprojlim^s \pi_*\mathbf{T}_E X = \varprojlim^s \pi_*\widehat{\mathbf{T}}_E X$ for all s. □

3 The Unstable Cobar Complex for E-theory

Section 2 identifies the E_2-page of the Bousfield-Kan spectral sequence for a ring spectrum with unit $\mathbf{E}$ as the homology of the cochain complex $ch(\pi_*\mathbf{T}_E X)$. However, for practical purposes, as in [5, 7], *etc.*, it is important to be able to reinterpret this in terms of a more manageable object, in practice as the homology of a sub-complex of the *stable* cobar complex, *i.e.*, as an Ext group over a more convenient (in particular, abelian) category.

We suppose for this section that $\mathbf{E}$ is a Landweber exact ring spectrum with unit and (largely for convenience) that $\mathbf{E}$ is p-local with coefficients E_* concentrated in even dimensions. Let $\mathcal{M}$ be the category of free, graded E_*-modules. Drawing on the results of [5, 6, 7] and those of sections 1 and 2, we introduce a certain associated abelian category $\mathcal{U}$. Our main theorem is the following.

Theorem 3.1 *Suppose* $\mathbf{E}$ *is a Landweber exact ring spectrum with unit. Suppose* $M \in \mathcal{M}$ *has* E_**-module generators only in odd degrees and suppose* X *is a space with* $E_*(X) \cong \Lambda(M)$ *as coalgebras, where* $\Lambda(M)$ *is the* E_**-Hopf algebra defined by letting* M *be the submodule of primitives,* i.e., $\Lambda(M)$ *is the exterior algebra on* M*. Then the* E_2*-term of the* E*-theory Bousfield-Kan spectral sequence of* X *can be identified as*

$$E_2^{s,t}(X) \cong \mathrm{Ext}^s_{\mathcal{U}}(E_*(S^t), M).$$

Example 3.2 Spaces X satisfying the hypotheses of the theorem include torsion free H-spaces and odd dimensional spheres.

We begin by defining functors G and $U\colon \mathcal{M} \to \mathcal{M}$. Here and below we draw on a number of the results of section 1 with $F = E$, *i.e.*, in this section we deal only with the coalgebraic ring $E_*(\underline{E}_*)$.

Definition 3.3 *For a free* E_**-module* M *define*

(a) $G(M)$ *to be* $E_*(\underline{EM}_0)$*, where* $\mathbf{EM}$ *denotes the spectrum realizing the homology theory* $E_*(-) \underset{E_*}{\otimes} M$*.*

(b) $U(M)$ *to be* $PG(M)$*, the primitive elements in* $G(M)$*.*

Both G and U are functorial; they take values in $\mathcal{M}$, the category of free E_* modules, by the results of section 2.

Remark 3.4 **(a)** As M is a free E_*-module, it is helpful to observe that $\underline{EM}_0 = \Omega^\infty \mathbf{EM}$ is a product of spaces in the Ω spectrum associated to $\mathbf{E}$ indexed by a set of generators of M. In particular, if $\{g_i\}$ are a set of E_* generators of M with g_i in dimension $|g_i|$,

$$\underline{EM}_0 = \Omega^\infty \left(\bigvee_i \Sigma^{-|g_i|} \mathbf{E} \right) = \prod_i \underline{E}_{-|g_i|}.$$

Moreover, with this notation, $M \cong \pi_* \left(\bigvee_i \Sigma^{-|g_i|} \mathbf{E} \right) = \pi_* \left(\prod_i \underline{E}_{-|g_i|} \right)$.

(b) Note that G is closely related to the functor $T_E \colon \mathcal{S} \to \mathcal{S}$ of section 2. For a space $X \in \mathcal{S}$ with $E_*(X) \in \mathcal{M}$, there is an isomorphism

$$G(E_*(X)) \cong E_*(T_E(X)).$$

(c) Note also that $U(M)$ is identical to the construction $U_E(M)$ of section 1. There is of course a similar functor $V \colon \mathcal{M} \to \mathcal{M}$ based on the indecomposable quotient of $G(M)$ and given by the construction $V_E(M)$ of section 1, but it will play no part in the proof of Theorem 3.1.

Proposition 3.5 *The unit and product in* $\mathbf{E}$ *respectively induce natural transformations*

$$\delta^G \colon G \to G^2 \qquad \epsilon^G \colon G \to I$$

making $(G, \delta^G, \epsilon^G)$ a cotriple on the category $\mathcal{M}$. There are similar natural transformations $\delta^U \colon U \to U^2$, $\epsilon^U \colon U \to I$ making $(U, \delta^U, \epsilon^U)$ also a cotriple on $\mathcal{M}$ and a sub-cotriple of $(G, \delta^G, \epsilon^G)$.

Proof The proof is essentially as in sections 6 and 7 of [5]; moreover, with the first observations of Remark 3.4 the maps δ^G and ϵ^G, for example, may be written explicitly. Alternatively, for Landweber exact **E**, given the definition (1.8) and Theorem 1.10, the result on $(U, \delta^U, \epsilon^U)$ also follows from the coaction formulæ for the b_i. □

Remark 3.6 As usual the cotriples define categories $\mathcal{G}$ and $\mathcal{U}$ of G, respectively U, coalgebras: writing C for either G or U, recall that a C coalgebra in $\mathcal{M}$ is an object $M \in \mathcal{M}$ with a map $\psi\colon M \to CM$ such that

$$\epsilon^C\psi = Id_M\colon M \to M \qquad \text{and} \qquad \delta^C\psi = (C\psi)\psi\colon M \to C^2M$$

(see [5, §5] for details).

In particular, recall that if $M \in \mathcal{M}$ then CM is naturally a C coalgebra with map ψ on $CM \to C^2M$ given by δ^C. There are adjoint functors

$$\mathcal{M} \underset{J}{\overset{C}{\rightleftarrows}} \mathcal{C}$$

where J denotes the forgetful functor. The adjunction gives natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(D, CM) \cong \mathrm{Hom}_{\mathcal{M}}(D, M)$$

for any $D \in \mathcal{C}$ (where we identify D with its image under the forgetful functor).

Strictly speaking, we shall abuse notation and write C not only for the functor $\mathcal{M} \to \mathcal{C}$ above, but also for the functor $JC\colon \mathcal{M} \to \mathcal{M}$ of the cotriple $(C, \delta^C, \epsilon^C)$ on $\mathcal{M}$ and for the other composite, $CJ\colon \mathcal{C} \to \mathcal{C}$, the functor of the adjoint *triple* (C, μ^C, η^C) on $\mathcal{C}$, as in [5, §5].

For $\mathcal{C} = \mathcal{G}$ or $\mathcal{U}$ and objects $W \in \mathcal{C}$ we recall the notions of *cosimplicial resolution* over $\mathcal{C}$, as in [8, 2.5] and [6, 2.2] and the resulting derived functors $\mathrm{Ext}_{\mathcal{C}}(E_*, W)$.

Definition 3.7 *A* cosimplicial resolution, $\mathbf{N}$, over $\mathcal{C}$, *of* $W \in \mathcal{C}$ *consists of objects* $N^n \in \mathcal{C}$ *for* $n \geqslant -1$ *and, for every pair of integers* (i, n) *with* $0 \leqslant i \leqslant n$*, coface and codegeneracy maps (in* $\mathcal{C}$*)*

$$d^i \colon N^{n-1} \to N^n, \qquad s^i \colon N^{n+1} \to N^n$$

satisfying the usual cosimplicial identities (cf. 2.3) *and such that*

(a) $N^{-1} = W$;

(b) *for* $n \geqslant 0$ *there is an* $M_n \in \mathcal{M}$ *with* $N^n = CM_n$;

(c) $H^n(J\mathbf{N}) = 0$ *for* $n \geqslant -1$.

Here $J \colon \mathcal{C} \to \mathcal{M}$ *is the forgetful functor and the homology of* $J\mathbf{N}$ *is the homology of the cochain complex with groups* JN^n *and boundary maps* $\Sigma(-1)^i Jd^i$.

The Ext groups $\mathrm{Ext}_{\mathcal{C}}(E_*, W)$ *are then defined as the homology of chain complex associated to* $\mathrm{Hom}_{\mathcal{C}}(E_*, \widetilde{J\mathbf{N}})$*, where* $\widetilde{J\mathbf{N}}$ *denotes the unaugmented complex*

$$0 \to JN_0 \to JN_1 \to JN_2 \to \cdots .$$

These are the derived functors of $\mathrm{Hom}_{\mathcal{C}}(E_*, -)$ *by* [8].

Example 3.8 The $\mathcal{C}$ *cobar complex* provides a standard example of a cosimplicial resolution. We illustrate it for $\mathcal{C} = \mathcal{U}$; the case of $\mathcal{G}$ is similar.

For $W \in \mathcal{U}$, consider the resolution with q^{th} module $U^{q+1}(W)$. The maps in the $\mathcal{U}$ resolution are displayed in the diagram of E_*-modules

$$W \xrightarrow{d^0} U(W) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^0} \end{array} \cdots$$

and are defined in terms of the triple (U, μ^U, η^U) by

$$\begin{aligned} d^i &= U^i \eta^U U^{n-i}: \quad U^n(W) \to U^{n+1}(W), \qquad 0 \leqslant i \leqslant n, \\ s^i &= U^i \mu^U U^{n-i}: \quad U^{n+2}(W) \to U^{n+1}(W), \quad 0 \leqslant i \leqslant n. \end{aligned}$$

The $\mathcal{U}$ cobar complex is then the complex

$$W \xrightarrow{\partial} U(W) \xrightarrow{\partial} U^2(W) \xrightarrow{\partial} \cdots$$

where $\partial = \sum_{i=0}^{n} (-1)^n d^i \colon U^n(W) \to U^{n+1}(W)$.

The embedding of the primitives in the stable cooperations, (1.7) and (1.10), shows that the acyclicity condition is satisfied since there is an extra codegeneracy $s^{-1} \colon U^{q+1}(W) \to U^q(W)$ induced by the counit in $E_*(E)$:

$$U^{q+1}(C) \to E_*(E) \otimes U^q(C) \xrightarrow{\epsilon \otimes 1} U^q(C).$$

In particular, again by (1.10), $\operatorname{Ext}_{\mathcal{U}}(E_*, W)$ is the homology of a sub-complex of the *stable* cobar complex.

These constructions and the link between the functors G and $\mathbf{T}_E$ of Remark 3.4**(b)** allow us to rewrite Theorem 2.8 as follows.

Theorem 3.9 *For* $\mathbf{E}$ *a ring spectrum with unit and* $X \in \mathcal{S}$ *such that* $E_*(X) \in \mathcal{M}$*, there is a natural isomorphism*

$$E_2^{s,t}(X) = \operatorname{Ext}^s_{\mathcal{G}}(E_*(S^t), E_*(X)). \qquad \square$$

Theorem 3.1 will now follow upon proving

Theorem 3.10 *Suppose* $\mathbf{E}$ *is a Landweber exact ring spectrum with unit. For* $M \in \mathcal{M}$ *with generators in odd degree and* $\Lambda(M)$ *denoting the exterior algebra on* M *with* $M \subset \Lambda(M)$ *the submodule of primitives, there is a natural isomorphism*

$$\mathrm{Ext}^s_{\mathcal{G}}(E_*(S^t), \Lambda(M)) \cong \mathrm{Ext}^s_{\mathcal{U}}(E_*(S^t), M).$$

Proof Let us write $\mathbf{U}M$ for the $\mathcal{U}$ cobar complex as in Example 3.8, *i.e.*, with q^{th} space $U^{q+1}(M)$. Applying the functor $\Lambda(-)$ gives a complex

$$\Lambda\mathbf{U}M\colon \qquad \Lambda(M) \to \Lambda(U(M)) \to \Lambda(U^2(M)) \to \cdots \ .$$

Now let

$$Y^q = G(U^q(M))$$

for $q \geqslant 0$. Since M is concentrated in odd degrees the same is true for $U^q(M)$. By the theorems (1.4) and (1.10) we have natural isomorphisms

$$G(U^q(M)) \cong \Lambda(U^{q+1}(M))$$

and we can identify the complex $\Lambda\mathbf{U}M$ as a complex

$$\mathbf{Y}\colon \qquad \Lambda(M) \to G(M) \to G(U(M)) \to G(U^2(M)) \to \cdots \ .$$

The maps in $\mathbf{Y}$ are coalgebra maps and $E_*(E)$-comodule maps. By [5, 7.3] the maps are in $\mathcal{G}$ (note that [5, 7.3] does not require the assumption [5, 7.7] that the homology of the spaces in the Ω-spectrum be cofree coalgebras – this is not satisfied in general). The extra codegeneracy in the $\mathcal{U}$ cobar complex passes via Λ to an extra codegeneracy in $\mathbf{Y}$, showing $\mathbf{Y}$ to be acyclic. Thus $\mathbf{Y}$ is a $\mathcal{G}$-resolution of $\Lambda(M)$.

The Ext groups $\mathrm{Ext}^s_{\mathcal{G}}(E_*(S^t),\Lambda(M))$ can be obtained using the complex $\mathbf{Y}$ by computing the homology of the complex

$$\mathrm{Hom}_{\mathcal{G}}(E_*(S^t),Y^s)=\mathrm{Hom}_{\mathcal{G}}(E_*(S^t),G(U^s(M))).$$

However, by the adjunction isomorphism mentioned in Remark 3.6 (applied twice), shows

$$\begin{aligned}\mathrm{Hom}_{\mathcal{G}}(E_*(S^t),G(U^s(M))) &= \mathrm{Hom}_{\mathcal{M}}(E_*(S^t),U^s(M)) \\ &= \mathrm{Hom}_{\mathcal{U}}(E_*(S^t),U^{s+1}(M)).\end{aligned}$$

Thus $\mathrm{Ext}_{\mathcal{G}}(E_*(S^t),\Lambda(M))$ is isomorphic to the homology of the $\mathcal{U}$-cobar complex which by definition is precisely $\mathrm{Ext}_{\mathcal{U}}(E_*(S^t),M)$. □

Remark 3.11 The results of section 1 on the algebra structure of $E_*(\underline{E}_*)$ allow further results to follow. For example, suppose for $M\in\mathcal{M}$ we write $\sigma^{-1}M$ for the isomorphic E_*-module with degrees shifted downward by one, *i.e.*, we let $\sigma^{-1}M_t = M_{t+1}$. Then Theorem 1.4 and its proof shows

$$\sigma^{-1}U(M)=QG(\sigma^{-1}M)\,.$$

If we take $M=E_*(S^{2n+1})$ then $E_*(\Omega S^{2n+1})=\sigma^{-1}M$ and an argument similar to that for BP-theory in [6], §6, shows that the complex $\mathbf{Y}$ used in the proof of Theorem 3.10 may also be used to compute the E_2-page of the E-theory Bousfield-Kan spectral sequence for ΩS^{2n+1}: for any odd dimensional sphere S^{2n+1} there is an isomorphism

$$E_2^{s,t-1}(\Omega S^{2n+1})\cong E_2^{s,t}(S^{2n+1})\,.$$

Acknowledgements Both authors are pleased to thank the Japan-U.S. Mathematics Institute (JAMI), Johns Hopkins University, and its organisers J.M. Boardman, D. Davis, J.-P. Meyer, J. Morava, G. Nishida, W.S. Wilson, and N. Yagita for support during Spring 2000 and at which this research was initiated. The first author thanks also Emmanuel Dror-Farjoun for helpful discussions on the material in section 2. The second author thanks the University of Leicester for sabbatical leave and the Leverhulme Foundation for a Research Fellowship during which most of this paper was written.

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