AN ALGEBRAIC MODEL FOR CHAINS ON $\Omega BG^\wedge_p$

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Abstract. We provide an interpretation of the homology of the loop space on the p-completion of the classifying space of a finite group in terms of representation theory, and demonstrate how to compute it. We then give the following reformulation. If f is an idempotent in kG such that $f.kG$ is the projective cover of the trivial module k, and $e = 1 - f$, then we exhibit isomorphisms for $n \geq 2$:

$$H_n(\Omega BG^\wedge_p; k) \cong \mathrm{Tor}^{e.kG.e}_{n-1}(kG.e, e.kG)$$

$$H^n(\Omega BG^\wedge_p; k) \cong \mathrm{Ext}^{n-1}_{e.kG.e}(e.kG, e.kG).$$

Further algebraic structure is examined, such as products and coproducts, restriction and Steenrod operations.

1. Introduction

This paper grew out of an attempt to understand the relationship between the work of Ran Levi [17, 18, 19, 20] (see also Cohen and Levi [6]) on $H_*(\Omega BG^\wedge_p; k)$, the homology of the loop space on the Bousfield–Kan p-completion of the classifying space of a finite group G with coefficients in a field k, and the papers [8, 9] of Dwyer, Greenlees and Iyengar. The result is a representation theoretic description of the homology of $\Omega BG^\wedge_p$. For reasonably small groups, this is explicitly computable, and to illustrate the methods we give the results of some computations using the computer algebra system MAGMA [4].

In order to describe our results, we introduce the following operation in modular representation theory. Let k be a field of characteristic p, and G be a finite group. Recall that $O^p(G)$ is the smallest normal subgroup of G such that the quotient is a p-group. If M is a finitely generated kG-module, we define $[O^p(G), M]$ to be the k-linear span of the set

$$\{g(m) - m \mid g \in O^p(G), m \in M\}.$$

This is the smallest submodule of M such that $O^p(G)$ acts trivially on the quotient, or equivalently the smallest submodule such that the quotient has a filtration where G acts trivially on the filtered quotients.

Inductively define $P_0 = N_0$ to be the projective cover $P(k)$ of the trivial kG-module k, and for $i \geq 1$, $M_{i-1} = [O^p(G), N_{i-1}]$, P_i is the projective cover of M_{i-1}, and $N_i = \Omega(M_{i-1})$, the kernel of $P_i \to M_{i-1}$. This construction gives us a complex of projective kG-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$

with the property that $O^p(G)$ acts trivially on the homology, and the P_i with $i \geq 1$ do not have the projective cover of k as a direct summand. We call this a *left k-squeezed resolution* for G, and we write $H^\Omega_*(G, k)$ for the homology of this complex. A more formal definition is given in Section 3.

The dual of this construction gives us a complex of injective kG-modules

$$0 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$$

such that I_0 is the injective hull of the trivial module (which is isomorphic to $P(k)$), $O^p(G)$ acts trivially on the homology, and the I_{-i} with $i \geq 1$ do not have $P(k)$ as a direct summand. We call this a *right k-squeezed resolution* for G, and we write $H^*_\Omega(G, k)$ for the homology of this complex, with the indexing negated to make it cohomological.

Our first theorem, which we prove in Section 5, gives a homological interpretation for the construction. Let f be an idempotent in kG such that $f.kG$ is isomorphic to the projective cover of the trivial module, and set $e = 1 - f$.

Theorem 1.1. *For $n \geq 2$, we have isomorphisms*

$$\mathrm{Tor}^{e.kG.e}_{n-1}(kG.e, e.kG) \cong H^\Omega_n(G, k)$$
$$\mathrm{Ext}^{n-1}_{e.kG.e}(e.kG, e.kG) \cong H^n_\Omega(G, k).$$

There are exact sequences

$$0 \to H^\Omega_1(G, k) \to kG.e \otimes_{e.kG.e} e.kG \to kG \to H^\Omega_0(G, k) \to 0$$
$$0 \to H^0_\Omega(G, k) \to kG \to \mathrm{Hom}_{e.kG.e}(e.kG, e.kG) \to H^1_\Omega(G, k) \to 0.$$

Notice that we can rewrite the left hand sides of the isomorphisms in Theorem 1.1 as

$$\mathrm{Tor}^{e.kG.e}_{n-1}(f.kG.e, e.kG.f) \quad \text{and} \quad \mathrm{Ext}^{n-1}_{e.kG.e}(e.kG.f, e.kG.f).$$

This makes computations slightly easier because $e.kG.f$ and $f.kG.e$ are smaller than $e.kG$ and $kG.e$.

Our second theorem gives a topological interpretation of $H^\Omega_*(G, k)$ and $H^*_\Omega(G, k)$.

Theorem 1.2. *For $i \geq 0$, $H^\Omega_i(G, k) \cong H_i(\Omega BG^\wedge_p; k)$ and $H^i_\Omega(G, k) \cong H^i(\Omega BG^\wedge_p; k)$.*

It is this theorem that we use for the MAGMA computations; examples can be found in the next section.

We prove Theorem 1.2 in Section 6 by making a model of $\Omega BG^\wedge_p$ that admits a free action of G with acyclic quotient. In Section 7, we provide another proof by interpreting the chains on $\Omega BG^\wedge_p$ as a best approximation to kG by chain complexes built from an injective resolution of k. This proof uses more machinery, but makes it easy to interpret the loop multiplication in $H_*(\Omega BG^\wedge_p; k)$ algebraically.

Combining the statements of Theorems 1.1 and 1.2, we obtain isomorphisms for $n \geq 2$:

$$\mathrm{Tor}^{e.kG.e}_{n-1}(f.kG.e, e.kG.f) \cong H_n(\Omega BG^\wedge_p; k) \tag{1.3}$$

$$\mathrm{Ext}^{n-1}_{e.kG.e}(e.kG.f, e.kG.f) \cong H^n(\Omega BG^\wedge_p; k). \tag{1.4}$$

Finally, in the last section we discuss some of the problems that motivated this work. These involve the boundary between polynomial and exponential growth for $H_*(\Omega BG^\wedge_p; k)$. We completely solve the problem in characteristic two for groups with elementary abelian Sylow 2-subgroup E. In this situation, we show that polynomial growth occurs if and only if $H^*(G, k)$ is a complete intersection. This condition is equivalent to the statement that $N_G(E)/O_{2'}N_G(E)$ is a direct product of copies of $\mathbb{Z}/2$, A_4 and $(\mathbb{Z}/2)^3 \rtimes F_{21}$, where F_{21} is a Frobenius group of order 21.

2. Examples

We illustrate the main theorems with some examples.

Example 2.1. Let $G = A_4$, the alternating group of degree four, and let k be a field of characteristic two. We write k, ω and $\bar{\omega}$ for the three simple kG-modules. The projective indecomposable kG-modules have the following structure:

$$\begin{array}{ccc} & k & \\ \omega & & \bar{\omega} \\ & k & \end{array} \qquad \begin{array}{ccc} & \omega & \\ \bar{\omega} & & k \\ & \omega & \end{array} \qquad \begin{array}{ccc} & \bar{\omega} & \\ k & & \omega \\ & \bar{\omega} & \end{array}$$

$$P(k) \qquad\qquad P(\omega) \qquad\qquad P(\bar{\omega})$$

We have $P_0 = N_0 = P(k)$,

$$M_0 = \begin{array}{ccc} \omega & & \bar{\omega} \\ & k & \end{array} \qquad P_1 = P(\omega) \oplus P(\bar{\omega}) \qquad N_1 = \begin{array}{ccccc} \bar{\omega} & & k & & \omega \\ & \omega & & \bar{\omega} & \end{array}$$

$$M_1 = \begin{array}{c} \bar{\omega} \\ \omega \end{array} \oplus \begin{array}{c} \omega \\ \bar{\omega} \end{array} \qquad P_2 = P(\bar{\omega}) \oplus P(\omega) \qquad N_2 = \begin{array}{c} k \\ \bar{\omega} \end{array} \oplus \begin{array}{c} k \\ \omega \end{array}$$

$$M_2 = \bar{\omega} \oplus \omega \qquad P_3 = P(\bar{\omega}) \oplus P(\omega) \qquad N_3 = \begin{array}{ccc} \bar{\omega} & & k \\ & \omega & \end{array} \oplus \begin{array}{ccc} k & & \omega \\ & \bar{\omega} & \end{array}$$

Thereafter, the construction has period two, so that $M_i \cong M_{i-2}$, $P_i \cong P_{i-2}$ and $N_i \cong N_{i-2}$. So according to Theorem 1.2, $H_n(\Omega BG_p^\wedge; k)$ has dimension one for $n = 0$ and $n = 1$, and it has dimension two if $n \geq 2$.

It is shown in Levi [19] that $H_*(\Omega BG_p^\wedge; k)$ does not always have polynomial growth. Using the computer system MAGMA [4], some examples of exponential growth were investigated using Theorem 1.2, and the following table gives the dimensions of the first few homology groups for these examples.

		Degree										
Group	p	0	1	2	3	4	5	6	7	8	9	10
$(\mathbb{Z}/2)^3 \rtimes \mathbb{Z}/7$	2	1	0	1	3	3	6	12	18	33	57	96
$(\mathbb{Z}/3)^2 \rtimes \mathbb{Z}/2$	3	1	1	5	12	32	84	220	576			
$(\mathbb{Z}/3)^2 \rtimes \mathbb{Z}/4$	3	1	1	3	6	12	22	42	80	150		
$(\mathbb{Z}/5)^2 \rtimes \mathbb{Z}/3$	5	1	1	3	6	14	30	64	138			

In each case the action in the semidirect product is faithful, and in the second entry the involution acts to invert every element of $(\mathbb{Z}/3)^2$.

Example 2.2. Take the group $(\mathbb{Z}/3)^2 \rtimes \mathbb{Z}/2$ given by the second entry in the table above, with k a field of characteristic three. Then $A = e.kG.e$ is the five dimensional commutative algebra

$$A = k[u, v, w]/(uw - v^2, u^2, uv, vw, w^2).$$

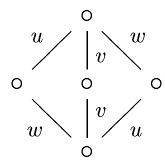
This is a Koszul algebra, whose Koszul dual is

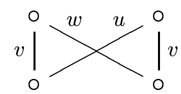
$$A^! = \mathrm{Ext}^*_A(k,k) = k\langle \alpha, \beta, \gamma\rangle/(\alpha\gamma + \gamma\alpha + \beta^2).$$

The module $M = e.kG.f$ is the four dimensional module given in terms of generators and relations as

$$M = (Ax \oplus Ay)/(vx - uy, wx - vy, ux, wy).$$

Here are pictures of A and M:

$A =$ 

$M =$ 

It is not hard to compute the syzygies of M as an A-module. Since A is a symmetric algebra with $J(A)^3 = 0$, each $\Omega^n M$ is an indecomposable module with $\mathrm{Rad}^2\Omega^n M = 0$ and $\mathrm{Soc}(\Omega^n M) = \mathrm{Rad}(\Omega^n M) \cong \Omega^{n-1}M/\mathrm{Rad}(\Omega^{n-1}M)$. Set $b_n = \dim_k \Omega^n M/\mathrm{Rad}(\Omega^n M)$ and $c_n = \dim_k \mathrm{Rad}(\Omega^n M)$. Then we have $b_0 = c_0 = 2$, $c_{n+1} = b_n$ and $b_{n+1} = 3b_n - c_n$.

This gives us the recurrence relation

$$b_0 = 2, \quad b_1 = 4, \quad b_{n+1} = 3b_n - b_{n-1} \quad (n \geq 2),$$

and hence the generating function

$$\sum_{n\geq 0} b_n t^n = \frac{2(1-t)}{1-3t+t^2} = 2 + 4t + 10t^2 + 26t^3 + 68t^4 + \cdots$$

Thus $b_n = 2F_{2n+1}$, where F_n is the nth Fibonacci number.

Lemma 2.3. *For every homomorphism from $\Omega^n M$ to M, for $n \geq 1$, the image lies in* $\mathrm{Rad}(M) = \mathrm{Soc}(M)$.

Proof (outline). The first step is to prove that there are short exact sequences

$$0 \to \Omega^n M \to \Omega^{n+1} M \to \Omega^n k \oplus \Omega^n k \to 0$$

for all $n \geq 0$. The second step is to prove by induction on n that for every $n \geq 0$, every endomorphism of $\Omega^n M$ can be written as a multiple of the identity plus an endomorphism whose image lies in $\mathrm{Rad}(\Omega^n M)$. The final step is to compose the given homomorphism from $\Omega^n M$ to M with the composite of the inclusions given by the short exact sequences above, $M \hookrightarrow \Omega M \hookrightarrow \cdots \hookrightarrow \Omega^n M$ to obtain an endomorphism of $\Omega^n M$, and examine the image. □

It follows from the lemma that for $n \geq 2$ there are short exact sequences

$$0 \to \mathrm{Ext}^{n-1}_A(M, M/\mathrm{Rad}(M)) \to \mathrm{Ext}^n_A(M, \mathrm{Rad}(M)) \to \mathrm{Ext}^n_A(M,M) \to 0.$$

Thus for $n \geq 2$ we have $\dim_k \mathrm{Ext}^n_A(M,M) = 2b_n - 2b_{n-1} = 4F_{2n}$. So

$$\sum_{n\geq 2} \dim_k \mathrm{Ext}^n_A(M,M) = -2b_1t^2 + 2(1-t)\sum_{n\geq 2} b_n t^n.$$

Finally, adjusting to give the correct dimensions for $n \le 2$, Theorem 1.1 gives us

$$\begin{aligned}\sum_{n\ge 0}\dim_k H^n(\Omega BG^\wedge_p;k)t^n &= 1+t+5t^2+2t(1-t)\sum_{n\ge 2} b_n t^n\\ &= \frac{(1-t+t^2)^2}{1-3t+t^2}\\ &= 1+t+5t^2+12t^3+32t^4+84t^5+220t^6+\cdots\end{aligned}$$

This function has poles at $(3\pm\sqrt{5})/2$; since one of these is in the interior of the unit circle in the complex plane, it follows that $\dim_k H^n(\Omega BG^\wedge_p;k)$ is growing exponentially with n. Alternatively, note that for $n \ge 3$ we have $\dim_k H^n(\Omega BG^\wedge_p;k) = 4F_{2n-2}$, and the Fibonacci numbers grow exponentially.

3. Squeezed resolutions

Let G be a finite group, and k be either a field of characteristic p. We also denote by k the trivial kG-module, namely the ring k regarded as a kG-module with trivial G-action. Thus if I_G is the augmentation ideal of kG, generated by the elements $g-1$ for $g \in G$, then $k \cong kG/I_G$.

If M is a kG-module, we define

$$[O^p(G), M] = \{g(m) - m \mid g \in O^p(G), m \in M\}.$$

This is the smallest submodule of M such that the quotient has a filtration with G acting trivially on the filtered quotients. We have

$$\mathrm{Hom}_{kG}([O^p(G),M],k) = 0.$$

Dually, we define $M^{O^p(G)}$ to be the fixed points of $O^p(G)$ on M. This is the largest submodule of M that has a filtration such that G acts trivially on the filtered quotients. We have

$$\mathrm{Hom}_{kG}(k, M/M^{O^p(G)}) = 0.$$

Definition 3.1. A *left k-squeezed resolution* is a complex

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

of kG-modules satisfying the following conditions.

(i) Each P_n is projective,

(ii) $H_n(P_* \otimes_{kG} k) \cong \begin{cases} k & n=0\\ 0 & n\ne 0,\end{cases}$

(iii) $[O^p(G), H_n(P_*)] = 0$ for all n.

We can construct a complex satisfying these conditions inductively by the process described in the introduction. The dual definition is as follows.

Definition 3.2. A *right k-squeezed resolution* is a complex

$$0 \to I_0 \xrightarrow{d_0} I_{-1} \xrightarrow{d_{-1}} I_{-2} \xrightarrow{d_{-2}} \cdots$$

of kG-modules satisfying the following conditions.

(i) Each I_n is injective,

(ii) $H_n \operatorname{Hom}_{kG}(k, I_*) \cong \begin{cases} k & n = 0 \\ 0 & n \neq 0, \end{cases}$

(iii) $[O^p(G), H_n(I_*)] = 0$ for all n.

We can construct a complex satisfying these conditions inductively by the dual of the process described in the introduction.

Theorem 3.3 (Comparison Theorem)**.** (i) *Let*

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$
$$\cdots \to P'_2 \to P'_1 \to P'_0 \to 0$$

be complexes together with augmentations $P_0 \to k$ and $P'_0 \to k$ satisfying:

(a) *each P_i a projective kG-module, and the augmentation $P_0 \to k$ induces an isomorphism $H_*(P_* \otimes_{kG} k) \to k$;*

(b) $[O^pG, H_*(P'_*)] = 0$.

Then there exists a map of complexes $P_* \to P'_*$ lifting the identity map on k:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \| & \\ \cdots \longrightarrow & P'_2 & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & k & \longrightarrow 0 \end{array}$$

Any two such comparison maps are chain homotopic.

(ii) *Let*

$$0 \to I_0 \to I_{-1} \to I_{-2} \to \dots$$
$$0 \to I'_0 \to I'_{-1} \to I'_{-2} \to \dots$$

be complexes together with coaugmentations $k \to I_0$ and $k \to I'_0$ satisfying:

(a) $[O^pG, H_*(I_*)] = 0$*;*

(b) *each I'_{-i} is an injective kG-module, and the augmentation $k \to I'_0$ induces an isomorphism $k \to H_* \operatorname{Hom}_{kG}(k, I'_*)$.*

Then there exists a map of complexes $I_* \to I'_*$ extending the identity map on k:

$$\begin{array}{ccccccccc} 0 \longrightarrow & k & \longrightarrow & I_0 & \longrightarrow & I_{-1} & \longrightarrow & I_{-2} & \longrightarrow \cdots \\ & \| & & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & k & \longrightarrow & I'_0 & \longrightarrow & I'_{-1} & \longrightarrow & I'_{-2} & \longrightarrow \cdots \end{array}$$

Any two such comparison maps are chain homotopic.

Proof. This is an exercise in diagram chasing. □

Corollary 3.4. (i) *Given two left k-squeezed resolutions P_* and P'_*, an isomorphism*

$$\mathrm{Hom}_{kG}(P_0, k) \cong \mathrm{Hom}_{kG}(P'_0, k),$$

lifts to a chain homotopy equivalence $P_* \to P'_*$, and hence to an isomorphism

$$H_*(P_*) \cong H_*(P'_*).$$

(ii) *Given two right k-squeezed resolutions I_* and I'_*, an isomorphism*

$$\mathrm{Hom}_{kG}(k, I_0) \cong \mathrm{Hom}_{kG}(k, I'_0),$$

extends to a chain homotopy equivalence $I_* \to I'_*$, and hence to an isomorphism

$$H_*(I_*) \cong H_*(I'_*).$$

Proof. This follows from the theorem. □

Definition 3.5. Given a finite group G, Corollary 3.4 implies that the homology of a left k-squeezed resolution P_* does not depend on the choice of resolution. So we define $H_n^\Omega(G,k) = H_n(P_*)$. Similarly, if I_* is a right k-squeezed resolution we define $H^n_\Omega(G,k) = H_{-n}(I_*)$.

4. Products, coproducts, subgroups, Steenrod operations

4.1. External coproducts. Let G and G' be finite group. If P_* and P'_* are left k-squeezed resolutions for G and G' then $P_* \otimes_k P'_*$ is a left k-squeezed resolution for $G \times G'$. This gives a Künneth theorem for homology. This gives

$$H_*^\Omega(G,k) \otimes_k H_*^\Omega(G',k) \cong H_*^\Omega(G \times G', k).$$

Dually, if I_* and I'_* are right k-squeezed resolutions for G and G' then $I_* \otimes_k I'_*$ is a right k-squeezed resolution for $G \times G'$. This gives

$$H^*_\Omega(G,k) \otimes_k H^*_\Omega(G',k) \cong H^*_\Omega(G \times G', k).$$

4.2. Restriction and corestriction maps. Let H be a subgroup of a finite group G. If P_* is a left k-squeezed resolution for G and Q_* is a left k-squeezed resolution for H then the Comparison Theorem 3.3 gives a map $Q_* \to \mathrm{res}_{G,H} P_*$, and hence a corestriction map

$$\mathrm{cores}_{H,G} \colon H_*^\Omega(H,k) \to H_*^\Omega(G,k).$$

Dually, there is a restriction map

$$\mathrm{res}_{G,H} \colon H^*_\Omega(G,k) \to H^*_\Omega(H,k).$$

4.3. Internal coproducts. Internal coproducts can be obtained by taking the external product for $G \times G$ and then restricting to the diagonal copy of G. This gives rise to a comparison map $P_* \to P_* \otimes P_*$, and hence to cocommutative coproducts

$$H_*^\Omega(G,k) \to H_*^\Omega(G,k) \otimes_k H_*^\Omega(G,k). \tag{4.4}$$

Dually, there are graded commutative products

$$H^*_\Omega(G,k) \otimes_k H^*_\Omega(G,k) \to H^*_\Omega(G,k). \tag{4.5}$$

4.6. Products. Using the Comparison Theorem 3.3, an element of $H_n^\Omega(G,k)$ can be lifted to a map of complexes

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & & & \\
\cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & 0
\end{array}$$

which is unique up to chain homotopy. This allows us to use composition of maps to define a product

$$H_*^\Omega(G,k) \otimes_k H_*^\Omega(G,k) \to H_*^\Omega(G,k). \tag{4.7}$$

This product is associative, but usually not graded commutative.

4.8. Steenrod operations. The comparison map (4.4) is E_∞. In particular, if k is a field, then there is an action of the Steenrod algebra on $H_*^\Omega(G,k)$ lowering degree. Similarly, there is an action of the Steenrod algebra on $H^*_\Omega(G,k)$ raising degree. We may develop this theory algebraically by analogy with the Evens norm map [1, Chapter 4] as follows.

Let k be a field of characteristic p. Suppose that G is a finite group, and consider the group $G\times\mathbb{Z}/p$. Then we have $O^p(G\times\mathbb{Z}/p) = O^p(G)$. So if J_* is a right k-squeezed resolution for G, then the tensor induced complex $J_*\overset{\uparrow}{\otimes}{}^{G\times Z/p}$ satisfies $[O^p(G), H_*(J_*\overset{\uparrow}{\otimes}{}^{G\times\mathbb{Z}/p})] = 0$. There is a subtle point here coming from the signs. If $p=2$ the signs are not a problem, but if p is odd then we must remark that the action of $G\times\mathbb{Z}/p$ on the cosets of G lies in the alternating group, and so the action of G on the homology of the tensor product is trivial.

It follows using the Comparison Theorem 3.3 that if I_* is a right k-squeezed resolution for G and Z_* is an injective resolution of k for $\mathbb{Z}/p$ then there is a comparison map $J_*\overset{\uparrow}{\otimes}{}^{G\times\mathbb{Z}/p} \to I_*\otimes_k Z_*$. So if $x\in H^n_\Omega(G,k) = H_{-n}(J_*)$ then $x^{\otimes p}$ defines an element of $H_{-np}(J_*\overset{\uparrow}{\otimes}{}^{G\times\mathbb{Z}/p})$ and hence by composition an element of $H_{-np}(I_*\otimes_k Z_*)$. It may be checked as in Lemma 4.1.1 of [1] that the resulting element is independent of the representative of the homology class, and so we obtain a well defined map

$$\mathrm{Norm}_{G,G\times\mathbb{Z}/p}\colon H^n_\Omega(G,k)\to H^*_\Omega(G,k)\otimes_k H^*(\mathbb{Z}/p,k).$$

As in Section 4.4 of [1], picking out coefficients of elements of $H^*(\mathbb{Z}/p,k)$ gives the Steenrod operations on $H^*_\Omega(G,k)$. For example if $k=\mathbb{F}_2$ then we have $H^*(\mathbb{Z}/2,\mathbb{F}_2)=\mathbb{F}_2[t]$ with $|t|=1$. So if $x\in H^n_\Omega(G,\mathbb{F}_2)$ then

$$\mathrm{Norm}_{G,G\times\mathbb{Z}/2}(x) = \sum_{j=0}^n \mathrm{Sq}^j(x)\otimes t^{n-j}$$

with $\mathrm{Sq}^j(x)\in H^{n+j}(G,\mathbb{F}_2)$. The corresponding formulae for p odd are more complicated, but are essentially the same as in [1, Proposition 4.4.3].

4.9. Yoneda interpretation. An n-fold k-squeezed extension for G is a complex of kG-modules

$$M_*\colon \qquad 0\to k\to M_{n-1}\to\cdots\to M_0\to k\to 0 \tag{4.10}$$

such that

(i) $[O^p(G), H_*(M_*)] = 0$,

(ii) the map $k \to M_{n-1}$ is injective, and

(iii) $M_0 \to k$ is surjective.

A map of n-fold k-squeezed extensions is a map of complexes which is the identity on the two copies of k. Two n-fold k-squeezed extensions are equivalent if there is a chain of maps connecting one with the other; in other words, this is the equivalence relation generated by the existence of a map of extensions.

Let P_* be a left k-squeezed resolution for G. Given an n-fold k-squeezed extension as above, the comparison theorem gives us a map of complexes

$$\begin{array}{ccccccccccc} & & & & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\ & & & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & k & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

and hence a map $\hat{\zeta}\colon \mathrm{Ker}(P_{n-1} \to P_{n-2}) \to k$. The map $\hat{\zeta}$ gives us a well defined element $\zeta \in H^n_\Omega(G, k)$ that is an invariant of the equivalence class of the k-squeezed extension. Writing L_ζ for the kernel of $\hat{\zeta}$, we obtain a map of k-squeezed extensions

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & k & \longrightarrow & P_{n-1}/L_\zeta & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & k & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

This way, we see that the equivalence classes of n-fold k-squeezed extensions are in one-one correspondence with elements of $H^n_\Omega(G, k)$.

4.11. The antipode. If $x \in H^n_\Omega(G, k)$ is represented by an n-fold k-squeezed extension (4.10), then we can form a dual extension

$$0 \to k \to M_0^* \to \cdots \to M_{n-1}^* \to k \to 0.$$

Here, $M^* = \mathrm{Hom}_k(M, k)$ is the k-linear dual of the kG-module M. Let $\tau(x) \in H^n_\Omega(G, k)$ be the class of this extension. Then $\tau\colon H^*_\Omega(G, k)$ is an antiautomorphism of order two for both the product and the coproduct.

The coproduct (4.4), product (4.7) and antipode (4.11) make $H^*_\Omega(G, k)$ into a commutative (but not cocommutative) Hopf algebra.

5. Homological reformulation

In this section we describe the homology of k-squeezed resolutions in terms of Tor and Ext over a finite dimensional algebra closely related to the group algebra. Let f be an idempotent in kG such that $f.kG$ is the projective cover of the trivial module, and let $e = 1 - f$. Let A denote the algebra $e.kG.e$. The following is Theorem 1.1 from the introduction.

Theorem 5.1. (i) *We have isomorphisms* $H_n^{\Omega}(G,k) \cong \mathrm{Tor}^A_{n-1}(kG.e, e.kG)$ *for* $n \geq 2$ *and an exact sequence*

$$0 \to H_1^{\Omega}(G,k) \to kG.e \otimes_A e.kG \to kG \to H_0^{\Omega}(G,k) \to 0.$$

(ii) *We have isomorphisms* $H^n_{\Omega}(G,k) \cong \mathrm{Ext}^{n-1}_A(e.kG, e.kG)$ *for* $n \geq 2$ *and an exact sequence*

$$0 \to H^0_{\Omega}(G,k) \to kG \to \mathrm{Hom}_A(e.kG, e.kG) \to H^1_{\Omega}(G,k) \to 0.$$

Proof. (i) Given a projective resolution of $e.kG$ as an A-module:

$$\cdots \to \bar{P}_2 \to \bar{P}_1 \to \bar{P}_0 \to 0$$

we consider the complex of kG-modules

$$\cdots \to kG.e \otimes_A \bar{P}_2 \to kG.e \otimes_A \bar{P}_1 \to kG.e \otimes_A \bar{P}_0 \to kG \to 0 \tag{5.2}$$

where the last non-zero map is the composite of the augmentation map and the multiplication map

$$kG.e \otimes_A \bar{P}_0 \to kG.e \otimes_A e.kG \to kG.$$

It is not hard to show that (5.2) is a left k-squeezed resolution. Define $P_0 = kG$ and $P_i = kG.e \otimes_A \bar{P}_{i-1}$ for $i \geq 1$. Then P_* is the mapping cone of the morphism of complexes $kG.e \otimes_A \bar{P}_* \to kG$. The long exact sequence of this mapping cone gives the required statements.

(ii) Dually, given a weakly injective resolution of $e.kG$ by projective A-modules:

$$0 \to \bar{I}_0 \to \bar{I}_{-1} \to \bar{I}_{-2} \to \dots$$

we consider the complex of kG-modules

$$0 \to kG \to \mathrm{Hom}_A(e.kG, \bar{I}_0) \to \mathrm{Hom}_A(e.kG, \bar{I}_1) \to \mathrm{Hom}_A(e.kG, \bar{I}_2) \to \cdots \tag{5.3}$$

where the first map is the composite

$$kG \to \mathrm{Hom}_A(e.kG, e.kG) \to \mathrm{Hom}_A(e.kG, \bar{I}_0).$$

It is not hard to show that (5.3) is a right k-squeezed resolution. Define $I_0 = kG$ and $I_{-i} = \mathrm{Hom}_A(e.kG, \bar{I}_{-i+1})$ for $i \geq 1$. Then I_* is the mapping cone of the morphism of complexes $kG \to \mathrm{Hom}_A(e.kG, \bar{I}_*)$. The long exact sequence of this mapping cone again gives the required statements. □

6. Loops on $BG^{\wedge}_p$

In this section, we give a homotopy theoretic interpretation of the squeezed resolutions described in Section 3. We begin by recalling that the Bousfield–Kan completion functor of [5] with respect to the coefficient ring $\mathbb{F}_p$ takes BG to a space which we denote $BG^{\wedge}_p$. There is a map $BG \to BG^{\wedge}_p$ which is a mod p homology equivalence, and we have $\pi_1(BG^{\wedge}_p) \cong G/O^p(G)$. Applying the Hurewicz theorem to the universal cover of $BG^{\wedge}_p$, we see that $\pi_2(BG^{\wedge}_p) \cong H_2(O^p(G), \mathbb{Z}_p)$.

Denote by $A_p(G)$ the homotopy fibre of the completion map $BG \to BG^\wedge_p$, and choose a basepoint in $A_p(G)$ mapping to the basepoint in BG. Then the long exact sequence of the fibration shows that there is an exact sequence

$$1 \to H_2(O^p(G), \mathbb{Z}_p) \to \pi_1(A_p(G)) \to O^p(G) \to 1$$

and since $A_p(G)$ is $\mathbb{F}_p$-acyclic, it follows that this is the universal p-central extension of $O^p(G)$.

Now form the pullback $L_p(G)$ as in the following diagram:

$$\begin{array}{ccccccc} & & L_p(G) & \longrightarrow & EG & & \\ & & \downarrow & & \downarrow & & \\ \Omega BG^\wedge_p & \longrightarrow & A_p(G) & \longrightarrow & BG & \longrightarrow & BG^\wedge_p \end{array}$$

Thus $L_p(G)$ is homotopy equivalent to $\Omega BG^\wedge_p$. The advantage of the space $L_p(G)$, though, is that it comes with a free action of G with connected acyclic quotient $A_p(G)$. So the chains $C_*(L_p(G); k)$ and the cochains $C^*(L_p(G); k)$ form complexes of free kG-modules.

Theorem 6.1. *$C_*(L_p(G); k)$ is a left k-squeezed resolution for G, and $C^*(L_p(G); k)$ is a right k-squeezed resolution for G.*

Proof. The fact that G acts freely on $L_p(G)$ gives condition (i) in Definitions 3.1 and 3.2. The fact that the quotient is connected and acyclic gives condition (ii). The fact that $O^p(G)$ acts trivially on $H_*(L_p(G); k)$ and on $H^*(L_p(G); k)$ gives condition (iii). □

Corollary 6.2. *There are natural isomorphisms*

$$H^\Omega_*(G, k) \cong H_*(\Omega BG^\wedge_p; k)$$
$$H^*_\Omega(G, k) \cong H^*(\Omega BG^\wedge_p; k).$$

Proof. This follows from Definition 3.5 and Theorem 6.1. □

This completes the proof of Theorem 1.2 of the introduction.

7. Approximation in $\mathsf{KInj}\,(kG)$

In this and the next section, we give another way of viewing the translation between the topology and the algebra.

We write $\mathsf{KInj}\,(kG)$ for the homotopy category of chain complexes of injective (or equivalently, projective) kG-modules. This category was studied in detail in Benson and Krause [2]. We recall some definitions from Dwyer and Greenlees [7], adapted to this context. Let ik denote an injective resolution of k, regarded as an object in $\mathsf{KInj}\,(kG)$, where it is well defined up to isomorphism. In $\mathsf{KInj}\,(kG)$, we say that an object N is *ik-trivial* if the chain complex of homomorphisms $\mathrm{Hom}_{kG}(ik, N) \simeq 0$. A map is an *$ik$-equivalence* if its mapping cone is ik-trivial. We say that X is *ik-torsion* if we have $\mathrm{Hom}_{kG}(X, N) \simeq 0$ for all ik-trivial N. We say that X is *ik-complete* if we have $\mathrm{Hom}_{kG}(N, X) \simeq 0$ for all ik-trivial N. We write ik-$\mathsf{Tors}(kG)$ and ik-$\mathsf{Comp}(kG)$ for the subcategories of ik-torsion and ik-complete complexes.

Remark 7.1. It is shown in Theorem 12.3 of [2] that localising subcategories of $\mathsf{KInj}\,(kG)$ are closed under filtered colimits. From this, it can be easily deduced that an object X in $\mathsf{KInj}\,(kG)$ is ik-torsion if and only if it is in the localising subcategory generated by ik.

Given any object X in $\mathsf{KInj}\,(kG)$, there exists an ik-torsion object $\mathsf{Cell}_{ik}(X)$ and an ik-equivalence $\mathsf{Cell}_{ik}(X) \to X$. Such an object and map are unique up to isomorphism.

Similarly, given any object X in $\mathsf{KInj}\,(kG)$, there exists an ik-complete object $X^{\wedge}_{ik}$ and an ik-equivalence $X \to X^{\wedge}_{ik}$. This is also unique up to isomorphism.

Lemma 7.2. (i) *Let I_* be an object in $\mathsf{KInj}\,(kG)$ which is bounded above and whose homology satisfies $[O^p(G), H_*(I_*)] = 0$. Then I_* is ik-torsion.*

(ii) *Let P_* be an object in $\mathsf{KInj}\,(kG)$ which is bounded below and whose homology satisfies $[O^p(G), H_*(P_*)] = 0$. Then P_* is ik-complete.*

Proof. (i) Let n be the largest integer such that $H_nI_* \neq 0$. Since I_* is bounded above, it is equivalent to a complex $I(0)_*$ such that $I(0)_j = 0$ for $j > n$. Let $X(0)_*$ be an injective resolution of $\mathrm{Soc}H_nI(0)_*$, shifted in degree by n, so that $X(0)_*$ is a direct sum of copies of shifts of ik. Then the isomorphism $H_nX(0)_* \to \mathrm{Soc}H_nI(0)_*$ extends to a map of complexes $X(0)_* \to I(0)_*$. Write $I(1)_*$ for the mapping cone, so that we have a triangle $X(0)_* \to I(0)_* \to I(1)_*$. Now $I(1)_*$ has smaller homology than $I(0)$, in the sense that either $H_jI(1)_* = 0$ for $j \geq n$, or $H_jI(1)_* = 0$ for $j > n$ and the radical length of $H_nI(1)_*$ is smaller than that of $H_nI(0)_*$.

So we repeat and get $X(1)_* \to I(1)_* \to I(2)_*$, and so on. By the octahedral axiom, we have an object $Y(1)_*$ and triangles $X(0)_* \to Y(1)_* \to X(1)_*$ and $Y(1)_* \to I(0)_* \to I(2)_*$. Proceeding inductively, with $Y(0)_* = X(0)_*$, we obtain objects $X(i)_*$, $Y(i)_*$ and $I(i)_*$ fitting into commutative diagrams

$$\begin{array}{ccccc}
Y(i-1)_* & = & Y(i-1)_* & & \\
\downarrow & & \downarrow & & \\
Y(i)_* & \longrightarrow & I(0)_* & \longrightarrow & I(i+1)_* \\
\downarrow & & \downarrow & & \downarrow \\
X(i)_* & \longrightarrow & I(i)_* & \longrightarrow & I(i+1)_*
\end{array}$$

Taking the homotopy colimit of the sequence of triangles

$$\begin{array}{ccccccc}
Y(0)_* & \longrightarrow & Y(1)_* & \longrightarrow & Y(2)_* & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
I(0)_* & = & I(0)_* & = & I(0)_* & = & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
I(1)_* & \longrightarrow & I(2)_* & \longrightarrow & I(3)_* & \longrightarrow & \cdots
\end{array}$$

we have $\varinjlim_i I(i)_* \simeq 0$ and so $I_* \simeq I(0)_* \simeq \varinjlim_i Y(i)_*$ is in the localising subcategory of $\mathsf{KInj}\,(kG)$ generated by ik.

(ii) The proof of this is dual. There is just one more point to be made, which is that generally the limit of an inverse system of triangles is not a triangle. However, the inverse systems in question are eventually constant in any particular degree, so the inverse limit of the triangles is indeed a triangle. □

Theorem 7.3. (i) *If I_* is a right k-squeezed resolution then $I_* \cong \mathsf{Cell}_{ik}(kG)$ in $\mathsf{KInj}\,(kG)$.*

(ii) *If P_* is a left k-squeezed resolution then $P_* \cong kG^\wedge_{ik}$ in $\mathsf{KInj}\,(kG)$.*

Proof. (i) Lemma 7.2 (i) shows that I_* is ik-torsion. There is a map $I_* \to kG$ which is an ik-equivalence, because its mapping cone is equivalent to a complex consisting of injective modules which admit no homomorphisms from k.

(ii) Lemma 7.2 (ii) shows that P_* is ik-complete. There is a map $kG \to P_*$ which is an ik-equivalence for the same reason as before. □

Now recall that if X is simply connnected, or k is a field of characteristic p and $\pi_1(X)$ is a finite p-group, then the Eilenberg–Moore construction gives equivalences

$$\mathbb{R}\mathrm{End}_{C^*(X;k)}(k) \cong C_*(\Omega X;k)$$
$$k \overset{\mathbb{L}}{\otimes}_{C^*(X;k)} k \cong C^*(\Omega X;k)$$

(see §4.22 of [8]). We can't apply this directly to BG with G a finite group, because $\pi_1(BG) = G$ is not necessarily a finite p-group. However, $\pi_1(BG^\wedge_p) = G/O^p(G)$ is a finite p-group, and we have $C^*(BG^\wedge_p;k) \simeq C^*(BG;k)$. In particular, we have

$$C_*(\Omega BG^\wedge_p;k) \cong \mathbb{R}\mathrm{End}_{C^*(BG;k)}(k) \tag{7.4}$$

$$C^*(\Omega BG^\wedge_p;k) \cong k \overset{\mathbb{L}}{\otimes}_{C^*(BG;k)} k. \tag{7.5}$$

Let $\mathcal{E}_G$ be the differential graded algebra $\mathrm{End}_{kG}(ik)$. Recall from Theorem 3.3 and Lemma 6.4 of Benson and Krause [2] that there is an adjunction

$$\mathsf{KInj}\,(kG) \underset{\mathrm{Hom}_{kG}(ik,-)}{\overset{-\overset{\mathbb{L}}{\otimes}_{\mathcal{E}_G} ik}{\rightleftarrows}} \mathsf{D}_{\mathrm{dg}}(\mathcal{E}_G) \simeq \mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$$

inducing an equivalence between $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ and $ik\text{-}\mathsf{Tors}(kG)$. The composite is the "cellularisation functor" in $\mathsf{KInj}\,(kG)$

$$\mathsf{Cell}_{ik}(X) = \mathrm{Hom}_{kG}(ik,X) \overset{\mathbb{L}}{\otimes}_{\mathcal{E}_G} ik.$$

We have $\mathrm{Hom}_{kG}(ik,kG) \cong k$ in $\mathsf{D}_{\mathrm{dg}}(\mathcal{E}_G)$, so that under this correspondence, the object k in $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ corresponds to $\mathsf{Cell}_{ik}(kG)$ in $ik\text{-}\mathsf{Tors}(kG)$.

Similarly, the functor $\mathrm{Hom}_{kG}(ik,-)\colon \mathsf{KInj}\,(kG) \to \mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ has a right adjoint, and this induces an equivalence between $\mathsf{D}_{\mathrm{dg}}(C^*(BG;k))$ and $ik\text{-}\mathsf{Comp}(kG)$. The composite is the "completion functor" in $\mathsf{KInj}\,(kG)$ sending X to $X^\wedge_{ik}$.

Theorem 7.6. *We have*

(i) $C_*(\Omega BG^\wedge_p;k) \cong \mathrm{End}_{kG}(\mathsf{Cell}_{ik}(kG)) \cong \mathrm{End}_{kG}(kG^\wedge_{ik})$,

(ii) $H_*(\Omega BG^\wedge_p;k) \cong H_*(kG^\wedge_{ik})$,

(iii) $H^*(\Omega BG^\wedge_p; k) \cong H_{-*}(\mathsf{Cell}_{ik}(kG))$.

Proof. Part (i) follows from the previous discussion.

(ii) Using the map $kG \to kG^\wedge_{ik}$, we have isomorphisms

$$H_*(kG^\wedge_{ik}) \cong \mathrm{Hom}_{kG}(kG, kG^\wedge_{ik})_* \cong \mathrm{Hom}_{kG}(kG^\wedge_{ik}, kG^\wedge_{ik})_*.$$

(iii) follows from (ii) by dualising both sides. □

Combining Theorems 7.3 and 7.6, we obtain our second proof of Theorem 1.2. The advantage of this proof is that it makes it clear that the product (4.7) in $H^\Omega_*(G,k)$ corresponds to the loop product in $H_*(\Omega BG^\wedge_p; k)$. This does not seem so apparent from the proof in Section 6.

8. Final remarks

Using work of Félix, Halperin and Thomas [12], Levi [19] proves that $H_*(\Omega BG^\wedge_p; k)$ has either polynomial or semi-exponential growth. Here, semi-exponential growth means that there exists $\lambda > 1$ such that for an infinite number of positive integers n the dimension of the nth homology group is at least $\lambda^{\sqrt{n}}$. In the case of polynomial growth, the homology is finitely generated, nilpotent, and a finite module over a central polynomial subalgebra (cf. Theorem A of [10], Theorem B of [11]). No examples are known where the growth is semi-exponential but not exponential. We close with some remarks on the boundary between polynomial and semi-exponential growth.

A theorem of Gulliksen [14, 15] states that if R is a commutative Noetherian local ring with residue field k then R is a complete intersection if and only if $\mathrm{Ext}^*_R(k,k)$ has polynomial growth.

In the case of connected graded commutative algebras, we have to modify the definition slightly to take account of the fact that odd degree elements anticommute instead of commuting. We say that a graded commutative ring R is *of polynomial type* over $R_0 = k$ if it can be generated by a set of elements $x_1, \dots, x_r$ of positive degree subject only to the relations coming from graded commutativity. Notice that if x_i is an element of odd degree then graded commutativity forces $2x_i^2 = 0$. If k is a field of characteristic two then this definition just gives polynomial rings. If k is a field of odd characteristic, on the other hand, then it defines a polynomial ring on the even degree generators tensored with an exterior algebra on the odd degree generators.

A homogeneous element y of a graded commutative ring R is said to be *regular* if the annihilator of y is $(y - (-1)^{|y|}y)$; i.e., it is the zero ideal if $|y|$ is even, and $(2y)$ if $|y|$ is odd. A sequence of elements $y_1, \dots, y_s$ of positive degree is said to be a *regular sequence* if each y_i is regular in $R/(y_1, \dots, y_{i-1})$.

A connected graded k-algebra is said to be a *complete intersection* if it can be written as a quotient of a k-algebra of polynomial type by a regular sequence. With this definition, Gulliksen's theorem continues to hold (Avramov and Iyengar, unpublished).

Theorem 8.1. *If $H^*(G,k)$ is a complete intersection then $H_*(\Omega BG^\wedge_p; k)$ has polynomial growth.*

Proof. The theorem follows from Gulliksen's theorem and the Eilenberg–Moore spectral sequence
$$\mathrm{Ext}^{**}_{H^*(G,k)}(k,k) \Rightarrow H_*(\Omega BG_p^\wedge;k). \qquad \square$$

We can prove the converse of Theorem 8.1 in the special case of an elementary abelian Sylow 2-subgroup in characteristic two. In this case, there is no ambiguity as to what is meant by a complete intersection since there are no signs to worry about.

Theorem 8.2. *Suppose that k is a field of characteristic two, and that G has an elementary abelian Sylow 2-subgroups E. Then the following are equivalent:*

(i) *$H_*(\Omega BG_2^\wedge;k)$ has polynomial growth.*

(ii) *$H^*(G,k)$ is a complete intersection.*

(iii) *$N_G(E)/C_G(E)$ is an odd order group generated by elements of order three whose fixed point set on E has index four.*

(iv) *$N_G(E)/O_{2'}N_G(E)$ is a direct product of copies of $\mathbb{Z}/2$, $A_4 = (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$ and $(\mathbb{Z}/2)^3 \rtimes F_{21}$, where F_{21} is the nonabelian group of order 21 and the action on $(\mathbb{Z}/2)^3$ is faithful.*

(v) *$G/O_{2'}(G)$ is a direct product of copies of $\mathbb{Z}/2$, A_4, $L_2(q)$ ($q = 4$ or $q \equiv 3,5 \pmod 8$), the Ree groups ${}^2G_2(3^{2m+1})$ ($m \geq 1$), $(\mathbb{Z}/2)^3 \rtimes F_{21}$, $L_2(8) \rtimes \mathbb{Z}/3$ and the Janko group J_1.*

Proof. First we prove that (i) is equivalent to (ii). If G has an elementary abelian Sylow 2-subgroup then $C^*(BG;k)$ is equivalent as a differential graded algebra to its cohomology. To see this, firstly it's true for $G = \mathbb{Z}/2$ since the reduced bar complex is the minimal resolution. For a more general elementary abelian 2-group, we use tensor products of copies of the reduced bar complex for the direct factors of the group to see that $C^*(BG;k)$ is equivalent to $H^*(G,k)$. In the case of a group with elementary abelian Sylow 2-subgroup E, $C^*(BG;k)$ is equivalent to the invariants of $N_G(E)/C_G(E)$ on $H^*(E,k)$.

It follows that the Eilenberg–Moore spectral sequence stops at the E_2 page, so that the dimension of $H_n(\Omega BG_p^\wedge;k)$ is equal to $\sum_{i+j=n}\dim_k \mathrm{Ext}^{i,j}_{H^*(G,k)}(k,k)$. Now we can use Gulliksen's theorem to see that (i) is equivalent to (ii).

Since $H^*(G,k) = H^*(E,k)^{N_G(E)/C_G(E)}$, the question of when (ii) holds reduces to the question of when the polynomial invariants of an odd order group over $\mathbb{F}_2$ form a complete intersection.

To prove that (ii) implies (iii), we remark that the general question of when polynomial invariants of a finite group form a complete intersection was studied by Gordeev [13], Kac and Watanabe [16] and Nakajima [21, 22, 23], among others. In particular, it is known that if the invariants form a complete intersection then the group is generated by elements whose fixed space has codimension at most two. Translating this into group theory, this means that $N_G(E)/C_G(E)$ should be generated by elements of order three whose fixed points on E have index four. This completes the proof that (ii) implies (iii).

The statement that (iv) implies (ii) is just a question of checking that the cohomology is a complete intersection in the cases given, and then using the Künneth formula for the cohomology of a product. The equivalence of (iv) and (v) is easy to deduce from John Walter's classification [24] of finite groups with an abelian Sylow 2-subgroup, together with Bombieri's proof [3] that the groups of Ree type are Ree groups.

It remains to prove that (iii) implies (iv). I am grateful to Geoff Robinson for supplying me with a proof of this. We use the phrase "3-reflection" for an element of order three in $\bar{G} = N_G(E)/C_G(E)$ whose centraliser in E has index four.

First we show that we may decompose $\bar{G}$ as a direct product of groups $\bar{G}_i$, and E as a direct product of E_i, so that $\bar{G}_i$ acts trivially on E_j for $j \neq i$, and $\bar{G}_i$ is generated by a single conjugacy class of 3-reflections. To see this, we note that if x and y are 3-reflections in $\bar{G}$ which generate non-conjugate subgroups then $\langle x, y\rangle$ is isomorphic to an odd order subgroup of $\mathrm{GL}(4,2)$ which is not isomorphic to F_{21}, and which must therefore be an elementary abelian 3-group. It follows that the conjugates of x and the conjugates of y generate mutually centralising normal subgroups of $\bar{G}$. Suppose that U is an irreducible summand of E on which both x and y act non-trivially. Setting $\mathbb{F} = \mathrm{End}_{\mathbb{F}_2\bar{G}}(U)$, we can regard U as an $\mathbb{F}\bar{G}$-module. We can write $U = U_1 \otimes_{\mathbb{F}} U_2$ so that x acts as scalars on U_1 and non-trivially on U_2, and y acts as scalars on U_2 and non-trivially on U_1. Since x and y are 3-reflections, this implies that either U_1 or U_2 is one dimensional; then we deduce that the other is also one dimensional. So x and y both act as scalars and U is one dimensional over $\mathbb{F}$. Thus y acts as x or x^{-1} on U and trivially on the remaining summands of E. So we can remove y from the generating set without loss. This completes the proof of the decomposition.

This means that we may assume that the action of $\bar{G}$ on E is faithful and irreducible. If $\bar{G}$ is trivial then $E \cong \mathbb{Z}/2$. Assuming we are not in this case, $\bar{G}$ is generated by a single conjugacy class of 3-reflections. The next step is to show that E is primitive. If it is induced from an $\mathbb{F}_2H$-module V for a proper subgroup H of $\bar{G}$ then x does not lie in the intersection of the conjugates of H, so V most be one dimensional, since otherwise x could not act as a 3-reflection. But then V is the trivial module, and this contradicts the irreducibility of E. Thus E is primitive.

As before, let $\mathbb{F} = \mathbb{F}_{2^e} = \mathrm{End}_{\mathbb{F}_2\bar{G}}(E)$, so that we may regard E as an $\mathbb{F}\bar{G}$-module. Since $\bar{G}$ contains a 3-reflection x, we have $e = 1$ or 2. If $e = 2$ then the fixed points of x on E have $\mathbb{F}$-codimension one. So for any $g \in \bar{G}$, $\langle x, x^g\rangle$ is isomorphic to an odd order subgroup of $GL(2,4) \cong \mathbb{Z}/3 \times A_5$ generated by elements of order three. Hence $\langle x, x^g\rangle$ is abelian. Since $\bar{G}$ is generated by the conjugates of x, it follows that $\bar{G}$ is cyclic, so $N_G(E)/O_{2'}N_G(E) \cong A_4$ and $E \cong (\mathbb{Z}/2)^2$. So if we are not in this case, we may assume that E is absolutely irreducible.

If E were absolutely primitive as an $\mathbb{F}_2\bar{G}$-module then using Clifford theory with respect to the Fitting subgroup $F(\bar{G})$ of the soluble group $\bar{G}$, we see that $C_{\bar{G}}(F(\bar{G})) = ZF(\bar{G})$ would be central in $\bar{G}$, and hence by Schur's lemma it would be trivial, a contradiction. So for some finite extension $\mathbb{F}$ of $\mathbb{F}_2$, $\mathbb{F}\otimes_{\mathbb{F}_2} E$ is induced from some proper maximal subgroup H of $\bar{G}$. If x is a 3-reflection in $\bar{G}$ then x must act as a single 3-cycle and possibly some fixed points on the cosets of H, or else the eigenvalues would be wrong. So the permutation action of $\bar{G}$ on the cosets of H is generated by 3-cycles and has odd order, so these 3-cycles commute. So G modulo the intersection of the conjugates of H is abelian, and hence H is normal of index 3. Since $\bar{G}$ contains 3-reflections, it follows that it's induced from a one dimensional representation of H, and so E is 3-dimensional. Thus $\bar{G}$ is isomorphic to a subgroup of $GL(3,2)$, and the only possibility is $\bar{G} \cong F_{21}$. □

Remark 8.3. The smallest simple group with elementary abelian Sylow 2-subgroups, and not satisfying the conditions of the theorem is $L_2(8)$. Since the Sylow 2-normalizer in this group is $(\mathbb{Z}/2)^3 \rtimes \mathbb{Z}/7$, this corresponds to the first row of the table in Section 2.

Conjecture 8.4. If G has abelian Sylow p-subgroups, then $H^*(G,k)$ is a complete intersection if and only if $H_*(\Omega BG_p^\wedge;k)$ has polynomial growth.

The conjecture fails badly when the Sylow p-subgroups are not abelian. For example, there are plenty of p-groups whose cohomology rings are not complete intersections. In this situation, we should use a "derived" notion of complete intersection. Dwyer, Greenlees and Iyengar [9] formulate the following definition, which we shall call "quasi complete intersection".

Definition 8.5. A differential graded augmented algebra $\Lambda \to k$ is said to be a quasi complete intersection if, given any object X in $D(\Lambda)$ with the augmentation ideal in its support, the thick subcategory it generates contains a non-zero object in the thick subcategory generated by Λ.

The following conjecture is due to John Greenlees:

Conjecture 8.6. For any finite group G, $C^*(BG;k)$ is a quasi complete intersection if and only if $H_*(\Omega BG_p^\wedge;k)$ has polynomial growth.

Finally, we reiterate and strengthen a conjecture made by Levi [17].

Conjecture 8.7. For any finite group G, $H_*(\Omega BG_p^\wedge;k)$ is a finitely generated k-algebra. Furthermore, either it has polynomial growth or it contains a free algebra on two generators.

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