CALCULUS OF FUNCTORS AND MODEL CATEGORIES

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ABSTRACT. The category of small covariant functors from simplicial sets to simplicial sets supports the projective model structure [5]. In this paper we construct various localizations of the projective model structure and also give a variant for functors from simplicial sets to spectra. We apply these model categories in the study of calculus of functors, namely for classification of polynomial and homogeneous functors. Finally we show that the n-th derivative induces a Quillen map between the n-homogeneous model structure on small functors from pointed simplicial sets to spectra and the category of spectra with Σ_n-action. We consider also a finitary version of the n-homogeneous model structure and the n-homogeneous model structure on functors from pointed finite simplicial sets to spectra. In these two cases the above Quillen map becomes a Quillen equivalence. This improves the classification of finitary homogeneous functors by T. G. Goodwillie [12].

1. Introduction

Calculus of homotopy functors applies to functors from spaces to spaces or spectra, which preserve weak equivalences. It interpolates between stable and unstable homotopy theory by analyzing carefully the rate of change of such functors. It was developed around 1990 by Thomas G. Goodwillie and has had spectacular applications to geometric topology [10, 11] and homotopy theory [1]. Although at the present time calculus of functors is a well developed and ramified theory, foundations of the subject remain technically involved. Part of the difficulty is the lack of a categorical framework. The problem is that the totality of functors from spaces to spaces does not form a legitimate category (the collections of morphisms need not be small).

In the current work we introduce a categorical approach to the foundations of the calculus of functors. We suggest to implement the ad-hoc machinery developed by Goodwillie as a part of a model category structure, which is a standard tool for describing an abstract homotopy theory. In order to overcome the set-theoretical difficulties we consider only small functors from spaces to spaces or from spaces to spectra, i.e. the functors that are determined as a left Kan extension by their values on a small subcategory only. For technical reasons, we use simplicial sets instead of topological spaces. This is justified by Kuhn's overview article [15], where first steps to an axiomatization of the theory are taken.

The projective model structure was constructed in [5]. In this paper we present several new model structures on the category of small functors, and each of these reflects certain aspect of Goodwillie's calculus.

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After necessary preliminaries on small functors in Section 2 we construct in Section 3 a localization of the projective model structure such that the new fibrant objects are precisely the projectively fibrant homotopy functors. This is the starting point for calculus of functors, since Goodwillie's machinery is intended for homotopy functors only. We make the interesting margin observation that any functor may be approximated by a homotopy functor in a universal way. It is worth mentioning that we do not have yet a general localization theory for model categories that are not cofibrantly generated. All localizations and colocalizations of model categories in this paper are constructed using the Bousfield-Friedlander localization technique, which is restricted to produce proper model categories only.

In Section 4 we localize the homotopy model structure on the category of small functors from spaces to spaces so that the new fibrant objects are precisely the n-excisive fibrant homotopy functors. This result may be viewed as a classification of n-polynomial functors. Goodwillie's n-th polynomial approximation P_n is equivalent to a fibrant replacement in our n-excisive model structure. An immediate advantage of having a model category structure is that the cofibrant replacement (equivalent to P_n) is universal, up to homotopy, with respect to maps into arbitrary n-excisive functor. This is an improvement of Goodwillie's result, which verifies the universal property only on the level of homotopy category [12, 1.8].

In the simpler category of functors from finite pointed spaces to all spaces, Lydakis has constructed the homotopy model structure as well as the 1-excisive (or stable) model structure (see [17], as well as its generalization [6] to more general model categories). Our work may be seen as a two-fold generalization of this work, since our results immediately apply to Lydakis' category. However, there are plenty of interesting small functors which are not finitary in the sense that they are determined by their values on finite spaces – for example, non-smashing Bousfield localizations.

In Section 5 we establish the stable projective, stable homotopy, and stable n-excisive model structures for small functors from (pointed) spaces to spectra. Then we recall and adapt several important definitions in Section 6. In Section 7 we colocalize the stable n-excisive model structure in order to obtain the n-homogeneous model structure. In this model structure, the bifibrant objects are precisely those projectively bifibrant homotopy functors which are n-homogeneous. This model structure may also be considered as a way to classify the n-homogeneous functors up to homotopy. T. Goodwillie has found another, simpler classification, but it applies only for finitary n-homogeneous functors or for a restriction of an arbitrary functor to finite spaces. Any such functor is determined by its n-th derivative, which is a spectrum with Σ_n-action.

In the final Section 8, we strengthen Goodwillie's classification. We introduce a finitary version our n-homogeneous model structure and an n-homogeneous model structure on the category of functors from pointed finite simplicial sets to spectra, and establish a Quillen equivalence between each of these model categories and the projective model structure on the category of spectra with Σ_n-action.

2. Preliminaries on small functors

Let $\mathcal{S}$ denote the category of simplicial sets. The object of study of this paper is homotopy theory of functors from simplicial sets to simplicial sets. The totality of these functors does not form a category in the usual sense – natural transformations between two functors need not form a set in general, but rather a proper class. On the other hand we are not eager to consider all such functors and would be satisfied with a treatment of a sufficiently large subcategory, which will be a category in the usual sense (with small hom-sets). The purpose of this section is to describe a satisfactory subcategory.

Definition 2.1. Let $\mathcal{C}$ be a (not necessarily small) simplicial category. A functor $\underset{\sim}{X}\colon \mathcal{C} \to \mathcal{S}$ is called *small* if $\underset{\sim}{X}$ is a small weighted colimit of representable functors. We denote the category of small functors as $\mathcal{S}^{\mathcal{C}}$.

Remark 2.2. If $\mathcal{C}$ is a small category then any functor from $\mathcal{C}$ to $\mathcal{S}$ is small. It is our point of view that small functors are the appropriate generalization, hence the notation. M.G. Kelly [14] calls small functors *accessible* and weighted colimits *indexed*. We emphasize that we are working in the enriched context: the colimits and left Kan extension are understood in the enriched sense.

Theorem 2.3 (For the proof see Prop. 4.83 of [14])**.** *A functor is small if and only if it is a left Kan extension from its restriction to a full small subcategory.*

Remark 2.4. Kelly proves also that small functors form an $\mathcal{S}$-category [14, 4.41], which is closed under small (weighted) colimits [14, Prop. 5.34]. That allows us to talk about simplicial function spaces $\hom(\underset{\sim}{X}, \underset{\sim}{Y})$ for any small functors $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ (in fact it suffices to demand that only $\underset{\sim}{X}$ is small). Existence of weighted colimits implies, in particular, that $\mathcal{S}^{\mathcal{C}}$ is tensored over $\mathcal{S}$, as the functor $-\otimes K$ may be viewed as a colimit over the trivial category weighted by $K \in \mathcal{S}$. Another immediate corollary of [14, Prop. 4.83] is that small functors are $\mathcal{S}$-functors, i.e. simplicial.

In order to initiate a discussion of homotopy theory we are bound to work in a category which is not only cocomplete, but also complete (at least under finite limits). It turns out that under some condition on $\mathcal{C}$ the category of small functors $\mathcal{S}^{\mathcal{C}}$ is complete.

P. Freyd [9] introduced the notion of *petty* and *lucid* set-valued functors. A set-valued functor is called *petty* if it is a quotient of a small sum of representable functors. Any small functor is clearly petty. A functor $F\colon \mathcal{A} \to \mathcal{S}ets$ is called *lucid* if it is petty and for any functor $G\colon \mathcal{A} \to \mathcal{S}ets$ and any pair of natural transformations $\alpha, \beta\colon G \rightrightarrows F$, the equalizer of α and β is petty. P. Freyd proved [9, 1.12] that the category of lucid functors from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{S}ets$ is complete if and only if $\mathcal{A}$ is *approximately* complete (that means that the category of cones over any small diagram in $\mathcal{A}$ has a weakly initial set).

J. Rosický proved [19, Lemma 1] that if the category $\mathcal{A}$ is approximately complete, then a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}ets$ is small if and only if it is lucid.

These results provide a full answer for the question when the category of small set-valued functors from a large category is complete. Unfortunately this is not sufficient for doing homotopy theory. We are interested in *simplicial* functors from a large simplicial category to simplicial sets. These results were partly generalized by S. Lack [16] to the enriched settings. Lack shows, in particular, that the category of small functors from $\mathcal{K}^{\mathrm{op}}$ to $\mathcal{S}$ is complete if $\mathcal{K}$ is a complete $\mathcal{S}$-category. Even more generally, the same result holds if one replaces $\mathcal{S}$ by a symmetric closed monoidal category $\mathcal{V}$, which is locally finitely presentable as a closed category. Existence of weighted limits implies, in particular, that $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ is cotensored over $\mathcal{S}$, as the functor $(-)^K$ may be viewed as a limit over the trivial category weighted by

$K \in \mathcal{S}$. The cotensor $(-)^K$ is the right adjoint to the tensor functor $- \otimes K$ by the usual commutation rules of weighted limits with the mapping spaces [14, 3.8].

Lack's results allow us to consider a model category structure on the category of small functors $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$. The simplest model structure on a category of functors is the projective model structure: weak equivalences and fibrations are levelwise. This model structure was established in [5].

The weak equivalences and fibrations in the projective model structure are "detected" by the maps from the representable functors: $R_A = \hom_{\mathcal{K}}(-, A)$ is the functor from $\mathcal{K}^{\mathrm{op}}$ to $\mathcal{S}$ represented in A. The following lemma is crucial for the proof of the existence of projective model structure.

Lemma 2.5 (For the proof see 3.1 in [5])**.** *The subcategory of representable functors is locally small in $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ (i.e., the inclusion functor satisfies the cosolution set condition).*

The category of small functors $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ carries the projective model structure with weak equivalences and fibrations being levelwise and cofibrations defined by the left lifting property with respect to trivial fibrations [5, Theorem 3.2].

Recall that a model category is class-cofibrantly generated if there are two classes I and J of generating cofibrations and generating trivial cofibrations, respectively, which admit the generalized small object argument [4] and generate the model structure in the usual sense: I-inj $=$ {trivial fibrations} and J-inj $=$ {fibrations}. Class-cofibrantly generated model categories share many nice properties with cofibrantly generated model categories. In particular, the category of functors from a small category to a class-cofibrantly generated model category may be equipped with the projective model structure (which is class-cofibrantly generated again).

The projective model category is class-cofibrantly generated with

$$I = \{R_A \otimes \partial\Delta^n \hookrightarrow R_A \otimes \Delta^n | A \in \mathcal{K},\, n \geq 0\} \tag{1}$$

$$J = \{R_A \otimes \Lambda^n_k \tilde{\hookrightarrow} R_A \otimes \Delta^n | A \in \mathcal{K},\, n > 0,\, 0 \leq k \leq n\}. \tag{2}$$

being classes of generating cofibration and trivial cofibrations respectively.

We summarize the properties of the projective model structure in the following

Proposition 2.6. *The projective model category structure on $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ is simplicial, proper and class-cofibrantly generated.*

Proof. Since $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ is enriched over $\mathcal{S}$, it suffices to verify SM7(a):

If $p\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ is a fibration (resp. trivial fibration) and $i\colon K \hookrightarrow L$ is a cofibration of simplicial sets, then the induced map $\hom(i,p)\colon \underset{\sim}{X}^L \to \underset{\sim}{X}^K \times_{\underset{\sim}{Y}^K} \underset{\sim}{Y}^L$ is a fibration (resp. trivial fibration).

But cotensor products and pullbacks are computed levelwise (as all weighted limits and colimits, since $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}}$ contains the representable functors), therefore the map $\hom(i,p)$ is an objectwise fibration (resp. trivial fibration), since the category of simplicial sets is simplicial. Hence $\hom(i,p)$ is a fibration (resp. trivial fibration) in the projective model structure on the category of small diagrams.

Properness follows in a similar manner from the properness of simplicial sets and the fact that pushouts and pullback are computed levelwise. This involves the fact that projective cofibrations are levelwise cofibrations. □

We are interested in the case $\mathcal{K}^{\mathrm{op}} = \mathcal{S}$. Then the category of small functors has another important property: it is closed under composition. We will need this property in the next section. In the covariant case here and in the rest of the paper let $R^A = \hom(A, \cdot\,)$ denote the enriched functor represented by A.

Lemma 2.7. *The category of small functors $\mathcal{S}^{\mathcal{S}}$ is closed under composition.*

Proof. Given two small functors $\underset{\sim}{X}, \underset{\sim}{Y} \in \mathcal{S}^{\mathcal{S}}$, we need to show that their composition $\underset{\sim}{X} \circ \underset{\sim}{Y}$ is a small functor again.

It suffices to verify that $R^A \circ \underset{\sim}{Y}$ is a small functor for any representable functor R^A, $A \in \mathcal{S}$, since $\underset{\sim}{X}$ is a weighted colimit of representable functors and small functors are closed under weighted colimits. But $(R^A \circ \underset{\sim}{Y})(\cdot\,) = R^A(\underset{\sim}{Y}(\cdot\,)) = \hom(A, \underset{\sim}{Y}(\cdot\,)) = \underset{\sim}{Y}^A$. And $\underset{\sim}{Y}^A$ is a small functor, since the category of small functors is closed under cotensor products (as under all weighted limits). □

3. Homotopy model structure on $\mathcal{S}^{\mathcal{S}}$

In this section we consider the case $\mathcal{K} = \mathcal{S}^{\mathrm{op}}$ and localize the projective model structure on $\mathcal{S}^{\mathcal{K}^{\mathrm{op}}} = \mathcal{S}^{\mathcal{S}}$ in such a way that fibrant objects after localization are exactly the projectively fibrant homotopy functors.

Note that small functors are simplicial. That implies, in particular, that simplicial homotopy equivalences are mapped into simplicial homotopy equivalences. Now, all simplicial sets are cofibrant and every weak equivalence between objects which are both fibrant and cofibrant is a simplicial homotopy equivalence [18, §2, Prop. 5]. In other words small functors preserve weak equivalences between fibrant objects.

We will construct the required localization by the method of Bousfield-Friedlander [2, A.7], which relies on existence of a coaugmented functor $Q\colon \mathcal{S}^{\mathcal{S}} \to \mathcal{S}^{\mathcal{S}}$, with coaugmentation $\eta\colon \mathrm{Id} \to Q$.

Let $\mathrm{fib}\colon \mathcal{S} \to \mathcal{S}$ be a small fibrant replacement functor. To construct it it suffices to take $\mathrm{fib} = \hat{R^*} = \hat{\mathrm{Id}}$ to be a fibrant replacement of the identity functor in the projective model structure on the category of small functors. The functor fib is equipped with a coaugmentation $\epsilon\colon \mathrm{Id} \to \mathrm{fib}$.

Define $Q\underset{\sim}{X} = \underset{\sim}{X} \circ \mathrm{fib}$ for all $\underset{\sim}{X} \in \mathcal{S}^{\mathcal{S}}$, then $Q\colon \mathcal{S}^{\mathcal{S}} \to \mathcal{S}^{\mathcal{S}}$ is a functor equipped with a coaugmentation η given by $\eta_{\underset{\sim}{X}} = \underset{\sim}{X} \circ \epsilon$. In this context a map $\underset{\sim}{X} \to \underset{\sim}{Y}$ is called a Q-equivalence, if it induces a weak equivalence $Q\underset{\sim}{X} \to Q\underset{\sim}{Y}$. Such a map will be called a Q-fibration if it has the right lifting property with respect to all projective cofibrations, that are also Q-equivalences.

Proposition 3.1. *Q is a coaugmented functor satisfying the following properties:*

- **(A.4):** *Q is a homotopy functor, i.e. it preserves levelwise weak equivalences;*
- **(A.5):** *Q is a homotopy idempotent functor, i.e. $\eta_{Q\underset{\sim}{X}}, Q\eta_{\underset{\sim}{X}}\colon Q\underset{\sim}{X} \rightrightarrows QQ\underset{\sim}{X}$ are levelwise weak equivalences;*
- **(A.6):** *For a pullback square*

$$\begin{array}{ccc} \underset{\sim}{A} & \xrightarrow{h} & \underset{\sim}{X} \\ \downarrow & & \downarrow j \\ \underset{\sim}{B} & \xrightarrow{k} & \underset{\sim}{Y} \end{array}$$

in $\mathcal{S}^{\mathcal{S}}$, if j is a Q-fibration and k is a Q-equivalence, then h is a Q-equivalence. (The dual condition was removed in [3]).

Proof. Given a levelwise weak equivalence, it is, in particular, a weak equivalence between fibrant entries, so applying Q, we obtain a weak equivalence again. Hence, (A.4).

(A.5) follows from the homotopy idempotence of the fibrant replacement functor in $\mathcal{S}$.

To verify that (A.6) is true note first that Q-equivalences are precisely the natural transformations of small functors which induce weak equivalences between fibrant entries. Since every Q-fibration is, in particular, a levelwise fibration, and the pullbacks in the category of small functors are computed objectwise and therefore the result follows from the right properness of $\mathcal{S}$. □

Theorem 3.2. *The category of small functors $\mathcal{S}^{\mathcal{S}}$ may be equipped with a proper simplicial model structure such that weak equivalences are Q-equivalences, cofibrations are projective cofibrations, and fibrations are Q-fibrations.*

Proof. It follows from Theorem A.7 of [2] and Proposition 3.1. □

Definition 3.3. The model structure on $\mathcal{S}^{\mathcal{S}}$ from theorem 3.2 will be called the homotopy model structure.

Corollary 3.4. (i) *The Q-equivalences are exactly those natural transformations between small functors that are weak equivalences on fibrant spaces.*

(ii) *A map $\underset{\sim}{X} \to \underset{\sim}{Y}$ is a Q-fibration if and only if it is a projective fibration such that the following square*

$$\begin{array}{ccc} \underset{\sim}{X} & \xrightarrow{\eta_{\underset{\sim}{X}}} & Q\underset{\sim}{X} \\ \downarrow & & \downarrow \\ \underset{\sim}{Y} & \xrightarrow{\eta_{\underset{\sim}{Y}}} & Q\underset{\sim}{Y} \end{array}$$

is a homotopy pullback square in the projective structure.

Proof. Part (i) follows directly from the definition of Q and part (ii) follows from the characterization theorem of Q-fibrations in [2]. □

Corollary 3.5. *Every small functor may be approximated by a homotopy functor in a universal, up to homotopy, way. In other words: for every small functor $\underset{\sim}{X} \in \mathcal{S}^{\mathcal{S}}$ there exists a functor $h\underset{\sim}{X}$ and a natural transformation $\iota\colon \underset{\sim}{X} \to h\underset{\sim}{X}$ such that for every projectively fibrant homotopy functor $\underset{\sim}{Y}$ and a natural transformation $\zeta \underset{\sim}{X} \to \underset{\sim}{Y}$ there exists a natural transformation $\xi\colon h\underset{\sim}{X} \to \underset{\sim}{Y}$, unique up to homotopy, such that $\zeta = \xi \circ \iota$.*

Proof. The functor $h\underset{\sim}{X}$ is obtained by factorization of the map $\underset{\sim}{X} \to *$ into a trivial cofibration followed by a fibration in the homotopy model structure. □

4. The n-excisive structure

In this section we localize the homotopy model structure on the category of small endofunctors of $\mathcal{S}$ in such a way that the fibrant replacement becomes the n-excisive part of a functor.

We begin with recalling necessary definitions from [12].

Definition 4.1. Let $\mathcal{P}(\underline{n})$ be the power set of the set $\underline{n} = \{1, \dots, n\}$ equipped with its canonical partial ordering. For later use we let $\mathcal{P}_0(\underline{n})$ be the complement of $\emptyset$ in $\mathcal{P}(\underline{n})$. An n-cubical diagram in $\mathcal{S}$ is a functor $\mathcal{P}(\underline{n}) \to \mathcal{S}$. A homotopy functor F is

- **excisive** if it takes homotopy pushouts to homotopy pullbacks,
- **reduced** if $F(*) \simeq *$,
- **linear** if it is both excisive and reduced.

A cubical diagram is

- **strongly homotopy cocartesian** if all of its two-dimensional faces are homotopy pushouts,
- **homotopy cartesian** if it is a 'homotopy pullback'.

A functor F is said to be **n-excisive** if it takes if it takes strongly homotopy cocartesian (n+1)-cubical diagrams to homotopy cartesian diagrams, see [12, 3.1].

For an arbitrary homotopy functor $\underset{\sim}{X}$ Goodwillie constructs an n-excisive approximation $p_{n,\underset{\sim}{X}}\colon \underset{\sim}{X} \to P_n\underset{\sim}{X}$, which is natural in $\underset{\sim}{X}$ and universal among all n-excisive functors under $\underset{\sim}{X}$. This is the n-excisive part of the Taylor tower of $\underset{\sim}{X}$. Since P_n is a simplicial functor, it has a natural extension to functors with values in spectra. We need the following properties from [12].

Lemma 4.2. *On the full subcategory of homotopy functors of $\mathcal{S}^{\mathcal{S}}$, the functor P_n commutes with finite homotopy limits and filtered homotopy colimits. The extension of P_n to homotopy functors with values in spectra commutes with arbitrary homotopy colimits.*

The functor P_n is not defined on all objects in $\mathcal{S}^{\mathcal{S}}$, but just on the homotopy functors. To remedy this we precompose P_n with our fibrant replacement functor $F \in \mathcal{S}^{\mathcal{S}}$, which we used in the construction of the homotopy model structure 3.1. This ensures that P_n gets applied to a homotopy functor. We do not want to introduce new notation, so the reader should remember that our P_n differs from Goodwillie's P_n.

Definition 4.3. Let $P_n\colon \mathcal{S}^{\mathcal{S}} \to \mathcal{S}^{\mathcal{S}}$ be the functor given by

$$\underset{\sim}{X} \mapsto P_n\underset{\sim}{X} := P_n(\underset{\sim}{X} \circ \mathrm{fib}).$$

It is a coaugmented functor with coaugmentation $\eta_{n,\underset{\sim}{X}} = p_n \circ \eta_{\underset{\sim}{X}}$.

Now we will show that we can perform the Bousfield-Friedlander localization using P_n, thus giving the desired n-excisive model structure.

Proposition 4.4. *The functor P_n satisfies the properties* **(A.4)**, **(A.5)** *and* **(A.6)** *from proposition* 3.1.

Proof. Condition **(A.4)** is fulfilled by construction, P_n preserves weak equivalences in the homotopy structure. Condition **(A.5)** is shown in [12, proof of 1.8]. It remains to prove condition **(A.6)**, but this follows directly from the fact that P_n preserves homotopy pullbacks. □

Definition 4.5. We call a map $\underset{\sim}{X} \to \underset{\sim}{Y}$ in $\mathcal{S}^{\mathcal{S}}$

(1) an n-excisive equivalence if $P_n\underset{\sim}{X} \to P_n\underset{\sim}{Y}$ is an equivalence in the homotopy model structure.
(2) an n-excisive fibration if it has the right lifting property with respect to all cofibrations, that are also n-excisive equivalences.

These classes of maps will be called the n-excisive structure on $\mathcal{S}^{\mathcal{S}}$.

Remark 4.6. A map between two homotopy functors is an equivalence in the homotopy model structure if and only if it is an objectwise weak equivalence. The functor $P_n\underset{\sim}{X}$ is a homotopy functor by definition, so $P_n\underset{\sim}{X} \to P_n\underset{\sim}{Y}$ is an equivalence in the homotopy model structure if and only if it is an objectwise weak equivalence.

The next theorem follows again from the Bousfield-Friedlander localization theorem [2].

Theorem 4.7. *The n-excisive structure on $\mathcal{S}^{\mathcal{S}}$ forms a proper simplicial model structure. A map $\underset{\sim}{X} \to \underset{\sim}{Y}$ is an n-excisive fibration if and only if it is a fibration in the homotopy structure, such that the diagram*

$$\begin{array}{ccc} \underset{\sim}{X} & \xrightarrow{\eta_{n,\underset{\sim}{X}}} & P_n\underset{\sim}{X} \\ \downarrow & & \downarrow \\ \underset{\sim}{Y} & \xrightarrow{\eta_{n,\underset{\sim}{Y}}} & P_n\underset{\sim}{Y} \end{array}$$

is a homotopy pullback square in the homotopy structure. Fibrant objects are exactly the objectwise fibrant n-excisive homotopy functors.

5. Homotopy theory of spectrum-valued functors

In this section we introduce a model category that describes homotopy theory of small functors with values in spectra. First of all we have to give a definition of small spectrum-valued functors. To streamline the exposition we will use the category of pointed spaces $\mathcal{S}_*$ as our underlying symmetric monoidal category. Note that all arguments of this paper go through for the category $\mathcal{S}_*^{\mathcal{S}_*}$ of small (enriched) endofunctors of pointed spaces if one replaces a construction by its pointed analogue. Let Sp denote the category of spectra in the sense of Bousfield-Friedlander [2]. More generally we use $\mathrm{Sp}(\mathcal{M})$ for a pointed simplicial model category $\mathcal{M}$.

Definition 5.1. An object in the category $\mathrm{Sp}(\mathcal{M})$ is a sequence $(X_0, X_1, ...)$ of objects in $\mathcal{M}$ together with bonding maps

$$\Sigma X_n \to X_{n+1},$$

for $n \geq 0$, where $\Sigma X_n := X_n \otimes \Delta^1/\partial\Delta^1$.

Definition 5.2. A functor from $\mathcal{S}_*$ to Sp is small if it is the left Kan extension of a functor defined on a small subcategory of $\mathcal{S}_*$. We remind the reader that this is to be understood in the enriched context, see remark 2.2 and theorem 2.3.

Definition 5.3. For each $n \geq 0$ let $\mathrm{Ev}_n\colon \mathrm{Sp} \to \mathcal{S}_*$ denote the functor taking a spectrum $X = (X_0, X_1, ...)$ to its n-th level X^n.

Lemma 5.4. *A functor $F\colon \mathcal{S}_* \to \mathrm{Sp}$ is small if and only if it is levelwise small, i.e. if $\mathrm{Ev}_n \circ F\colon \mathcal{S}_* \to \mathcal{S}_*$ is small for each $n \geq 0$.*

Proof. The evaluation functors Ev_n are simplicial and have enriched right adjoints, which therefore commute with enriched left Kan extensions. □

Lemma 5.5. *The evident functors give an equivalence $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*}) \cong \mathrm{Sp}^{\mathcal{S}_*}$ of categories.*

Proof. This follows directly from lemma 5.4. □

Using lemma 5.5 we will identify the categories $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*})$ and $\mathrm{Sp}^{\mathcal{S}_*}$. This shows in particular that the category $\mathrm{Sp}^{\mathcal{S}_*}$ is complete. Now we want to lift the projective model structure, where a weak equivalence is given objectwise, to the spectrum-valued setting. Our strategy is the following: We take the projective model structure on $\mathcal{S}_*^{\mathcal{S}_*}$ and then consider spectrum objects over this category. Using results from [20] we obtain a model structure on $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*})$, which is the desired one.

Equally well, we could use the pointed category $(\mathcal{S}^{\mathcal{S}})_* \cong \mathcal{S}_*^{\mathcal{S}}$ to describe a theory of small functors originating in unpointed spaces. This would give a model for the category of small functors from unpointed spaces to spectra, but we will not pursue this now.

In [20] the stable model structure on spectra is obtained analogously as in [2], but the construction Q used there is adaptable to more general situations. In lemma 1.3.2. of [20] the properties are listed, which have to be satisfied in order to make the machinery work. Although in our case the underlying model structure on $\mathcal{S}_*^{\mathcal{S}_*}$ is not cofibrantly generated, we are still able to prove the statements of this lemma. The reason is that our category is class-cofibrantly generated, the only deficiency being the (possible) lack of functorial factorization. Here is the adapted version of the relevant part (a) of the cited lemma.

Lemma 5.6. *Let $X \to Y$ be a termwise (trivial) fibration between sequences in the category $\mathcal{S}_*^{\mathcal{S}_*}$. Then the induced map $\operatorname{colim} X \to \operatorname{colim} Y$ is a (trivial) fibration. In particular, sequential colimits preserve weak equivalences.*

Proof. The proof for the case of fibration and trivial fibration is literally the same except that one uses the different test classes I or J from (1). Since source and target of the generating classes I and J are small, we get the following liftings

$$\begin{array}{ccccc} R^A \otimes K & \longrightarrow & X^k & \longrightarrow & \operatorname{colim} X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle (\simeq)} & & \downarrow \\ R^A \otimes L & \longrightarrow & Y^k & \longrightarrow & \operatorname{colim} Y \end{array}$$

where i is either in I or J. This proves the statement. □

For the definition of the coaugmented functor $Q\colon \mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*}) \to \mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*})$ we refer to [20, p. 93]. For each K in $\mathcal{S}_*$ the spectrum $(Q\underset{\sim}{X})(K)$ is weakly equivalent to usual Ω-spectrum $Q(\underset{\sim}{X}(K))$ in the Bousfield-Friedlander sense.

Definition 5.7. A map $\underset{\sim}{X} \to \underset{\sim}{Y}$ in $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*})$ will be called

(i) a stable projective cofibration if the map $\underset{\sim}{X}^0 \to \underset{\sim}{Y}^0$ and for each $n \geq 0$ the maps $\underset{\sim}{X}^n \vee_{\underset{\sim}{X}^{n-1}} \underset{\sim}{Y}^{n-1} \to \underset{\sim}{Y}^n$ are projective cofibrations.

(ii) a stable projective equivalence if for all $n \geq 0$ the maps $Q\underset{\sim}{X}^n \to Q\underset{\sim}{Y}^n$ are projective equivalences.

(iii) a stable projective fibration if for all $n \geq 0$ the maps $Q\underset{\sim}{X}^n \to Q\underset{\sim}{Y}^n$ are projective fibrations and the squares

$$\begin{array}{ccc} \underset{\sim}{X}^n & \longrightarrow & Q\underset{\sim}{X}^n \\ \downarrow & & \downarrow \\ \underset{\sim}{Y}^n & \longrightarrow & Q\underset{\sim}{Y}^n \end{array}$$

are homotopy pullback squares in the projective structure.

We call these classes of maps the stable projective model structure on $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*})$.

Theorem 5.8. *The stable projective model structure on* $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*}) \cong \mathrm{Sp}^{\mathcal{S}_*}$ *is a simplicial proper model structure.*

We point out again, that we do not claim that this model structure has (or has not) functorial factorization. The proof of this theorem is as in [20].

To localize this model structure in order to obtain the homotopy model structure we observe that as in the unstable setting small functors in $\mathrm{Sp}^{\mathcal{S}_*}$ are simplicial. Hence they preserve simplicial homotopies and therefore map weak equivalences between fibrant spaces to weak equivalences. So we can use the same method as in Section 3 to obtain the homotopy structure on $\mathrm{Sp}^{\mathcal{S}_*}$.

Definition 5.9. A map $\underset{\sim}{X} \to \underset{\sim}{Y}$ in $\mathrm{Sp}^{\mathcal{S}_*}$ will be called
(i) a stable equivalence in the homotopy structure if $\underset{\sim}{X}(K) \to \underset{\sim}{Y}(K)$ is a stable equivalence of spectra for all fibrant spaces K.
(ii) a stable fibration in the homotopy structure if $\underset{\sim}{X} \to \underset{\sim}{Y}$ is a stable projective fibration and the square

$$\begin{array}{ccc} \underset{\sim}{X} & \longrightarrow & \underset{\sim}{X} \circ \mathrm{fib} \\ \downarrow & & \downarrow \\ \underset{\sim}{Y} & \longrightarrow & \underset{\sim}{Y} \circ \mathrm{fib} \end{array}$$

is a homotopy pullback square in the stable projective structure. Here $\mathrm{fib}\colon \mathcal{S}_* \to \mathcal{S}_*$ is a small fibrant replacement functor.
We call these classes of maps the stable homotopy model structure on $\mathrm{Sp}(\mathcal{S}_*^{\mathcal{S}_*})$.

As in Section 3 we obtain the following theorem.

Theorem 5.10. *The stable homotopy model structure on* $\mathrm{Sp}^{\mathcal{S}_*}$ *is a simplicial proper model structure. A functor in* $\mathrm{Sp}^{\mathcal{S}_*}$ *is a homotopy functor if and only if it is weakly equivalent in the stable projective structure to a fibrant object in the stable homotopy structure.*

There are different characterizations of weak equivalences, here we give one.

Lemma 5.11. *A map* $\underset{\sim}{X} \to \underset{\sim}{Y}$ *is a weak equivalence in the stable homotopy structure if and only if for each* $n \geq 0$ *the maps* $Q\underset{\sim}{X}^n \to Q\underset{\sim}{Y}^n$ *are weak equivalences in the homotopy structure on* $\mathcal{S}_*^{\mathcal{S}_*}$.

Proof. This follows from the natural equivalence $Q(\underset{\sim}{X} \circ \mathrm{fib}) \cong (Q\underset{\sim}{X}) \circ \mathrm{fib}$. □

Again in the same way we arrive at the n-excisive structure; we localize along the coaugmented functor P_n.

Theorem 5.12. *The stable n-excisive model structure on* $\mathrm{Sp}^{\mathcal{S}_*}$ *is a simplicial proper model structure. A functor in* $\mathrm{Sp}^{\mathcal{S}_*}$ *is an n-excisive homotopy functor if and only if it is weakly equivalent in the stable projective structure to a fibrant object in the stable n-excisive structure.*

6. The Taylor tower and homogeneous functors

The natural map $p_n\colon \underset{\sim}{X} \to P_n\underset{\sim}{X}$ induces a categorical localization in the homotopy category associated to the homotopy model structure. The local objects with respect to this localization functor are the n-excisive functors. Since $(n-1)$-excisive functors are also n-excisive, there is a map $P_n\underset{\sim}{X} \to P_{n-1}\underset{\sim}{X}$ under $\underset{\sim}{X}$ in the homotopy category. But Goodwillie [12, p. 664] gives a model for this map in $\mathcal{S}^{\mathcal{S}}$ and calls it

$$q_{n,\underset{\sim}{X}}\colon\ P_n\underset{\sim}{X} \to P_{n-1}\underset{\sim}{X}.$$

These maps fit into a tower under $\underset{\sim}{X}$, which is called the Taylor tower of $\underset{\sim}{X}$. The fibers of this tower are of special interest. Let us give a new definition first.

Definition 6.1. A functor $\underset{\sim}{X}$ is called n-reduced if $\underset{\sim}{X}$ is weakly contractible in $(n-1)$-excisive structure, i.e. $P_{n-1}\underset{\sim}{X} \simeq *$ in the homotopy structure. A functor is called n-homogeneous if it is n-reduced and n-excisive.

To introduce the homogeneous part $D_n\underset{\sim}{X}$ of a small functor $\underset{\sim}{X}$ we have to consider fibers and homotopy fibers. Henceforth we will work in the category of small functors on $\mathcal{S}_*$ with values in pointed spaces or spectra.

Definition 6.2. For an object Z in a pointed simplicial model category we define the simplicial path object by

$$WZ := Z^{\Delta^1} \times_{(Z\times Z)} (* \times Z),$$

where the map $Z^{\Delta^1} \to Z \times Z$ is induced by $d_0 \vee d_1\colon\ \Delta^0 \vee \Delta^0 \to \Delta^1$. The projection $\mathrm{pr}_2\colon\ Z \times Z \to Z$ induces a map $PZ \to Z$. Note, that if Z is fibrant then this map is a fibration. Note that PZ is simplicially contractible.

Definition 6.3. We define for each small functor $\underset{\sim}{X}$ a new functor $D_n\underset{\sim}{X}$ by the following pullback square:

$$\begin{array}{ccc}
D_n\underset{\sim}{X} & \longrightarrow & W(P_{n-1}\underset{\sim}{X}) \\
{\scriptstyle d_{n,\underset{\sim}{X}}}\big\downarrow & & \big\downarrow \\
P_n\underset{\sim}{X} & \xrightarrow[q_{n,\underset{\sim}{X}}]{} & P_{n-1}\underset{\sim}{X}
\end{array}$$

We call $D_n\underset{\sim}{X}$ the n-homogeneous part of $\underset{\sim}{X}$.

Remark 6.4. The map $q_n\underset{\sim}{X}$ is an equivalence in the $(n-1)$-excisive structure, and therefore is $D_n\underset{\sim}{X}$ $(n-1)$-excisively contractible, hence n-reduced. The map $d_{n,\underset{\sim}{X}}$ is the base change of an n-excisive fibration, therefore $D_n\underset{\sim}{X}$ is n-excisively fibrant. So $D_n\underset{\sim}{X}$ really is n-homogeneous. We also point out that the defining square has homotopy meaning in the homotopy structure.

We will need the following properties, which are given in [12, Prop. 1.18].

Proposition 6.5. *The functor $D_n\colon\ \mathcal{S}_*^{\mathcal{S}_*} \to \mathcal{S}_*^{\mathcal{S}_*}$ commutes with finite homotopy limits and filtered homotopy colimits in the homotopy structure. On the category of spectrum-valued homotopy functors D_n commutes with arbitrary homotopy colimits.*

7. Classification of n-homogeneous functors

In this section we construct the n-homogeneous model structure on $\mathrm{Sp}^{\mathcal{S}_*}$, where the homotopy types correspond bijectively to n-homogeneous spectrum-valued functors. This model structure classifies *all* n-homogeneous functors as homotopy types (cf. [15, Remark 4.13]). We will give an interpretation of Goodwillie's classification of *finitary* homogeneous functors in terms of Quillen equivalence between model categories in the next section.

The n-homogeneous model structure is obtained by a colocalization which is dual to the Bousfield-Friedlander localization. This process has similarity to the ordinary Postnikov tower of spaces. One obtains the stages by localizing with respect to S^n, which kills all homotopy above degree n. Then one can colocalize with respect to S^{n-1}, where cofibrant replacement would be given by the connected covers. This kills everything below degree $n-1$. Here we are going to do the same, we will colocalize with respect to the n-reduced part of a functor.

Definition 7.1. For each small functor $\underset{\sim}{X}$ we define a new functor $M_n\underset{\sim}{X}$ by the following pullback square:

$$\begin{array}{ccc} M_n\underset{\sim}{X} & \longrightarrow & P(P_{n-1}\underset{\sim}{X}) \\ {\scriptstyle m_{n,\underset{\sim}{X}}}\downarrow & & \downarrow \\ \underset{\sim}{X} & \xrightarrow[p_{n-1,\underset{\sim}{X}}]{} & P_{n-1}\underset{\sim}{X} \end{array}$$

The augmented functor $M_n\colon \mathcal{S}_*^{\mathcal{S}_*} \to \mathcal{S}_*^{\mathcal{S}_*}$ is called the n-reduced part of $\underset{\sim}{X}$.

Remark 7.2. It follows that $M_n\underset{\sim}{X}$ is the homotopy pullback of $\underset{\sim}{X} \to P_{n-1}\underset{\sim}{X} \leftarrow P(P_{n-1}\underset{\sim}{X})$ in the projective structure, as well as in the homotopy structure and in the $n-1$-excisive structure. The functor $M_n\underset{\sim}{X}$ is weakly contractible in the $(n-1)$-excisive structure, and therefore n-reduced. For each $\underset{\sim}{X}$ we have a square

$$\begin{array}{ccc} M_n\underset{\sim}{X} & \longrightarrow & D_n\underset{\sim}{X} \\ {\scriptstyle m_{n,\underset{\sim}{X}}}\downarrow & & \downarrow{\scriptstyle d_{n,\underset{\sim}{X}}} \\ \underset{\sim}{X} & \xrightarrow[p_n\underset{\sim}{X}]{} & P_n\underset{\sim}{X} \end{array}$$

which is a pullback as well as a homotopy pullback square in the homotopy structure. The construction M_n preserves homotopy pullbacks, since it is the homotopy fiber of functors preserving homotopy pullbacks. Of course, $M_n\underset{\sim}{X}$ is not a homotopy functor unless $\underset{\sim}{X}$ is one.

We want to colocalize along the functor M_n. In fact, we will observe that we can colocalize the n-excisive structure, as well as the homotopy structure resulting in the n-homogeneous structure 7.7 and the n-reduced structure 7.8. We have to prove the dual set of the Bousfield-Friedlander axioms. For the proof of the left properness condition $\mathbf{(A.6)^{dual}}$ we have to use, that D_n commutes with homotopy pushouts. So our construction only works for spectrum-valued functors.

Lemma 7.3. *The functor M_n satisfies* $\mathbf{(A.4)^{dual}}$=$\mathbf{(A.4)}$.

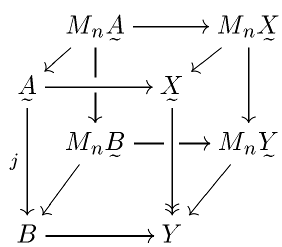
Proof. Since M_n is defined as a homotopy fiber in the homotopy model structure of the functors id and P_{n-1}, which preserve weak equivalences in the homotopy structure, M_n preserves them also. It follows that M_n also preserves n-equivalences by noticing that $P_n M_n \simeq D_n \simeq M_n P_n$. □

Lemma 7.4. *The maps $m_{n,M_n \underset{\sim}{X}}$ and $M_n m_{n,\underset{\sim}{X}}\colon M_n M_n \underset{\sim}{X} \to M_n \underset{\sim}{X}$ are weak equivalences in the homotopy structure and, in particular, n-excisive weak equivalences. This is axiom* $\textbf{(A.5)}^{\textbf{dual}}$.

Proof. To show that $m_{n,M_n \underset{\sim}{X}}$ is an equivalence, just apply P_{n-1} to the defining square of $M_n \underset{\sim}{X}$. The resulting square is again a homotopy pullback square and $P_{n-1} M_n \underset{\sim}{X} \simeq *$. For $M_n m_{n,\underset{\sim}{X}}$ observe, that M_n preserves homotopy pullbacks and $M_n P_{n-1} \underset{\sim}{X} \simeq *$. This last equivalence follows from the definition of $M_n P_{n-1} \underset{\sim}{X}$ and the fact, that $P_{n-1} P_{n-1} \underset{\sim}{X} \simeq P_{n-1} \underset{\sim}{X}$. □

Lemma 7.5. *The functor M_n satisfies* $\textbf{(A.6)}^{\textbf{dual}}$.

Proof. Consider the following diagram



where j is a cofibration between cofibrant objects. If the map $M_n A \to M_n X$ is a weak equivalence in the homotopy structure, the left properness of the homotopy structure implies that $M_n B \to M_n Y$ is a weak equivalence in the homotopy structure.

Now suppose $M_n A \to M_n X$ is an n-excisive equivalence. We have to show that the map $M_n B \to M_n Y$ is also an n-excisive equivalence. To test for n-excisive equivalence we apply P_n to the second square and by using the equivalence $P_n M_n \simeq D_n$ we obtain the square

$$\begin{array}{ccc} D_n\underset{\sim}{A} & \xrightarrow{\simeq} & D_n\underset{\sim}{X} \\ \downarrow & & \downarrow \\ D_n\underset{\sim}{B} & \longrightarrow & D_n\underset{\sim}{Y} \end{array}$$

where $D_n \underset{\sim}{A} \to D_n \underset{\sim}{X}$ is an equivalence by assumption. Since the spectrum-valued D_n commutes with arbitrary homotopy colimits and the original square is a pushout as well as a homotopy pushout square, the latter square is a homotopy pushout. The homotopy structure is left proper, so $D_n \underset{\sim}{B} \to D_n \underset{\sim}{Y}$ is an equivalence, proving that $M_n \underset{\sim}{B} \to M_n \underset{\sim}{Y}$ is an n-excisive equivalence. □

Definition 7.6. Let $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ be a map in $\mathrm{Sp}^{\mathcal{S}_*}$ and let $\widetilde{\underset{\sim}{X}} \to \widetilde{\underset{\sim}{Y}}$ be a replacement of f between homotopy functors. We call f an n-homogeneous equivalence if the map

$$D_n \widetilde{\underset{\sim}{X}} \to D_n \widetilde{\underset{\sim}{Y}}$$

is a weak equivalence. We call f an n-homogeneous cofibration if f has the left lifting property with respect to all maps which are n-excisive fibrations and n-homogeneous equivalences. We call this the n-homogeneous structure on $\mathrm{Sp}^{\mathcal{S}_*}$.

Observe that $M_n(f)$ is an n-excisive weak equivalence if and only if $D_n(f)$ is an n-excisive weak equivalence. The dual of the Bousfield-Friedlander machinery then proves the following theorem.

Theorem 7.7. *On the category $\mathrm{Sp}^{\mathcal{S}_*}$ the n-homogeneous structure exists and is simplicial and proper. The bifibrant objects are exactly the projectively bifibrant n-homogeneous homotopy functors. In particular, the homotopy types correspond bijectively to the homotopy types of n-homogeneous functors from $\mathcal{S}_*$ to* Sp.

Note that this theorem applies to all, not just the finitary n-homogeneous functors.

For a small functor $\underset{\sim}{X}$ the object $P_n\underset{\sim}{X}$ is not exactly the localization of $\underset{\sim}{X}$ in the sense of [13, 3.2.16], since the coaugmentation $p_{n,\underset{\sim}{X}}$ is usually not a trivial cofibration. But $P_n\underset{\sim}{X}$ is not far away from that, it is weakly equivalent to the localization in the underlying model structure, here the homotopy structure. The same is true for $D_n\underset{\sim}{X}$: The maps $\underset{\sim}{X} \to P_n\underset{\sim}{X} \leftarrow D_n\underset{\sim}{X}$ are not a fibrant approximation followed by a cofibrant approximation, but $D_n\underset{\sim}{X}$ is weakly equivalent in the homotopy structure to a fibrant and cofibrant replacement of $\underset{\sim}{X}$ in the n-homogeneous structure. In fact, since both functors $D_n\underset{\sim}{X}$ and the replacement of $\underset{\sim}{X}$ are homotopy functors, they are even weakly equivalent in the projective structure on $\mathcal{S}_*^{\mathcal{S}_*}$.

Finally it is worth remarking that we can colocalize along the functor M_n starting directly from the homotopy structure without going first to the n-excisive structure. The arguments given above can easily be seen to justify the following theorem.

Theorem 7.8. *The category $\mathrm{Sp}^{\mathcal{S}_*}$ may be equipped with the n-reduced model structure. The resulting model category is simplicial and proper. The cofibrant objects are exactly the projectively cofibrant n-reduced functors.*

8. Spectra with Σ_n-action and n-homogeneous functors

A functor is called finitary if it commutes with filtered homotopy colimits. In [12] it is shown that a finitary n-homogeneous functor $F(X)$ from pointed spaces to spectra is weakly equivalent to the functor $X \mapsto (E \wedge X^{\wedge n})_{\mathrm{h}\Sigma_n}$, where E is a spectrum with Σ_n-action. This spectrum is called the n-th derivative of F at $*$. The same relation holds for an arbitrary n-homogeneous functor F, provided that X is a finite space. This result is true, of course, in our framework (for small functor from pointed simplicial sets to Bousfield-Friedlander spectra). In this generality it was proven in [15].

In this section we give an alternative construction of the n-th derivative (and of the n-th cross effect), which allows us to interpret the above result as a Quillen map between the n-homogeneous model category and the category of spectra with Σ_n-action. Moreover, we supply two more model categories Quillen equivalent to the category of spectra with Σ_n-action. These model categories correspond to the two alternative conditions of Goodwillie's theorem: in the first the bifibrant objects are finitary n-homogeneous small functors, while the second model category is analogous to the n-homogeneous model structure, but the underlying category is the category of functors from *finite* pointed simplicial sets to spectra.

Consider the category of small functors from pointed simplicial sets to pointed simplicial sets $\mathcal{S}_*^{\mathcal{S}_*}$. The **finitary projective** model structure on this category is given by:

- A natural transformation $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ is a weak equivalence or a fibration if $f(K)\colon \underset{\sim}{X}(K) \to \underset{\sim}{Y}(K)$ is a weak equivalence or a fibration of simplicial sets for every finite simplicial set K.
- Cofibrations are given by the left lifting property with respect to trivial fibrations.

This model category is generated by the set $\{R^K \mid K \in \mathcal{S}_*, K \text{ finite}\}$ of orbits (in the sense of Dwyer and Kan [8]) and it is Quillen equivalent to the projective model structure on the category of functors from the pointed finite simplicial sets $\mathcal{S}_*^f$ to the pointed simplicial sets $\mathcal{S}_*$. We start from these two Quillen equivalent model categories and perform all the localizations, the stabilization, and the colocalization we have applied on the projective model structure on the category of small functors. Some of these procedures may be simplified, since these model categories are cofibrantly generated. At the end we will obtain two n-homogeneous model categories Quillen equivalent to the category of spectra with Σ_n-action.

Say that a functor F is **pre-finitary** if $F \circ \text{fib}$ finitary.

Every cellular object in the finitary projective model structure on $\mathcal{S}_*^{\mathcal{S}_*}$ is pre-finitary as a homotopy colimit of pre-finitary functors R^K, for $K \in \mathcal{S}_*$ being finite. Every cofibrant object in this model category is a retract of a cellular object, therefore a pre-finitary functor. The procedures of localization and colocalization preserve this property, since they either preserve or reduce the class of cofibrant objects. Stabilization of the projective model structure also preserves the property that all cofibrant functors are pre-finitary, as they are levelwise pre-finitary (therefore homotopy colimits are preserved up to strict equivalence).

The name "finitary" is justified, since after passing to the homotopy model structure bifibrant objects become finitary functors. Therefore we will use the adjective finitary for all the derivatives from the finitary projective model structure.

The methods used in the preceding sections of this paper apply equally well to the finitary projective (resp., projective) model structure on $\mathcal{S}_*^{\mathcal{S}_*^{(f)}}$, so after stabilization, finitary homotopy (resp., homotopy) and finitary n-excisive (resp., n-excisive) model structure we finally obtain the finitary n-homogeneous (resp., n-homogeneous) model structure on $\mathrm{Sp}^{\mathcal{S}_*^{(f)}}$. We would like to note only that the projective model structure on $\mathcal{S}_*^{\mathcal{S}_*^f}$ (or $\mathrm{Sp}_*^{\mathcal{S}_*^f}$) admits a set of maps $F = \{R^L \to R^K \mid K \tilde{\to} L \text{ is a w.e. of fin. simp. sets}\}$, such that the localization with respect to F produces a homotopy model structure. The advantage of this method is that the resulting model category is cofibrantly generated again. The same applies to the n-excisive model structure; the corresponding set of maps was constructed in [7].

Let us denote the finitary n-homogeneous model structure on $\mathcal{S}_*^{\mathcal{S}_*}$ by $\mathcal{M}$ and the n-homogeneous model structure on $\mathrm{Sp}_*^{\mathcal{S}_*^f}$ by $\mathcal{N}$.

In the rest of this section we will argue simultaneously for all three model categories under consideration: n-homogeneous model structure on $\mathrm{Sp}_*^{\mathcal{S}_*}$, $\mathcal{M}$ and $\mathcal{N}$.

We have to state explicitly what type of equivariant model structure on spectra with Σ_n-action we use. View the group Σ_n as a category with one object and then consider presheaves on this category with values in spectra. We equip spectra with the Bousfield-Friedlander model structure and take the projective model structure on presheaves over it. Thus, weak equivalences or fibrations are just given by weak equivalences or fibrations of the underlying spectra. More generally one can put such a Σ_n-equivariant model structure on presheaves with values in any cofibrantly generated model category [13], and it is easy to check that it can be promoted to any class-cofibrantly generated model category.

The category of small functors with projective model structure is class-cofibrantly generated [5], therefore the category of small functors with Σ_n-action may be given the projective model structure similarly to the cofibrantly generated case. The other two model structures under consideration are cofibrantly generated and the category of Σ_n-presheaves over them also may be equipped with the projetive model structure.

First we need to define a homotopy invariant version of the smash product in $\mathcal{S}_*^{\mathcal{S}_*^{(f)}}$. The smash product can be defined formally just the same as for spaces, but the map $R^K \vee R^L \to R^{K\vee L}$ is not a projective cofibration in general. So the smash product might fail to be cofibrant. We have to do this Σ_n-equivariantly, so for $K_1, \dots, K_n \in \mathcal{S}_*$ we define

$$\left(\bigwedge_{i=1}^{n} R^{K_i}\right)_{\mathrm{cof}} \to \bigwedge_{i=1}^{n} R^{K_i}$$

to be a Σ_n-equivariant projective cofibrant replacement. Here the right hand side has the Σ_n-action, which permutes the factors in the smash product. In particular, let

$$(\mathrm{id}^n)_{\mathrm{cof}} := \left(\bigwedge_{i=1}^{n} R^{S^0}\right)_{\mathrm{cof}}$$

We define a pair of adjoint functors. The left adjoint $\lambda_n \colon \mathrm{Sp}^{\Sigma_n} \to \mathrm{Sp}^{\mathcal{S}_*}$ is given by:

$$\lambda_n E := (E \wedge (\mathrm{id}^n)_{\mathrm{cof}})_{\Sigma_n}$$

Note that for cofibrant E these are actually the homotopy orbits, since the action is free. To describe the right adjoint we first define a functor

$$\mathrm{hom}\colon (\mathcal{S}_*^{\mathcal{S}_*^{(f)}\times\Sigma_n})^{\mathrm{op}} \times \mathrm{Sp}^{\mathcal{S}_*^{(f)}} \cong \left((\mathcal{S}_*^{\mathrm{op}})^{\Sigma_n} \times \mathrm{Sp}\right)^{\mathcal{S}_*^{(f)}} \to \mathrm{Sp}^{\Sigma_n}$$

for $\underset{\sim}{K} \in \mathcal{S}_*^{\mathcal{S}_*^{(f)}}$ and $\underset{\sim}{X} \in \mathrm{Sp}^{\mathcal{S}_*^{(f)}}$ by levelwise prolongation of the mapping space functor of the simplicial enrichment of $\mathcal{S}_*^{\mathcal{S}_*^{(f)}}$:

$$\mathrm{Ev}_k \mathrm{hom}(\underset{\sim}{K}, \underset{\sim}{X}) := \mathrm{map}(\underset{\sim}{K}, \mathrm{Ev}_k \underset{\sim}{X})$$

The right hand side inherits its Σ_n-action from $\underset{\sim}{K}$. Observe that we have a natural adjunction

$$\mathrm{map}(\underset{\sim}{L} \wedge \underset{\sim}{K}, \underset{\sim}{X}) \cong \mathrm{map}(\underset{\sim}{L}, \mathrm{hom}(\underset{\sim}{K}, \underset{\sim}{X}))$$

for $\underset{\sim}{L} \in \mathcal{S}_*^{\mathcal{S}_*^{(f)}}$ and that there is an enriched Yoneda isomorphism

$$\mathrm{hom}(R^K, \underset{\sim}{X}) \cong \underset{\sim}{X}(K)$$

for $K \in \mathcal{S}_*$. Then the right adjoint of λ_n is the functor $\rho_n \colon \mathrm{Sp}^{\mathcal{S}_*^{(f)}} \to \mathrm{Sp}^{\Sigma_n}$ given by:

$$\rho_n F := \mathrm{hom}((\mathrm{id}^n)_{\mathrm{cof}}, F)$$

The spectrum $\rho_n F$ obtains a Σ_n-action through the action on $(\mathrm{id}^n)_{\mathrm{cof}}$.

We can relate the smash product to the n-th cross effect as defined in [12, p. 676] or [15, 5.8.]. Recall from 4.1 that $\mathcal{P}_0(\underline{n}) = \mathcal{P}(\underline{n}) - \emptyset$.

Lemma 8.1. (i) *For any F there is the following natural equivalence:*

$$\hom\left(\bigwedge_{i=1}^{n} R^{K_i}, F\right) \cong \mathrm{fib}\left[F\left(\bigvee_{i=1}^{n} K_i\right) \to \lim_{T\in\mathcal{P}_0(\underline{n})} F\left(\bigvee_{\underline{n}-T} K_i\right)\right]$$

(ii) *For projectively fibrant F we have:*

$$\begin{aligned}\hom\left(\left(\bigwedge_{i=1}^{n} R^{K_i}\right)_{\mathrm{cof}}, F\right) &\simeq \mathrm{hofib}\left[F\left(\bigvee_{i=1}^{n} K_i\right) \to \operatorname*{holim}_{T\in\mathcal{P}_0(\underline{n})} F\left(\bigvee_{\underline{n}-T} K_i\right)\right]\\ &\cong \mathrm{cr}_n F(K_1, \dots, K_n)\end{aligned}$$

Proof. Part (ii) follows from part (i), because here source and target have homotopy meaning. Part (i) follows by adjunction from the following representation of an iterated smash product:

$$\begin{array}{ccc} \operatorname*{colim}_{T\in\mathcal{P}_0(\underline{n})} \bigvee_{i\in\underline{n}-T} R^{K_i} & \longrightarrow & \prod_{i=1}^{n} R^{K_i} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bigwedge_{i=1}^{n} R^{K_i} \end{array}$$

□

The spectrum $\partial^{(n)}F(*)$ for any homotopy functor F was introduced in [12, p. 686]; see also [15, pp. 14-15]. There $\partial^{(n)}F(*)$ is called the n-th derivative of F at $*$ and identified as $\mathrm{cr}_n F(S^0, \dots, S^0)$. From 8.1 we deduce a natural Σ_n-equivariant weak equivalence

$$\rho_n F \simeq \mathrm{cr}_n F(S^0, \dots, S^0) \cong \partial^{(n)}F(*).$$

Theorem 8.2. *The functors $\lambda_n \colon \mathrm{Sp}^{\Sigma_n} \leftrightarrows \mathrm{Sp}^{\mathcal{S}_*^{(f)}} : \rho_n$ form a Quillen pair, where Sp^{Σ_n} has the projective equivariant model structure and $\mathrm{Sp}^{\mathcal{S}_*}$ has either the n-homogeneous model structure, or the finitary n-homogeneous model structure, or $\mathrm{Sp}^{\mathcal{S}_*^f}$ has the n-homogeneous model structure.*

If either the finitary n-homogeneous model structure on $\mathrm{Sp}^{\mathcal{S}_*}$, or the n-homogeneous model structure on $\mathrm{Sp}^{\mathcal{S}_*^f}$ is considered, then this Quillen pair becomes a Quillen equivalence.

Proof. The functor ρ_n maps stable projective (trivial) fibrations to (trivial) fibrations, since $(\mathrm{id}^n)_{\mathrm{cof}}$ is projectively cofibrant. Therefore λ_n and ρ_n form a Quillen pair for the stable projective structure to the projective Σ_n-equivariant Bousfield-Friedlander structure. They also from a Quillen pair for the homotopy and the n-excisive structure, since ρ_n still preserves (trivial) fibrations. The same conclusion holds for the finitary version of these model structures and for the analogous model structures on functors from pointed finite simplicial sets to spectra.

We claim that λ_n maps (trivial) cofibrations to n-homogeneous (trivial) cofibrations, which shows that λ_n and ρ_n form a Quillen pair for the n-homogeneous model structure on $\mathrm{Sp}^{\mathcal{S}_*}$ (resp., $\mathcal{M}$ or $\mathcal{N}$) and the Σ_n-equivariant projective model structure on Sp^{Σ_n}.

We know already from the previous step that λ_n maps trivial Σ_n-equivariant projective cofibrations to trivial n-homogeneous cofibrations (resp., trivial cofibrations of $\mathcal{M}$ or $\mathcal{N}$), which are the same as trivial n-excisive cofibrations. Furthermore, λ_n maps Σ_n-equivariant cofibrations to stable projective cofibrations between n-reduced functors, i.e. there is a projective weak equivalence $M_n\lambda_n E \to \lambda_n E$. This shows that for a cofibration $A \to B$ of Σ_n-spectra the square

$$\begin{array}{ccc} M_n\lambda_n A & \longrightarrow & M_n\lambda_n B \\ \simeq\downarrow & & \downarrow\simeq \\ \lambda_n A & \longrightarrow & \lambda_n B \end{array}$$

is a homotopy pushout and therefore $\lambda_n A \to \lambda_n B$ is an n-homogeneous cofibration (resp., a cofibration in $\mathcal{M}$ or $\mathcal{N}$).

Now we show that this is actually a Quillen equivalence if we consider $\mathcal{M}$ or $\mathcal{N}$. Let E be a cofibrant spectrum, and let F be a fibrant object of $\mathcal{M}$ or $\mathcal{N}$. Without loss of generality we may assume that F is also cofibrant (this assumption is redundant for $\mathcal{N}$), hence $F \in \mathcal{M}$ is a finitary functor. It suffices to show that a map $E \to \rho_n F$ is a weak equivalence if and only if the corresponding map $\lambda_n E \to F$ is a weak equivalence.

A map $E \to \rho_n F$ is a weak equivalence if and only if the map $\lambda_n E \to \lambda_n\rho_n F$ is a projective weak equivalence, since the n-homogeneous functors in the image of λ_n are determined by their coefficient spectrum. Any n-homogeneous homotopy functor is projectively equivalent to its n-homogeneous part, so $\lambda_n E \to \lambda_n\rho_n F$ is a projective weak equivalence if and only if $D_n\lambda_n E \to D_n\lambda_n\rho_n F$ is a projective weak equivalence. By [12, p. 686] and Lemma 8.1 for every finitary F or for every finite $K \in \mathcal{S}_*$ we have

$$D_n\lambda_n\rho_n F(K) \cong D_n F(K).$$

Since one of the two conditions is necessarily satisfied in either $\mathcal{M}$, or $\mathcal{N}$, we conclude that $E \to \rho_n F$ is a weak equivalence iff $D_n\lambda_n E \to D_n F$ is a projective weak equivalence. Finally, $\lambda_n E \to F$ is a weak equivalence in $\mathcal{M}$ or $\mathcal{N}$ if and only if $D_n\lambda_n E \to D_n F$ is a projective weak equivalence. □

References

[1] G. Arone and M. Mahowald. The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. *Invent. Math.*, 135(3):743–788, 1999.

[2] Bousfield and Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric Applications of Homotopy Theory II*, number 658 in Lecture Notes in Mathematics. Springer, 1978.

[3] A. K. Bousfield. On the telescopic homotopy theory of spaces. *Trans. Amer. Math. Soc.*, 353(6):2391–2426 (electronic), 2001.

[4] B. Chorny. A generalization of Quillen's small object argument. *Journal of Pure and Applied Algebra*, 204:568–583, 2006.

[5] B. Chorny and W. G. Dwyer. Homotopy theory of small diagrams over large categories. Preprint, 2005.

[6] B. I. Dundas, O. Röndigs, and P. A. Østvær. Enriched functors and stable homotopy theory. *Doc. Math.*, 8:409–488 (electronic), 2003.

[7] W. G. Dwyer. Localizations. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 3–28. Kluwer Acad. Publ., Dordrecht, 2004.

[8] W. G. Dwyer and D. M. Kan. Singular functors and realization functors. *Proc. Kon. Nederl. Acad. 87(2); Indag. Math. 46(2)*, pages 147–153, 1984.

[9] P. Freyd. Several new concepts: Lucid and concordant functors, pre-limits, pre-completeness, the continuous and concordant completions of categories. In *Category Theory, Homology*

Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three), pages 196–241. Springer, Berlin, 1969.
[10] T. G. Goodwillie. Calculus. I. The first derivative of pseudoisotopy theory. *K-Theory*, 4(1):1–27, 1990.
[11] T. G. Goodwillie. Calculus. II. Analytic functors. *K-Theory*, 5(4):295–332, 1991/92.
[12] T. G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711 (electronic), 2003.
[13] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
[14] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.
[15] N. J. Kuhn. Goodwillie towers and chromatic homotopy: an overview. In *Proc. Kinosaki Conf. in Alg. Topology*, Geometry and Topology monographs, Japan, 2003. To appear.
[16] S. Lack. Limits of small functors. Preprint, 2002.
[17] M. Lykadis. Simplicial functors and stable homotopy theory. Preprint, available via Hopf archive, 1998.
[18] D. G. Quillen. *Homotopical Algebra*. Lecture Notes in Math. 43. Springer-Verlag, Berlin, 1967.
[19] J. Rosický. Cartesian closed exact completions. *J. Pure Appl. Algebra*, 142(3):261–270, 1999.
[20] S. Schwede. Spectra in model categories and applications to the algebraic cotangent complex. *J. Pure Appl. Algebra*, 120(1):77–104, 1997.

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