THE BOUSFIELD-KAN SPECTRAL SEQUENCE FOR PERIODIC HOMOLOGY THEORIES

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Abstract. We construct the Bousfield-Kan (unstable Adams) spectral sequence based on certain non-connective periodic homology theories E such as complex periodic K-theory, and define an E-completion of a space X. For $X = S^{2n+1}$ and $E = K$ we calculate the E_2-term and show that the spectral sequence converges to the homotopy groups of the K-completion of the sphere. This also determines all of the homotopy groups of the (unstable) K-theory localization of S^{2n+1} including three divisible groups in negative stems.

1. Introduction

In [3] the first author with E. Curtis and H. Miller generalized the Bousfield-Kan construction of an unstable Adams spectral sequence based on homology with coefficients in a ring to generalized homology based on certain connective ring spectra E. In this paper we extend that generalization to the case of certain non-connective periodic ring spectra such as the K-theory spectrum.

In the case of K-theory, localized at an odd prime p, we calculate the E_2-term of the resulting spectral sequence for a sphere and show that it converges to a K-theory completion, whose simply connected cover is the unstable K-theory Bousfield localization. Of particular interest is the way in which the divisible groups in the negative stems arise in connection with higher derived functors of the primitive element functor, and also in connection with the failure of Bousfield localization to commute with loops and fiber sequences.

In [19] the second author with Mark Mahowald computed the mod p-homotopy groups of the Bousfield localization of an unstable sphere with respect to K, and also the integral homotopy groups in dimensions greater than or equal to the dimension of the sphere. As a corollary

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of the results of the present paper, we obtain the remaining homotopy groups in dimensions less than the dimension of the sphere.

We understand that many of the results here have been obtained independently by A. K. Bousfield using a somewhat different approach to the K-theory unstable Adams spectral sequence. The authors would like to extend our thanks to all the people who have helped us understand various aspects of this material. This includes, but is not limited to: Pete Bousfield, John Hunton, Hal Sadofsky, Mark Mahowald, Haynes Miller, Tom Hunter, Emmanuel Dror-Farjoun, and Roland Kargl.

2. The E-completion

We begin by briefly recalling the construction from [3] of an Unstable Adams Spectral Sequence (UASS) based on a generalized homology theory associated to a ring spectrum E. This construction works for any ring spectrum. See [5] for further details. This construction is a generalization of the one given by Bousfield and Kan in [12] so we will also call it a Bousfield-Kan spectral sequence (BKSS) and use these terms interchangeably.

For a ring spectrum, E, and a space X, define $E(X)$ to be the 0^{th}-space in the Ω-spectrum associated to $E \wedge \Sigma^\infty X$, i.e.

$$E(X) = \Omega^\infty(E \wedge \Sigma^\infty X)$$

For $i \geq 0$, $\pi_i(E(X)) \approx E_i(X)$ (we use $E_*(X)$ to denote reduced E-homology). We define the first derived functor, $D_1(X)$ of X to be the fibre of the Hurewicz map

$$\eta : X \to E(X)$$

In general $D_n(X)$ is the fibre of the map

$$D_{n-1}(\eta) : D_{n-1}(X) \to D_{n-1}(E(X)).$$

The UASS based on E-homology is the spectral sequence associated to the tower of fibrations:

$$
\begin{tikzcd}
\vdots \arrow[d] & \\
D_2(X) \arrow[r] \arrow[d] & D_n(E(X)) \\
D_1(X) \arrow[r] \arrow[d] & D_1(E(X)) \\
X \arrow[r] & E(X)
\end{tikzcd}
\tag{2.1}
$$

Specifically, $E_1^{s,t} = \pi_{t-s} D_s(E(X))$ for $t - s \geq 0$ and $E_1^{s,t} = 0$ for $t - s < 0$. In [3] it is shown that for any unital connective spectrum E which admits a unit preserving map $E \to H$, where H is the integral Eilenberg-Mac Lane spectrum, and any simply connected space X, the UASS based on E converges to the homotopy groups of X. It then follows by standard arguments that if E is any 'nice' ring spectrum, and X is a simply connected space, the UASS base on E converges to the unstable E-localization of X. See [9] for the definition and construction of unstable localization with respect to a homology theory. For example if we fix a prime p and let E be the BP spectrum for the prime p, then we obtain the Unstable Novikov Spectral Sequence (UNSS), which converges to the homotopy groups of X localized at p ([3]).

In order to approach the convergence question in greater generality, we will instead consider towers under X. Following Bousfield in the stable situation (see [10]) we will construct a tower under X for suitable X and E and define an E-completion of X to be the homotopy inverse limit of this tower. This E-completion will be the target of the spectral sequence. For those ring spectra for which this can be done, the spectral sequence will be the same as the one described above.

We begin with some general remarks concerning towers over versus towers under. Suppose we are given an arbitrary tower under X:

$$\begin{tikzcd} & X \arrow[d] \\ & \vdots \arrow[d] \\ F_2 \arrow[r] & X_2 \arrow[d] \\ F_1 \arrow[r] & X_1 \arrow[d] \\ F_0 \arrow[r, "="] & X_0 \end{tikzcd}$$

Define an associated homotopy spectral sequence with $E_1^{s,t} = \pi_{t-s} F_s$. If we suppose further that X is the homotopy inverse limit of the tower, then under suitable conditions we expect this spectral sequence to converge to the homotopy groups of X. For example, the following result is proved in [11, Ch IX,5.4].

Proposition 2.2. *Given a tower as above, where* $X = \operatorname{holim}_{\leftarrow} X_n$ *and* $i \geq 1$. *Suppose that*

$$\lim_{\substack{\leftarrow \\ r}}{}^1 E_r^{s,s+i} = 0 = \lim_{\substack{\leftarrow \\ r}}{}^1 E_r^{s,s+i+1}$$

for all $s \geq 0$. *Then* $\{E_r\}$ *converges completely to* $\pi_i X$.

Remark 2.3. *The hypothesis of Proposition 2.2 is automatically satisfied if the spectral sequence has a* horizontal vanishing line, *i.e. if there exists* (N, r) *such that* $E_r^{s,t} = 0 \quad \forall \quad s \geq N$.

Now suppose we have an arbitrary tower over X:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X^2 \arrow[r] \arrow[d] & D^2 \\
 & X^1 \arrow[r] \arrow[d] & D^1 \\
X \arrow[r, "="] & X^0 \arrow[r] & D^0
\end{tikzcd}$$

This yields a spectral sequence with $E_1^{s,t} = \pi_{t-s} D^s$ and under suitable conditions one hopes that this spectral sequence converges to $\pi_* X$.

Suppose $\{X_s\}$ is a given tower under X. We can construct a tower over X by defining X^{s+1} to be the homotopy fiber of the map $X \to X_s$. The diagram

$$\begin{tikzcd}
X^{s+1} \arrow[r] \arrow[d, "="'] & X^s \arrow[r] \arrow[d] & F_s = D_s \arrow[d] \\
X^{s+1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X_s \arrow[d] \\
* \arrow[r] & X_{s-1} \arrow[r, "="] & X_{s-1}
\end{tikzcd}$$

shows that the two spectral sequences are the same.

Stably, as in [10], one can go back and forth between the two notions since fibre sequences and cofibre sequences are the same thing. However unstably, given a tower over X, the most we can say in general is that, by taking fibres, one gets a tower under ΩX. In particular it is not clear that the tower over X given by (2.1) comes from an associated tower under X.

In [11] it is shown that the tower in (2.1) comes from an associated tower under X in the case where E is ordinary homology with coefficients in a ring R. In this case the functor $E(X)$ can be represented by a functor on the category of topological spaces, and not merely on the associated homotopy category. The multiplication in R is used to show that $E(X)$ is the functor of a triple. See Section 4 for the definition of a triple. There is an associated cosimplicial space $\mathbf{E}X$, and the tower results by considering a filtration on the total space of the cosimplicial space $\mathbf{E}X$. The resulting spectral sequence is the Bousfield-Kan Spectral Sequence (BKSS) based on R-homology (see [11, Ch X]).

In order to define the total space of a cosimplicial space and its associated filtration (see [11, Ch X,3.2]), it is crucial that $\mathbf{E}X$ be a cosimplicial object in the category of topological spaces, and not merely a cosimplicial object in the homotopy category of topological spaces. For an arbitrary ring spectrum E, the most we can say is that the multiplication on E makes $E(X)$ into the functor of a triple on the homotopy category of spaces. We are thus naturally led to the notion of an S-algebra as studied in [13].

Recall that in [13] it is shown that one can work stably in a category of spectra with concrete point-set topological models representing the objects, *before* passing to the associated stable homotopy category. Thus in this setting it makes sense to talk about strictly commuting diagrams of spectra, and not just diagrams of spectra commuting up to homotopy. Furthermore, it is shown that this stable homotopy category is equivalent to the stable homotopy category associated to the category $\mathcal{M}_s$ of S-modules. The latter is a category of spectra which possesses a strictly commutative, associated and unital smash product. Within $\mathcal{M}_s$ is the subcategory of S-algebras, which are spectra with a strictly associative, unital multiplication. In older parlance, an S-algebra is an E_∞-ring spectrum with a strict unit.

In May's stable category, every spectrum is an "Ω spectrum", and Ω^∞ is a functor from the category of spectra to the category of spaces (no homotopies involved).

The following proposition now follows from a straightforward check of the definitions.

Proposition 2.4. *Let E be represented as an S-algebra. Then $E(X) = \Omega^\infty(E \wedge \Sigma^\infty X)$ is the functor of a triple on the category of topological spaces.*

This immediately leads to the following definition. Compare this with Proposition 5.5 of [10].

Definition 2.5. *Let E be represented as an S-algebra. Let X be a space. By Proposition 2.4 there is a cosimplicial space $\mathbf{E}X$. Define the unstable E-completion of X to be $Tot_\infty \mathbf{E}X$, which we denote by $E\hat{}X$. As in* [11]*, one can also define a tower $Tot_s \mathbf{E}X$, and we have*

$$E\hat{}X = \underset{\leftarrow}{\text{holim}}\ Tot_s \mathbf{E}X.$$

Using the same argument as in [11], one sees that the resulting spectral sequence agrees with that of (2.1).

Classical Thom spectra such as MU are representable as S-algebras [20]. In [16] it is shown that BP is an S-algebra. May and McClure have shown that K is an S-algebra. It would be interesting to know if the Johnson-Wilson spectra $E(n)$ are S-algebras for all n.

3. K-theory and the sphere

In what follows, everything is understood to be localized at a fixed odd prime p. We now describe our main example, in which $E = E(1)$, the summand of p-local complex periodic K-theory, and $X = S^{2n+1}$. As usual, $q = 2(p-1)$. The coefficient ring is given by $E(1)_* = Z_{(p)}[v_1, v_1^{-1}]$, where $|v_1| = q$. For each integer k, Let $\nu(k)$ stand for the exponent in the highest power of p that divides k. Thus $k = ap^{\nu(k)}$ where $p \nmid a$. First recall that the E_2-term of the $E(1)$-based stable ASS for the sphere spectrum is given by

$$E_2^{s,t} = \text{Ext}^{s,t}_{E(1)_*E(1)}(E(1)_*, E(1)_*)$$

and can be computed using calculations from [21]: there is a cyclic group of order $p^{\nu(k)+1}$ in $E_2^{1,qk}(S)$ for all integers k except $k = 0$. There is a $Q/Z_{(p)}$ in $E_2^{2,0}(S)$ and a $Z_{(p)}$ in $E_2^{0,0}(S)$. All other groups are zero. See also Theorem 8.10 and its proof in [25].

The following theorem describes the unstable E_2-term for each odd dimensional sphere as well as the double suspension homomorphisms $E_2^{s,t}(S^{2n-1}) \to E_2^{s,t}(S^{2n+1})$.

Theorem 3.1. (i) *For each $n \geq 1$, $t-s \geq 1$, the E_2-term of the $E(1)$-BKSS for $X = S^{2n+1}$ is given by*

$$E_2^{s,t}(S^{2n+1}) = \begin{cases} 0 & \text{if } s \geq 4, \\ Q/Z_{(p)} & \text{if } s=3,\ t-s=2n-2, \\ Q/Z_{(p)} \oplus Q/Z_{(p)} & \text{if } s=2,\ t-s=2n-1, \\ Z/p^{\min(\nu(k)+1,n)} & \text{if } s=2,\ t-s=2n+qk-1,\ k>0, \\ Z/p^{\min(\nu(k)+1,n+k(p-1))} & \text{if } s=2,\ t-s=2n+qk-1,\ k<0, \\ Z/p^{\min(\nu(k)+1,n)} & \text{if } s=1,\ t-s=2n+qk,\ k>0, \\ Z/p^{\min(\nu(k)+1,n+k(p-1))} & \text{if } s=1,\ t-s=2n+qk,\ k<0, \\ Z_{(p)} & \text{if } s=0,\ t-s=2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *In filtration $s=1$ (in the $qk-1$ stem) the double suspension is an injection on the groups of order less than the maximal order of $p^{\nu(k)+1}$, and is an isomorphism on the groups of maximal order.*
(iii) *In filtration $s=2$ (in the $qk-2$ stem) the double suspension is an injection on the groups of order less than the maximal order of $p^{\nu(k)+1}$, and is multiplication by p on the groups of maximal order.*
(iv) *In the -2 stem, the stable $Q/Z_{(p)}$ desuspends to S^3, and double suspension is multiplication by p on the unstable $Q/Z_{(p)}$. It is multiplication by p on the unstable $Q/Z_{(p)}$ in the -3 stem.*

This will be proved in the subsequent sections of the paper. It is immediate that the spectral sequence collapses for dimensional reasons. The fact that the E_2-term has a horizontal vanishing line allows us to deduce the following convergence fact.

Theorem 3.2. *The spectral sequence described above converges to the homotopy groups of $E(1)\hat{\ }S^{2n+1}$, and these homotopy groups, in positive dimensions, are given by the E_2-term.*

The first step in the proof of Theorem 3.1 is the calculation of the mod p E_2-term. This result is an unstable analogue of the Morava change of rings theorem.

Theorem 3.3 (unstable change of rings). *For each $n \geq 1$, $t-s \geq 1$, we have*

$$E_2^{s,t}(S^{2n+1}; Z/p) = \begin{cases} 0 & \text{if } s \geq 3, \\ Z/p & \text{if } s = 2,\ t-s = 2n+qk-1, \\ Z/p \oplus Z/p & \text{if } s = 1,\ t-s = 2n+qk, \\ Z/p & \text{if } s = 0,\ t-s = 2n+1+qk, \\ 0 & \text{otherwise.} \end{cases}$$

This will be proved in Section 5. Since the unstable change of rings theorem yields a horizontal vanishing line, this is all that is needed to prove Theorem 3.2.

Proof of Theorem 3.2. Since $E(1)$ is representable as an S-algebra, Definition 2.5 applies and $E(1)\hat{\ }S^{2n+1}$ occurs as the homotopy inverse limit of the tower. By the Bockstein sequence, the horizontal vanishing line of the E_2-term mod p implies a horizontal vanishing line of the E_2-term. Applying Remark 2.3 to this situation yields the theorem. □

Recall that in [19], the second author and M. Mahowald studied the Bousfield localization of S^{2n+1} with respect to $E(1)$, which we denote by L_1S^{2n+1}. The homotopy groups of L_1S^{2n+1} were computed mod p, but integrally only the homotopy groups in non-negative stems were computed. The remaining cases in negative stems is resolved by the present work.

Theorem 3.4. *L_1S^{2n+1} simply connected, and $\pi_i(L_1S^{2n+1})$, for $i \geq 2$, is given by Theorem 3.1.*

Proof. That L_1S^{2n+1} is simply connected follows from Corollary 2.8 of [23]. Since $E(1)\hat{\ }S^{2n+1}$, by construction, is $E(1)$-local, there is an induced map $L_1S^{2n+1} \to E(1)\hat{\ }S^{2n+1}$. By Theorem 3.2 above and Theorem 1.2 of [19], the mod p homotopy groups of the source and target are the same in dimensions greater than or equal to 2. By considering positive stems only, and using v_1-periodicity, it is easy to check that the map induces an isomorphism in mod p-homotopy. The map is clearly a rational equivalence, and the result follows. □

4. Description of the E_2-term

In this section we will recall from [3] the cosimplicial description of the E_2-term of the BKSS based on E, and the relevant homological algebra of non-additive functors needed to compute this E_2-term.

First we recall the conditions on E which must hold in order for the techniques of [3] to work.

As usual, the Hopf algebroid (E_*, E_*E) will be denoted by (A, Γ). In what follows we will assume that E is a ring spectrum which is represented as an Ω-spectrum $\{\mathbf{E}_k\}$ satisfying the following conditions:

Hypothesis 4.1. (i) *E is a homotopy associative, homotopy commutative, CW ring spectrum with a unit.*

(ii) *For each $k \geq 0$, $E_*(\mathbf{E}_k)$ is free as an A-module.*

(iii) *Let $PE_*(\mathbf{E}_k)$ denote the primitives in the coalgebra $E_*(\mathbf{E}_k)$. Then $PE_*(\mathbf{E}_k)$ is A-free and the composite*

$$PE_*(\mathbf{E}_k) \to E_*(\mathbf{E}_k) \xrightarrow{\sigma_*} \Gamma$$

is injective, where σ_* is the homology suspension.

We do not assume connectivity of the spectrum E as in [3]. In that paper connectivity is used primarily to deduce a convergence theorem, and we are approaching the convergence issue in a totally different way. There is another condition (assumption 7.7) from [3] which we do not assume: we do not require that $E_*(\mathbf{E}_k)$ be the cofree coalgebra on $PE_*(\mathbf{E}_k)$. We discuss this in greater detail later in this section and in Section 8.

The above conditions hold of course for ordinary mod p homology. As stated in [3], the work of Wilson [26] shows that they hold for BP. In [15], Hopkins and Hunton compute the Hopf ring of a spectrum representing a Landweber exact cohomology theory. Their result implies that Hypothesis 4.1 is satisfied by any complex oriented ring spectrum which represents a Landweber exact multiplicative cohomology theory such that the coefficients are concentrated in even dimensions and are a free R-module of countable rank for some subring $R \subset Q$. This includes for example BP, K, elliptic cohomology Ell, and the Johnson-Wilson spectra $E(n)$.

We turn now to the cosimplicial description of the E_2-term of the E-based BKSS.

Let $\mathcal{C}$ be an arbitrary category and let $T : \mathcal{C} \to \mathcal{A}$ be a functor to an abelian category. Let $G : \mathcal{C} \to \mathcal{C}$ be the functor of a *triple* on $\mathcal{C}$. This means that there are natural transformations $\mu : G^2 \to G$ and $\eta : 1 \to G$ such that the following diagrams commute:

$$\begin{tikzcd} G \arrow[r, "G\eta"] \arrow[dr, "="'] & G^2 \arrow[d, "\mu"] & G \arrow[l, "\eta G"'] \arrow[dl, "="] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} G^3 \arrow[r, "G\mu"] \arrow[d, "\mu G"'] & G^2 \arrow[d, "\mu"] \\ G^2 \arrow[r, "\mu"] & G \end{tikzcd}$$

We can define the derived functors of T with respect to G, or more simply, the G-derived functors of T, as follows: the triple G induces a functor from $\mathcal{C}$ to the category of cosimplicial objects over $\mathcal{C}$. For each $C \in \mathcal{C}$, $\mathbf{G}C$ is the cosimplicial object over $\mathcal{C}$ defined by $\mathbf{G}C^n = G^{n+1}(C)$. We have a diagram

$$\mathbf{G}C^0 \quad \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \end{array} \quad \mathbf{G}C^1 \quad \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^1} \\ \xrightarrow{d^2} \end{array} \quad \cdots \tag{4.2}$$

with maps defined by

$$\begin{aligned} d^i &= G^i \eta G^{n-i} : G^n(C) \to G^{n+1}(C), \quad 0 \le i \le n \\ s^i &= G^i \mu G^{n-i} : G^{n+2}(C) \to G^{n+1}(C), \quad 0 \le i \le n \end{aligned}$$

and satisfying the cosimplicial identities

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, & i < j, \\ s^j d^i &= d^i s^{j-1}, & i < j, \\ &= id, & i = j, j+1, \\ &= d^{i-1} s^j, & i > j+1, \\ s^j s^i &= s^{i-1} s^j, & i > j. \end{aligned}$$

Applying the functor T to diagram 4.2 yields a cosimplicial object $T\mathbf{G}C$ over $\mathcal{A}$. If $\mathbf{X}$ is any cosimplicial object over the abelian category $\mathcal{A}$, we define a chain complex $\mathrm{ch}X$ by $\mathrm{ch}X^n = X^n$ and $\delta = \sum(-1)^i d^i : \mathrm{ch}X^n \to \mathrm{ch}X^{n+1}$. Then the right G-derived functors of T applied to the object C are defined by

$$R^i_G T(C) = H^i(\mathrm{ch}T\mathbf{G}C).$$

In our situation the triple we use arises as the adjoint of a cotriple: A *cotriple* on a category $\mathcal{C}$ is a functor $G : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\delta : G \to G^2$ and $\epsilon : G \to 1$ such that the following diagrams commute:

$$\begin{tikzcd}
G & G^2 \arrow[l, "G\epsilon"'] \arrow[r, "\epsilon G"] & G & G^3 & G^2 \arrow[l, "G\delta"'] \\
& G \arrow[ul, "="] \arrow[u, "\delta"'] \arrow[ur, "="'] & & G^2 \arrow[u, "\delta G"] & G \arrow[l, "\delta"'] \arrow[u, "\delta"']
\end{tikzcd}$$

Given a cotriple G, a *G-coalgebra* is defined to be an object $C \in \mathcal{C}$ with a map $\psi : C \to GC$ making these diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "\psi"] & GC \arrow[d, "\epsilon"] & G^2C & GC \arrow[l, "\delta"'] \\
& C \arrow[ul, "="] & GC \arrow[u, "G\psi"] & C \arrow[l, "\psi"'] \arrow[u, "\psi"']
\end{tikzcd}$$

A map of G-coalgebras is a map $f : C \to D$ such that

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\psi"'] & D \arrow[d, "\psi"] \\
GC \arrow[r, "Gf"] & GD
\end{tikzcd}$$

commutes. Denote the category of G-coalgebras by $\mathcal{C}(G)$. It can be readily checked from the definitions that for any $C \in \mathcal{C}$, GC is naturally a G-coalgebra, with $\psi_{GC} = \delta_C$. Thus we have functors $G : \mathcal{C} \to \mathcal{C}(G)$ and $J : \mathcal{C}(G) \to \mathcal{C}$ with J being the forgetful functor. The functors G and J are adjoint, $JG : \mathcal{C} \to \mathcal{C}$ is the functor of a cotriple (the original cotriple G), and $GJ : \mathcal{C}(G) \to \mathcal{C}(G)$ is the functor of a triple. By abuse of notation, GJ will also be denoted simply by G. In case we have $T : \mathcal{C}(G) \to \mathcal{A}$, we sometimes write $R^*_G T$ as $R^*_{\mathcal{C}(G)}$.

Now we apply the above abstractions to the case at hand. Assume E satisfies Hypothesis 4.1. Let $\mathcal{M}$ denote the category whose objects are graded modules over $A = E_*$ which are free over A. Define a functor $G : \mathcal{M} \to \mathcal{M}$ as follows: for $M \in \mathcal{M}$, let F be the spectrum such that $\pi_*(F) = M$. Then let $G(M)$ be defined to be $E_*(\Omega^\infty(F))$. By the same proof as in [3], G is the functor of a cotriple on $\mathcal{M}$. As above, $\mathcal{M}(G)$ denotes the category of G-coalgebras. As in [3], for any space X such that E_*X is A-free, $E_*X \in \mathcal{M}(G)$.

Let $A[t]$ denote $E_*S^t \in \mathcal{M}(G)$. Then $\mathrm{Hom}_{\mathcal{M}(G)}(A[t], M)$ defines a functor from $\mathcal{M}(G)$ to abelian groups. We let $\mathrm{Ext}_{\mathcal{M}(G)}(A[t], M)$ denote the G-derived functors of $\mathrm{Hom}_{\mathcal{M}(G)}(A[t], M)$. For a space X, we note that the cosimplicial G-resolution 4.2 is obtained by applying E_* to the cosimplicial space $\mathbf{E}(X)$ of section 2.

Theorem 4.3. *Suppose X is CW-complex for which $E_*X \in \mathcal{M}$, where E satisfies Hypothesis 4.1. Then*

$$E_2^{s,t}(X) = \mathrm{Ext}^s_{\mathcal{M}(G)}(A[t], E_*X) \quad \textit{for} \quad t - s > 0$$

The proof is the same as in [3]. The only difference is that the E_2-term of the spectral sequence is zero for $t - s < 0$ since spaces have no negative dimensional homotopy groups, whereas the Ext groups defined above are possibly nonzero for $t - s < 0$. It is not clear what relevance, if any, these negative dimensional Ext groups have to the spectral sequence. There is an isomorphism between the E_1-term of the spectral sequence and the chain complex whose homology is the Ext groups for $t - s \geq 0$, and this induces an isomorphism in homology for $t - s > 0$. The E_2 of the spectral sequence for $t - s = 0$ could be bigger than the Ext group in this dimension.

The category $\mathcal{M}(G)$ is somewhat mysterious. We can study it by considering functors from $\mathcal{M}(G)$ to more concrete categories, such as the category of unstable Γ-comodules, which we now define.

As in [3], an object $M \in \mathcal{M}(G)$ is a coalgebra over the graded ground ring A, and we let PM denote the module of primitive elements. Let U be the endofunctor on $\mathcal{M}$ defined by $U = PG$. The Hypotheses of 4.1 imply that U is the functor of a cotriple on $\mathcal{M}$. If we let $\mathcal{A}$ denote the category of all graded A modules, not necessarily free, then U can be extended to a functor (of a cotriple) on all of $\mathcal{A}$ as follows. For $M \in \mathcal{A}$, let

$$F_1 \xrightarrow{d} F_0 \to M \to 0$$

be a resolution with each F_i a free module. Define

$$U(M) = \mathrm{coker}(U(d) : U(F_1) \to U(F_0)).$$

From this we get $\mathcal{A}(U)$, the category of U-coalgebras. A module with such a U-structure will be called an unstable Γ-comodules.

If M happens to be a Γ-comodule which is free as an A-module then an unstable Γ-comodule structure can be described as follows. M has a comodule structure map $\psi : M \to \Gamma \otimes_A M$. There is an increasing filtration on Γ defined by $F_k\Gamma = \sigma_*(PE_*(E_k))$ and $\Gamma = \bigcup F_k\Gamma$. Then M is an unstable Γ-comodule if $\psi(m) = \Sigma\gamma_i \otimes m_i$ implies $\gamma_i \in F_{|m_i|}\Gamma$

for each i. There is an injection $U(M) \to \Gamma \otimes_A M$ and a factorization

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr] & \Gamma \otimes_A M \\ & U(M) \arrow[u] \end{tikzcd}$$

If M has torsion, then the map $U(M) \to \Gamma \otimes_A M$ is not injective, and one cannot describe an unstable Γ-comodule structure in terms of a factorization of a stable Γ-comodule structure.

If $M \in \mathcal{A}(U)$, U defines a cosimplicial A-module $\mathbf{U}(M)$ and we can apply $\mathrm{Hom}_{\mathcal{A}(U)}$ to get

$$C^{s,t}(M) = \mathrm{Hom}_{\mathcal{A}(U)}(A[t], \mathbf{U}(M)^s),$$

the *unstable cobar complex* for M. Recalling the adjointness isomorphism

$$\mathrm{Hom}_{\mathcal{A}(U)}(A[t], U(M)) = \mathrm{Hom}_{\mathcal{A}}(A[t], M)$$

we get

$$C^{s,t}(M) = U^s(M)_t.$$

If M is free over A, then this is a sub-chain complex of the stable cobar complex for M over (A, Γ). For example, the stable cobar complex for S^0 is filtered by the sub-complexes $C^{*,*}(S^n)$.

In many cases the E_2-term of the spectral sequence can be computed as the homology of an unstable cobar complex. We will say more about this shortly, but first we give an explicit description of the unstable cobar complex for many Landweber exact cohomology theories.

Now fix a prime p. In the case of $E = BP$, the Hopf algebroid (A, Γ) is given by

$$BP_* = Z_{(p)}[v_1, v_2, ...],$$

where the v_i are the Hazewinkel generators, and

$$\Gamma = BP_*[h_1, h_2, ...],$$

where $h_i = c(t_i)$, c the canonical anti-isomorphism of Γ. We have $|v_i| = |h_i| = 2(p^i - 1)$. There are right and left unit maps

$$\eta_R : A \to \Gamma \qquad \text{and} \qquad \eta_R : A \to \Gamma.$$

In [17] Landweber gives a criterion on a p-local spectrum E which implies that E-homology is obtained from BP-homology according to the formula

$$E_*(X) = E_* \otimes_{BP_*} BP_*(X).$$

This implies

$$E_*(E) = E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_*$$

where BP_* acts on the right on E_* in the first tensor product and on the left in the second.

An example of such a Landweber exact homology theory is $E(n)$. The spectrum $E(n)$ is obtained from BP by killing off v_i, $i > n$, and inverting v_n, both on the right and the left. Specifically,

$$A = E(n)_* = Z_{(p)}[v_1, v_2, ...v_n, v_n^{-1}]$$

and (see for example [25])

$$\Gamma = E(n)_*(E(n)) \cong E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*$$

The spectrum $E(1)$ is a summand of p-local K-theory.

For $E = BP$, the work of Ravenel and Wilson ([24]) allows us to describe the functor U. In [24] it is shown that the best way to understand $BP_*(\mathbf{BP}_k)$ for all k is to consider the global bigraded object $BP_*(\mathbf{BP}_*)$. This is a sequence of Hopf algebras $BP_*(\mathbf{BP}_k)$ together with a pairing (called the circle product)

$$\circ : BP_*(\mathbf{BP}_k) \otimes BP_*(\mathbf{BP}_j) \to BP_*(\mathbf{BP}_{k+j})$$

which makes $BP_*(\mathbf{BP}_*)$ into a graded ring object in the category of graded coalgebras. Such a thing is called a Hopf ring. The Hopf ring $BP_*(\mathbf{BP}_*)$ is computed in [24] and this is used in [3] to describe U.

In [15] the Hopf ring $E_*(\mathbf{E}_*)$ is computed for any p-local complex oriented ring spectrum which represents a Landweber exact multiplicative cohomology theory, such that the coefficients E_* are concentrated in even dimensions and are free of countable rank over some subring R of Q. The result is that

$$E_*(\mathbf{E}_*) = E_* \otimes_{BP_*} BP_*(\mathbf{BP}_*) \otimes_{BP_*[BP^*]} BP_*[E^*]. \tag{4.4}$$

Here $F_*[G^*]$ denotes the sub-Hopf ring of $F_*(\mathbf{G}_*)$ obtained by applying F_* to elements in $\pi_*G = G^{-*}$.

This gives the following result. Let $I = (i_1, i_2, ...)$ be a sequence of non-negative integers. Let h^I denote $h_1^{i_1} h_2^{i_2} ....$

Proposition 4.5. (i) *Let E satisfy 4.4. If M is a free left A-module, then $U(M)$ is the A-span of*

$$\{h^I \otimes m \mid 2(i_1 + i_2 + ...) < |m|\} \subset \Gamma \otimes_A M.$$

(ii) *Suppose M is an unstable Γ-comodule, free as an A-module, with coaction $\psi : M \to U(M)$. Then the unstable cobar complex is the chain complex $C^{s,t}(M) = U^s(M)_t$ with differential given by*

$$\begin{aligned} d([\gamma_1 \mid \gamma_2 \mid ... \mid \gamma_s]m) =& [1 \mid \gamma_1 \mid ... \mid \gamma_s]m \\ &+ \sum_{j=1}^{s} (-1)^j [\gamma_1 \mid ... \mid \gamma_j' \mid \gamma_j'' \mid ... \mid \gamma_s]m \\ &+ (-1)^{s+1} \sum [\gamma_1 \mid ... \mid \gamma_s \mid \gamma']m'', \end{aligned}$$

where $\gamma_j \in \Gamma$, $\psi(\gamma_j) = \sum \gamma_j' \otimes \gamma_j''$ and $\psi(m) = \sum \gamma' \otimes m''$.

In what follows we will also need a functor $V : \mathcal{M} \to \mathcal{M}$ which is defined by $V = QG$ where Q is the quotient module of indecomposables. Like U, the functor V can be extended to a functor on all of $\mathcal{A}$. From [24] and [15] we get

Proposition 4.6. *Let E satisfy 4.4. If M is a free left A-module, then $V(M)$ is the A-span of*

$$\{h^I \otimes m \mid 2(i_1 + i_2 + ...) \leq |m|\} \subset \Gamma \otimes_A M.$$

For a general M let

$$F_0 \to F_1 \xrightarrow{d} M \to 0$$

be a resolution with each F_i a free A-module. Then $V(M) = \operatorname{coker}(V(d))$.

Notice that

$$U(M(2n+1)) = V(M(2n+1)) = V(M(2n)). \tag{4.7}$$

At this point we will depart somewhat from the development in [3]. There it was shown that there is a composite functor spectral sequence

$$E_2^{r,s} = \operatorname{Ext}^r_{\mathcal{A}(U)}(A[t], R^s_G PM) \Longrightarrow \operatorname{Ext}^{r+s}_{\mathcal{M}(G)}(A[t], M). \tag{4.8}$$

Furthermore, it is assumed in [3] that $E_*(\mathbf{E}_k)$ is a cofree coalgebra (assumption 7.7), from which it follows that the G-derived functors of P are the same as the S-derived functors of P, where S is the cofree cocommutative coassociative coalgebra functor. In the case of $M = M(2n+1) = E_*(S^{2n+1})$ for example the S-derived functors of P obviously vanish for $s > 0$ and the composite functor spectral sequence collapses. One then concludes that the E_2-term of the E-based BKSS for $X = S^{2n+1}$ is given by the homology of the unstable cobar complex.

This argument works for BP, but for spectra like $E(n)$ the above hypothesis on $E_*(\mathbf{E}_k)$ doesn't hold. The E_2-term for a sphere still turns out to be the homology of the unstable cobar complex but we have to prove this another way.

Theorem 4.9. *Let E be a Landweber exact ring spectrum satisfying 4.4. Let $M = M(2n_1+1, 2n_2+1, \dots, 2n_k+1)$ be a free A-module on a sequence of odd dimensional generators $\{x_{2n_1+1}, \dots, x_{2n_k+1}\}$. Let X be an H-space such that $E_*(X) \cong A[x_{2n_1+1}, \dots, x_{2n_k+1}]$ the free commutative algebra. Then*

$$\mathrm{Ext}^s_{\mathcal{M}(G)}(A[t], E_*(X)) = \mathrm{Ext}^s_{\mathcal{A}(U)}(A[t], M).$$

For example we could have $X = S^{2n+1}$ and $M = M(2n+1)$. Before proving this we need one more bit of machinery (see [8] and [4]). Returning to the general situation of a cotriple G on a category $\mathcal{C}$, the cosimplicial object $\mathbf{G}C$, defined for $C \in \mathcal{M}(G)$, is a large canonical resolution which is used to define the G derived functors of a functor T from $\mathcal{M}$ to an abelian category. We need to know that derived functors can be computed using more general resolutions. For simplicity, we assume that $\mathcal{C}$ is an additive category like $\mathcal{M}$, the category of graded A-modules.

Definition 4.10. *Consider the adjoint functors $G : \mathcal{C} \to \mathcal{C}(G)$ and $J : \mathcal{C}(G) \to \mathcal{C}$. Define the models in $\mathcal{C}(G)$ to be the objects of the form GC, for $C \in \mathcal{C}$. For $X \in \mathcal{C}(G)$, a cosimplicial resolution of X by models is an augmented cosimplicial object $\mathbf{Y}$ over $\mathcal{C}(G)$ such that*

(i) $\mathbf{Y}^{-1} = X$.
(ii) *For each $q \geq 0$, $\mathbf{Y}^q$ is a model $\mathcal{C}(G)$.*
(iii) *For each model M in $\mathcal{C}(G)$, the chain complex*

$$\mathrm{Hom}_{\mathcal{C}(G)}(ch(\mathbf{Y}), M)$$

is acyclic.

Note that by adjointness of J and G, the last condition above can be replaced by the condition that the cochain complex $\mathrm{ch}(J\mathbf{Y})$ be acyclic.

If we let $\tilde{\mathbf{Y}}$ denote $\mathbf{Y}$ without the augmentation then we have

Theorem 4.11.

$$R^s_G T(X) = H^s(ch(T\tilde{\mathbf{Y}}))$$

This is proved in [8].

Proof of 4.9. This argument closely resembles the proof of Theorem 6.1 of [4]. Recall that $M = M(2n_1+1, \dots, 2n_k+1)$. Construct a cosimplicial object $\mathbf{Y}^q = G(V^q(M))$. Define an augmentation

$$A[x_{2n_1+1}, \dots, x_{2n_k+1}] = \mathbf{Y}^{-1} \to \mathbf{Y}^0 = G(M)$$

induced by η_M. Then $\mathbf{Y}$ is an augmented cosimplicial object over $\mathcal{M}(G)$. Each $\mathbf{Y}^q$, for $q \geq 0$ is a model, by definition. To see that $\mathbf{Y}$ is a cosimplicial resolution of M, filter each $\mathbf{Y}^q$ by powers of the augmentation ideal. Let S' stand for the free associative graded commutative algebra functor. By [24] and [15] $E_*(\mathbf{E}'_k)$ is a free graded commutative algebra and so there are natural isomorphisms

$$\begin{aligned} E_0(\mathbf{Y}^q) &\cong S'(Q(\mathbf{Y}^q)) \\ &\cong S'(Q(G(V^q(M)))) \\ &\cong S'(V^{q+1}(M)) \\ &\cong S'(\mathbf{V}(M)). \end{aligned}$$

Since S' applied to an acyclic complex is acyclic, $E_0(\mathbf{Y})$ is acyclic, and so $\mathbf{Y}$ is acyclic. Thus

$$\begin{aligned} \mathrm{Ext}_{\mathcal{M}(G)}(A[t], A[x_{2n_1+1}, \ldots, x_{2n_k+1}]) &\cong H^*(\mathrm{Hom}_{\mathcal{M}(G)}(A[t], \mathbf{Y})) \\ &\cong H^*(\mathrm{Hom}_{\mathcal{M}(G)}(A[t], GV^q(M))) \\ &\cong H^*(\mathrm{Hom}_{\mathcal{M}}(A[t], V^q(M))) \\ &\cong H^*(\mathrm{Hom}_{\mathcal{M}}(A[t], U^q(M))) \qquad \text{by 4.7} \end{aligned}$$

and this last is the homology of the unstable cobar complex. □

Next we observe that the analogue of Theorem 6.1 of [4] holds in our setting.

Theorem 4.12. *Let E satisfy the same hypotheses as Theorem 4.9. Then*

$$E_2^{s,t-1}(\Omega S^{2n+1}) \cong E_2^{s,t}(S^{2n+1}).$$

Proof. The proof is the same as the BP case [4] since this only uses the fact that $E_*(\mathbf{E_n})$ is a free algebra, which is true in the current setting, and does not use the coalgebra structure of $E_*(\mathbf{E_n})$ (which is cofree for BP but not for $E(n)$). □

Corollary 4.13. *K-theory completion commutes with loops for $X = S^{2n+1}$.*

Finally we note a result concerning fiber sequences which is a corollary of the preceding results.

Theorem 4.14. *Let E be as above and let*

$$F \to X \to B$$

be a fiber sequence such that

(i) *Each of the three spaces satisfies*

$$\mathrm{Ext}^s_{\mathcal{M}(G)}(A[t], E_*(Y)) = \mathrm{Ext}^s_{\mathcal{A}(U)}(A[t], PE_*(Y)).$$

(ii) *Two of out the three spaces has a horizontal vanishing line in the E-based BKSS.*

(iii) *There is a short exact sequence*

$$0 \to PE_*(F) \to PE_*(X) \to PE_*(B) \to 0.$$

Then the spectral sequence converges to the completion for all three spaces and there is a fiber sequence

$$E\hat{}F \to E\hat{}X \to E\hat{}B.$$

Example 4.15. *Consider the fiber sequence*

$$SU(n) \to SU(n+1) \to S^{2n+1}.$$

Then the hypothesis of Theorem 4.9 is satisfied for each of the three spaces, hence by induction Theorem 4.14 holds for this example.

5. The unstable change of rings

We now turn to the proof of Theorem 3.1. As a first step we compute the mod p E_2-term of the $E(1)$ based BKSS for S^{2n+1}. Recall from [21] that $\alpha_{k/n} = d(v_1^k)/p^n$ is an integral cycle in the stable cobar complex in filtration $s = 1$, and by [3] it desuspends to S^{2n+1}.

The following result gives the homology of the $E(1)$-unstable cobar complex for S^{2n+1} tensored with Z/p, including a specific basis.

Theorem 5.1. (i) *For each $n \geq 1$, $t - s \geq 1$, we have*

$$E_2^{s,t}(S^{2n+1}; Z/p) = \begin{cases} 0 & \text{if } s \geq 3, \\ Z/p & \text{if } s = 2,\ t-s = 2n+qk-1, \\ Z/p \oplus Z/p & \text{if } s = 1,\ t-s = 2n+qk, \\ Z/p & \text{if } s = 0,\ t-s = 2n+1+qk, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *A vector space basis over $Z/p[v_1, v_1^{-1}]$ for the elements in filtration $s = 0, 1$ is given by $\{v_1^k,\ v_1^{k-1}h_1,\ v_1^{k-n-1}ph_1^{n+1}\}$*

(iii) *To write down a basis for filtration $s = 2$ proceed as follows: Let $t \geq n-1$ (so that $\alpha_{p^t/n}$ is just born on S^{2n+1}). Then*

$$v_1^{k-p^t-1}\alpha_1 \otimes \alpha_{p^t/n}$$

is a mod p cycle in filtration 2 which is nonzero in $E_2(S^{2n+1}; Z/p)$.

The basic idea of the proof is to use v_1-periodicity to reduce the calculation to a range in which it agrees with the v_1-periodic unstable Novikov spectral sequence studied in [7]. We set some notation and terminology. For a free BP_* module M let:

- $(A, \Gamma) = (BP_*, BP_*BP)$
- $\tilde{\Gamma} = \mathrm{Ker}(\epsilon : \Gamma \to A)$
- $U_\Gamma(M)$ is the A-span of $\{h^I \otimes_A m |\ 2(i_1 + i_2 \cdots) < |m|\} \subset \Gamma \otimes_A M$
- $\tilde{U}_\Gamma(M) = U_\Gamma(M) \cap (\tilde{\Gamma} \otimes_A M)$
- $(B, \Sigma) = (E(1)_*, E(1)_* \otimes_A \Gamma \otimes_A E(1)_*)$
- $\tilde{\Sigma} = E(1)_* \otimes_A \tilde{\Gamma} \otimes_A E(1)_*$
- $\bar{M} = B \otimes_A M$
- $U_\Sigma(\bar{M})$ is the B span of
$$\begin{aligned}\{h^I \otimes_B \bar{m} |\ 2(i_1 + i_2 \cdots) < |\bar{m}|\} &\subset \Sigma \otimes_B \bar{M} \\ &= B \otimes_A \Gamma \otimes_A \bar{M}\end{aligned}$$
- $\tilde{U}_\Sigma(\bar{M}) = U_\Sigma(\bar{M}) \cap (\tilde{\Sigma} \otimes_B \bar{M})$

Now let $M = M(2n+1) = BP_*(S^{2n+1})$, $n \geq 0$. From Section 4 we know that $U_\Gamma(M)$ is the module of primitives in $BP_*(\mathbf{BP}_{2n+1})$ and $U_\Sigma(\bar{M})$ is the module of primitives in $E(1)_*(\mathbf{E(1)}_{2n+1})$. Recall that $\mathrm{Ext}_{U_\Gamma}(BP_*, M)$ can be defined as follows: Let
$$0 \to M \to U_\Gamma(M_0) \to U_\Gamma(M_1) \to \cdots$$
be an exact sequence in the category of unstable Γ-comodules. Then $\mathrm{Ext}_{U_\Gamma}(BP_*, M) = H_*(M_k)$ where the differentials are given by
$$M_k \cong \mathrm{Hom}_{U_\Gamma}(BP_*, U_\Gamma(M_k)) \to \mathrm{Hom}_{U_\Gamma}(BP_*, U_\Gamma(M_{k+1})) \cong M_{k+1}$$

In the following theorem, the right hand side is the mod p E_2-term of the $E(1)$-based BKSS. The left hand side is the mod p E_2-term of the unstable Novikov spectral sequence with v_1-inverted. The latter is computed in [7], and this is what is given in 5.1. Thus the first part of Theorem 5.1 is equivalent to the following statement, which is an unstable version of the Morava change of rings theorem (see [22]).

Theorem 5.2. *We have*
$$\mathrm{Ext}_{U_\Gamma}(BP_*, v_1^{-1}M/pM) \cong \mathrm{Ext}_{U_\Sigma}(E(1)_*, \bar{M}/p\bar{M})$$

Proof. There is a chain complex
$$\begin{aligned}0 \to v_1^{-1}M/pM \to v_1^{-1}U_\Gamma(\bar{M}/p\bar{M}) &\to v_1^{-1}U_\Gamma(\tilde{U}_\Sigma(\bar{M}/p\bar{M})) \\ &\to v_1^{-1}U_\Gamma(\tilde{U}^2_\Sigma(\bar{M}/p\bar{M})) \to \cdots\end{aligned}$$
where the maps are sums of maps of the form:

- $U_\Gamma(\text{the structure map for}\, \Sigma)$
- $U_\Gamma(N) \to U_\Gamma(U_\Gamma(N)) \to U_\Gamma(\tilde{U}_\Sigma(N))$ where $N = \tilde{U}^k_\Sigma(\bar{M}/p\bar{M})$.

The differentials end up in the reduced cobar complex for the usual reason.

So the theorem will follow once we show that the above complex is exact. To do this we show that, as an $E(1)_*$-module, $U_\Gamma(\tilde{U}_\Sigma(N))$ is a direct sum of copies of $U_\Sigma(\tilde{U}_\Sigma(N))$. Hence the above complex is a direct sum of unstable Σ-comodule *resolutions* and therefore it is acyclic.

To this end we need the Ravenel-Wilson basis. In addition to the $h_i's$ with $i > 0$ we need a "suspension operator" h_0 of degree zero. Then a BP_* basis for $QBP_*(\mathbf{BP}_{2n}) \cong PBP_*(\mathbf{BP}_{2n+1})$ is given by the set of all

$$h_0^{j_0} h_1^{j_1} \cdots \circ v_1^{i_1} v_2^{i_2} \cdots$$

such that if $J = (j_0, j_1, \dots)$ can be written as

$$J = p\Delta_{k_1} + p^2\Delta_{k_2} \cdots p^n \Delta_{k_n} + J'$$

where $k_1 \leq k_2 \leq \cdots$ and J' is a non-negative sequence then $i_n = 0$. (Δ_i is the sequence with a 1 in the i^{th} place and 0 elsewhere.)

In our case we are killing any v_k, $k > 1$. So for N, an $E(1)_*$ module, there are two families of basis elements left for the unstable part of $\Gamma \otimes_{BP_*} N$:

- $h_0^{j_0} h_1^{j_1} \cdots \circ v_1^{i_1}$ with all $j_s < p$, and $2(\Sigma\, j_k) - 2i_1(p-1) = n$
- $h_0^{j_0} h_1^{j_1} \cdots$ with $2(\Sigma\, j_k) = n$

Since we are inverting v_1 only the first type of basis element is left. (We can multiply the second type by v_1 and v_1^{-1} to write the second type of basis element in terms of the first.) Since this is a basis mod p, we have a decomposition:

$$v_1^{-1} BP_*\{h_0^{j_0} h_1^{j_1} \cdots \circ v_1^{i_1}\} \otimes N \cong \mathbf{Z}/p[v_2, v_3 \cdots] \otimes U_\Sigma(N)$$

and the result follows. □

Now we prove parts (*ii*) and (*iii*) of Theorem 5.1. The elements v_1^k and $v_1^{k-1}h_1$ are just desuspensions of the stable basis. The calculation given in section 9 of [3] and repeated in section 6 here shows that $v_1^{p^t-n-1} h_1^{n+1}$, $t >> n$, is a cycle on S^{2n+3} mod S^{2n+1}. Hence $v_1^{p^t-n-1} p h_1^{n+1}$ is a mod p cycle on S^{2n+1}; it is nonzero mod p in E_1 since it it not divisible by p on S^{2n+1}, and it is nonzero mod p in E_2 since there is nothing in filtration 0 to hit it. It is unstable mod p since it is divisible by p on S^{2n+3}. Finally, v_1-periodicity mod p gives the result in any stem.

For part (*iii*), first note that by Theorem 4.1 of [6], the element $\alpha_1 \otimes \alpha_{p^t/n}$ is nonzero. Since the group in this stem has order p, it must be the generator, and hence the generator mod p. Finally, we use v_1-periodicity to obtain a mod p generator in any stem.

6. Proof of Theorem 3.1–the finite groups

To prove Theorem 3.1, first note that for the elements in filtration one in positive stems, the argument is identical to that given in [3], which shows that the element of order p^n in the stable $qk-1$ stem desuspends to S^{2n+1} and not to S^{2n-1}. Specifically, if $n \leq \nu(k)+1$ then modulo $E_1(S^{2n-1})$:

$$\begin{aligned} d(v_1^k)/p^n &= \sum_{j=1}^{k}(-1)^j \binom{k}{j} v_1^{k-j} p^{j-n} h_1^j \\ &\equiv \sum_{j=n}^{k}(-1)^j \binom{k}{j} v_1^{k-j} p^{j-n} h_1 j \\ &\equiv \sum_{j=n}^{k}(-1)^j \binom{k}{j} v_1^{k-n} h_1^n \\ &\equiv -\sum_{j=0}^{n-1}(-1)^j \binom{k}{j} v_1^{k-n} h_1^n \\ &\equiv -v_1^{k-n} h_1^n. \end{aligned}$$

This last class is in $E_1^{1,*}(S^{2n+1})$ and not in $E_1^{1,*}(S^{2n-1})$. For the elements in negative stems, the following lemma is useful for dealing with negative powers of v_1:

Lemma 6.1. *For $k \in Z$,*

$$d(v_1^k) = -v_1^k d(v_1^{-k})\eta_R(v_1^k) = -v_1^k d(v_1^{-k}) \otimes v_1^k.$$

Proof.

$$\begin{aligned} d(v_1^k) &= \eta_R(v_1^k) - v_1^k \\ &= [1 - v_1^k \eta_R(v_1^{-k})]\eta_R(v_1^k) \\ &= v_1^k[v_1^{-k} - \eta_R(v_1^{-k})]\eta_R(v_1^k) \\ &= -v_1^k d(v_1^{-k})\eta_R(v_1^k). \end{aligned}$$

□

Now, for $k<0$, working modulo S^{2n-1}, with $n \leq m$, we have

$$\begin{aligned} d(v_1^k)/p^n \otimes \iota_{2m+1} &= -v_1^k d(v_1^{-k})/p^n \otimes v_1^k \iota_{2m+1} \\ &\equiv -v_1^k(-v_1^{-k-n} h_1^n) \otimes v_1^k \iota_{2m+1} \\ &\equiv v_1^{-n} h_1^n \otimes v_1^k \iota_{2m+1} \end{aligned}$$

which is defined as long as $2n-qk+1 \leq 2m+1$. Hence the element of order p^n in the $qk-1$ stem desuspends to $S^{2n-qk+1}$ and not to $S^{2n-qk-1}$ in E_1.

Since we are working on the one line, and everything on the zero line desuspends all the way, the desuspension pattern described above holds for E_2 as well as E_1. This also shows that the suspension map is injective on E_2 as claimed. Finally, note that the generators of the cyclic groups described above are not any further divisible for, if they were, then by multiplying by v_1^{-k} on the right and on the left, we would obtain divisibility of $d(v_1^{-k})/p^n$ on S^{2n+1} which is a contradiction. This completes the proof of Theorem 3.1 for the one line.

For the elements on the two line, we will use the relation which results from killing v_2. In the following, we are using the Hazewinkel generators $\{v_i\}$.

Lemma 6.2. *Suppose $\bar{i} \leq i$, $j > 0$, so that $v_1^a h_1^{\bar{i}} \otimes v_1^{-j} \iota_{jq+2i+1}$ is defined. Then, mod p,*

$$v_1^a h_1^{\bar{i}} \otimes v_1^{-j} i_{jq+2i+1} = v_1^{a-p} h_1^{\bar{i}+p-1} \otimes v_1^{-j+1} \iota_{jq+2i+1}$$

Iterating this,

$$\begin{aligned} &v_1^a h_1^{\bar{i}} \otimes v_1^{-j} i_{jq+2i+1} \\ =&v_1^{a-jp} h_1^{\bar{i}+j(p-1)} \otimes \iota_{jq+2i+1} \end{aligned}$$

Proof. We use the following relation which arises from killing v_2:

$$v_1^p h_1 = h_1^p \eta_R(v_1) + \epsilon p h_2 \quad \epsilon = \text{unit}$$

We have

$$\begin{aligned} v_1^a h_1^{\bar{i}} \otimes v_1^{-j} \iota_{jq+2i+1} &= v_1^a h_1^{\bar{i}-1} h_1 \otimes v_1^{-j} \iota_{jq+2i+1} \\ &= v_1^a h_1^{\bar{i}-1} [v_1^{-p} h_1^p \eta_R(v_1) + v_1^{-p} \epsilon p h_2] \otimes v_1^{-j} \iota_{jq+2i+1} \\ &= v_1^{a-p} h_1^{\bar{i}+p-1} \otimes v_1^{-j+1} \iota_{jq+2i+i} \quad \text{mod } p \end{aligned}$$

□

Lemma 6.3. *Suppose $p \leq a \leq b$. Then, mod p,*

$$h_1^a \otimes \iota_{2b+1} = \begin{cases} v_1^{p-1} h_1^{a-(p-1)} \otimes \iota_{2b+1} & \text{if } a < b \\ (v_1^{p-1} h_1^{a-(p-1)} + v_1^{-1} p h_1^{a+1}) \otimes \iota_{2b+1} & \text{if } a = b \end{cases}$$

Proof.

$$\begin{aligned}
h_1^a \otimes \iota_{2b+1} &= v_1^{-1}(v_1 h_1^a) \otimes \iota_{2b+1} \\
&= v_1^{-1}[h_1^a \eta_R(v_1) + p h_1^{a+1}] \otimes \iota_{2b+1} \\
&= v_1^{-1}[h_1^{a-p} h_1^p \eta_R(v_1) + p h_1^{a+1}] \otimes \iota_{2b+1} \\
&= v_1^{-1}[h_1^{a-p}[v_1^p h_1 - \epsilon p h_2] + p h_1^{a+1}] \otimes \iota_{2b+1} \\
&= \begin{cases} v_1^{p-1} h_1^{a-(p-1)} \otimes \iota_{2b+1} & \text{if } a < b \\ (v_1^{p-1} h_1^{a-(p-1)} + v_1^{-1} p h_1^{a+1}) \otimes \iota_{2b+1} & \text{if } a = b \end{cases}
\end{aligned}$$

□

For what follows we must know the mod p reduction of $\alpha_{k/m} \otimes \iota_{2n+1}$, the element order p^m in the $qk-1$ stem on S^{2n+1}. That is the content of the next lemma. For simplicity, abbreviate $\nu(k)$ by ν. ϵ is a unit in $Z_{(p)}$. Recall the generators of the one line mod p given in Theorem 5.1, part (ii).

Lemma 6.4. *The mod p reduction of $\alpha_{k/m} \otimes \iota_{2n+1}$ ($m \le \nu+1$, $m \le n$, $k \ne 0$) is given by:*

$$\begin{cases}
v_1^{k-1} h_1 & \text{if } m = \nu+1,\ n > m,\ k > 0 \\
 & \text{or if } m = \nu+1,\ n > m-(p-1)k,\ k < 0 \\
v_1^{k-n-1} p h_1^{n+1} & \text{if } m < \nu+1,\ n = m,\ k > 0 \\
 & \text{or if } m < \nu+1,\ n = m-(p-1)k,\ k < 0 \\
v_1^{k-1} h_1 + \epsilon v_1^{k-n-1} p h^{n+1} & \text{if } m = \nu+1,\ n = m,\ k > 0 \\
 & \text{or if } m = \nu+1 > 1,\ n = m-(p-1)k, \\
 & k < 0 \\
v_1^{k-1} h_1 + (1+\epsilon) v_1^{k-n-1} p h^{n+1} & \text{if } m = \nu+1 = 1,\ n = m-(p-1)k, \\
 & k < 0
\end{cases}$$

Fortunately the last case, in which we don't resolve whether or not the coefficient is a unit, is the one case we don't need in the sequel.

Proof. The first four cases are easy in both positive and negative stems. In the first two cases, the group is of maximal order for that stem, and the generator is at least a double suspension and the result follows. In the next two cases the group has less than maximal order for that stem, and the generator suspends to an element which is divisible by p, hence mod p it suspends to zero and the result follows.

In the next case we are looking at the first sphere in which the group has maximal order for the stem it is in. This case is proved for $k > 0$ in [6]. For $k < 0$ we need to use the relation provided by v_2.

Using Lemma 6.4 and Lemma 6.1 for positive stems we have

$$\begin{aligned}
\alpha_{k/\nu+1} \otimes \iota_{2(\nu+1)-qk+1} &= v_1^k(\alpha_{-k/\nu+1})\eta_R(v_1^k) \otimes \iota_{2(\nu+1)-qk+1} \\
&= v_1^k(\alpha_{-k/\nu+1}) \otimes v_1^k \iota_{2(\nu+1)-qk+1} \\
&= v_1^k(v_1^{-k-1}h_1 \\
&\qquad + \epsilon v_1^{-k-\nu-2} p h_1 \nu + 2) \otimes v_1^k \iota_{2(\nu+1)-qk+1} \quad \bmod p
\end{aligned}$$

First we deal with

$$\begin{aligned}
* &= v_1^{-1}h_1 \otimes v_1^k \iota_{2(\nu+1)-qk+1} \\
&= v_1^{-1+kp} h_1^{1-k(p-1)} \otimes \iota_{2(\nu+1)-qk+1} \qquad \text{by Lemma 6.2}
\end{aligned}$$

There are two cases to consider. In the first case, $\nu + 1 > 1$:

$$\begin{aligned}
* &= v_1^{-1+kp} h_1^{1-k(p-1)} \otimes \iota_{2(\nu+1)-qk+1} \\
&= v_1^{-1+kp} v_1^{-k(p-1)} h_1 \otimes \iota_{2(\nu+1)-qk+1} \qquad \text{by Lemma 6.3} \\
&= v_1^{k-1} h_1 \otimes \iota_{2(\nu+1)-qk+1}
\end{aligned}$$

In the second case, $\nu + 1 = 1$, using the second part of Lemma 6.3:

$$* = [v_1^{k-1}h_1 + v_1^{-2+kp} p h_1^{2-k(p-1)}] \otimes \iota_{3-qk}$$

Now for the second term:

$$\begin{aligned}
** &= v_1^k \epsilon v_1^{-k-\nu-2} p h_1^{\nu+2} \otimes v_1^k \iota_{2(\nu+1)-qk+1} \\
&= \epsilon v_1^{-\nu-2} p h_1^{\nu+2} \otimes v_1^k \iota_{2(\nu+1)-qk+1} \\
&= \epsilon v_1^{-\nu-2} (ph_1) h_1^{\nu+1} \otimes v_1^k \iota_{2(\nu+1)-qk+1} \\
&= \epsilon v_1^{-\nu-2} (v_1 - \eta_R(v_1)) h_1^{\nu+1} \otimes v_1^k \iota_{2(\nu+1)-qk+1} \\
&= \epsilon v_1^{-\nu-1} h_1^{\nu+1} \otimes v_1^k \iota_{2(\nu+1)-qk+1} - \epsilon v_1^{-\nu-2} h_1^{\nu+1} \otimes v_1^{k+1} \iota_{2(\nu+1)-qk+1} \\
&= \epsilon v_1^{-\nu-1+pk} h_1^{\nu+1-k(p-1)} \otimes \iota_{2(\nu+1)-qk+1} \\
&\qquad - \epsilon v_1^{-\nu-2+kp} h_1^{\nu+1-k(p-1)} \otimes v_1^1 \iota_{2(\nu+1)-qk+1} \qquad \text{by Lemma 6.2} \\
&= \epsilon v_1^{-\nu-2+pk} p h_1^{\nu+2-k(p-1)} \otimes_{2(\nu+1)-qk+1} .
\end{aligned}$$

Putting these together we get Lemma 6.4 for $k < 0$. $\square$

Now we consider Theorem 3.1 for the two line. In positive stems the result follows immediately from the calculations of [7], which in turn are obtained from [6] by inverting v_1. For negative stems, consider the $qk-2$ stem, $k < 0$. Abbreviate $\nu(k)$ by ν. The maximum order which occurs in this stem is $p^{\nu+1}$. The Theorem states that this order first occurs on $S^{2(\nu+1)-qk+1}$. We will write down an integral cycle representing the generator of this group, which is killed by $p^{\nu+1}$. We show that it is not further divisible by computing its mod p reduction which is not zero. The shows that this group is a finite cyclic group (as opposed to an infinitely divisible group, a scenario which must be ruled out) with order no more than as claimed.

Define a cycle in $E_1^{2,qk}(S^{2(\nu+1)-qk+1})$ as follows: Choose $w >> 0$ and consider

$$z = \alpha_{-p^w+k/\nu+1}\alpha_{p^w/\nu+1-(p-1)k} \otimes \iota_{2(\nu+1)-qk+1}.$$

Thus z is a cycle which is just born, has maximal order for its stem, and lives on the first sphere were this maximal order occurs. We need to show (see Lemma 6.4 and Theorem 5.1 (ii)) that this reduces mod p to the same thing as

$$v^{-p^w+k-1}\alpha_1\alpha_{p^w/\nu+1-(p-1)k} \otimes \iota_{2(\nu+1)-qk+1}.$$

In what follows we will not specify the exact power of v_1 that appears if it is not relevant to the argument. By Lemma 6.4 we know that mod p

$$\alpha_{p^w/\nu+1-(p-1)k} \otimes \iota_{2(\nu+1)-qk+1} = v_1^{\bullet} p h_1^{\nu-(p-1)k+2} \otimes \iota_{2(\nu+1)-qk+1}$$

and next to this class

$$\alpha_{-p^w+k/\nu+1} = v_1^{-p^w+k-1}h_1 + \lambda v_1^{\bullet} p h_1^{\nu-(p-1)(-p^w+k)+2}$$

Thus mod p

$$z = (v_1^{-p^w+k-1}h_1 + \lambda v_1^{\bullet} p h_1^{\nu-(p-1)(-p^w+k)+2}) v_1^{\bullet} p h_1^{\nu-(p-1)k+2} \otimes \iota_{2(\nu+1)-qk+1}.$$

We can evaluate the coefficient

$$x = (v_1^{-p^w+k-1}h_1 + \lambda v_1^{\bullet} p h_1^{\nu-(p-1)(-p^w+k)+2}) v_1^{\bullet}$$

by stabilizing mod p, in which case it is non-zero. Hence unstably it must have the form

$$x = v_1^{\bullet} h_1 + \lambda' v_1^{\bullet} p h_1^{p(\nu-(p-1)k+2)}$$

for some λ'. This gives

$$z = (v_1^{\bullet} h_1 + \lambda' v_1^{\bullet} p h_1^{p(\nu-(p-1)k+2)}) p h_1^{\nu-(p-1)k+2} \otimes \iota_{2(\nu+1)-qk+1}.$$

The following is easy to check

$$d(p^2 h_2^{\nu-(p-1)k+2}) = p^2 h_1^{p(\nu-(p-1)k+2)} \otimes h_1^{\nu-(p-1)k+2}$$

which leaves mod p

$$z = v_1^{\bullet} h_1 p h_1^{\nu-(p-1)k+2} \otimes \iota_{2(\nu+1)-qk+1}$$

which is the desired class.

This takes care of the upper bound in the case of the first sphere where the group has maximal order for the stem it is in. The same calculation works on higher spheres for the element just born having maximal order for the stem. Now, by reducing mod p^r, for various r, and applying v_1-periodicity, we see that the groups have order at least as great as claimed, hence exactly as claimed. Also, periodicity shows that the suspension pattern is as claimed.

Now we address the case where the dimension of the sphere is less than $2(\nu+1)-qk+1$. In this case the order of the group is less than maximal for the stem. The main thing we need to check is that the element of order p on $S^{2(\nu+1)-qk+1}$ desuspends all the way to the three sphere. Then, since double suspension is zero mod p on the two line, the double suspension of an element is always divisible by p, so the orders of the groups are at least as great as claimed in the Theorem. However, they can be no larger than this, for again this would contradict the fact that double suspension is zero mod p on the two line.

To this end, suppose we are on $S^{2(\nu+1)-qk+1}$ where a group of maximal order for the $qk-2$ stem first occurs. By part (*ii*) of Lemma 6.4, the element on the one line mod p which is the mod p reduction of an integral class is the sum of the stable class $v_1^\bullet h_1$ and the unstable class $v_1^\bullet p h_1^{\nu-(p-1)k+2}$. Applying the Bockstein yields

$$\beta(v_1^\bullet h_1) = \beta(v_1^\bullet p h_1^{\nu-(p-1)k+2}).$$

The latter term represents the class of order p in filtration 2, but is hard to calculate. The former is easy: $v_1^\bullet h_1$ desuspends all the way, so $\beta v_1^\bullet h_1$ does as well, and this is the element of order p in the $qk-2$ stem, so this proves the result.

It worth making a note of why the above proof doesn't work for $k=0$, i.e. the -2 stem. In this case $\nu(k)$ is not defined and there is no maximal order for this stem. This case is taken care of in the next section.

7. Calculation of the -2 and -3 stems

For the remaining part of the proof of Theorem 3.1 it will be convenient to consider the cobar complex with coefficients in the group $Q/Z_{(p)}$. From the short exact sequence of coefficient groups

$$0 \to Z_{(p)} \to Q \to Q/Z_{(p)} \to 0$$

we get the long exact sequence

$$\dots E_2^{s,t}(S^{2n+1}) \to E_2^{s,t}(S^{2n+1};Q) \to$$
$$E_2^{s,t}(S^{2n+1};Q/Z_{(p)}) \to E_2^{s+1,t}(S^{2n+1}) \to \dots$$

It is easy to show that $E_2^{s,t}(S^{2n+1};Q)=0$ except for $E_2^{0,0}(S^{2n+1};Q)=Q$. Thus Theorem 3.1, part (i), is equivalent to the following.

Theorem 7.1.

$$E_2^{s,t}(S^{2n+1};Q/Z_{(p)}) = \begin{cases} 0 & \text{if } s\geq 3,\\ Q/Z_{(p)} & \text{if } s=2,\ t-s=2n-1,\\ Q/Z_{(p)}\oplus Q/Z_{(p)} & \text{if } s=1,\ t-s=2n,\\ Q/Z_{(p)} & \text{if } s=0,\ t-s=2n+1,\\ Z/p^{\min(\nu(k)+1,n)} & \text{if } s=1,\ t-s=2n+qk,\ k>0,\\ Z/p^{\min(\nu(k)+1,n+k(p-1))} & \text{if } s=1,\ t-s=2n+qk,\ k<0,\\ Z/p^{\min(\nu(k)+1,n)} & \text{if } s=0,\ t-s=2n+qk+1,\ k>0,\\ Z/p^{\min(\nu(k)+1,n+k(p-1))} & \text{if } s=0,\ t-s=2n+qk+1,\ k<0,\\ 0 & \text{otherwise.}\end{cases}$$

There are also evident analogues of parts (ii)–(iv) of Theorem 3.1 with coefficients in $Q/Z_{(p)}$. Note that the groups we are studying have lower homological degree than their integral counterparts, which is the reason that $Q/Z_{(p)}$ coefficients are convenient for the calculation we are about to make.

From the short exact sequence of coefficient groups

$$0\to Z/p \xrightarrow{i} Q/Z_{(p)}\to Q/Z_{(p)}\to 0$$

we get a "Bockstein" long exact sequence

$$\cdots\to E_2^{s,t}(S^{2n+1};Z/p)\xrightarrow{i_*} E_2^{s,t}(S^{2n+1};Q/Z_{(p)}) \\ \to E_2^{s,t}(S^{2n+1};Q/Z_{(p)})\to E_2^{s+1,t}(S^{2n+1};Z/p)\to\dots$$

Now consider the -1 and -2 stems (with $Q/Z_{(p)}$ coefficients). There are two possibilities. Either the generator of $E_2^{2,2n+1}(S^{2n+1};Z/p)$ is in the kernel of the Bockstein i_*, or it is not. In the former case, there would be a finite cyclic group in the -1 stem, which represents some portion of the stable $Q/Z_{(p)}$ desuspended to the $2n+1$ sphere. In the latter case the entire stable $Q/Z_{(p)}$ desuspends to the three sphere, and there is an additional $Q/Z_{(p)}$ in the -2 stem. It is this latter case which occurs, as stated in Theorem 7.1. Note that in either case there is one unstable $Q/Z_{(p)}$ in the -1 stem accounting for the fact that the element in $E_2^{0,2n+1}(S^{2n+1};Z/p)$ is the reduction of an integral class, namely the generator of the 0 stem.

Since the statement of the Theorem is equivalent to the stable $Q/Z_{(p)}$ desuspending to S^3, it suffices to do the calculation on the three sphere (algebraically we actually calculate on a two dimensional class in light of the fact that $V(A[2n]) = U(A[2n+1])$). The generator of $E_2^{2,2n+1}(S^{2n+1}; Z/p)$ is represented by $v_1^{-2}h_1 \otimes h_1$. Thus we need to show that

$$\frac{v_1^{-2}h_1 \otimes h_1}{p} \otimes \iota_3 \neq 0$$

in the homology of the cobar complex for S^3 with $Q/Z_{(p)}$ coefficients.

The p-typical formal group law complicates the calculation. (An alternative to the following proof would be to give a direct proof that the stable $Q/Z_{(p)}$ desuspends to the three sphere. We have not been able to do so. It is left as an exercise to desuspend the class of order p^2 to S^3.) In order to show that $\dfrac{v^{-2}h_{1\otimes}h_1}{p} \otimes \iota_3$ is not zero it is easier to work with the K-theory spectrum. We recall the results of [2] and [1] and explain how to convert their notation to BP notation. Write $u \in \pi_2(K)$ for the Bott generator. u and $1 \cdot u \in K_2(K)$ for its image under the left and right unit maps. ([2] and [1] denote the left action by u and the right action by v. We also identify the Bott class in homotopy with its K-theory Hurewicz image.) Let $\underline{K}_{2i}$ denote the $2i$-th space in the Ω-spectrum for K-theory. The Bott map gives a homotopy equivalence $\underline{K}_{2i} \longrightarrow BU$. The image of the map,

$$K_*(CP^\infty) \longrightarrow K_*(BU) \tag{7.2}$$

induced by the inclusion, generates $K_*(BU)$ as a ring (the product is induced by the Whitney sum). The K-theory of CP^∞ is described in [2, (1.3)] .

Definition 7.3. *A polynomial $f(\omega) \in Q[\omega]$ is said to be numerical if $f(n)$ is an integer for every integer, n. Let A denote the set of all numerical polynomials.*

A is a subring of $Q[\omega]$ which contains $Z[\omega]$.

Let

$$\binom{\omega}{n} = \frac{\omega(\omega-1)\cdots(\omega-n+1)}{n!}. \tag{7.4}$$

Then $\left\{\binom{\omega}{0}, \binom{\omega}{1}, \binom{\omega}{2} \cdots\right\}$ is a Z basis for A. It is shown in [2] that A is isomorphic to $K_0(CP^\infty)$ as rings where the ring structure on $K_0(CP^\infty)$ is induced by the tensor produce of complex line bundles. $K_0(K)$ is the direct limit of iterated Bott maps, $B_* : K_0(BU) \longrightarrow K_0(BU)$.

The Bott map annihilates decomposables, and the following diagram commutes up to decomposables ([2]):

$$
\begin{array}{ccccccccc}
A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
K_0(BU) & \xrightarrow{B_*} & K_0(BU) & \xrightarrow{B_*} & K_0(BU) & \xrightarrow{B_*} & \cdots & \longrightarrow & K_0(K)
\end{array}
$$

where the maps $A \longrightarrow A$ are multiplication by ω . Hence $K_0(K) \approx A[\omega^{-1}]$. Since the multiplication on $K_0(K)$ is induced by the tensor product of bundles, the isomorphism is as rings. There is the relation

$$\omega\binom{\omega}{k} = k\binom{\omega}{k} + (k+1)\binom{\omega}{k+1}$$

which describes the Bott map. In $K_0(K)$ ω is identified with $u^{-1} \cdot u$. The coproduct

$$K_0(CP^\infty) \longrightarrow K_0(K) \otimes K_0(CP^\infty)$$

is induced by

(7.5) $$\omega \longrightarrow \omega \otimes \omega.$$

There is another description of the generators. We view BU as the 0-space in the Ω spectrum for K theory. Recall that $K^*(CP^\infty) \approx K^*[[x]]$ where $|x| = 2$. We define $\beta_n \in K_{2n}(CP^\infty)$ by

$$\langle x^i, \beta_n \rangle = \begin{cases} 1 & \text{if } \ i = n \\ 0 & \text{if } \ i \neq n \end{cases}$$

$x \in K_2(CP^\infty)$ corresponds to a map

$$x : CP^\infty \to K_2 \approx BU$$

which induces (7.2). Define $\widetilde{\beta}_n$ to be $(x)_*(\beta_n)$. Then $K_*(BU) \approx K_*\left[\widetilde{\beta}_1, \widetilde{\beta}_2, \cdots\right]$. The two sets of generators are related by $\widetilde{\beta}_n = u^n\binom{w}{n}$. We have homomorphisms

(7.6) $$K_q(BU) \approx K_q(\underline{K}_{2m}) \xrightarrow{\sigma_m} K_{q-2m}(K)$$

Denote the class, $\widetilde{\beta}_n$, in $K_q(\underline{K}_{2m})$ by $\beta_{n,m}$. Then $K_*(K)$ is generated over $Z[u, u^{-1}, 1 \cdot u^{-1}]$ by $\sigma_0(\beta_{n,o})$ with $\sigma_m(\beta_{n,m}) = \sigma_0(\beta_{n,o}) \cdot u^{-m}$. In $K_n(K)$ we use b_n^s to denote $\sigma_2(\beta_{n+1,2})$. This agrees the names for the generators defined in [28] (who does not use the s superscript).

There is the Hopf Ring description of $K_q(\underline{K}_{2m})$ (which is equivalent to the usual description of the terms in the unstable cobar complex) in [24], [15]. They denote by b_i the classes $\beta_{i,2} \in K_{2i}(\underline{K}_2)$. We also have classes $[\alpha] \in K_0(K_{-|\alpha|})$ defined as follows. If $\alpha \in K_{-r}$ then $\alpha \in \pi_0(\underline{K}_r)$. $[\alpha]$ is the K-theory Hurewicz image of α. The tensor product of virtual bundles induces the ring structure in $K_*(K)$ which induces a $\circ$-product

$$K_*(K_m) \otimes K_*(K_n) \to K_*(K_{m+n})$$

$K_*(K_m)$ also has a $*$-product inducted by the Whitney sum. As usual, Q denotes the indecomposables functor with respect to the $*$-product. Then [24], [15] show that $QK_*(K_{2m})$ is generated by the b_i's and the $[u]$'s using the $\circ$-product. Since $K_*(K_m)$ is generated as a $Z[u, u^{-1}, 1 \cdot u^{-1}]$-module by $\{b_i\}$ it follows that $\circ$-products of b_i's must be a sum of b_i's over $Z[u, u^{-1}, 1 \cdot u^{-1}]$. In the unstable cobar complex, we identify $QK_*(K_{2m})$ with its image in $K_*(K)$. In particular if $\delta = \sigma_m(\gamma)$, $\gamma \in QK_*(K_{2m})$ we will write $\delta \otimes \iota_{2m}$ for γ . For example $b_n = \beta_n \cdot u^{-1} \otimes \iota_2$ and $b_n^s = \beta_{n+1} \cdot u^{-1}$. The Hopf ring class, $[\alpha]$, suspends to $1 \cdot \alpha$ and is written $1 \otimes \alpha\iota_{-|u|}$ (which has dimension 0, as it should on $K_{-|u|}$). Notice that the shift in degree in (7.6) is realized by deleting the $\otimes\iota_{2m}$ in the unstable cobar notation. The b_n's and $[\alpha]$ enjoy simple coaction formulas. Let $\mathbf{b} = \Sigma_{i\geq 0}\, b_i^s$ then

$$\begin{aligned} \psi(b) &= \Sigma_{j\geq 0}\mathbf{b}^j \otimes b_j, \\ \psi([\alpha]) &= 1 \otimes [\alpha] \end{aligned} \tag{7.7}$$

It is an amusing exercise to show that (7.5) is compatible with (7.7).

We will calculate on the 2-dimensional class. The first few terms of the K-theory unstable cobar complex for S^{2n} are:

$$K_*(S^{2n}) \overset{\to}{\underset{\to}{}} K_*(BU) \overset{\to}{\underset{\to}{\to}} K_*(K(BU)) \cdots \tag{7.8}$$

Recall that $K(BU)$ is the space defined in Section 2 for $E = K$ and is the space whose homotopy groups are $K_*(BU)$. There is a homotopy equivalence

$$K(BU) \approx \prod BU_{(\beta_n)} \times Y$$

where the copies of BU are indexed by the K_* generators, β_n and Y is a product of copies of BU indexed by the $*$-decomposable generators.

On the two sphere the differential $K_*(S^2) \to K_*(BU)$ is given by

$$d(u^n \otimes \iota_2) = (1 \cdot u^n - u^n) \otimes \iota_2 = u^n(\omega^n - 1) \otimes \iota_2.$$

The description of the generators in terms of numerical polynomials gives us formulas for the right action of u^n in terms of the b's.

Example 7.9. *Let $p = 3$. Then*

$$1 \cdot u^2 = u^2 + 6(\widetilde{\beta}_3 + u\widetilde{\beta}_2)$$

An instructive way to check the validity of the right action formula such as this is to substitute $\omega = 0, 1, 2 \cdots$. In our example we assert that

$$u^2(\omega^2 - 1) = 1 \cdot u^2 - u^2 = 6(\widetilde{\beta}_3 + u\widetilde{\beta}_2) = 6u^2(\binom{\omega}{3} + \binom{\omega}{2})\omega^{-1}$$

These are readily seen to be equal using the fact that

$$\binom{\omega}{k} = 0 \text{ for } \omega = 0, 1, \cdots k - 1$$

and $\binom{k}{k} = 1$.

We now consider (7.8) tensored with $Q/Z_{(p)}$. An element $f \in Q[\omega^{-1}]$ represents something in the rational unstable cobar complex on S^{2n} if and only if $f(\omega)\omega^n$ is a polynomial in $Q[\omega]$, and such a polynomial $f(\omega)\omega^n \in Q[\omega]$ is zero if and only if $f(\omega)\omega^n$ is a numerical polynomial. More generally we may write an element in filtration k of the unstable cobar complex as $f(\omega_1, \omega_2, \cdots \omega_k)$. f is zero if and only if

$$f(\omega_1, \omega_2, \cdots \omega_k)(\omega_1\omega_2 \cdots \omega_k)^n$$

is integral.

We now consider filtration 2. For integers, a and b we have the map $\Phi_{a,b} :\ Q[\omega_1,\ \omega_2\ ] \to Q$ defined by sending $\omega_1^s\omega_2^t$ to a^sb^t . To clarify the notation, ω_1 denotes $\omega \otimes 1$ and ω_2 denotes $1 \otimes \omega$. $\Phi_{a,b}$ maps $A \otimes A$ to $Z_{(p)}$. So we have a well defined map

$$\Theta_{a,b} : \mathcal{C}_2 \otimes Q/Z_{(p)} \xrightarrow{(\omega_1\omega_2)^n} A \otimes A \otimes Q/Z_{(p)} \xrightarrow{\Phi_{a,b}} Q/Z_{(p)}$$

We now restrict to $n = 1$ (i.e. S^2) and internal degree 0 and write $\mathcal{C}$ for $\mathcal{C}_2 \otimes Q/Z_{(p)}$. We say a polynomial, $f(\omega_1, \omega_2)$ is symmetric if $\Theta_{a,b} = \Theta_{b,a}$ for all integers a, b .

Theorem 7.10. $\dfrac{v^{-2}h_{1\otimes}h_1}{p}$ *is a non-zero homology class on S^2.*

Proof.

$$\begin{aligned} h_1 &= -d(v)/p \\ &= -d(u^{p-1})/p \\ &= u^{p-1}(1 - \omega^{p-1})/p \end{aligned}$$

so

$$f = \frac{v^{-2}h_1 \otimes h_1}{p} = \omega^{p-1}(1 - \omega^{p-1}) \otimes (1 - \omega^{p-1})/p^3$$

which is not symmetric ($\Theta_{p,p-1}(f) = 0$, $\Theta_{p-1,p}(f) \neq 0 \mod Q/Z_{(p)}$). Now consider any $f(\omega) \in Q[\omega]$ in filtration 1 with internal degree 0 (i.e. there are no coefficients). We have

$$d(f(\omega)) = 1 \otimes f(\omega) - f(\omega \otimes \omega) + f(\omega) \otimes 1$$

which is symmetric (since there are no coefficients to pass through the tensor). So $\dfrac{v^{-2}h_{1\otimes}h_1}{p}$ cannot be in the image of the differential and this proves the theorem. $\square$

Remark 7.11. *On S^4 we have the differential $d(v^{-2}h_1^2/p) = \dfrac{v^{-2}h_{1\otimes}h_1}{p}$. It is interesting to note how the above proof for S^2 does not work on S^4. We want to show that $\Theta_{a,b} = \Theta_{b,a}$ on the 4 sphere. To see this notice that*

$$\begin{aligned}\omega^{p+1}(1-\omega^{p-1}) \otimes \omega^2(1-\omega^{p-1})/p^3 =& \omega^2(1-\omega^{p-1}) \otimes \omega^2(1-\omega^{p-1})/p^3 \\ &- \omega^2(1-\omega^{p-1})^2 \otimes \omega^2(1-\omega^{p-1})/p^3\end{aligned}$$

and the second term on the right, $\omega^2(1-\omega^{p-1})^2 \otimes \omega^2(1-\omega^{p-1})/p^3$, is numerical (i.e.

$$\Phi_{a,b}(\omega^2(1-\omega^{p-1})^2 \otimes \omega^2(1-\omega^{p-1})/p^3) \in Z_{(p)}$$

for all integers (a, b)) and is therefore 0 with $Q/Z_{(p)}$ coefficients. The first term on the right, $\omega^2(1-\omega^{p-1}) \otimes \omega^2(1-\omega^{p-1})/p^3$, is clearly symmetrical.

Remark 7.12. *The method of numerical polynomials gives an alternative proof of the stable result Theorem 4.2 (b) of* [21]. *In terms of $Q[\omega]$, their class y is $\ln(\omega)$, the formal natural log series. So $dy = 0$ follows from our formula for $d(f(\omega))$.*

8. The double suspension spectral sequence

The double suspension sequence in homotopy is the long exact sequence of homotopy groups

$$\cdots \to \pi_i(S^{2n-1}) \to \pi_{i+2}(S^{2n+1}) \to \pi_{i-1}(W(n)) \to \dots$$

where $W(n)$ is defined to be the homotopy fiber of the double suspension map $S^{2n-1} \to \Omega^2 S^{2n+1}$.

The double suspension sequence appears at the level of the E_2-term of the classical unstable Adams spectral sequence:

$$\cdots \to E_2^{s,t}(S^{2n-1}) \to E_2^{s,t}(S^{2n+1}) \to E_2^{s,t}(\Lambda(W(n))) \to E_2^{s+1,t}(S^{2n-1}) \to \dots.$$

This is obtained by filtering the Λ-algebra by the odd spheres and defining $E_2^*(\Lambda(W(n)))$ for various n to be the homology of the subquotients of the filtration (with a suitable choice of indexing). See [18] and [14] for details. Note that $E_2^*(\Lambda(W(n)))$ is not actually the E_2-term of the unstable Adams spectral sequence for the *space* $W(n)$.

The double suspension sequence appears at the E_2-level of the BP-based BKSS, i.e. the unstable Novikov spectral sequence:

(8.1)

$$\cdots \to E_2^s(S^{2n-1}) \xrightarrow{\sigma^2} E_2^s(S^{2n+1}) \xrightarrow{H_2} \mathrm{Ext}^{s-1}_{\mathcal{A}(U)}(W(n)) \xrightarrow{P_2} E_2^{s+1}(S^{2n-1}) \to \dots$$

where

- $$W(n) \approx BP_*/p\{x_{2pn-1}, x_{2p^2n-1}, ...\}$$

  with Γ-coaction given by

  $$\psi(x_{2p^in-1}) = \sum p^{k-i} h_{k-i}^{np^i} \otimes x_{2p^in-1},$$
- If $z = \gamma \otimes x_{2pn-1} \in \mathrm{Ext}(W(n))$, then

  $$P_2(z) = d(\gamma \otimes ph^n) \otimes \iota_{2n-1},$$

  (In general if $z = \sum \gamma_i \otimes x_{2p^in-1}$ then $P_2(z) = d(\sum \gamma_i \otimes p^{k-1}h_k^n)$.)
- If $x \in E_2^s(S^{2n+1})$ is represented in the unstable cobar complex by $\gamma \otimes h^n \otimes \iota_{2n+1}$ modulo terms which desuspend, and γ is a double suspension with respect to the dimension of $h^n \otimes \iota_{2n+1}$ (i.e. γ is defined on the $2n(p-1)+2n-1$ sphere) then

  $$H_2(x) = \gamma \otimes x_{2pn-1}$$

  (In general x can be represented by a cocyle of the form $\sum \gamma_k \otimes p^{k-1}h_k^n$ and $H_2(x) = \sum \gamma_k \otimes x_{2p^kn-1}$.)

See [4] and [5] for details.

The double suspension sequence for BP comes about as a special case of the composite functor spectral sequence 4.8 applied to the unstable Γ-comodule $BP_*(\Omega S^{2n+1})$. In [4] it is shown that the machinery of [8] generalizes to the case of coalgebras over the graded ground ring BP_*. Since $BP_*(\mathbf{BP_n})$ is cofree as a coalgebra, the G-derived functors of P are the same as the derived functors of P in the category of BP_*-coalgebras. The calculations of [4] then show that the higher derived functors of $BP_*(\Omega S^{2n+1})$ vanish (specifically $R^i = 0$ for $i > 1$) and the CFSS collapses to two rows. After various identifications, this yields 8.1.

In this section we make a few remarks concerning the fact that this is *not* what happens for $E(1)$. In particular the above CFSS does not reduce to two rows: instead of a double suspension long exact sequence there is a double suspension spectral sequence.

First we recall the explicit double complex construction given in [5] of the CFSS for $E_*(\Omega S^{2n+1})$. Let E satisfy 4.4. From Proposition 4.6 we have the functor V, defined as the indecomposables in G. From [24] we know that $G(M(2n))$ is isomorphic to the polynomial algebra $E_*[V(M(2n))]$. For simplicity, in what follows denote $M(2n)$ by M. Like U, there are maps $V^2 \to V$ and $1 \to V$ making V into a triple. Hence there is a cosimplicial resolution

$$(8.2) \qquad M \xrightarrow{d^0} V(M) \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \end{array} V^2(M) \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^1} \\ \xrightarrow{d^2} \end{array} \cdots$$

Apply the polynomial algebra functor to this resolution to obtain a resolution

$$(8.3) \qquad E_*(\Omega S^{2n+1}) \xrightarrow{d^0} G(M) \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \end{array} G(VM) \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{s^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^1} \\ \xrightarrow{d^2} \end{array} \cdots$$

Now apply the primitive element functor to the un-augmented cosimplicial object from 8.3 to give a cosimplicial object, $\mathcal{U}(2n)$,

$$(8.4) \qquad U(M) \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d^1} \end{array} U(V(M)) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} \cdots$$

and the homology of the resulting chain complex is, by definition, $R^i_G PE_*(\Omega S^{2n+1})$.

The cosimplicial object 8.4 fits into a bi-cosimplicial object defined by

$$D^{i,j} = U^i V^j(M), \quad i > 0, \quad j \geq 0$$

which we represent by the following diagram.

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
\bullet & \to & \bullet & \to & \\
\uparrow\ \uparrow\ \uparrow & & \uparrow\ \uparrow\ \uparrow & & \\
U(U(M)) & \begin{array}{c}\to\\ \to\end{array} & U(U(V(M)) & \begin{array}{c}\to\\ \to\\ \to\end{array} & \cdots \\
\uparrow\ \uparrow & & \uparrow\ \uparrow & & \\
(U(M)) & \begin{array}{c}\to\\ \to\end{array} & (UV(M)) & \begin{array}{c}\to\\ \to\\ \to\end{array} & \cdots
\end{array}
$$

The associated bi-complex generates the composite functor spectral sequence. The complex associated to $U^{(k)}\mathcal{U}(2n)$ is the k-th row. If we take vertical homology first, the result is concentrated in filtration zero and is the chain complex associated to

$$
M \quad \begin{array}{c}\to\\ \to\end{array} \quad V(M) \quad \begin{array}{c}\to\\ \to\\ \to\end{array} \quad V^2(M) \quad \cdots
$$

Taking homology horizontally, we get $\mathrm{Ext}_{\mathcal{A}(V)}(M(2n))$ which is the same as $\mathrm{Ext}_{\mathcal{A}(U)}(M(2n+1)) = E_2(S^{2n+1})$ by 4.7 and 4.9. By taking homologies in the opposite order we obtain a spectral sequence

$$
Ext_{\mathcal{A}(U)}(R^i_G P(E_*(\Omega S^{2n+1}))) \Longrightarrow E_2(S^{2n+1}).
$$

The term $\mathrm{Ext}_{\mathcal{A}(U)}(R^0_G PE_*(\Omega S^{2n+1}))$ is just

$$
\mathrm{Ext}_{\mathcal{A}(U)}(M(2n)) = \mathrm{Ext}_{\mathcal{A}(U)}(M(2n-1)) = E_2(S^{2n-1}).
$$

We have $R^1_G PE_*(\Omega S^{2n+1}) = W(n)$ by definition.

In the case of $E = BP$, that is it. The G-derived functors of $BP_*(\Omega S^{2n+1})$ are computed in [4] and shown to vanish for $i > 1$. Hence the above spectral sequence degenerates to a long exact sequence. This cannot happen for K-theory. For example, $W(n)$ is a mod-p vector space and if there were a double suspension long exact sequence of E_2-terms, this would imply that $E_2(S^{2n+1})$ is bounded by p^n as an abelian group ($W(1)$ is essentially the three sphere). This contradicts the existence of unbounded torsion in $E_2(S^{2n+1})$, specifically the three groups isomorphic to Q/Z.

For this reason, the divisible groups can be thought of as obstructions to the vanishing of higher G-derived functors of P in the category $\mathcal{M}(G)$. They can also be thought of as corresponding in some sense to the Eilenberg-Mac Lane spaces that measure the failure of localization to commute with fiber sequences.

Several things appear to be going on here. For one, $E(1)_*(\mathbf{E(1)_{2n}})$ is not cofree as a coalgebra, so the G-derived functors of P are not necessarily the same as the derived functors of P in the category of coalgebras over $E(1)_*$. From [15] one sees that the spaces in the Ω spectrum are limits of Wilson spaces [27]. But the category of coalgebras over a non-connective ground ring is subtle, and even $E(1)_*$ of Wilson spaces are not cofree, e.g. $E(1)_*(CP^\infty)$. (Notice that $E^*(CP^\infty)$ is not a free algebra for $E = E(n)$, K and even BP.)

Secondly, even if there were some relationship between the G-derived functors of P, and the derived functors of P in the category of coalgebras over a non-connective ground ring, the methods of [8], generalized to BP in [4], rely on *irreducibility* of the coalgebras under study, a condition which fails to hold for the K-theory of non-connected spaces. In particular the spaces $\mathbf{E(n)_m}$ are non-connected. It would be useful and interesting to have a better understanding of $\mathcal{M}(G)$, along with some techniques for calculating these derived functors.

References

1. J. F. Adams, A. S. Harris, and R. M. Switzer, *Hopf algebras of cooperations for real and complex K-theory*, Proc. London Math. Soc. (3) **23** (1971), 385–408.
2. Andrew Baker, Francis Clarke, Nigel Ray, and Lionel Schwartz, *On the Kummer congruences and the stable homotopy of BU*, Trans. Amer. Math. Soc. **316** (1989), 385–432.
3. M. Bendersky, E. B. Curtis, and H. R. Miller, *The unstable Adams spectral sequence for generalized homology*, Topology **17** (1978), 229–248.
4. M. Bendersky, E. B. Curtis, and D. Ravenel, *EHP sequences in BP theory*, Topology **21** (1982), no. 4, 373–391.
5. Martin Bendersky, *The derived functors of the primitives for $BP_*(\Omega S^{2n+1})$*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 599–619.
6. ———, *Unstable towers in the odd primary homotopy groups of spheres*, Trans. Amer. Math. Soc. **287** (1985), no. 2, 529–542.
7. ———, *The v_1-periodic unstable Novikov spectral sequence*, Topology **31** (1992), no. 1, 47–64.
8. A. K. Bousfield, *Nice homology coalgebras*, Trans. Amer. Math. Soc. **148** (1970), 473–489.
9. ———, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.
10. ———, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281.
11. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, 1972.
12. ———, *The homotopy spectral sequence of a space with coefficients in a ring*, Topology **11** (1972), 79–106.

13. A.D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May, *Modern foundations for stable homotopy theory*, Handbook of Algebraic Topology (I.M. James, ed.), North-Holland, Amsterdam, 1995, pp. 213–253.
14. J. R. Harper and H. R. Miller, *On the double suspension homomorphism at odd primes*, Trans. Amer. Math. Soc. **273** (1982), no. 1, 319–331.
15. M. J. Hopkins and J. R. Hunton, *On the structure of spaces representing a Landweber exact cohomology theory*, Topology (19xx), xxx–xxx.
16. I. Kriz, *Towers of E_∞ ring spectra with an application to BP*, preprint.
17. P. S. Landweber, *Homological properties of comodules over MU_*MU and BP_*BP*, Amer. J. Math. **98** (1976), 591–610.
18. Mark Mahowald, *On the double suspension homomorphism*, Trans. Amer. Math. Soc. **214** (1975), 169–178.
19. Mark Mahowald and Robert Thompson, *The K-theory localization of an unstable sphere*, Topology **31** (1992), no. 1, 133–141.
20. J.P. May, N. Ray, F. Quinn, and J. Tornehave, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math., vol. 577, Springer-Verlag, 1977.
21. H. R. Miller, D. C. Ravenel, and W. S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. **106** (1977), 469–516.
22. Haynes R. Miller and Douglas C. Ravenel, *Morava stabilizer algebras and the localization of Novikov's E_2-term*, Duke Math. J. **44** (1977), no. 2, 433–447.
23. G. Mislin, *Localizations with respect to K-theory*, J. Pure Appl. Algebra **10** (1977), 201–213.
24. D. C. Ravenel and W. S. Wilson, *The Hopf ring for complex cobordism*, J. Pure Appl. Algebra **9** (1977), 241–280.
25. Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351–414.
26. W. S. Wilson, *The Ω-spectrum for Brown-Peterson cohomology, part I*, Comment. Math. Helv. **48** (1973), 45–55.
27. ———, *The Ω-spectrum for Brown-Peterson cohomology, part II*, Amer. J. Math. **97** (1975), 101–123.
28. W. Stephen Wilson, *Brown-peterson homology, an introduction and sampler*, Regional Conference Series in Math, vol. 48, AMS, Providence RI, 1980.

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