ALGEBRAIC INVARIANTS FOR HOMOTOPY TYPES

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Abstract. We define a sequence of purely algebraic invariants – namely, classes in the Quillen cohomology of the Π-algebra $\pi_*\mathbf{X}$ – for distinguishing between different homotopy types of spaces. Another sequence of such cohomology classes allows one to decide whether a given abstract Π-algebra can be realized as the homotopy Π-algebra of a space.

1. Introduction

The usual Postnikov system for a (simply-connected) CW complex $\mathbf{X}$ serves to determine its homotopy type. One begins with purely algebraic data, consisting of the homotopy groups $(\pi_n\mathbf{X})_{n=2}^{\infty}$. However, in order to construct the succesive approximations $\mathbf{X}^{(n)}$ $(n \geq 2)$, with $\mathbf{X} \simeq \operatorname{holim} \mathbf{X}^{(n)}$, one must specify a sequence of cohomology classes $k_n \in H^{n+2}(\mathbf{X}^{(n)}; \pi_{n+1}\mathbf{X})$ (see [W, IX, §2]). These can hardly qualify as *algebraic* invariants, since their description involves the cohomology groups of topological spaces. In this paper we show that if one is willing to invest the graded group $\pi_*\mathbf{X} := (\pi_n\mathbf{X})_{n=1}^{\infty}$ with some further algebraic structure, the additional information needed to determine the homotopy type of $\mathbf{X}$ can be described in purely algebraic terms.

The structure needed on $\pi_*\mathbf{X}$ is that of a Π-algebra – i.e., a graded group equipped with an action of the primary homotopy operations (Whitehead products and compositions). In this context, the additional data needed consists of cohomology classes in the Quillen cohomology of this Π-algebra – which can be defined as usual in algebraic terms (see §4.1 below). We show:

Theorem A. *Given two realizations $\mathbf{X}$ and $\mathbf{X}'$ of a Π-algebra J_*, there is a successively defined sequence of "difference obstructions" $\delta_n \in H^{n+1}(J_*, \Omega^n J_*)$, taking value in the Quillen cohomology groups of J_*, with coefficients in the J_*-module $\Omega^n J_*$, whose vanishing implies that $\mathbf{X} \simeq \mathbf{X}'$.*

(See Theorems 4.18 and 4.21 below). The $(n+1)$-st cohomology class is defined whenever the n-th Postnikov section of the simplicial space resolutions of the spaces $\mathbf{X}$ and $\mathbf{X}'$, respectively, agree, up to homotopy. Even though the obstructions are defined in terms of a specific choice of Π-algebra resolution of J_*, in fact they depend only on the homotopy type of the Postnikov sections.

Moreover, these cohomology groups can also be used to determine the realizability of an abstract Π-algebra as the homotopy groups of some space:

Theorem B. *Given a Π-algebra J_*, there is a successively defined sequence of "characteristic classes" $\xi \in H^{n+2}(J_*, \Omega^n J_*)$, which vanish if and only if J_* is realizable by a topological space.*

(See Theorems 4.8 and 4.15 below). The vanishing requirement should be understood in the sense of an obstruction theory: if any such sequence of cohomology classes vanishes, the Π-algebra is realizable; if one reaches a non-trivial obstruction, one must back-track, and try to

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vary the choices involved in order to obtain a realization. These choices again depend only the homotopy type of a suitable Postnikov section – this time, of a simplicial resolution we are trying to construct for the putative topological space $\mathbf{X}$ realizing J_*. See Proposition 4.10 below.

The theory is greatly simplified if we are only interested in the *rational* homotopy type of a simply-connected space $\mathbf{X}$. In that case, a rational Π-algebra is simply a graded Lie algebra over $\mathbb{Q}$, and the cohomology theory in question reduces to the usual cohomology of Lie algebras. Theorem A thus provides an integral version of (the dual to) the Halperin-Stasheff obstruction theory for rational homotopy types (see [HS] and §4.22 below).

It is in order to be able to deal with this case, too (and other possible variants – see §2.14 below), that we have stated our results for a general model catgory $\mathcal{C}$ (subject to certain somewhat restrictive simplifying assumptions on $\mathcal{C}$ – not all of which are really necessary). For technical convenience we have chosen to describe the ordinary topological version of our theory within the framework of simplicial groups, rather than topological spaces (see §4.12 below).

1.1. **notation and conventions.** $\mathcal{T}$ will denote the category of topological spaces, and $\mathcal{T}_*$ that of pointed connected topological spaces with base-point preserving maps. The base-point will be written $* \in X$.

The category of groups is denoted by $\mathcal{G}p$, that of graded groups by $gr\mathcal{G}p$, that of (left) R-modules by R-$\mathcal{M}od$, and that of sets by $\mathcal{S}et$.

1.2. Definition. $\mathbf{\Delta}$ is the category of ordered sequences $\mathbf{n} = \langle 0, 1, \dots, n\rangle$ $(n \in \mathbb{N})$, with order-preserving maps. $\mathbf{\Delta}^{op}$ is the opposite category. As usual, a *simplicial object* over any category $\mathcal{C}$ is a functor $X : \mathbf{\Delta}^{op} \to \mathcal{C}$; more explicitly, it is a sequence of objects $\{X_n\}_{n=0}^{\infty}$ in $\mathcal{C}$, equipped with *face maps* $d_i : X_n \to X_{n-1}$ and *degeneracies* $s_j : X_n \to X_{n+1}$ $(0 \leq i, j \leq n)$, satisfying the usual simplicial identities ([May, §1.1]). We usually denote such a simplical object by $X_\bullet$. The category of simplicial objects over $\mathcal{C}$ is denoted by $s\mathcal{C}$. The standard embedding of categories $c(-)_\bullet : \mathcal{C} \to s\mathcal{C}$ is defined by letting $c(X)_\bullet \in s\mathcal{C}$ denote the constant simplicial object on any $X \in \mathcal{C}$ (with $c(X)_n = X$, $d_i = s_j = id_X$).

The category of simplical sets will be denoted by $\mathcal{S}$, rather than $s\mathcal{S}et$, that of pointed connected simplicial sets by $\mathcal{S}_*$, and that of simplicial groups by $\mathcal{G}$. If we consider a simplicial object $X_\bullet$ over $\mathcal{G}$, say, we shall sometimes call n in $X_1, \dots, X_n, \dots$ the *external* simplicial dimension, written $(-)^{ext}_n$, in distinction from the *internal* simplicial dimension k, inside $\mathcal{G}$, denoted by $(-)^{int}_k$. In this case we shall sometimes write $(X_\bullet)^{int}_k \in s\mathcal{G}p$, in contrast with $X_n \in \mathcal{G}$, to emphasize the distinction.

The standard n simplex in $\mathcal{S}$ is denoted by $\Delta[n]$, generated by $\sigma_n \in \Delta[n]_n$, with $\Lambda^k[n]$ the subobject generated by $d_i\sigma_n$ for $i \neq k$.

If we denote by $\mathbf{\Delta}\langle n\rangle$ the category obtained from $\mathbf{\Delta}$ by omitting the objects $\{\mathbf{k}\}_{k=n+1}^{\infty}$, the category of functors $(\mathbf{\Delta}\langle n\rangle)^{op} \to \mathcal{C}$ is called the category of *n-simplicial objects* over $\mathcal{C}$ – written $s_{\langle n\rangle}\mathcal{C}$. If $\mathcal{C}$ has enough colimits, the obvious truncation functor $\mathrm{tr}_n : s\mathcal{C} \to s_{\langle n\rangle}\mathcal{C}$ has a left adjoint $\rho_n : s_{\langle n\rangle}\mathcal{C} \to s\mathcal{C}$, and the composite $\mathrm{sk}_n := \rho_n \circ \mathrm{tr}_n : s\mathcal{C} \to s\mathcal{C}$ is called the *n-skeleton* functor.

1.3. **organization.** In section 2 we review some background material on closed model category structures for categories of simplicial objects and show how certain convenient CW resolutions may be constructed therein. In section 3 we construct Postnikov systems for such resolutions, and define the action of the fundamental group on them; and in section 4 we explain how these resolutions are determined in terms of appropriate cohomology classes, which may also be used to determine the realizability of a (generalized) Π-algebra (Theorems 4.8 and 4.15), as well as to distinguish between different possible realizations (Theorems 4.18 and 4.21).

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It should be noted that Baues had previously constructed the first difference obstruction of Theorem A, lying in $H^2(J_*, \Omega J_*)$, by different methods, and has since extended his construction to the full range of invariants we define here: see [Ba3]. Yet a third description of these invariants, more in the spirit of the original approach of Dwyer, Kan, and Stover, is planned in [BG].

2. MODEL CATEGORIES OF SIMPLICIAL OBJECTS

We first review some background material on model category structures for categories of simplicial objects, in particular a slightly expanded version of structure defined in [DKS2], and show how one can construct CW resolutions in such a context.

2.1. **model categories.** A *model category* in the sense of Quillen (see [Q1]) is a category $\mathcal{C}$ equipped with three distinguished classes of morphisms: $\mathfrak{W}$ (weak equivalences), $\mathfrak{C}$, and $\mathfrak{F}$, satisfying the following assumptions:

(1) $\mathcal{C}$ has all small limits and colimits.
(2) $\mathfrak{W}$ is a class of quasi-isomorphisms (i.e., there is some functor $F : \mathcal{C} \to \mathcal{D}$ such that $f \in \mathfrak{W} \Leftrightarrow F(f)$ is an isomorphism).
(3) Any morphism $f : A \to B$ in $\mathcal{C}$ has a factorization $A \xrightarrow{i} C \xrightarrow{p} B$ $(f = p \circ i)$ with $i \in \mathfrak{C} \cap \mathfrak{W}$ and $p \in \mathfrak{F}$; moreover, this factorization is unique up to weak equivalence, in the sense that if $A \xrightarrow{i'} C' \xrightarrow{p'} B$ is another such factorization of f $(i' \in \mathfrak{C} \cap \mathfrak{W},\ p' \in \mathfrak{F})$, then there is a map $h : C \to C'$ such that $h \circ i = i'$ and $p' \circ h = p$.
(4) Similarly, any morphism $f : A \to B$ in $\mathcal{C}$ has a factorization $A \xrightarrow{i} C \xrightarrow{p} B$ $(f = p \circ i)$ with $i \in \mathfrak{C}$ and $p \in \mathfrak{F} \cap \mathfrak{W}$ – again unique up to weak equivalence.
(5) We will assume here that the factorizations above may be chosen *functorially* (though this is not included in the original definition in [Q1, I, §1]).

We call the closures under retracts of $\mathfrak{C}$ and $\mathfrak{F}$ the classes of *cofibrations* and *fibrations*, respectively. The definition given here is then equivalent to the original one of Quillen in [Q1, Q3] (see [Bl5, §2]).

An object $X \in \mathcal{C}$ is called *fibrant* if $X \to *_f$ is a fibration, where $*_f$ is the final object of $\mathcal{C}$; similarly X is *cofibrant* if $*_i \to X$ is a cofibration ($*_i$ = initial object). If $X \in \mathcal{C}$ is cofibrant and $Y \in \mathcal{C}$ is fibrant, we denote by $[X, Y]_{\mathcal{C}}$ (or simply $[X, Y]$) the set of homotopy equivalence classes $[f]$ of maps $f : X \to Y$. For this to be defined we in fact need only require X to be cofibrant *or* Y to be fibrant (cf. [Q1, I,§1]). A map in $\mathfrak{W} \cap \mathfrak{F}$ is called a *trivial fibration*, and one in $\mathfrak{W} \cap \mathfrak{C}$ a *trivial cofibration*.

Given a model category $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$, one can "invert the weak equivalences" to obtain the associated *homotopy category* $ho\mathcal{C}$, in which the set of morphisms from X to Y is just $[X, Y]$ (at least when X and Y are both fibrant and cofibrant). See [Q1, I], [Q3, II,§1], or [Hi, ch. IX-XI] for some basic properties of model categories.

2.2. **pointed model categories.** In a *pointed* model category $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ – i.e., one with a zero object, denoted by 0 or $*$ $(= *_f = *_i)$ – we may define the *fiber* of a map (usually: a fibration) $f : X \to Y$ to be the pullback of $X \xrightarrow{f} Y \leftarrow *$, and the *cofiber* of a map (usually: a cofibration) $i : A \to B$ to be the pushout of $* \leftarrow A \xrightarrow{i} B$. The *suspension* ΣA of a (cofibrant) object $A \in \mathcal{C}$ is then defined to be the cofiber of $A \amalg A \to A \times I$, where $A \times I$ is any cylinder object for A (cf. [Q1, I,1,Def. 4]); it is unique up to homotopy equivalence. Similarly, the *loops* ΩX of a fibrant object X is the fiber of $X^I \to X \times X$, where X^I is a path object for X (ibid.). Finally, the *cone* CA of a (cofibrant) object $A \in \mathcal{C}$ is the cofiber of either map $A \hookrightarrow A \times I$. See [Q1, I,2.8-9].

2.3. **simplicial objects.** For any category $\mathcal{C}$ with coproducts, one has a *simplicial structure* (cf. [Q1, II, §1]) on the category $s\mathcal{C}$ of simplicial objects over $\mathcal{C}$, defined as usual by:

(i) For any simplicial set $A \in \mathcal{S}$ and $X \in \mathcal{C}$, we define $X \hat{\otimes} A \in s\mathcal{C}$ by $(X \hat{\otimes} A)_n := \coprod_{a \in A_n} X$, with the face and degeneracy maps induced from those of A. We denote the cofiber of $A \hat{\otimes} * \to A \hat{\otimes} X$ by $A \wedge X$.

Now for $X_\bullet \in s\mathcal{C}$ we define $X_\bullet \otimes A \in s\mathcal{C}$ by $(X_\bullet \otimes A)_n := \coprod_{a \in A_n} X_n$ (the diagonal of the bisimplicial object $X_\bullet \hat{\otimes} A$).

(ii) For any $X_\bullet, Y_\bullet \in s\mathcal{C}$ we define the function complex $\operatorname{map}(X_\bullet, Y_\bullet)$ by

$$\operatorname{map}(X_\bullet, Y_\bullet)_n := \operatorname{Hom}_{s\mathcal{C}}(X_\bullet \otimes \Delta[n], Y_\bullet),$$

where $\Delta[n] \in \mathcal{S}$ denotes the standard simplicial n-simplex.

2.4. Definition. For any complete category $\mathcal{C}$, the *matching object* functor $M : \mathcal{S}^{op} \times s\mathcal{C} \to \mathcal{C}$, written $M_A X_\bullet$ for a (finite) simplicial set $A \in \mathcal{S}$ and any $X_\bullet \in s\mathcal{C}$, is defined by requiring that $M_{\Delta[n]} X_\bullet := X_n$, and if $A = \operatorname{colim}_i A_i$ then $M_A X_\bullet = \lim_i M_{A_i} X_\bullet$ (see [DKS2, §2.1]). This may be defined by adjointness, via:

$$\operatorname{Hom}_{s\mathcal{C}}(Z \otimes A, X_\bullet) \cong \operatorname{Hom}_C(Z, M_A X_\bullet)$$

for $X_\bullet \in s\mathcal{C}$ and $Z \in \mathcal{C}$.

In particular, we write $M_n^k X_\bullet$ for $M_A X_\bullet$ where A is the subcomplex of $\operatorname{sk}_{n-1} \Delta[n]$ generated by the last $(n-k+1)$ faces $(d_k \sigma_n, \ldots, d_n \sigma_n)$. When $\mathcal{C} = \mathcal{S}et$ or $\mathcal{G}p$, for example, this reduces to:

$$M_n^k X_\bullet = \{(x_k, \ldots, x_n) \in (X_{n-1})^{n+1} \mid d_i x_j = d_{j-1} x_i \text{ for all } k \leq i < j \leq n\}, \tag{2.5}$$

and the map $\delta_n^k : X_n \to M_n^k X_\bullet$ induced by the inclusion $A \hookrightarrow \Delta[n]$ is defined $\delta_n(x) = (d_k x, \ldots, d_n x)$. The original matching object of [BK, X,§4.5] was $M_n^0 X_\bullet = M_{\partial\Delta[n]} X_\bullet$, which we shall further abbreviate to $M_n X_\bullet$; note that each face map $d_k : X_{n+1} \to X_n$ factors through $\delta_n := \delta_n^0$. See also §3.1 below and [Hi, XVII, 87.17].

The dual construction yields the colimit $L_n X_\bullet$, sometimes called the "n-th latching object" of $X_\bullet$ – see [DKS1, §2.3(i)]. For $X_\bullet \in \S$, for example, we have $L_n X_\bullet := \coprod_{0 \leq i \leq n-1} X_{n-1} / \sim$, where for any $x \in X_{n-k-1}$ and $0 \leq i \leq j \leq n-1$ we set $s_{j_1} s_{j_2} \ldots s_{j_k} x$ in the i-th copy of X_{n-1} equivalent to $s_{i_1} s_{i_2} \ldots s_{i_k} x$ in the j-th copy of X_{n-1} whenever the simplicial identity

$$s_i s_{j_1} s_{j_2} \ldots s_{j_k} = s_j s_{i_1} s_{i_2} \ldots s_{i_k}$$

holds (so in particular $s_j x \in (X_{n-1})_i$ is equivalent to $s_i x \in (X_{n-1})_{j+1}$ for all $0 \leq i \leq j \leq n-1$). The map $\sigma_n : L_n X_\bullet \to X_n$ is defined $\sigma_n(x)_i = s_i x$, where $(x)_i \in (X_{n-1})_i$.

There are (at least) two ways to extend a given model category structure on $\mathcal{C}$ to $s\mathcal{C}$:

2.6. Definition. In the *Reedy model structure* on $s\mathcal{C}$ (see [R] or [Hi, XVII, §88]), a simplicial map $f : X_\bullet \to Y_\bullet$ is

(i) a weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence in $\mathcal{C}$ for each $n \geq 0$;

(ii) a (trivial) cofibration if $f_n \amalg \sigma_n : X_n \amalg_{L_n X_\bullet} L_n Y_\bullet \to Y_n$ is a (trivial) cofibration in $\mathcal{C}$ for each $n \geq 0$;

(iii) a (trivial) fibration if $f_n \times \delta_n : X_n \to Y_n \times_{M_n Y_\bullet} M_n X_\bullet$ is a (trivial) fibration in $\mathcal{C}$ for each $n \geq 0$.

Note that these definitions imply that $X_\bullet \in s\mathcal{C}$ is fibrant if and only if the maps $\delta_n : X_n \to M_n X_\bullet$ are fibrations (in $\mathcal{C}$) for all n.

We shall require another structure, originally called the "E^2-model category" (see [DKS1, §3] and §4.20 below), defined under the following

2.7. Assumption. Assume that $\langle \mathcal{C}; \mathfrak{W}, \mathfrak{C}, \mathfrak{F} \rangle$ is a pointed cofibrantly generated model category, in which every object is fibrant (this holds, for example, if $\mathcal{C} = \mathcal{T}_*$ or $\mathcal{C} = \mathcal{G}$). Let $\mathcal{F} = F_\mathcal{C}$ be a small full subcategory of $\mathcal{C}$ with the following properties:

(i) There is a subset $\{\mathbf{M}\langle\alpha\rangle\}_{\alpha \in \hat{\mathcal{F}}} \subset Obj\mathcal{F}$ consisting of cogroup objects for $\mathcal{C}$ – so there is a natural group structure on $\mathrm{Hom}_\mathcal{C}(\mathbf{M}\langle\alpha\rangle, Y)$ for any $Y \in \mathcal{C}$.

(ii) $\mathcal{F}$ is closed under coproducts, and every object $Z \in \mathcal{F}$ is weakly equivalent to some (possibly infinite) coproduct $\coprod_i \mathbf{M}\langle\alpha_i\rangle$ with $\alpha_i \in \hat{\mathcal{F}}$ – so Z is a *homotopy* cogroup object (i.e., $[Z, Y]_\mathcal{C}$ has a natural group structure). However, we do not require the morphisms in $\mathcal{F}$ to respect the cogroup structure, even up to homotopy.

(iii) $\mathcal{F}$ is closed under suspensions – that is, for each $X \in \mathcal{F}$, there is a model for ΣX in $\mathcal{F}$. We also assume $C\mathbf{M}\langle\alpha\rangle \in \mathcal{F}$ for every $\alpha \in \hat{\mathcal{F}}$ (§2.2).

We now wish to define an algebraic model for the collection of sets of homotopy classes of maps $\{[X, Y]_\mathcal{C}\}_{X \in \mathcal{F}}$, for a given object $Y \in \mathcal{C}$. This is provided by the following

2.8. Definition. Given $\mathcal{F} \subset \mathcal{C}$ as in §2.7, we define a $\Pi_\mathcal{F}$-*algebra* to be a functor $ho(\mathcal{F})^{op} \to \mathcal{S}et$, which takes coproducts in $\mathcal{F}$ to products in $\mathcal{S}et$ (compare [Dr]).

The category of all $\Pi_\mathcal{F}$-algebras will be denoted by $\Pi_\mathcal{F}$-$\mathcal{A}lg$, and the functor $[ho\mathcal{F}, -] : \mathcal{C} \to \Pi_\mathcal{F}$-$\mathcal{A}lg$ defined $([\mathcal{B}, Y])_{\mathcal{B} \in ho\mathcal{F}}$ will be denoted by $\pi_\mathcal{F}$. $\Pi_\mathcal{F}$-$\mathcal{A}lg$ is a category of universal graded algebras, or CUGA, in the sense of [BS, §2.1]. In particular, the *free* $\Pi_\mathcal{F}$-algebras are those isomorphic to $\pi_\mathcal{F} X$ for some $X \in \mathcal{F}$. If we assume that $X \simeq \coprod_{\alpha \in \hat{\mathcal{F}}} \coprod_{t \in T_\alpha} \mathbf{M}\langle\alpha\rangle_t$ for some $\hat{\mathcal{F}}$-graded set T_*, we say that $\pi_\mathcal{F} X$ is the free $\Pi_\mathcal{F}$-algebra *generated by* T_*.

If $f : X \to Y$ is a morphism in $\mathcal{C}$, the induced morphism of $\Pi_\mathcal{F}$-algebras, $\pi_\mathcal{F} f : \pi_\mathcal{F} X \to \pi_\mathcal{F} Y$, will be denoted simply by $f_\#$.

2.9. Remark. Since all objects in $\mathcal{F}$ are homotopy equivalent to coproducts of objects from the set $\hat{\mathcal{F}}$, a $\Pi_\mathcal{F}$-algebra may be thought of more concretely as an $\hat{\mathcal{F}}$-graded group – i.e., a collection of groups $(G_\alpha)_{\alpha \in \hat{\mathcal{F}}}$ – equipped with a (contravariant) action of the homotopy classes of morphisms in $\mathcal{F}$ on them, modeled on the action of such homotopy classes on $\{[\mathbf{M}\langle\alpha\rangle, Y]\}_{\alpha \in \hat{\mathcal{F}}}$ by precomposition (cf. [W, XI, §1]).

We shall write $\pi_\alpha X$ for $(\pi_\mathcal{F} X)_\alpha := [\mathbf{M}\langle\alpha\rangle, X]$, and $\pi_{\alpha+k} X$ for $[\Sigma^k \mathbf{M}\langle\alpha\rangle, X]$.

2.10. Definition. As usual, a $\Pi_\mathcal{F}$-algebra X is called *abelian* if $\mathrm{Hom}_{\Pi_\mathcal{F}\text{-}\mathcal{A}lg}(X, A)$ has a natural abelian group structure for any $A \in \Pi_\mathcal{F}$-$\mathcal{A}lg$ (see [BS, §5.1] for an explicit description.). In particular, for any $X \in \Pi_\mathcal{F}$-$\mathcal{A}lg$, its *abelianization* X_{ab} may be defined as in [BS, §5.1.4] as a suitable quotient of X. Another abelian $\Pi_\mathcal{F}$-algebra which may be defined for any X is its *loop algebra* ΩX, defined by $\Omega X(\mathcal{B}) := X(\Sigma\mathcal{B})$ (cf. [DKS2, §9.4]; recall that $\mathcal{F}$ is closed under suspension). The fact that it is abelian follows as in [Gr, Prop. 9.9]. The (abelian) category of abelian $\Pi_\mathcal{F}$-algebras will be denoted by $\Pi_\mathcal{F}$-$\mathcal{A}lg_{ab}$.

2.11. Example. In $\mathcal{C} = \mathcal{T}_*$, let $\mathcal{F}$ denote the subcategory whose objects are wedges of spheres of various dimensions; then for any space $X \in \mathcal{T}_*$, the functor $\pi_\mathcal{F} X$ is determined up to isomorphism by $\pi_* X$, the homotopy Π-*algebra* of X – that is, its homotopy groups, together with the action of the primary homotopy operations (Whitehead products and compositions) on them. See [Bl2, §2] or [St, §4]. In particular, the abelian Π-algebras are those for which all Whitehead products are trivial (cf. [Bl2, §3]).

2.12. Remark. This example does not quite fit our assumptions (§2.7), since the spheres are only co-H-spaces, i.e., *homotopy* cogroup objects in $\mathcal{T}_*$. This does not affect the arguments at this stage – in fact, this is the original example of an "E^2-model category" in [DKS1]. However, for our purposes $\mathcal{G}$ appears to be more convenient than $\mathcal{T}_*$ as a model for the homotopy category of (connected) spaces (see [K2]; also, e.g., [Bl6, §5]).

In fact, in all the examples we have in mind the objects in $\mathcal{C}$ will have an (underlying) group structure, so it will be convenient to add to §2.7 the following additional

2.13. Assumption. $\mathcal{C}$ is equipped with a faithful forgetful functor $\hat{U}:\mathcal{C}\to\mathcal{D}$ – where $\mathcal{D}$ is one of the "categories of groups" $\mathcal{D}=\mathcal{G}p$, $gr\mathcal{G}p$, $\mathcal{G}$, R-$\mathcal{M}od$, or sR-$\mathcal{M}od$, for some ring R – and the cogroup objects $\mathbf{M}\langle\alpha\rangle\in\hat{\mathcal{F}}$ of §2.7(i) are in the image of its adjoint $\hat{F}$, with the group structure on $\mathrm{Hom}_{\mathcal{C}}(\mathbf{M}\langle\alpha\rangle,X)$ induced from that of $\hat{U}(X)$. When $\mathcal{D}=\mathcal{G}$ or $\mathcal{D}=sR$-$\mathcal{M}od$, the objects $\mathbf{M}\langle\alpha\rangle$ must actually lie in the image of the composite $\hat{F}\circ F':\mathcal{S}\to\mathcal{C}$, where $F':\mathcal{S}\to\mathcal{D}$ is adjoint to the forgetful functor $U':\mathcal{D}\to\mathcal{S}$.

We also assume that the adjoint pair $(\hat{U},\hat{F})$ *create* the model category structure on $\mathcal{C}$ in the sense of [Bl5, §4.13] – so in particular $\hat{U}$ creates all limits in $\mathcal{C}$ (cf. [Mc1, V,§1]).

2.14. Remark. In fact, the categories $\mathcal{C}$ in which shall be interested are the following:

- $\mathcal{C}=\mathcal{G}$, so $s\mathcal{C}$, the category of bisimplicial groups, is a model for simplicial spaces;
- $\mathcal{C}=\mathcal{G}p$, so $s\mathcal{C}=\mathcal{G}$ is a model for the homotopy category of connected topological spaces of the homotopy type of a CW complex;
- $\mathcal{C}=d\mathcal{L}$, the category of differential graded Lie algebras (or equivalently, $\mathcal{C}=s\mathcal{L}ie$), so $s\mathcal{C}$ is a model for simplicial rational spaces;
- $\mathcal{C}=\mathcal{L}ie$, the category of Lie algebras, so $s\mathcal{L}ie$ is a model for (simply connected) rational spaces (cf. [Q3, II,§4-5]);
- $\mathcal{C}=R$-$\mathcal{M}od$, the category of (left) modules over a not-necessarily commutative, possibly graded, ring R, so $s\mathcal{C}$ is a model for chain complexes over R.

and it is the desire to give a unified treatment for these five cases that forces upon us the somewhat unnatural set of assumptions we have made in §2.7 and here.

2.15. Definition. A map $f:V_\bullet\to Y_\bullet$ in $s\mathcal{C}$ is called *$\mathcal{F}$-free* if for each $n\geq 0$, there is

a) a cofibrant object W_n which is weakly equivalent to an object in $\mathcal{F}$;
b) a map $\varphi_n:W_n\to Y_n$ in $\mathcal{C}$ which induces a trivial cofibration $(V_n\amalg_{L_nV_\bullet}L_nY_\bullet)\amalg W_n\to Y_n$.

2.16. **The resolution model category.** Given a model category $\mathcal{C}$ and a subcategory $\mathcal{F}$ as in §2.7, we define the *resolution model category structure* on $s\mathcal{C}$, with respect to $\mathcal{F}$ by setting a simplicial map $f:X_\bullet\to Y_\bullet$ to be

(i) a *weak equivalence* if $\pi_{\mathcal{F}}f$ is a weak equivalence of $\hat{\mathcal{F}}$-graded simplicial groups (§2.9).
(ii) a *cofibration* if it is a retract of an $\mathcal{F}$-free map;
(iii) a *fibration* if it is a Reedy fibration (Def. 2.6(iii)) and $\pi_{\mathcal{F}}f$ is a (levelwise) fibration of simplicial groups (that is, for each $B\in\mathcal{F}$ and each $n\geq 0$, the group homomorphism $[B,X_n]\xrightarrow{[B,f_n]}[B,Y_n]^{ext}$ is an epimorphism (where for $G_\bullet:=[B,Y_\bullet]\in\mathcal{G}$, $G_\bullet^{ext}$ denotes the connected component of the identity) – see [Q1, II,3.8].

This was originally called the "E^2-model category structure" on $s\mathcal{C}$. See [DKS1, §5] for further details.

2.17. Example. Let $\mathcal{C}=\mathcal{G}p$ with the *trivial* model category structure: i.e., only isomorphisms are weak equivalences, and every map is both a fibration and a cofibration. Let $\mathcal{F}_{\mathcal{G}p}$ be the category of all free groups (which are the cogroup objects in $\mathcal{G}p$ – cf. [K1]). The resulting resolution model category structure on $\mathcal{G} := s\mathcal{G}p$ is the usual one (cf. [Q1, II, §3]). This observation is due to Pete Bousfield. We can then iterate the process by letting $\mathcal{F}_{\mathcal{G}}$ be the category of (coproducts of) the $\mathcal{G}$-spheres, defined: $\mathbf{S}^n := FS^{n-1} \in \mathcal{G}$ – see [Mi] – (with $\mathbf{S}^0 := GS^0$), and obtain a resolution model category structure on $s\mathcal{G}$ (bisimplicial groups).

Note that if we tried to do the same for $\mathcal{C} = \mathcal{S}et$, there are no nontrivial cogroup objects, while in $\mathcal{S}$ not all objects are fibrant (see §2.7). The category $\mathcal{T}_*$ of pointed topological spaces, which is the main example we actually have in mind, does not quite fit our assumptions (but see §2.12 above).

Motivation for the name of "resolution model category" is provided by the following

2.18. Definition. A *resolution* of an object $X_\bullet \in s\mathcal{C}$ (relative to $\mathcal{F}$) is a *cofibrant replacement* for $X_\bullet$ in the resolution model category on $s\mathcal{C}$ determined by $\mathcal{F}$: that is, it is any cofibrant object $Q_\bullet$, equipped with a weak equivalence to $X_\bullet$, which may be obtained from the factorization of $* \to X_\bullet$ as $* \xrightarrow{\text{cof}} Q_\bullet \xrightarrow{\text{fib+w.e.}} X_\bullet$ – and is thus unique up to weak equivalence, by §2.1(4)).

More classically, a (simplicial) resolution for an object $X \in \mathcal{C}$ is a resolution of the constant simplicial object $c(X)_\bullet$ (cf. §1.2) in $s\mathcal{C}$.

2.19. **Functorial resolutions.** The construction of [St, §2] provides canonical resolutions in $s\mathcal{C}$, defined as follows: consider the comonad $L : \mathcal{C} \to \mathcal{C}$ given by

$$LY = \coprod_{\alpha \in \hat{\mathcal{F}}} \coprod_{\phi \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{M}\langle\alpha\rangle, Y)} \mathbf{M}\langle\alpha\rangle_\phi \bigcup \coprod_{\alpha \in \hat{\mathcal{F}}} \coprod_{\Phi \in \mathrm{Hom}_{\mathcal{C}}(C\mathbf{M}\langle\alpha\rangle, Y)} C\mathbf{M}\langle\alpha\rangle_\Phi, \tag{2.20}$$

by which we mean the the coproduct, over all $\phi : \mathbf{M}\langle\alpha\rangle \to Y$, of the colimits of the various diagrams consisting of an inclusion $\mathbf{M}\langle\alpha\rangle_\phi \to C\mathbf{M}\langle\alpha\rangle_\Phi$ for each $\Phi : C\mathbf{M}\langle\alpha\rangle \to Y$ such that $\Phi|_{\mathbf{M}\langle\alpha\rangle} = \phi$. The counit $\varepsilon : LY \to Y$ is "evaluation of indices", and the comultiplication $\vartheta : LY \hookrightarrow L^2Y$ is the obvious "tautological" one. Note that $LY \in \mathcal{F}$ for any $Y \in \mathcal{C}$ by our assumptions on $\mathcal{F}$ (§2.7).

Given $X \in \mathcal{C}$, we define its *canonical resolution* $Q_\bullet \to X$ by $Q_n := L^{n+1}X$, with the degeneracies and face maps induced as usual by ε and ϑ (see [Gd, App., §3]).

The construction can be modified so as to yield resolutions for arbitrary $Y_\bullet \in s\mathcal{C}$, and not only $c(X)_\bullet$. Moreover, it has the advantage that $\pi_{\mathcal{F}} g : \pi_{\mathcal{F}} Q_n \to \pi_{\mathcal{F}} Y_n$ is clearly surjective for all n, so g can be changed into a fibration (Def. 2.16(iii)) by simply changing each Q_n up to homotopy, which yields the factorization needed for §2.1(3).

An alternative (noncanonical) construction of a resolution is given in Proposition 2.41 below.

2.21. **representing objects for $s\mathcal{C}$.** Just as the spheres "represent" the weak equivalences in the usual model structure on $\mathcal{T}_*$, for example, in the sense that a map $f : X \to Y$ is a weak equivalence if and only if it induces an isomorphism $f_* : [S^n, X] \to [S^n, Y]$ for each $n \geq 0$, we may similarly define representing objects for the resolution model category (compare [DKS2, §5.1]):

2.22. Definition. Given a model category $\mathcal{C}$ and a subcategory $\mathcal{F}$ as above, for each $n \geq 0$, the *n-dimensional simplicial $\mathcal{F}$-sphere*, denoted by $\mathcal{S}^n_{\mathcal{F}}$, is the subcategory $\Sigma^n\mathcal{F}$ of $s\mathcal{C}$, whose objects are of the form $\Sigma^n X := X \wedge S^n$ for $X \in \mathcal{F}$, where $S^n = \Delta[n]/\dot{\Delta}[n]$ is the usual simplicial n-sphere (see §2.3(i)).

Note that each such $\Sigma^n X$ is cofibrant (in fact, free) in the resolution model category $s\mathcal{C}$. Moreover, by the definition of the simplicial structure on $s\mathcal{C}$ (§2.3), $\Sigma^n X$ is also a cogroup object in $s\mathcal{C}$.

Given $Y_\bullet \in s\mathcal{C}$, choose some *fibrant replacement* $X_\bullet$ (that is, factor $Y_\bullet \to *$ as $Y_\bullet \xrightarrow{\text{cof+w.e.}} X_\bullet \xrightarrow{\text{fib}} *$, using §2.1(3)) and define $\hat{\pi}_n Y_\bullet$ (also written $[\mathbb{S}^n_{\mathcal{F}}, Y_\bullet]$) to be the $\hat{\mathcal{F}}$-graded set $\pi_0 \operatorname{map}(\mathbb{S}^n_{\mathcal{F}}, X_\bullet)$. This definition is independent of the choice of $X_\bullet$.

We define a map $f : X_\bullet \to Y_\bullet$ in $s\mathcal{C}$ to be an $\mathcal{F}$*-equivalence* if it induces isomorphisms in $\hat{\pi}_n(-)$ for all $n \geq 0$.

2.23. **fibration sequences.** Let $\mathcal{F} \subset \mathcal{C}$ be as in §2.7, and $X_\bullet \to Y_\bullet$ a fibration in the resolution model category $s\mathcal{C}$ (§2.16), with fiber $F_\bullet$ (§2.2). Then as usual we have the *long exact sequence of the fibration*:

$$\cdots \to \hat{\pi}_{n+1} Y_\bullet \xrightarrow{\partial_*} \hat{\pi}_n F_\bullet \to \hat{\pi}_n X_\bullet \to \hat{\pi}_n Y_\bullet \to \cdots \to \hat{\pi}_0 Y_\bullet, \tag{2.24}$$

(see [Q1, I,3.8]), which in fact may be constructed in this case as for $\mathcal{S}_*$ (see [May, 7.6]).

2.25. Definition. Given $X_\bullet \in s\mathcal{C}$, , we define the *n-cycles* object of $X_\bullet$, written $Z_n X_\bullet$, to be the fiber of $\delta_n : X_n \to M_n X_\bullet$ (see §2.4), so $Z_n X_\bullet = \{x \in X_n \mid d_i x = 0 \text{ for } i = 0, \ldots, n\}$ (cf. [Q1, I,§2]). Of course, this definition really makes sense only when δ_n is a fibration in $\mathcal{C}$. Similarly, the *n-chains* object of $X_\bullet$, written $C_n X_\bullet$, is defined to be the fiber of $\delta^1_n : X_n \to M^1_n X_\bullet$.

Note that for any $W \in \mathcal{C}$ and fibrant $X_\bullet \in s\mathcal{C}$ we have natural adjunction isomorphisms $\operatorname{Hom}_{s\mathcal{C}}(W \wedge S^n, X_\bullet) \cong \operatorname{Hom}_{\mathcal{C}}(W, Z_n X_\bullet)$ and $\operatorname{Hom}_{s\mathcal{C}}(W \wedge D^n, X_\bullet) \cong \operatorname{Hom}_{\mathcal{C}}(W, C_n X_\bullet)$ (where $D^n := \Delta^n / \Lambda^0[n] \in \mathcal{S}$ is a simplicial model for the n-disc).

If $X_\bullet$ is fibrant, the map $\mathbf{d}_0 = \mathbf{d}^n_0 := d_0|_{C_n X_\bullet} \colon C_n X_\bullet \to Z_{n-1} X_\bullet$ is the pullback of $\delta_n : X_n \to M_n X_\bullet$ along the inclusion $\iota : Z_{n-1} X_\bullet \to M_n X_\bullet$ (where $\iota(z) = (z, 0, \ldots, 0)$), so $\mathbf{d}_0$ is a fibration (in $\mathcal{C}$), fitting into a fibration sequence

$$\cdots \Omega Z_{n-1} X_\bullet \to Z_n X_\bullet \xrightarrow{j^X_n} C_n X_\bullet \xrightarrow{\mathbf{d}_0} Z_{n-1} X_\bullet \tag{2.26}$$

(see [DKS2, Prop. 5.7]). Moreover, there is an exact sequence of $\Pi_{\mathcal{F}}$-algebras

$$\pi_{\mathcal{F}} C_{n+1} X_\bullet \xrightarrow{(\mathbf{d}_0)_\#} \pi_{\mathcal{F}} Z_n X_\bullet \xrightarrow{q} \hat{\pi}_n X_\bullet \to 0, \tag{2.27}$$

(see [DKS2, Prop. 5.8]), which provides a (relatively) explicit way to recover $\hat{\pi}_n X_\bullet$ from $X_\bullet$.

Finally, the composition of the boundary map $\partial_* : \Omega Z_{n-1} X_\bullet \to Z_n X_\bullet$ of the fibration sequence (2.26) with $\Omega \mathbf{d}_0$ is trivial, so by (2.27) it induces a map of $\Pi_{\mathcal{F}}$-algebras from $\hat{\pi}_{n-1} \Omega X_\bullet \cong \Omega \hat{\pi}_{n-1} X_\bullet$ (§2.10) to $\pi_{\mathcal{F}} Z_n X_\bullet$ which, composed with the map q in (2.27), defines a "shift map" $s : \Omega \hat{\pi}_{n-1} X_\bullet \to \hat{\pi}_n X_\bullet$ (see [DKS2, Prop. 6.2]).

2.28. **the simplicial** $\Pi_{\mathcal{F}}$**-algebra.** Applying the functor $\pi_{\mathcal{F}}$ dimensionwise to any simplicial object $X_\bullet \in s\mathcal{C}$ yields a simplicial $\Pi_{\mathcal{F}}$-algebra $G_\bullet = \pi_{\mathcal{F}} X_\bullet$, which is in particular an $\hat{\mathcal{F}}$-graded simplicial group; its homotopy groups form a sequence of $\hat{\mathcal{F}}$-graded groups which we denote by $(\pi_n \pi_{\mathcal{F}} X_\bullet)_{n=0}^\infty$, and each $\pi_n \pi_{\mathcal{F}} X_\bullet$ is a $\Pi_{\mathcal{F}}$-algebra.

Note that as for any (graded) simplicial group, the homotopy groups of $G_\bullet$ may be computed using the Moore chains $C_* G_\bullet$, defined $C_n G_\bullet := \cap_{i=0}^n \operatorname{Ker}\{d_i : G_n \to G_{n-1}\}$ (cf. §2.25 and [May, 17.3]), and we have the following version of [Bl8, Prop. 2.11]

2.29. Lemma. *For any fibrant* $X_\bullet \in s\mathcal{C}$, *the inclusion* $\iota : C_n X_\bullet \hookrightarrow X_n$ *induces an isomorphism* $\iota_\star : \pi_* C_n X_\bullet \cong C_n(\pi_* X_\bullet)$ *for each* $n \geq 0$.

Proof. (a) First, note that any *trivial* cofibration $j : A \hookrightarrow B$ in $\mathcal{S}$ induces a fibration $j^* : M_B X_\bullet \to M_A X_\bullet$ in $\mathcal{C}$.

To see this, by assumption 2.13 it suffices to consider $\mathcal{C} = \mathcal{D}$, (since by [Bl5, Def. 4.13], f is a fibration in $\mathcal{C}$ if and only if Uf is a fibration in $\mathcal{D}$), and in fact the only nontrivial case is when $\mathcal{D} = \mathcal{G}$ (where the fibrations are maps which surject onto the identity component – see [Q1, II, 3.8]). Note that in internal simplicial dimension k we have $(M_A X_\bullet)^{int}_k \cong \mathrm{Hom}_{s\mathcal{G}p}(FA, (X_\bullet)^{int}_k)$ (see §1.2) for $A \in \mathcal{S}$, where F denotes the (dimensionwise) free group functor. Since FA is fibrant in $s\mathcal{G}p$, $Fj : FA \hookrightarrow FB$ has a left inverse $r : FB \to FA$, so $j^* : (M_B X_\bullet)^{int}_k \to (M_A X_\bullet)^{int}_k$ has a right inverse r^*, so in particular is onto. Since this is true in each simplicial dimension k, $j^* : M_B X_\bullet \to M_A X_\bullet$ is a fibration in $\mathcal{G}$. (Note that $d_i : X_n \to X_{n-1}$ is always a fibration.)

(b) In addition, $\psi^k_n = j^* : M^0_n X_\bullet \to M^k_n X_\bullet$ is a fibration for all $0 \leq k \leq n$, as one can see by considering (2.5) (since δ_{n-1} surjects onto the identity component by assumption).

(c) Given $\eta \in C_n(\pi_\alpha X_\bullet)$, represented by $h : \mathbf{M}\langle\alpha\rangle \to Y_n$, with $d_j h \sim 0$ for $1 \leq j \leq n$, note that for $1 \leq k \leq n$, $M^k_n X_\bullet$ is the pullback of

$$M^{k+1}_n X_\bullet \xrightarrow{(d_k,\dots,d_k)} M^k_{n-1} X_\bullet \xleftarrow{\delta^k_{n-1}} X_{n-1},$$

in which $(d_k, \dots, d_k)$ is a fibration by (a) if $k \geq 1$, so this is in fact a homotopy pullback square (see [Mat, §1]). By descending induction on $1 \leq k \leq n$, (starting with $\delta^n_n = d_n$), we may assume $\delta^{k+1}_n \circ h : \mathbf{M}\langle\alpha\rangle \to M^{k+1}_n X_\bullet$ is nullhomotopic in $\mathcal{C}$, as is $d_k \circ h$, so the induced pullback map, which is just $\delta^k_n \circ h : \mathbf{M}\langle\alpha\rangle \to M^k_n X_\bullet$, is also nullhomotopic by the universal property. We conclude that $\delta^1_n \circ h \sim 0$, and since $\delta^1_n : X_n \to M^1_n X_\bullet$ is a fibration by (b), we can replace h by a homotopic map $h' : \mathbf{M}\langle\alpha\rangle \to X_n$ such that $\delta_n h' = 0$. Thus h' lifts to $Z_n Y_\bullet = \mathrm{Fib}(\delta_n)$, so $\iota_\star$ is surjective.

(d) Even though the retraction $r : F\Delta[n] \to F\Lambda^0_n$ in (a) is not canonical, it may be chosen independently of the internal simplicial dimension k to yield a section r^* for $\delta^1_n = j^* : X_n \twoheadrightarrow M^1_n X_\bullet$. The long exact sequence in $[\mathbf{M}\langle\alpha\rangle, -]$ for the fibration sequence $C_n Y_\bullet \xrightarrow{i} Y_n \xrightarrow{\delta'_n} M^1_n Y_\bullet$ (cf. [Q1, I,§3]) then implies that $i_\#$ is monic, so $\iota_\star$ is, too. The argument lifts from $\mathcal{D} = \mathcal{G}$ to $\mathcal{C}$ because the objects $\mathbf{M}\langle\alpha\rangle$ are in the image of the adjoint of $U : \mathcal{C} \to \mathcal{D}$, by assumption 2.13. □

This Lemma, together with (2.27), yields a commuting diagram:

$$\begin{tikzcd}
\pi_{\mathcal{F}} C_{n+1} X_\bullet \arrow[r, "(\mathbf{d}_0)_\#"] \arrow[d, "\iota_\star"', "\cong"] & \pi_{\mathcal{F}} Z_n X_\bullet \arrow[r, two heads] \arrow[d, "\hat{\iota}_\star"'] & \hat{\pi}_n X_\bullet \arrow[d, dotted, "h"] \\
C_{n+1}(\pi_{\mathcal{F}} X_\bullet) \arrow[r, "d_0^{\pi_{\mathcal{F}} X_\bullet}"] & Z_n(\pi_{\mathcal{F}} X_\bullet) \arrow[r, two heads] & \pi_n \pi_{\mathcal{F}} X_\bullet
\end{tikzcd}$$

FIGURE 1

which defines the dotted morphism of $\Pi_{\mathcal{F}}$-algebras $h : \hat{\pi}_n X_\bullet \to \pi_n(\pi_{\mathcal{F}} X_\bullet)$ (this was called the "Hurewicz map" in [DKS2, 7.1]). Note that for $n = 0$ the map $\hat{\iota}_\star$ is an isomorphism, so h is, too.

2.30. **An exact couple.** If $X_\bullet \in s\mathcal{C}$ is Reedy fibrant, the long exact sequences (2.24) for the fibrations $\mathcal{C}_{n+1} X_\bullet \to Z_n X_\bullet$ fit into an $(\mathbb{N}, \hat{\mathcal{F}})$-bigraded exact couple $(D^1_{*,\alpha}, E^1_{*,\alpha})$ with $D^1_{k,\alpha} \cong \pi_\alpha Z_k X_\bullet$ and $E^1_{k,\alpha} \cong \pi_\alpha C_k X_\bullet$ for $k \geq 0$ and $\mathbf{M}\langle\alpha\rangle \in \hat{F}$. As in [DKS2, §8] the derived couple has $D^2_{k,\alpha} \cong (\hat{\pi}_k X_\bullet)_\alpha$ and $E^2_{k,\alpha} \cong \pi_k(\pi_\alpha X_\bullet)$ (using Lemma 2.29), which fit into a "spiral exact sequence"

$$\cdots \to \pi_{n+1}\pi_{\mathcal{F}} X_\bullet \xrightarrow{\partial} \Omega\hat{\pi}_{n-1}X_\bullet \xrightarrow{s} \hat{\pi}_n X_\bullet \xrightarrow{h} \pi_n\pi_{\mathcal{F}} X_\bullet \to \cdots \hat{\pi}_0 X_\bullet \xrightarrow{h} \pi_0\pi_{\mathcal{F}} X_\bullet \to 0 \tag{2.31}$$

as in [DKS2, 8.1], so by Reedy fibrant replacement (§2.22), one has such an exact sequence for any $Y_\bullet \in s\mathcal{C}$. Of course, $\hat{\pi}_{-1}X_\bullet := 0$; and at the right hand end we have $h : \hat{\pi}_0 X_\bullet \cong \pi_0\pi_{\mathcal{F}} X_\bullet$, as noted above.

We immediately deduce the following

2.32. Proposition. *A map $f : X_\bullet \to Y_\bullet$ in $s\mathcal{C}$ is a weak equivalence in the resolution model category – i.e., induces an isorphism in $\pi_n\pi_{\mathcal{F}}$ for all $n \geq 0$ (§2.16(i)) – if and only if it is an $\mathcal{F}$-equivalence – i.e., induces an isomorphism in $\hat{\pi}_n$ for all $n \geq 0$ (see §2.22).*

2.33. **Resolutions.** By Definition 2.18, a *resolution* of an object $X \in \mathcal{C}$ is a simplicial object $Q_\bullet$ over $\mathcal{C}$ which is cofibrant and has a weak equivalence $f : Q_\bullet \to c(X)_\bullet$. Note that such an f is detemined by an *augmentation* $\varepsilon : Q_0 \to X$ in $\mathcal{C}$ (with $d_0 \circ \varepsilon = d_1 \circ \varepsilon$); by Proposition 2.32, f is a weak equivalence if and only if the augmented $\hat{\mathcal{F}}$-graded simplicial group $\varepsilon_* : \pi_{\mathcal{F}} Q_\bullet \to \pi_{\mathcal{F}} X$ is acyclic (i.e., has vanishing homotopy groups in all dimensions ≥ 0).

The long exact sequence (2.31) then implies that

$$\hat{\pi}_n Q_\bullet \cong \Omega^n \pi_{\mathcal{F}} X \quad \text{for all } n \geq 0. \tag{2.34}$$

2.35. Definition. A *CW complex* over a pointed category $\mathcal{C}$ is a simplicial object $R_\bullet \in s\mathcal{C}$, together with a sequence of objects $\bar{R}_n$ $(n = 0, 1, \dots)$ – called a *CW basis* for $R_\bullet$ – such that $R_n = \bar{R}_n \amalg L_n R_\bullet$ (§2.4), and $d_i|_{\bar{R}_n} = 0$ for $1 \leq i \leq n$. The morphism $\bar{d}^n_0 : \bar{R}_n \to Z_{n-1}R_\bullet$ is called the n-th *attaching map* for $R_\bullet$ (compare [Bl1, §5]).

A *CW resolution* of a simplicial $\Pi_{\mathcal{F}}$-algebra $A_\bullet$ is a CW complex $G_\bullet \in s\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg$, with CW basis $(\bar{G}_n)_{n=0}^\infty$ such that each $\bar{G}_n$ is a free $\Pi_{\mathcal{F}}$-algebra, together with a weak equivalence $\phi : G_\bullet \to A_\bullet$.

2.36. Definition. In the situation of §2.7, a simplicial object $R_\bullet \in s\mathcal{C}$ is called a *CW resolution* of $X_\bullet \in s\mathcal{C}$ if $R_\bullet$ is a CW complex with each $\bar{R}_n$ in $\mathcal{F}$, up to homotopy (so in particular $R_\bullet$ is indeed cofibrant), equipped with a weak equivalence $f : R_\bullet \to X_\bullet$.

2.37. Remark. It is easy to see that one can inductively construct a CW resolution for every simplicial $\Pi_{\mathcal{F}}$-algebra $A_\bullet$, since in order for $\phi : G_\bullet \to A_\bullet$ to be a weak equivalence it is necessary and sufficient that $Z_n\phi$ take $Z_n G_\bullet$ onto a set of representatives of $\pi_n A_\bullet$ in $Z_n A_\bullet$, and the attaching map $\bar{d}^n_0$ map $\bar{G}_n$ onto a set of representatives for $\mathrm{Ker}(\pi_n\phi)$ in $Z_{n-1}G_\bullet$. Thus we can let $\bar{G}_n$ be the free $\Pi_{\mathcal{F}}$-algebra (§2.8) generated by union of the underlying sets of $Z_n A_\bullet$ and $\mathrm{Ker}(Z_{n-1}f)$, say.

The "topological" version of this requires a little more care. In particular, [Bl8, Remark 3.16] implies that not every free simplicial $\Pi_{\mathcal{F}}$-algebra $A_\bullet$ is *realizable* in the sense that there is a $R_\bullet \in s\mathcal{C}$ with $\pi_{\mathcal{F}} R_\bullet \cong A_\bullet$. In order to see what can be said on this context, assume given a fibrant and cofibrant simplicial object $P_\bullet$ with an augmentation $\varepsilon : P_0 \to X$. For each $\alpha \in \hat{\mathcal{F}}$, consider the long exact sequence

$$\dots \pi_{\alpha+1} C_m P_\bullet \xrightarrow{(\mathbf{d}^m_0)_\#} \pi_{\alpha+1} Z_{m-1} P_\bullet \xrightarrow{\partial_{m-1}} \pi_\alpha Z_m P_\bullet \xrightarrow{(j_m)_\#} \pi_\alpha C_m P_\bullet \dots \tag{2.38}$$

for the fibration $\mathbf{d}^m_0$, where $Z_0 P_\bullet := P_0$. By definition, $P_\bullet \to X$ is a resolution if and only if $\pi_i \pi_{\mathcal{F}} P_\bullet = 0$ for each $i \geq 0$, where the homotopy groups are understod in the augmented sense

– that is, $\pi_0\pi_{\mathcal{F}}P_\bullet := \operatorname{Ker}((\mathbf{d}_0^0)_\# : C_0\pi_{\mathcal{F}}P_\bullet \to Z_{-1}\pi_{\mathcal{F}}P_\bullet)/\operatorname{Im}((\mathbf{d}_0^1)_\# : C_1\pi_{\mathcal{F}}P_\bullet \to Z_0\pi_{\mathcal{F}}P_\bullet)$. The key technical fact we shall need in this context is contained in the following

2.39. Lemma. *An fibrant and cofibrant $P_\bullet \in s\mathcal{C}$ with an augmentation $P_\bullet \to X$ is a resolution of X if and only if for each $m > 0$:*

(a) *There is a short exact sequence* $0 \to \operatorname{Im}(\partial_{m-1}) \hookrightarrow \pi_{\mathcal{F}}Z_mP_\bullet \xrightarrow{(j_m)_\#} Z_m\pi_{\mathcal{F}}P_\bullet \to 0$, *and*
(b) $\partial_m|_{\operatorname{Im}(\partial_{m-1})}$ *is one-to-one, and surjects onto* $\operatorname{Im}(\partial_m)$, *and* $\operatorname{Im}\partial_0 \cong \Omega\pi_{\mathcal{F}}X$.

Note that since ∂_m shifts degrees by one, (a) and (b) together imply that $\operatorname{Im}(\partial_m) \cong \Omega^{m+1}\pi_{\mathcal{F}}X$ for each m.

Proof. For any $P_\bullet$, the inclusion $j_m : Z_mP_\bullet \to C_mP_\bullet$ induces a map of $\Pi_{\mathcal{F}}$-algebras $(j_m)_\# : \pi_{\mathcal{F}}Z_mP_\bullet \to \pi_{\mathcal{F}}C_mP_\bullet \cong C_m\pi_{\mathcal{F}}P_\bullet$ (see Lemma 2.29), which factors through $Z_m\pi_{\mathcal{F}}P_\bullet$. Denote the boundary map for the chain complex $C_*\pi_{\mathcal{F}}P_\bullet$ (which computes $\pi_*\pi_{\mathcal{F}}P_\bullet$) by $D_m := (j_{m-1})_\# \circ (\mathbf{d}_0^m)_\#$.

If $P_\bullet \to X$ is a resolution, we must have $\operatorname{Im}((j_m)_\# \circ (\mathbf{d}_0^{m+1})_\#) = \operatorname{Im}(D_{m+1}) = \operatorname{Ker}(D_m)$ for each $m \geq 0$, so in particular $(j_m)_\#$ maps onto $Z_{m-1}\pi_{\mathcal{F}}P_\bullet$. Moreover, since $\pi_{\mathcal{F}}C_1P_\bullet \to \pi_{\mathcal{F}}P_0 \twoheadrightarrow \pi_{\mathcal{F}}X \to 0$ is exact, $\operatorname{Im}\partial_0 \cong \Omega\pi_{\mathcal{F}}X$ and so if we assume by induction that (b) holds for $m-1$, we see that $\operatorname{Ker}(j_m)_\# = \operatorname{Im}\partial_{m-1}$ is isomorphic to $\Omega^m\pi_{\mathcal{F}}X$, which proves (a). Moreover, if $0 \neq \gamma \in \operatorname{Ker}\partial_m = \operatorname{Im}(\mathbf{d}_0^{m+1})_\#$, and $\gamma \in \operatorname{Im}\partial_{m-1} = \operatorname{Ker}(j_m)_\#$, then we have $\beta \in \pi_{\mathcal{F}}C_{m+1}P_\bullet$ with $(\mathbf{d}_0^{m+1})_\#(\beta) = \gamma \neq 0$ but $D_{m+1}(\beta) = 0$ – contradicting (a) for $m+1$. Finally, if $(j_m)_\#(\gamma) \neq 0$, there is a $\beta \in \pi_{\mathcal{F}}C_{m+1}P_\bullet$ with $D_m(\beta) = (j_m)_\#(\gamma)$, by the acyclicity of $\pi_{\mathcal{F}}P_\bullet$, so $\gamma - (\mathbf{d}_0^{m+1})_\#(\beta) \in \operatorname{Ker}(j_m)_\# = \operatorname{Im}\partial_{m-1}$, and $\partial_m(\gamma - (\mathbf{d}_0^{m+1})_\#(\beta)) = \partial_m(\gamma)$, which proves (b) for m. The identification of $\operatorname{Im}\partial_0$ is immediate from (2.38).

Conversely, if (a) and (b) are satisfied for all m, for any element in $\zeta \in Z_m\pi_{\mathcal{F}}P_\bullet$, we have $\zeta = (j_m)_\#(\gamma)$ for some $\gamma \in \pi_{\mathcal{F}}Z_mP_\bullet$. Thus there is a $\theta \in \pi_{\mathcal{F}}Z_{m-1}P_\bullet$ with $\partial_m(\partial_{m-1}(\theta)) = \partial_m(\gamma)$, by (b), so $\gamma \cdot \partial_{m-1}(\theta)^{-1}$ is in $\operatorname{Ker}\partial_m = \operatorname{Im}(\mathbf{d}_0^{m+1})_\#$; thus $(j_m)_\#(\gamma \cdot \partial_{m-1}(\theta)^{-1}) = \zeta$ bounds, and $\pi_{\mathcal{F}}P_\bullet$ is acyclic. $\square$

It should be pointed out that the fundamental short exact sequence

$$0 \to \Omega^m\pi_{\mathcal{F}}X \cong \operatorname{Im}(\partial_{m-1}) \hookrightarrow \pi_{\mathcal{F}}Z_mP_\bullet \xrightarrow{(j_m)_\#} Z_m\pi_{\mathcal{F}}P_\bullet \to 0 \tag{2.40}$$

for a resolution $P_\bullet$ is actually *split*, as a sequence of graded groups, because $(j_m)_\#|_{\operatorname{Im}(\mathbf{d}_0^{m+1})_\#} = (j_m)_\#|_{\operatorname{Ker}\partial_m}$ is one-to-one, by (b), and surjects onto $Z_m\pi_{\mathcal{F}}P_\bullet$ by the acyclicity. However, $\operatorname{Im}(\mathbf{d}_0^{m+1})_\# = \operatorname{Ker}\partial_m$ need not be a sub-$\Pi_{\mathcal{F}}$-algebra of $\pi_{\mathcal{F}}Z_mP_\bullet$, since ∂_m is not a morphism of $\Pi_{\mathcal{F}}$-algebras.

With the aid of Lemma 2.39 we can now show:

2.41. Proposition. *Under the assumptions of §2.7 and 2.13, any $X \in \mathcal{C}$ has a CW resolution $R_\bullet \in s\mathcal{C}$.*

Proof. Let $Q_\bullet \in s\mathcal{C}$ be the functorial resolution of §2.19; we may assume that the augmentation $\varepsilon^Q : Q_0 \to X$ is a fibration.

We start off by choosing a set $T_*^0 \subseteq \pi_{\mathcal{F}}Q_0$ of $\Pi_{\mathcal{F}}$-algebra generators (§2.8), such that if we let $R'_0 := \coprod_{\alpha\in\hat{\mathcal{F}}} \coprod_{\beta\in T_\alpha^0} \mathbf{M}\langle\alpha\rangle_\beta$, then $\varepsilon_\#^Q$ maps the free $\Pi_{\mathcal{F}}$-algebra $\pi_{\mathcal{F}}R'_0 \subset \pi_{\mathcal{F}}Q_0$ onto $\pi_{\mathcal{F}}X$. We may assume T_*^0 is *minimal*, in the sense that no sub-graded set generates a free $\Pi_{\mathcal{F}}$-algebra surjecting onto $\pi_{\mathcal{F}}X$ – so that $\varepsilon_\#^Q(\beta) \neq 0$ for all $\beta \in T_*^0$.

The inclusion $\phi : \pi_{\mathcal{F}}R'_0 \hookrightarrow \pi_{\mathcal{F}}Q_0$ defines a map $f'_0 : R'_0 \to Q_0$ with $(f'_0)_\# = \phi$, and we let $\varepsilon^{R'} := \varepsilon^Q \circ f'_0$; factoring $\varepsilon^{R'}$ by 2.1(3) as $R'_0 \xrightarrow{i} R_0 \xrightarrow{\varepsilon^R} X$ and usng the LLP for i and ε^Q yields $f_0 : R_0 \to Q_0$ commuting with ε.

Now assume by induction that we have constructed a fibrant and cofibrant $R_\bullet$ through simplicial dimension $n-1 \geq 0$, together with a map $\mathrm{tr}_{n-1} f : \mathrm{tr}_{n-1} R_\bullet \to \mathrm{tr}_{n-1} Q_\bullet$ which induces an embedding of $\Pi_{\mathcal{F}}$-algebras $(\mathrm{tr}_{n-1} f)_\#$. We assume that $R_\bullet$ satisfies (a) and (b) of Lemma 2.39 for $0 < m < n$ (and of course $Q_\bullet$ satisfies them for all $m > 0$). If we map the short exact sequence (a) for $R_\bullet$ to the corresponding sequence for $Q_\bullet$ by f_*, we see that $Z_{n-1}(f_\#) = Z_{n-1}\phi : Z_{n-1}\pi_{\mathcal{F}}R_\bullet \to Z_{n-1}\pi_{\mathcal{F}}Q_\bullet$ is one-to-one, so $(Z_{n-1}f)_\# : \pi_{\mathcal{F}}Z_{n-1}R_\bullet \to \pi_{\mathcal{F}}Z_{n-1}Q_\bullet$ is, too.

Any non-zero element in $Z_{n-1}\pi_\alpha R_\bullet$ is represented by $\gamma \in \pi_{\mathcal{F}}Z_{n-1}R_\bullet$, by (2.40) for R_{n-1}. Let $g : \mathbf{M}\langle\alpha\rangle \to Z_{n-1}Q_\bullet$ represent $f_\#\gamma \in \pi_\alpha Z_{n-1}Q_\bullet$, with $\mathbf{M}\langle\alpha\rangle_{(g)}$ the corresponding coproduct summand of $Q_n = LQ_{n-1}$ in (2.20), with $i_{(g)} : \mathbf{M}\langle\alpha\rangle_{(g)} \to Q_n$ the inclusion. Then $d_i \circ i_{(g)} = i_{(d_{i-1}g)}$ for $1 \leq i \leq n$ (in the same notation) and $d_0 \circ i_{(g)} = g$, by §2.19. Thus the $\Pi_{\mathcal{F}}$-algebra generator $\langle i_{(g)}\rangle \in \pi_\alpha Q_n$ is in $\mathcal{C}_n\pi_{\mathcal{F}}Q_\bullet$, and $(\mathbf{d}_0^n)_\#\langle i_{(g)}\rangle = f_\#\gamma$.

Thus if we choose a set T_*^n of $\Pi_{\mathcal{F}}$-algebra generators for $Z_{n-1}\pi_{\mathcal{F}}R_\bullet$ and set

$$\bar{R}_n := \coprod_{\alpha\in\hat{\mathcal{F}}} \coprod_{\beta\in T_\alpha^n} \mathbf{M}\langle\alpha\rangle_{(\beta)}, \tag{2.42}$$

we have maps $\bar{f}_n : \bar{R}_n \to C_nQ_\bullet$ and $\bar{d}_0 : \bar{R}_n \to Z_{n-1}R_\bullet$ such that $(j_{n-1})_\# \circ (\mathbf{d}_0^Q)_\# \circ (\bar{f}_n)_\# = (j_{n-1})_\# \circ (Z_{n-1}f)_\# \circ (\bar{d}_0)_\#$. Now (2.40) implies that $(j_{n-1})_\#$ is one-to-one on $\operatorname{Im}\mathbf{d}_0$, so $(\mathbf{d}_0^Q)_\# \circ (\bar{f}_n)_\# = (Z_{n-1}f)_\# \circ (\bar{d}_0)_\#$. Because $(\mathbf{d}_0^Q)_\#$ is a fibration and $\pi_{\mathcal{F}}\bar{R}_n$ is free, this implies that one can choose $\bar{f}_n$ so that $\mathbf{d}_0^Q \circ \bar{f}_n = Z_{n-1}f \circ \bar{d}_0$. Since $L_nf : L_nR_\bullet \to L_nQ_\bullet$ exists by the induction hypothesis, one can define $f_n : R_n \simeq L_nR_\bullet \amalg \bar{R}_n \to Q_n$ extending $\mathrm{tr}_{n-1} f$ to $\mathrm{tr}_n f : \mathrm{tr}_n R_\bullet \to \mathrm{tr}_n Q_\bullet$, with $\delta_n^R : R_n \to M_nR_\bullet$ a fibration. Since $\pi_i\pi_{\mathcal{F}}P_\bullet = 0$ then holds for $i \leq n-1$, (2.42) and (2.40) hold for $m = n$. □

2.43. Remark. We have actually proved a little more: given any *minimal* simplicial CW resolution of $\Pi_{\mathcal{F}}$-algebra's $A_\bullet \to \pi_{\mathcal{F}}X$ (§2.35) of a realizable $\Pi_{\mathcal{F}}$-algebra, one can find a CW resolution $R_\bullet \to X$ realizing it: that is, $\pi_{\mathcal{F}}R_\bullet \cong A_\bullet$. (Minimality here is understood to mean that we allow no unnecessary $\Pi_{\mathcal{F}}$-algebra generators in each $\bar{A}_n$, beyond those needed to map onto $Z_{n-1}A_\bullet$.)

By a more careful analysis, as in [Bl8, Thm. 3.19], one could in fact show that *any* CW resolution of $\pi_{\mathcal{F}}X$ is realizable. However, this will follow from Corollary 4.11 below.

3. Postnikov systems and the fundamental group action

We now describe Postnikov systems for simplicial objects in the resolution model category, and the fundamental group action on them.

3.1. Definition. If $\mathcal{C}$ is a category satisfying the assumptions of §2.7, a *Postnikov system* for an object $Y_\bullet \in s\mathcal{C}$ is a sequence of objects $P_nX_\bullet \in s\mathcal{C}$, together with maps $\varphi^n : X_\bullet \to P_nX_\bullet$ and $p^n : P_{n+1}X_\bullet \to P_nX_\bullet$ (for $n \geq 0$), such that $\hat{\pi}_kp^n$ and $\hat{\pi}_k\varphi^n$ are isomorphisms for all $k \leq n$, and $\hat{\pi}_kP_nX_\bullet = 0$ for $k \geq n+1$

3.2. Remark. In general, such Postnikov towers may be constructed for fibrant $X_\bullet$ using a variant of the standard construction for simplicial sets (cf. [May, §8]) due to Dwyer and Kan in [DK2, §1.2], and for arbitrary $X_\bullet$ by using a fibrant approximation.

Note that if $Q_\bullet \in s\mathcal{C}$ is a resolution of some $X \in \mathcal{C}$ (see §2.33), then by (2.34) $\hat{\pi}_iP_nQ_\bullet \cong \Omega^i\pi_{\mathcal{F}}X$ for $n \geq i \geq 0$, and $\hat{\pi}_iP_nQ_\bullet = 0$ for $i > n$; so (2.31) implies that

$$\pi_i\pi_{\mathcal{F}}P_nQ_\bullet \cong \begin{cases} \pi_{\mathcal{F}}X & \text{for } i = 0 \\ \Omega^{n+1}\pi_{\mathcal{F}}X & \text{for } i = n+2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

3.4. **Postnikov towers for resolutions.** It is actually easier to construct a cofibrant version of the Postnikov tower for a resolution than it is to construct the resolution itself: Given a CW resolution $Q_\bullet$ of an object $X \in \mathcal{C}$, (constructed as in Proposition 2.41), with CW basis $(\bar{Q}_k)_{k=0}^{\infty}$, we construct a CW cofibrant approximation $Y_\bullet \to Q_\bullet^{(n)}$ as follows.

Let $J_* := \pi_{\mathcal{F}} X$, and choose some $G \in ho\mathcal{F}$ (i.e., $G \simeq \coprod_{\alpha\in\hat{\mathcal{F}}} \coprod_{T_\alpha} \mathbf{M}\langle\alpha\rangle$) having a surjection of $\Pi_{\mathcal{F}}$-algebras $\phi : \pi_{\mathcal{F}} G \twoheadrightarrow \Omega^{n+1} J_*$. Set $\bar{Y}_{n+2} := \bar{Q}_{n+2} \amalg G$, with $(d_0|_G)_\# = \phi$, mapping onto $\Omega^{n+1} J_* \cong \mathrm{Im}(\partial_n) \hookrightarrow \pi_{\mathcal{F}} Z_{n+1} Q_\bullet = \pi_{\mathcal{F}} Z_{n+1} Y_\bullet$ (see (2.40)). This defines $Y'_{n+2} := \bar{Y}_{n+1} \amalg L_{n+2} Y_\bullet \xrightarrow{\delta_{n+2}} M_{n+2} Y_\bullet$, which we then change into a fibration. Since $(\mathbf{d}_0^{n+2})_\# : \pi_{\mathcal{F}} C_{n+2} Y_\bullet \twoheadrightarrow \pi_{\mathcal{F}} Z_{n+1} Y_\bullet$ is surjective, we may assume by induction on $k \geq n+2$ that

$$(j_k)_* : \pi_{\mathcal{F}} Z_k Y_\bullet \xrightarrow{\cong} Z_k \pi_{\mathcal{F}} Y_\bullet \qquad \text{and} \qquad \partial_{k-1} = 0, \tag{3.5}$$

and thus we may choose $\bar{Y}_{k+1} \in ho\mathcal{F}$ with $\bar{d}_0 : \pi_{\mathcal{F}} \bar{Y}_{k+1} \twoheadrightarrow \pi_{\mathcal{F}} Z_k \pi_{\mathcal{F}} Y_\bullet$, and see that (3.5) holds for $k+1$ by (2.40).

Note that $Y_\bullet \simeq Q_\bullet^{(n)}$ is constructed by "attaching cells" to $Q_\bullet$, as in the traditional method for "killing homotopy groups" (cf. [Gr, §17]), so we have a natural *embedding* $\rho : Q_\bullet \hookrightarrow Y_\bullet$, rather than a fibration. In fact, it is helpful to think of $P_n X_\bullet$ as a homotopy-invariant version of the $(n+1)$-skeleton of $X_\bullet$: starting with $\mathrm{tr}_{n+1} X_\bullet$, one completes it to a full simplicial object by a functorial construction which (unlike the skeleton) depends only on the homotopy type of $X_\bullet$.

3.6. Π**-algebras and the fundamental group.** Under our assumptions, the category $\mathcal{C} = \Pi_{\mathcal{F}}\text{-}\mathcal{A}lg$ is a *CUGA*, or category of universal graded algebras (see [BS, §2.1] and [Mc1, V,§6]), so that $s\mathcal{C}$, the category of simplicial $\Pi_{\mathcal{F}}$-algebras, has a model category structure defined by Quillen (see [Q1, II, §4]). Equivalently, one could take the resolution model category on $s\mathcal{C}$, starting with the trivial model category structure on $\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg$, and letting $\mathcal{F}_{\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg}$ be the subcategory of all free $\Pi_{\mathcal{F}}$-algebras – as in §2.17. One thus has a concept of "spheres" in $s\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg$ – namely, $\pi_{\mathcal{F}} \Sigma^n \mathbf{M}\langle\alpha\rangle$, for $\alpha \in \hat{\mathcal{F}}$ (cf. §2.22) – and $(\pi_n A_\bullet)_\alpha \cong [\Sigma^n \mathbf{M}\langle\alpha\rangle, A_\bullet]_{s\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg}$ for any simplicial $\Pi_{\mathcal{F}}$-algebra $A_\bullet$. Thus if we take homotopy classes of maps between (coproducts of) these spheres as the primary homotopy operations (see [W, XI, §1]), we can endow the homotopy groups $\pi_* A_\bullet = (\pi_i A_\bullet)_{i=0}^{\infty}$ of $A_\bullet$ with an additional structure: that of a $(\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg)$-$\Pi$-algebra, in the (somewhat unfortunate, in this case) terminology of [BS, §3.2]. By definition, this structure is a homotopy invariant of $A_\bullet$.

In our situation, however, because we are dealing with Postnikov sections, by (3.3) we only need the very simplest part of that structure – namely, the action of the fundamental group $\pi_0 A_\bullet$ on each of the higher homotopy groups $\pi_n A_\bullet$.

Observe that because $\mathcal{C}$ has an underlying group structure, by assumption 2.13, the indexing of the homotopy groups of an object in $s\mathcal{C}$ should be shifted by one compared with the usual indexing in $\mathcal{T}_*$, so that $\pi_0 A_\bullet$ is indeed the fundamental group, and in fact the action we refer to is a straightforward generalization of the usual action of the fundamental group of a simplicial group (or topological space) on the higher homotopy groups.

3.7. J_***-modules and** J_***-algebras.** We shall be interested in an algebraic description of this action: that is, we would like a category of universal algebras which model this action, in the same sense that Π-algebras model the action of all the primary homotopy operations on the homotopy groups of a space. Just as in the case of ordinary Π-algebras, the action in question is determined by the homotopy classes of maps of simplicial $\Pi_{\mathcal{F}}$-algebras.

Thus we are led to consider two distinct "varieties of algebras", in the terminology of [Mc1, V, §6]): one modeled on the homotopy classes of maps, and one on the actual maps.

3.8. Definition. Given a $\Pi_{\mathcal{F}}$-algebra J_*, let J_*-$\mathcal{M}od$ denote the category of universal algebras whose operations are in one-to-one correspondence with homotopy classes of maps $\pi_{\mathcal{F}}\Sigma^n\mathbf{M}\langle\alpha\rangle \to \pi_{\mathcal{F}}(\Sigma^n\mathbf{M}\langle\alpha'\rangle \amalg \Sigma^0\mathbf{M}\langle\alpha''\rangle)$, and whose universal relations correspond to the relations holding among these homotopy class in $ho(s\mathcal{C})$. These model $\pi_n A_\bullet$, with the action of $\pi_0 A_\bullet$, for $A_\bullet \in s\Pi_{\mathcal{F}}$-$\mathcal{A}lg$.

An object $K_* \in J_*$-$\mathcal{M}od$ is itself a $\Pi_{\mathcal{F}}$-algebra, equipped with an action of an operation $\lambda : J_{\alpha''} \times K_{\alpha'} \to K_\alpha$ for each $\lambda \in [\pi_{\mathcal{F}}\Sigma^n\mathbf{M}\langle\alpha\rangle, \pi_{\mathcal{F}}(\Sigma^n\mathbf{M}\langle\alpha'\rangle \amalg \Sigma^0\mathbf{M}\langle\alpha''\rangle)]$. Such a K_* will be called a *J_*-module*, even though in general the category of such objects, which we shall denote by J_*-$\mathcal{M}od$, need not be abelian (and it could depend on n). However, in the cases that interest us, J_*-$\mathcal{M}od$ will be abelian, and will not depend on $n > 0$.

3.9. Definition. Given a $\Pi_{\mathcal{F}}$-algebra J_*, let J_*-$\mathcal{A}lg$ denote the category of universal algebras whose operations are in one-to-one correspondence with *actual maps* $\pi_{\mathcal{F}}\Sigma^n\mathbf{M}\langle\alpha\rangle \to \pi_{\mathcal{F}}(\Sigma^n\mathbf{M}\langle\alpha'\rangle \amalg \Sigma^0\mathbf{M}\langle\alpha''\rangle)$ as above, and whose universal relations correspond to the relations holding among these maps in $s\mathcal{C}$. The objects in J_*-$\mathcal{A}lg$, which are again $\Pi_{\mathcal{F}}$-algebras with additional structure, will be called *J_*-algebras.*

The category J_*-$\mathcal{A}lg$ is generally very complicated; it is not abelian, and we cannot expect to know much about it, even for $\mathcal{C} = \mathcal{G}$, say. In particular, one may well have a different category for each $n > 0$ (although we surpress the dependence on n to avoid excessive notation). Note, however, that maps $\ell : \pi_{\mathcal{F}}\Sigma^n\mathbf{M}\langle\alpha\rangle \to A_\bullet$, for any simplicial $\Pi_{\mathcal{F}}$-algebra $A_\bullet$, correspond to elements in $Z_n A_\bullet$, so that the A_0-algebra structure on A_n restricts to an action of of $Z_0 A_\bullet = A_0$ on $Z_n A_\bullet$.

3.10. Remark. Let $Q_\bullet$ be a resolution (in $s\mathcal{C}$) of some object $X \in \mathcal{C}$, with $J_* := \pi_{\mathcal{F}} X$, and $Y_\bullet \simeq P_n Q_\bullet$ its n-th Postnikov approximation. Then we have an action of $\pi_0\pi_{\mathcal{F}} Y_\bullet \cong J_*$ on $\pi_{n+2}\pi_{\mathcal{F}} Y_\bullet \cong \Omega^{n+1} J_*$ which is a homotopy invariant of $Y_\bullet$, and thus in turn of $Q_\bullet$, so of X. It is not clear on the face of it whether the J_*-module $\Omega^n J_*$ depends only on J_*, though we shall see (in §4.5 below) this holds for $n = 1$, and hope to show in [BG] that in fact this holds for all n. In any case it is describable purely in terms of the primary $\Pi_{\mathcal{F}}$-algebra-structure of J_*.

In general, for any simplicial object $X_\bullet \in s\mathcal{C}$, there is an action of $\hat{\pi}_0 X_\bullet \cong \pi_0\pi_{\mathcal{F}} X_\bullet$ on the higher $\Pi_{\mathcal{F}}$-algebras $\hat{\pi}_n X_\bullet$, defined similarly via homotopy classes of maps $[\mathcal{S}^n_{\mathcal{F}}, \mathcal{S}^0_{\mathcal{F}} \amalg \mathcal{S}^n_{\mathcal{F}}]_{s\mathcal{C}}$ (see §2.22); but there is no reason why this should define the same category of "$\hat{\pi}_0 X_\bullet$-modules" as that defined above. Thus we do not know (2.31)to be a long exact sequence of $\hat{\pi}_0 X_\bullet$-modules. However, in our case, when $X_\bullet = Q_\bullet$ is a resolution, the isomorphism of (abelian) $\Pi_{\mathcal{F}}$-algebras $\pi_{n+2}\pi_{\mathcal{F}} Y_\bullet \cong \Omega^{n+1} J_*$ is *defined* inductively by means of the connecting homomorphism of (2.31), and this yields the J_*-module structure on $\Omega^n J_*$.

3.11. Assumption. Under mild assumptions on the category $\mathcal{C}$ one may show that for any $A_\bullet \in s\Pi_{\mathcal{F}}$-$\mathcal{A}lg$ and $n \geq 1$, the $\Pi_{\mathcal{F}}$-algebra $\pi_n A_\bullet$ is abelian (see [BS, Lemma 5.2.1]).

However, we shall need to assume more than this: namely, that J_*-$\mathcal{M}od$ as defined above is in fact an abelian category. We also assume that when $A_\bullet$ is a simplicial $\Pi_{\mathcal{F}}$-algebra, the action of $\pi_0 A_\bullet$ on each $\pi_n A_\bullet$ is induced by an action of A_0 on A_n, and if $A_\bullet = \pi_{\mathcal{F}} Q_\bullet$, then this in turn is induced by an action of Q_0 on Q_n. Moreover, $Z_n A_\bullet$ and $C_n A_\bullet$ are sub-A_0-algebras of A_n, and $\mathbf{d}_0$ is a homomorphism of A_0-algebras.

3.12. Proposition. *These assumptions are satisfied for the categories listed in §2.14.*

Proof. As we shall see, all the categories in question are essentially special cases of the first:

(I) When $\mathcal{C} = \mathcal{G}$, the fundamental group action has an explicit description as follows:

We define the *generalised Samelson product* of two elements $x \in X_{p,k}$ $y \in X_{q,\ell}$ (where, as in §1.2, p is the "external" dimension, k the "internal" dimension in a a bisimplicial group $X_{\bullet,\bullet} \in s\mathcal{G}$) to be the element $\langle\langle x, y\rangle\rangle \in X_{p+q,k+\ell}$

$$(3.13)\quad \langle\langle x, y\rangle\rangle := \prod_{(\sigma,\rho)\in S_{p,q}} \left(\prod_{(\varphi,\psi)\in S_{k,\ell}} (s^{ext}_{\rho_q}\dots s^{ext}_{\rho_1} s^{int}_{\psi_\ell}\dots s^{int}_{\psi_1} x, s^{ext}_{\sigma_q}\dots s^{ext}_{\sigma_1} s^{int}_{\varphi_\ell}\dots s^{int}_{\varphi_1} y)^{\varepsilon(\varphi)} \right)^{\varepsilon(\sigma)}.$$

Here $S_{p,q}$ is the set of all (p,q)-shuffles – that is, partitions of $\{0,1,\dots,p+q-1\}$ into disjoint sets $\sigma_1 < \sigma_2 < \cdots < \sigma_p$, $\rho_1 < \rho_2 < \cdots < \rho_q$ – and $\varepsilon(\sigma)$ is the sign of the permutation corresponding to (σ,ρ) (see [Mc2, VIII, §8]); $S_{p,q}$ is ordered by the reverse lexicographical ordering in σ. (a,b) denotes the commutator $a\cdot b\cdot a^{-1}\cdot b^{-1}$ (where $\cdot$ is the group operation). When $p=q=0$, $\langle\langle x,y\rangle\rangle$ is just the usual Samelson product $\langle x,y\rangle$ in $X_{0,\bullet}\in\mathcal{G}$ (cf. [C, §11.11]).

We are mainly interested here in the case $p=0$, so $\langle\langle x,y\rangle\rangle := \langle \hat{x}, y\rangle$ for $\hat{x} := s_{q-1}\cdots s_0 x \in X_{q,k}$. It is sometimes convenient to think of this as an "action" of x on y, setting $t_x(y) := \langle\langle x,y\rangle\rangle\cdot y$ (cf. [W, X, (7.4)]).

The simplicial identities imply that if $d^{int}_i x = d^{int}_i y = 0$ for all i, the same holds for $\langle x,y\rangle$, and if $x = d^{int}_0 z$ for some $z\in C^{int}_{k+1}X_{p,\bullet}$, then $\langle x,y\rangle = d^{int}_0\langle z,y\rangle$, so that $\langle\langle\ ,\ \rangle\rangle$ induces a well-defined operation $\langle\langle\ ,\ \rangle\rangle : \pi^{int}_k X_{p,\bullet}\times\pi^{int}_\ell X_{q,\bullet}\to\pi^{int}_{k+\ell}X_{p+q,\bullet}$, which is defined for any simplicial Π-algebra $A_{\bullet *}$, with $\alpha\in A_{p*}$ and $\beta\in A_{q*}$, by:

$$(3.14)\qquad \langle\langle\alpha,\beta\rangle\rangle := \prod_{(\sigma,\rho)\in S_{p,q}} \langle s_{\rho_q}\dots s_{\rho_1}\alpha, s_{\sigma_q}\dots s_{\sigma_1}\beta\rangle^{\varepsilon(\sigma)} \in A_{p+q*}.$$

Again when $p=0$ we write $\tau_\alpha(\beta) := \langle\langle\alpha,\beta\rangle\rangle\cdot\beta$, so that $\tau_\alpha : A_{q*}\to A_{q*}$ is a group homomorphism in each degree (if $\alpha\in Z_pA_{\bullet *}$, $\beta\in Z_qA_{\bullet *}$, then $\langle\langle\alpha,\beta\rangle\rangle\in Z_{p+q}A_{\bullet *}$).

Now let $X_{\bullet,\bullet} := \Sigma^0\mathbf{S}^k \amalg \Sigma^n\mathbf{S}^\ell$, (where $\mathbf{S}^k$ is the k-sphere for $\mathcal{G}$ – §2.17) and let $\iota_{0,k}$ and $\iota_{n,\ell}$ be Π-algebra-generators for $\pi_k X_{0,\bullet}$ and $\pi_\ell\mathbf{S}^\ell\subseteq\pi_*X_{n,\bullet}$, respectively, so $\pi_*X_{n,\bullet}$ is generated by $\{\hat{\iota}_{0,k},\iota_{n,\ell}\}$. Since $d_j\iota_{n,\ell}=0$ $(0\le j\le n)$, we have a short exact sequence of Π-algebras

$$(3.15)\qquad 0\to Z_n\pi_*X_{\bullet,\bullet}\to\pi_*(\mathbf{S}^k\amalg\mathbf{S}^\ell)\to\pi_*\mathbf{S}^k\to 0.$$

When $k,\ell>0$, by [H, Theorem A] any element $x\in\pi_*X_{n,\bullet}\cong\pi_*(\mathbf{S}^k\amalg\mathbf{S}^\ell)$ can be written as a sum of elements of the form $\zeta^\#\omega(\hat{\iota}_{0,k},\iota_{n,\ell})$ (where $\omega(x,y)=\langle\dots\langle x,y\rangle,\dots\rangle$ is some iterated Samelson product), so x can be obtained by means of the "internal" Π-$\mathcal{A}lg$ operations from expressions of the form $\tau_\alpha(\iota_{n,\ell})$ (for $\alpha\in\pi_*X_{0,\bullet}$).

By passing to universal covers we have a similar description when $\ell>k=0$, since then any $x\in\pi_jX_{n,\bullet}$ $(j\ge 1)$ can be written as a sum of elements of the form $\zeta^\#\omega(\tau_{\alpha_1}(\iota_{n,\ell}),\dots,\tau_{\alpha_r}(\iota_{n,\ell})$ (for $\alpha_i\in\pi_*X_{0,\bullet}$), and any other $\alpha\in\pi_*X_{0,\bullet}$ acts on this by permuting the generators $\tau_{\alpha_i}(\iota_{n,\ell}$, so again $tau_\alpha(-)$ is a group homomorphism. When $k=\ell=0$, we are reduced to the case $\mathcal{C}=\mathcal{G}p$ (see (II) below).

When $k>0$ and $\ell=0$, let us write $\varphi_\alpha(\beta) := \langle\langle\beta,\alpha\rangle\rangle$ for $\alpha\in\pi_*X_{0,\bullet}$ and $\beta\in\pi_0X_{n,\bullet}$, so that we are thinking of the usual (internal) action of the fundamental group $\pi_0X_{n,\bullet}$ as a function of β. This is *not* a homomorphism, since we have $\varphi_\alpha(\beta\cdot\gamma) = \varphi_\alpha(\beta)+\varphi_\alpha(\gamma)+\langle\langle\beta,\langle\langle\gamma,\hat{\alpha}\rangle\rangle\rangle\rangle$ by [W, III, (1.7) & X, (7.4)].

But $\langle\langle\alpha,\beta\rangle\rangle$ is a cycle (i.e., in $Z_n\pi_*X_{\bullet,\bullet}$), by (3.15), so $\langle\langle\langle\langle\alpha,\beta\rangle\rangle,\gamma\rangle\rangle\sim 0$ in $\pi_n\pi_*X_{\bullet,\bullet}$ for any $\gamma\in\pi_0X_{n,\bullet}$ by [BS, 5.2.1], which means that φ_α induces a homomorphism on $\pi_n\pi_*X_{\bullet,\bullet}$.

In summary, an J_*-*algebra* (§3.9), for any $J_*\in\Pi$-$\mathcal{A}lg$, is just a Π-algebra K_* together with an action of each $\alpha\in J_*$, which may be expressed in terms of the (degree-shifting) homomorphisms τ_α, or the functions φ_α, respectively, satisfying whatever relations hold among these (and the internal Π-algebra operations) in $\pi_* X_{n,\bullet}$.

A J_*-*module*, on the other hand, is an *abelian* Π-algebra K_*, together with homomorphisms $\tau_\alpha : K_* \to K_*$ or $\varphi_\alpha : K_* \to K_*$ for each $\alpha \in J_*$, satisfying the identities occuring in $\pi_n \pi_* X_{\bullet,\bullet}$.

These identities could be described more or less explicitly in the category Π-$\mathcal{A}lg$, in terms of suitable Hopf invariants (cf. [Ba1, II, §3]). Compare [Ba2, §3]).

(II) When $\mathcal{C} = \mathcal{G}p$, $s\mathcal{C}$ models the homotopy theory of (connected) topological spaces, and J_*-$\mathcal{M}od$, defined (as noted above) through the usual action of the fundamental group, is equivalent to the category of (left) modules over the group ring $\mathbb{Z}[\pi_0 A_\bullet]$ (for $A_\bullet \in s\mathcal{C} \approx \mathcal{G}$).

(III) When $\mathcal{C} = \mathcal{L}ie$, the situation is similar to $\mathcal{C} = \mathcal{G}p$, with Samelson products replaced by Lie brackets.

(IV) When $\mathcal{C} = d\mathcal{L} \approx s\mathcal{L}ie$, one has a generalized Lie bracket defined for bisimplicial Lie algebras as in (3.13), with commutators replaced by Lie brackets (see [Bl7, §2.6]).

(V) When $\mathcal{C} = R$-$\mathcal{M}od$, $s\mathcal{C}$ is equivalent to the category of chain complexes over R, so there is no action of $\pi_0 = H_0$ on the higher groups.

□

3.16. Remark. It is possible to write down general conditions on category of universal algebras (or CUGA) $\mathcal{C}$, defined in terms of operations and relations, which suffice to ensure that assumptions 3.11 hold: all one really needs is a suitable Hilton-Milnor theorem in $s\mathcal{C}$ (see, e.g., [Go]). However, it seems simpler to state the conditions needed as above, and verify them directly in any particular case of interest.

4. Cohomology of $\Pi_{\mathcal{F}}$-algebras

In this section we complete the description of the algebraic invariants used to distinguish homotopy types. To do so, we recall Quillen's definition of cohomology in a model category, in the context of $\Pi_{\mathcal{F}}$-$\mathcal{A}lg$:

4.1. Definition. Let $\mathcal{C}$ be a model category with an *abelianization* functor $\mathcal{A}b : \mathcal{C} \to \mathcal{C}_{ab}$, where $\mathcal{C}_{ab}$ denotes of course the full category of abelian objects in $\mathcal{C}$; we shall usually write X_{ab} for $\mathcal{A}b(X)$ (see §2.10). In [Q1, II, §5] (or [Q4, §2]), Quillen defines the *homology* of an object $X \in \mathcal{C}$ to be the total left derived functor $\mathbf{L}\mathcal{A}b$ of $\mathcal{A}b$, applied to X (cf. [Q1, I, §4]). Likewise, given an object $M \in \mathcal{C}_{ab}/X$, the *cohomology of X with coefficients in M* is $\mathbf{R}\,\mathrm{Hom}_{\mathcal{C}_{ab}/X}(X, M) := \mathrm{Hom}_{\mathcal{C}_{ab}/X}(\mathbf{L}\mathcal{A}bX, M)$.

4.2. **Quillen cohomology of $\Pi_{\mathcal{F}}$-algebras.** When $J_* \in \mathcal{C} = \Pi_{\mathcal{F}}$-$\mathcal{A}lg$, we have the model category structure defined in §3.6 above, so we can choose a resolution $A_\bullet \to J_*$ in $s\Pi_{\mathcal{F}}$-$\mathcal{A}lg$ as in §2.33, and define the i-th homology group of J_* to be the i-th homotopy group $\pi_i(\mathcal{A}bA_\bullet)$ of the $\hat{\mathcal{F}}$-graded simplicial abelian group $(A_\bullet)_{ab}$ – i.e., of the associated chain complex (cf. [D, §1]). One must verify, of course, that this definition is independent of the choice of the resolution $A_\bullet \to J_*$.

Similarly, if K_* is an abelian J_*-algebra, then the i-th cohomology group of J_* with coefficients in K_*, written $H^i(J_*; K_*)$, is that of the cochain complex corresponding to the cosimplicial $\hat{\mathcal{F}}$-graded abelian group $\mathrm{Hom}_{J_*\text{-}\mathcal{A}lg}(A_\bullet, K_*)$.

4.3. Remark. Here $\mathrm{Hom}_{J_*\text{-}\mathcal{A}lg}(A, B)$ is the group of $\Pi_{\mathcal{F}}$-algebra homomorphisms which respect the J_*-action; because we are mapping into an abelian object K_*, $\mathrm{Hom}_{J_*\text{-}\mathcal{A}lg}(A_\bullet, K_*) \cong \mathrm{Hom}_{J_*\text{-}\mathcal{A}lg}((A_\bullet)'_{ab}, K_*)$ (where A'_{ab} denotes the abelianization of $A \in J_*$-$\mathcal{A}lg$ as an J_*-algebra).

However, in the simplicial abelian J_*-algebra $(A_\bullet)'_{ab}$ we have a direct product decomposition $(A_k)'_{ab} = (\hat{A}'_k)_{ab} \oplus (L_k A_\bullet)'_{ab}$ for $k \geq 0$, where $(\hat{A}_k)'_{ab} := C_k(A_\bullet)'_{ab}$ is the the sub-abelian J_*-algebra of $(A_k)'_{ab}$ generated by $(\bar{A}_k)'_{ab}$ (cf. [May, Cor. 22.2]) – and in fact $(\hat{d}_0)'_{ab} : (\hat{A}_n)'_{ab} \to (A_{n-1})'_{ab}$ factors through a map $\hat{\partial}_n : (\hat{A}'_n)_{ab} \to (\hat{A}_{n-1})'_{ab}$ (see [May, p. 95(i)]).

Thus the n-cochains split as:

$$\mathrm{Hom}_{J_*\text{-}\mathcal{A}lg}((A_n)'_{ab}, K_*) \cong \mathrm{Hom}_{J_*\text{-}\mathcal{A}lg}((\hat{A}_n)'_{ab}, K_*) \oplus \mathrm{Hom}_{J_*\text{-}\mathcal{A}lg}(L_n(A_\bullet)'_{ab}, K_*),$$

so by [BK, X, §7.1] any cocycle representing a cohomology class in $H^n(J_*; K_*)$ may be represented uniquely either by a map of abelian A_0-algebras $\hat{f} : (\hat{A}_n)'_{ab} \to K_*$, or by a map of A_0-algebras $f : A_n \to K_*$.

Since $C_n A_\bullet$ contains the sub-A_0-algebra of A_n generated by $\bar{A}_n$ (by assumption 3.11), f determines its restriction $f|_{C_n A_\bullet}\colon C_n A_\bullet \to K_*$, which determines $\hat{f}$, which determines f in turn. We have thus shown that $H^*(J_*; K_*)$ may be calculated as the cohomology of the (abelian) cochain complex $\mathrm{Hom}_{A_0\text{-}\mathcal{A}lg}(C_* A_\bullet, K_*)$ (even though $C_* A_\bullet$ is not in general a homotopy invariant of $A_\bullet$, in non-abelian categories).

4.4. **obstructions to existence of resolutions.** Given an object $X \in \mathcal{C}$, and a (suitable) simplicial resolution $A_\bullet \to J_*$ of the $\Pi_{\mathcal{F}}$-algebra $J_* := \pi_{\mathcal{F}} X$, we have seen in Section 2 that one can construct a resolution $Q_\bullet$ of X (in the resolution model category $s\mathcal{C}$) realizing $A_\bullet$, in the sense that $\pi_{\mathcal{F}} Q_\bullet \cong A_\bullet$. It is thus natural to ask whether any simplicial $\Pi_{\mathcal{F}}$-algebra – or at least, any resolution $A_\bullet$ of an abstract $\Pi_{\mathcal{F}}$-algebra J_* – is realizable in $s\mathcal{C}$.

One approach to this question in the topological setting (i.e., for $\mathcal{C} = \mathcal{G}$), in terms of higher homotopy operations, was given in [Bl3]. However, a glance at the proof of Proposition 2.41 shows that one can instead consider obstructions to extending $\mathrm{tr}_n Q_\bullet$ to the next simplicial dimension. For a homotopy-invariant description, we state this in terms of successive Postnikov approximations to $Q_\bullet$, since it is clear that, once we have constructed $\mathrm{tr}_n Q_\bullet$, it is always possible to obtain $Y_\bullet \simeq Q_\bullet^{(n-1)}$ from it by successive choices of free objects $\bar{Y}_{k+1} \in ho\mathcal{F}$ $(k = n, \dots)$ mapping to $Z_k Y_\bullet$ by a $\Pi_{\mathcal{F}}$-algebra surjection.

4.5. **constructing the obstruction.** Assume given a CW resolution $A_\bullet \in s\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg$ of J_*, with CW basis $(\bar{A}_n)_{n=0}^{\infty}$, and choose corresponding free objects $\bar{Q}_n \in \mathcal{F} \subset \mathcal{C}$ with $\pi_{\mathcal{F}} \bar{Q}_n \cong \bar{A}_n$. We begin the induction with $\mathrm{tr}_1 Q_\bullet$, and thus $Q_\bullet^{(0)}$, constructed as in the proof of Proposition 2.41. Note that to obtain $\mathrm{tr}_1 Q_\bullet$ we do not in fact need to know $X \in \mathcal{C}$ with $\pi_{\mathcal{F}} X \cong J_*$ – or even to know that such an object exists! This implies that the J_*-module structure on ΩJ_* is uniquely determined.

In the inductive stage we assume given $\mathrm{tr}_n Q_\bullet$ (equivalently: $Q_\bullet^{(n-1)}$), satisfying 2.39(a) and (b) for $0 < m \leq n$. Our strategy is to try to attach $(n+1)$-dimensional "cells" to $\mathrm{tr}_n Q_\bullet$ in such a way as to guarantee acyclicity of the resulting $\mathrm{tr}_{n+1} Q_\bullet$ in one more simplicial dimension – using Lemma 2.39 above. The key to the construction of $\mathrm{tr}_{n+1} Q_\bullet$ from $\mathrm{tr}_n Q_\bullet$ thus lies in the extension of A_0-algebras (2.40) (for $Q_\bullet$, rather than $P_\bullet$), in which the two ends are given to us. Observe that this extension determines the A_0-algebra structure on $\Omega^n J_*$, if more than one is possible.

We want this extension to be "trivial" (that is, split as a semi-direct product of A_0-algebras), in order to be able to lift the given map of A_0-algebras $\bar{d}_0^A : \bar{A}_{n+1} \to Z_n A_\bullet$ to a map $\bar{d}_0^Q : \bar{Q}_{n+1} \to Z_n Q_\bullet$, so the question is reduced from one about *simplicial* objects over $\mathcal{C}$ to one of algebraic objects, namely: A_0-algebras. There is a close analagy to the classical theory of group extensions, where the triviality of an extension $E :\ 0 \to A \to B \to G$ is measured by the characteristic class $\xi(E) \in H^2(G; A)$ (compare [Mc2, IV, §6]). Similarly, in our case the triviality of the extension is measured by the vanishing of a suitable cohomology class in $H^{n+2}(J_*; \Omega^n J_*)$, defined as follows:

Because $(j_n)_\# : \pi_{\mathcal{F}} Z_n Q_\bullet \twoheadrightarrow Z_n \pi_{\mathcal{F}} Q_\bullet \cong Z_n A_\bullet$ is surjective, and $\bar{A}_{n+1}$ is a free $\Pi_{\mathcal{F}}$-algebra, we can choose a lifting λ in the following diagram:

$$\begin{tikzcd}
 & & \bar{A}_{n+1} \arrow[r, "\bar{d}_0^{A_{n+1}}"] \arrow[d, dotted, "\lambda"] & Z_n A_\bullet \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \Omega^n J_* \arrow[r, "i"] & \pi_{\mathcal{F}} Z_n Q_\bullet \arrow[r, "(j_n^Q)_\#"] & Z_n \pi_{\mathcal{F}} Q_\bullet \arrow[r] & 0
\end{tikzcd}$$

FIGURE 2

and we can find a map $\ell : \bar{Q}_{n+1} \to Z_n Q_\bullet$ realizing λ (again, because $\bar{A}_{n+1} = \pi_{\mathcal{F}} \bar{Q}_{n+1}$ is free). Combined with the "tautological map" $L_{n+1} Q_\bullet \to M_{n+1} Q_\bullet$ (see §2.4), which depends only on $\operatorname{tr}_n Q_\bullet$, by setting $Q_{n+1} := \bar{Q}_{n+1} \amalg L_{n+1} Q_\bullet$ we obtain an extension $d_0 : Q_{n+1} \to Q_n$ of ℓ (which is a map of Q_0-algebras), and thus an $(n+1)$-truncated simplicial object $\operatorname{tr}_{n+1} Q_\bullet$ over $\mathcal{C}$, with $Q_{n+1} := \bar{Q}_{n+1} \amalg L_{n+1} Q_\bullet$, and $\pi_{\mathcal{F}} \operatorname{tr}_{n+1} Q_\bullet \cong \operatorname{tr}_{n+1} A_\bullet$. In particular, $\mathbf{d}_0^{Q_{n+1}} : C_{n+1} Q_\bullet \to Z_n Q_\bullet$ induces a map $\hat{\lambda}$ from $\pi_{\mathcal{F}} C_{n+1} Q_\bullet = C_{n+1} A_\bullet$ to $\pi_{\mathcal{F}} Z_n Q_\bullet$ extending (and determined by) the lifting $\lambda : \bar{A}_{n+1} \to \pi_{\mathcal{F}} Z_n Q_\bullet$ of $\bar{d}_0^{A_{n+1}}$. This is a map of $A_0 = \pi_{\mathcal{F}} Q_0$-algebras, by Assumption 3.11.

Since $(j_n^Q)_\# \circ (\hat{\lambda}|_{Z_{n+1} A_\bullet}) = 0$, the map $\hat{\lambda}|_{Z_{n+1} A_\bullet}$ factors through $\mu : Z_{n+1} A_\bullet \to \operatorname{Ker}(j_n^Q)_\# = \Omega^n J_*$, and composing μ with $\mathbf{d}_0^{A_{n+2}} : C_{n+2} A_\bullet \to Z_{n+1} A_\bullet$ defines $\xi : C_{n+2} A_\bullet \to \Omega^n J_*$ – again, a map of A_0-algebras:

$$\begin{tikzcd}
C_{n+2} A_\bullet \arrow[r, "\mathbf{d}_0^{A_{n+2}}"] \arrow[dr, "\xi"'] & Z_{n+1} A_\bullet \arrow[r, "j"] \arrow[d, "\mu"] & C_{n+1} A_\bullet \arrow[r, "\mathbf{d}_0^{A_{n+1}}"] \arrow[d, "\hat{\lambda}"] & Z_n A_\bullet \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \Omega^n J_* \arrow[r, "i"] & \pi_{\mathcal{F}} Z_n Q_\bullet \arrow[r, "(j_n^Q)_\#"] & Z_n \pi_{\mathcal{F}} Q_\bullet \arrow[r] & 0
\end{tikzcd}$$

The cochain $\xi = \mu \circ \mathbf{d}_0^{A_{n+2}}$ is clearly a cocycle in the cochain complex $\operatorname{Hom}_{J_*\text{-}\mathcal{M}od}(A_\bullet, \Omega J_*)$, so it represents a cohomology class $\chi_n \in H^{n+2}(J_*; \Omega^n J_*)$, called the *characteristic class of the extension.*

4.6. Lemma. *The cohomology class* χ_n *is independent of the choice of lifting* λ.

Proof. Assume that we want to replace λ in §4.5 by a different lifting $\lambda' : \bar{A}_{n+1} \to \pi_{\mathcal{F}} Z_n Q_\bullet$, and choose maps $\ell, \ell' : \bar{Q}_{n+1} \to Z_n Q_\bullet$ realizing λ, λ' respectively; their extensions to maps $Q_{n+1} \to Q_n$ (which we may denote by d_0, d_0') agree on $L_{n+1} Q_\bullet$. We correspondingly having $\mu' : Z_{n+1} A_\bullet \to \Omega^n J_*$ and $\xi' := \mu' \circ \mathbf{d}_0^{A_{n+2}}$.

Because $Q_{n+1} := \bar{Q}_{n+1} \amalg L_{n+1} Q_\bullet$ is a coproduct of the form $\coprod_i \mathbf{M}\langle \alpha_i \rangle$, by §2.13 the underlying group structure on any $X \in \mathcal{C}$ induces a group structure on

$$\operatorname{Hom}_{\mathcal{C}}(Q_{n+1}, X) \tag{4.7}$$

(and similarly for $\operatorname{Hom}_{\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg}(A_{n+1}, \pi_{\mathcal{F}} X)$).

Therefore, we can set $h := (d_0)^{-1} \cdot (d_0') : Q_{n+1} \to Q_n$, and h induces a map $\eta : C_{n+1} A_\bullet \to \pi_{\mathcal{F}} Z_n Q_\bullet$ such that $\eta|_{\bar{A}_{n+1}} = \lambda^{-1} \cdot \lambda'$. Moreover, because d_0 and d_0' agree outside of $\bar{Q}_{n+1}$, $(j_n^Q)_\# \circ \eta = 0$. Thus η factors through $\zeta : C_{n+1} A_\bullet \to \Omega^n J_*$, which is a map of A_0-algebras because $\Omega^n J_*$ is an abelian A_0-algebra (actually, a J_*-module), and ζ is induced by group operations from the A_0-algebra maps $\mathbf{d}_0$ and $\mathbf{d}'_0$.

Moreover, $\zeta|_{Z_{n+1}A_\bullet} = \mu - \mu'$ in the abelian group structure on $\mathrm{Hom}_{J_*\text{-}\mathcal{M}od}(-, \Omega^n J_*)$ (which corresponds to the group structure of (4.7)). Thus $\xi' - \xi = \hat{\eta} \circ \mathbf{d}_0^{A_{n+2}}$ is a coboundary. $\square$

4.8. Theorem. *$\chi_n = 0$ if and only if one can extend $Q_\bullet^{(n-1)}$ to an n-th Postnikov approximation $Q_\bullet^{(n)}$ of a resolution of X.*

Proof. First assume that there exists $Y_\bullet \simeq Q_\bullet^{(n+1)}$ with $\mathrm{tr}_n Y_\bullet \cong \mathrm{tr}_n Q_\bullet$: by Lemma 2.39 we know $(j_n^Q)_\#|_{\mathrm{Im}(\mathbf{d}_0^{n+1})_\#}$ is one-to-one (and onto $Z_n \pi_{\mathcal{F}} Q_\bullet$), for $\mathbf{d}_0^{n+1} : C_{n+1} Y_\bullet \to Z_n Y_\bullet = Z_n Q_\bullet$, and thus $\mathrm{Im}(\mathbf{d}_0^{n+1})_\# \cap \mathrm{Im}\, \partial_{n-1}^Q = \{0\}$. But then we can choose $\lambda : \bar{A}_{n+1} \to \pi_{\mathcal{F}} Z_n Q_\bullet$ to factor through $\mathrm{Im}(\mathbf{d}_0^{n+1})_\#$, (and this will induce a map of A_0-algebras because of §3.10), so that $\mu = 0$ and thus $\xi = 0$.

Conversely, if $\chi_n = 0$, we can represent it by a coboundary $\xi = \vartheta \circ \mathbf{d}_0^{A_{n+2}}$ for some A_0-algebra map $\vartheta : C_{n+1} A_\bullet \to \Omega^n J_*$, and thus get $i \circ \vartheta|_{\bar{A}_{n+1}} : \bar{A}_{n+1} \to \pi_{\mathcal{F}} Z_n Q_\bullet$. If we set $\lambda' := \lambda \cdot (i \circ \vartheta|_{\bar{A}_{n+1}})^{-1}$, we have $\mathrm{Im}\, \lambda' \cap \Omega^n J_* = \{0\}$. We can therefore choose $\bar{d}_0^{Q_{n+1}} : \bar{Q}_{n+1} \to Z_n Q_\bullet$ realizing λ', and then $(\mathbf{d}_0^{Q_{n+1}})_\#$ avoids $\mathrm{Im}(\partial_{n-1}^Q) \cong \Omega^n J_*$, so that $\mathrm{tr}_{n+1} Q_\bullet$ so constructed yields $Q_\bullet^{(n+1)}$, as required. In particular, this determines a choice of J_*-module structure on $\Omega^{n+2} J_*$ (if more than one is possible), via (2.40) for $n+1$. $\square$

4.9. **notation.** If we wish to emphasize the dependence on the choice of λ, we shall write $Q_\bullet^{(n+1)}[\lambda]$ for the extension of $Q_\bullet^{(n)}$ so constructed.

4.10. Proposition. *The class χ_n depends only on the homotopy type of $Q_\bullet^{(n-1)}$ in $s\mathcal{C}$.*

Proof. Assume $Q_\bullet^{(n-1)}$ has been constructed, realizing a simplicial resolution of $\Pi_{\mathcal{F}}$-algebras $A_\bullet \to J_*$ through simplicial dimension n, and let $B_\bullet \to J_*$ be any other $\Pi_{\mathcal{F}}$-algebra resolution: we then have a weak equivalence $\varphi : B_\bullet \to A_\bullet$ in $s\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg$. Assume by induction on $0 \le m < n$ that we have constructed an m-truncated simplicial object $\mathrm{tr}_m R_\bullet$ over $\mathcal{C}$, and a map $f : \mathrm{tr}_m R_\bullet \to \mathrm{tr}_m Q_\bullet^{(n-1)}$ realizing $\mathrm{tr}_m \varphi$. Moreover, assume that we have a map of the (split) short exact sequences (2.40) (in dimension m) for $R_\bullet$ and $Q_\bullet$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^m J_* \arrow[r, "i"] \arrow[d, "="] & \pi_{\mathcal{F}} Z_m R_\bullet \arrow[r, "(j_R^m)_\#"] \arrow[d, "(Z_n f)_\#"] & Z_m \pi_{\mathcal{F}} R_\bullet \arrow[r] \arrow[d, "Z_n(f_\#) = Z_n \varphi"] & 0 \\
0 \arrow[r] & \Omega^m J_* \arrow[r, "i"] & \pi_{\mathcal{F}} Z_m Q_\bullet \arrow[r, "(j_m^Q)_\#"] & Z_m \pi_{\mathcal{F}} Q_\bullet \arrow[r] & 0
\end{tikzcd}$$

Now, in order to extend f to dimension $n+1$, we must choose the map $(\bar{d}_0^{R_{m+1}})_\# : \pi_{\mathcal{F}} \bar{R}_{m+1} \to \pi_{\mathcal{F}} Z_m R_\bullet$ (lifting $\bar{d}_0^{B_{m+1}} : \bar{B}_{m+1} \to Z_m B_\bullet$) in such a way that $(Z_m f)_\# \circ (\bar{d}_0^{R_{m+1}})_\# = (\bar{d}_0^{R_{m+1}})_\# \circ Z_m \varphi$. Since $\bar{B}_{m+1} = \pi_{\mathcal{F}} \bar{R}_{m+1}$ is free, it suffices to show that the obvious map from $\pi_{\mathcal{F}} Z_m R_\bullet$ to the pullback of $\pi_{\mathcal{F}} Z_m Q_\bullet \xrightarrow{(j_m^Q)_\#} Z_m \pi_{\mathcal{F}} Q_\bullet = Z_m A_\bullet \xleftarrow{Z_m \varphi} Z_m B_\bullet$ is a surjection: given $(a, b) \in \pi_{\mathcal{F}} Z_m Q_\bullet \times Z_m B_\bullet$ with $(j_m^Q)_\#(a) = \varphi(b)$, for any $z \in \pi_{\mathcal{F}} Z_m R_\bullet$ with $(j_m^R)_\#(z) = b$ we have an $\omega \in \Omega^{m+1} J_* \subset \pi_{\mathcal{F}} Z_m R_\bullet$ such that $(Z_m f)_\#(z \cdot \omega) = (Z_m f)_\#(z') \cdot \omega = b$ in the diagram above (where $\cdot$ is the group operation), so $z \cdot \omega$ maps to (a, b). Thus we can choose $\bar{d}_0^{R_{m+1}} : \bar{R}_{m+1} \to Z_m R_\bullet$ in such a way that we can define $\mathrm{tr}_{m+1} R_\bullet$, together with a map $\mathrm{tr}_{m+1} f : \mathrm{tr}_{m+1} R_\bullet \to \mathrm{tr}_{m+1} Q_\bullet$ realizing $\mathrm{tr}_{m+1} \varphi$.

Because φ was a weak equivalence of *resolutions*, it is actually a homotopy equivalence, with homotopy inverse $\psi : A_\bullet \to B_\bullet$, say, and the above argument also yields a homotopy inverse for $f^{(m)}$ (or $\mathrm{tr}_{m+1} f$). Moreover, the characteristic classes we defined are clearly functorial with respect to maps in $s\mathcal{C}$; since the characteristic class $\chi_{m+1} \in H^{m+3}(J_*; \Omega^{m+1} J_*)$, defined for the resolution $A_\bullet \to J_*$ by means of the lift $\bar{d}_0^{Q_{m+1}}$, must vanish, by Theorem 4.8, the same holds for $R_\bullet$, so by Theorem 4.8 again we can extend $R_\bullet^{(m)}$ to $R_\bullet^{(m+1)}$, and continue the induction as long as $m < n$. □

We deduce the following generalization of Proposition 2.41:

4.11. Corollary. *Given $X \in \mathcal{C}$, any CW $\Pi_{\mathcal{F}}$-algebra resolution $A_\bullet \to \pi_{\mathcal{F}} X$ is realizable as a resolution $Q_\bullet \to X$ in $s\mathcal{C}$.*

One could further extend Proposition 4.10 to obtain a statement about the naturality of the characteristic classes with respect to morphisms of $\Pi_{\mathcal{F}}$-algebras $\psi : J_* \to L_*$. However, such a statement would be somewhat convoluted, in our setting, and it seems better to defer it to a more general discussion of the realization of simplicial $\Pi_{\mathcal{F}}$-algebras, in [BG].

4.12. **realization of Π-algebras.** If $G : \mathcal{S}_* \to \mathcal{G}$ denotes Kan's simplicial loop functor (cf. [May, Def. 26.3]), with adjoint $\bar{W} : \mathcal{G} \to \mathcal{S}_*$ the Eilenberg-Mac Lane classifying space functor (cf. [May, §21]), and $S : \mathcal{T}_* \to \mathcal{S}_*$ is the singular set functor, with adjoint $\| - \| : \mathcal{S}_* \to \mathcal{T}_*$ the geometric realization functor (see [May, §1,14]), then functors

$$\mathcal{T}_* \underset{\|-\|}{\overset{S}{\rightleftarrows}} \mathcal{S}_* \underset{\bar{W}}{\overset{G}{\rightleftarrows}} \mathcal{G} \tag{4.13}$$

induce isomorphisms of the corresponding homotopy categories (see [Q1, I, §5]), so any homotopy-theoretic question about topological spaces may be translated to one in $\mathcal{G}$. In particular, in order to find a topological space $\mathbf{X}$ having a specified homotopy Π-algebra $J_* \cong \pi_* \mathbf{X}$, it suffices to find the corresponding simplicial group $X \in \mathcal{G}$ (with the $\Pi_{\mathcal{F}}$-algebra J_* suitably re-indexed). If J_* is realizable by such an X, any free simplicial resolution $Q_\bullet \to X$ evidently provides a Π-algebra resolution $\pi_* Q_\bullet$ of $J_* = \pi_* X$. But the converse is also true: if $Q_\bullet \in s\mathcal{G}$ realizes some (abstract) Π-algebra resolution $A_\bullet \in s\Pi\text{-}\mathcal{A}lg$ of J_*, then the collapse of the Quillen spectral sequence of [Q2], with

$$E^2_{s,t} = \pi_s(\pi_t Q_\bullet) \Rightarrow \pi_{s+t} \operatorname{diag} Q_\bullet \tag{4.14}$$

converging to the diagonal $\operatorname{diag} Q_\bullet \in \mathcal{G}$ (defined $(\operatorname{diag} Q_\bullet)_k = (Q_k)_k^{int}$) implies that $\pi_* \operatorname{diag} Q_\bullet \cong J_*$. Thus J_* is realizable by a simplicial group (or topological space) if and only if some Π-algebra resolution $A_\bullet \to J_*$ is realizable.

The characteristic classes $(\chi_n)_{n=0}^\infty$ (whose existence was promised in [DKS2, §1.3] under the name of the "k-invariants for J_*), thus provide a more succinct (if less explicit) version of the theory described in [Bl3, §5-6] (as simplified in [Bl6, §6]), for determining the realizablity of a Π-algebra in terms of higher homotopy operations – which we summarize in

4.15. Theorem. *Given an (abstract) Π-algebra J_*, the following conditions are equivalent:*

1. *J_* is realizable as $\pi_* \mathbf{X}$ for some topological space $\mathbf{X} \in \mathcal{T}_*$.*
2. *Any CW Π-algebra resolution $A_\bullet \to J_*$ is realizable by a simplicial space $Q_\bullet$.*
3. *The (inductively defined) characteristic classes $\chi_n \in H^{n+2}(J_*; \Omega^n J_*)$ $(n = 0, 1, \dots)$ all vanish.*

Of course, the characteristic class χ_{n+1} is determined by the choice of some extension $Q_\bullet^{(n)}$ of $Q_\bullet^{(n-1)}$, so as usual our obstruction theory requires back-tracking if at some stage we find $\chi_n \neq 0$. We shall now show how we can use other cohomology classes to determine the choices of extensions at each stage:

4.16. **distinguishing between different resolutions.** A more interesting question, perhaps, is how one can distinguish between non-equivalent realizations $Q_\bullet, R_\bullet \in s\mathcal{C}$ of a fixed $\Pi_{\mathcal{F}}$-algebra resolution $A_\bullet \to J_*$ of a *realizable* $\Pi_{\mathcal{F}}$-algebra $J_* \cong \pi_{\mathcal{F}} X$. Of course, if $Q_\bullet$ and $R_\bullet$ are both resolutions (in the resolution model category $s\mathcal{C}$) of weakly equivalent objects $X \simeq Y$ in the model category $\mathcal{C}$, then by definition $Q_\bullet$ is weakly equivalent (actually: homotopy equivalent) to $R_\bullet$. Thus we are looking for a way to distinguish between objects in $\mathcal{C}$, using the iterative construction of a resolution $Q_\bullet \to X$ (or equivalently, the Postnikov system for $Q_\bullet$).

There are a number of possible approaches to this question: one could try to construct a homotopy equivalence $Q_\bullet \to R_\bullet$ by induction on the Postnikov tower for $R_\bullet$, using an adaptation to $s\mathcal{C}$ of the classical obstruction theory for spaces (cf. [W, V, §5]). Alternatively, one could try directly to construct a map $Q_\bullet \to Y$ realizing the augmentation $\pi_{\mathcal{F}} A_\bullet \to J_*$ (see [Bl3, §7], and compare [B, §5]). A description more in this spirit will be given in [BG].

Here our strategy is similar to that of §4.4: rather than assuming that we are given X and Y to begin with, we try to construct all different realizations (up to homotopy equivalence in $s\mathcal{C}$) of a given simplicial $\Pi_{\mathcal{F}}$-algebra $A_\bullet$ (which is assumed to be a resolution of a realizable $\Pi_{\mathcal{F}}$-algebra J_*). We start our construction as in §4.5, and in the induction step we have assume given $\operatorname{tr}_n Q_\bullet$ – or equivalently $Q_\bullet^{(n-1)}$, satisfying the assumptions of §4.5 (see the proof of Proposition 2.41). We ask in how many different ways we can attach $(n+1)$-dimensional "cells" to extend the realization one further dimension.

Again the key lies in the extension of $\Pi_{\mathcal{F}}$-algebras of (2.40). Of course, we may assume that the characteristic class $\chi_n \in H^{n+2}(J_*; \Omega^n J_*)$ vanishes, so that it is possible to find "splittings" for (2.40), given by various liftings λ in Figure 2 – all of which yield the same cohomology class χ_n by Lemma 4.6. As in the classical case of groups, we find that the difference between two such "semi-direct products" is represented by suitable cohomology classes, in dimension lower by one than the characteristic classes (see [Mc2, IV, §2]).

4.17. Definition. Assume given two liftings $\lambda, \lambda' : \bar{A}_{n+1} \to \pi_{\mathcal{F}} Z_n Q_\bullet$ in Figure 2 above, which define extensions of $\operatorname{tr}_n Q_\bullet$ – so that, as in the proof of Theorem 4.8, we may assume without loss of generality that the corresponding maps $\mu, \mu' : Z_{n+1} A_\bullet \to \Omega^n J_*$ vanish. As in the proof of Lemma 4.6, we extend λ, λ' to face maps $d_0, d'_0 : Q_{n+1} \to Q_n$, define $\eta : C_{n+1} A_\bullet \to \pi_{\mathcal{F}} Z_n Q_\bullet$ with $(j_n^Q)_\# \circ \eta = 0$, and lift to a map of A_0-algebras $\zeta : C_{n+1} A_\bullet \to \Omega^n J_*$. Again $\zeta|_{Z_{n+1} A_\bullet} = \mu - \mu'$, which is zero, so ζ is a cocycle in $\operatorname{Hom}_{J_*\text{-}\mathcal{M}od}(A_\bullet, \Omega J_*)$, representing a cohomology class $\delta_{\lambda,\lambda'} \in H^{n+1}(J_*, \Omega^n J_*)$, which we call the *difference obstruction* for the corresponding Postnikov sections $Q_\bullet^{(n)}[\lambda]$ and $Q_\bullet^{(n)}[\lambda']$ (in the notation of §4.9).

Just as in the proof of Proposition 4.10, one can show that the classes $\delta_{\lambda_{n+1},\lambda'_{n+1}}$ in question do not in fact depend on the choice of $\Pi_{\mathcal{F}}$-algebra resolution $A_\bullet \to J_*$, but only on the homotopy type of $Q_\bullet^{(n-1)}$ in $s\mathcal{C}$. Their significance is indicated by the following

4.18. Theorem. *If $\delta_{\lambda,\lambda'} = 0$ then the corresponding Postnikov sections $Q_\bullet^{(n)}[\lambda]$ and $Q_\bullet^{(n)}[\lambda']$ are weakly equivalent.*

Proof. If ζ is a coboundary, there is a map $\vartheta : C_n A_\bullet \to \Omega^n J_*$ such that $\zeta = \vartheta \circ \mathbf{d}_0^{A_n}$. Composing with the inclusion $i : \Omega^n J_* \hookrightarrow \pi_{\mathcal{F}} Z_n Q_\bullet$ yields a morphism of A_0-algebras $\varphi : A_n \to \pi_{\mathcal{F}} Z_n Q_\bullet$. If, as in the proof of Proposition 2.41, we set $Q'_n := \bar{Q}_n \amalg L_n Q_\bullet$, we may realize φ by a map $z' : Q'_n \to Z_n Q_\bullet$. Since we assumed Q'_n is actually a coproduct of objects in $\hat{\mathcal{F}}$, it is a cogroup object in $\mathcal{C}$ by §2.7(i), so using the resulting group structure on $\operatorname{Hom}_{\mathcal{C}}(Q'_n, Q_n)$ we may set $s' := k \cdot z : Q'_n \to Q_n$, where $k : Q'_n \hookrightarrow Q_n$ is the inclusion. Since k is a trivial cofbration and Q_n is fibrant in $\mathcal{C}$, we have a retraction $r : Q_n \to Q'_n$ (which is a weak equivalence). Let $s := s' \circ r : Q_n \to Q_n$.

Recall from §2.13 that we have a faithful forgetful functor $\hat{U}:\mathcal{C}\to\mathcal{D}$, where for simplicity we may assume $\mathcal{D}=\mathcal{G}$ or $\mathcal{D}=sR\text{-}\mathcal{M}od$ (the other cases are trivial). We therefore have a further forgetful functor $U':\mathcal{D}\to\mathcal{S}$, and we denote $U'\circ\hat{U}$ simply by $U:\mathcal{C}\to\mathcal{S}$. The group operation map, while not a morphism in $\mathcal{C}$ or $\mathcal{D}$, is a map $m:UQ_n\times UQ_n\to UQ_n$ in $\mathcal{S}$. Thus the following diagram commutes in $\mathcal{S}$:

$$\begin{tikzcd}
UZ_nQ_\bullet \arrow[rr, "id\top U(z'\circ r\circ j)"] \arrow[d, "Uj"'] & & UZ_nQ_\bullet\times UZ_nQ_\bullet \arrow[r, "m|_{Z_nQ_\bullet}"] \arrow[d, "U(j\times j)"] & UZ_nQ_\bullet \arrow[d, "Uj"] \\
UQ_n \arrow[r, "Ur", "\simeq"'] \arrow[rrr, bend right, "Us"'] & UQ'_n \arrow[r, "Uk\top U(j\circ z')"] & UQ_n\times UQ_n \arrow[r, "m"] & UQ_n
\end{tikzcd}$$

Since U is faithful, this implies that $s\circ j:Z_nQ_\bullet\to Q_n$ factors through a map $t: Z_nQ_\bullet\to Z_nQ_\bullet$ in $\mathcal{C}$. Moreover, because we assumed that each $\mathbf{M}\langle\alpha\rangle\in\hat{\mathcal{F}}$ is of the form $\mathbf{M}\langle\alpha\rangle=F\mathbf{M}\langle\alpha\rangle'$ for some $\mathbf{M}\langle\alpha\rangle'\in\mathcal{S}$ (where $F=\hat{F}\circ F'$ is adjoint to $U:\mathcal{C}\to\mathcal{S}$), any map $b:\mathbf{M}\langle\alpha\rangle\to Z_nQ_\bullet$ corresponds under the adjunction isomorphism to $\hat{b}:\mathbf{M}\langle\alpha\rangle'\to UZ_nQ_\bullet$, and thus $t_\#\beta=\beta\cdot(\zeta\circ(j^Q_n)_\#\beta)$ for any $\beta\in\pi_{\mathcal{F}}ZS_nQ_\bullet$ (since the group operation $\cdot$ in $\pi_{\mathcal{F}}Z_nQ_\bullet$ is induced by m – cf. [Gr, Prop. 9.9]).

Now if $\ell:\bar{Q}_{n+1}\to Z_nQ_\bullet$ realizes λ, we have $(t\circ\ell)_\#=(\ell\cdot(z'\circ\ell))_\#=\lambda\cdot(\vartheta\circ(j^Q_n)_\#\circ\lambda)=\lambda\cdot(\lambda^{-1}\cdot\lambda')=\lambda':\bar{A}_{n+1}\to\pi_{\mathcal{F}}Z_nQ_\bullet$. Thus we have a comutative diagram

$$\begin{tikzcd}
\pi_{\mathcal{F}}\bar{Q}_{n+1}=\bar{A}_{n+1} \arrow[r, "\lambda'"] \arrow[d, "id"'] & \pi_{\mathcal{F}}Z_nQ_\bullet \arrow[r, "(j^Q_n)_\#"] \arrow[d, "t_\#"] & Z_n\pi_{\mathcal{F}}Q_\bullet=Z_nA_\bullet \arrow[d, "id"] \\
\pi_{\mathcal{F}}\bar{Q}_{n+1}=\bar{A}_{n+1} \arrow[r, "\lambda"] & \pi_{\mathcal{F}}Z_nQ_\bullet \arrow[r, "(j^Q_n)_\#"] & Z_n\pi_{\mathcal{F}}Q_\bullet=Z_nA_\bullet
\end{tikzcd}$$

which yields a map of $(n+1)$-truncated objects $\rho:\operatorname{tr}_{n+1}Q_\bullet[\lambda]\to\operatorname{tr}_{n+1}Q_\bullet[\lambda']$ (or equivalently, $Q^{(n)}_\bullet[\lambda]\to Q^{(n)}_\bullet[\lambda']$). Clearly ρ induces an isomorphism in $\pi_k\pi_{\mathcal{F}}$ for $k\leq n+1$.

Now for any choice of lifting λ we have $\pi_{n+2}\pi_{\mathcal{F}}Q^{(n)}_\bullet[\lambda]\cong\operatorname{Im}(\partial^Q_n)$, and since

$$(\vartheta\circ(j^Q_n)_\#)|_{\operatorname{Im}(\partial^Q_{n-1})}=0,$$

we find $(t_\#)|_{\operatorname{Im}(\partial^Q_{n-1})}=id$, so by 2.39(b) the diagram

$$\begin{tikzcd}
\pi_{\alpha+1}Z_nQ_\bullet \arrow[r, "\partial^Q_n"] \arrow[d, "t_\#"'] & \pi_\alpha Z_{n+1}Q_\bullet \arrow[d, "id"] \\
\pi_{\alpha+1}Z_nQ_\bullet \arrow[r, "\partial^Q_n"] & \pi_\alpha Z_{n+1}Q_\bullet
\end{tikzcd}$$

commutes. Thus ρ induces an isomorphism on $\operatorname{Im}(\partial^Q_n)$, so that $(\rho)_*:Q^{(n)}_\bullet[\lambda]\to Q^{(n)}_\bullet[\lambda']$ is a weak equivalence. □

4.19. Remark. Given a (realizable) $\Pi_{\mathcal{F}}$-algebra J_*, a CW resolution $A_\bullet\in s\Pi_{\mathcal{F}}\text{-}\mathcal{A}lg$ of J_*, and a fixed (but arbitrary) choice object $X\in\mathcal{C}$ with $\pi_{\mathcal{F}}X\cong J_*$, by Corollary 4.11 we have a corresponding resolution $Q_\bullet\to X$. If $X'\in\mathcal{C}$ is another realization of J_* with corresponding

$Q'_\bullet \to X'$, we may assume without loss of generality that $Y'_\bullet := (Q'_\bullet)^{(n)} \simeq Y_\bullet := Q_\bullet^{(n)}$ for some $n \geq 0$, with $\lambda, \lambda' : A_{n+2} \to \pi_{\mathcal{F}} Z_{n+1} Q_\bullet \cong \pi_{\mathcal{F}} Z_{n+1} Q'_\bullet$ the respective liftings.

4.20. **different realizations of a Π-algebra.** Assume given an abstract Π-algebra J_*, which is known to be realizable (e.g., by the cohomological criterion of Theorem 4.15). We wish to distinguish between the various non-weakly equivalent realizations of J_* by topological spaces (or simplicial groups). The spectral sequence (4.14) implies that in order for two such $X, X' \in \mathcal{G}$ (with $\pi_* X \cong J_* \cong \pi_* X'$) to be weakly equivalent, it suffices that their corresponding resolutions $Q_\bullet \to X$ and $Q'_\bullet \to X'$ be weakly equivalent (and thus homotopy equivalent) in the resolution model category. This is in fact the main reason for considering this model category structure on $s\mathcal{G}$ in the first case (and justifies its original name of "E^2-model category" in [DKS1]).

Note, however, that this is not a necessary condition; an alternative model structure on $s\mathcal{S}$ (or $s\mathcal{G}$), defined in [Mo], has as weak equivalences precisely those maps in $s\mathcal{C}$ inducing an equivalence on the realizations.

The difference obstructions $\delta_{\lambda,\lambda'}$, which yield an inductive procedure for distinguishing between various realizations of a given Π-algebra resolution $A_\bullet \to J_*$, thus again provide an alternative to the theory described in [Bl3, §7] (as simplified in [Bl4, §4.9]) for distinguishing between different realizations of a given Π-algebra, in terms of higher homotopy operations.

To state this explicitly, assume given an (abstract) Π-algebra J_*, a CW resolution $A_\bullet \in s\Pi\text{-}\mathcal{A}lg$ of J_*, and two realizations $Q_\bullet, Q'_\bullet \in s\mathcal{G}$ of $A_\bullet$, determined as in §4.16 by successive choices of lifts $\lambda_{k+1} : \bar{A}_{k+1} \to \pi_{\mathcal{F}} Z_k Q_\bullet$ and $\lambda'_{k+1} : \bar{A}_{k+1} \to \pi_{\mathcal{F}} Z_k Q'_\bullet$. By §4.12, we know that the realizations $X := \operatorname{diag} Q_\bullet$ and $X' := \operatorname{diag} Q'_\bullet$ are two realizations of J_*. If $\delta_{\lambda_0,\lambda'_0} = 0$, there is a weak equivalence $f_0 : (Q'_\bullet)^{(0)} \simeq Q_\bullet^{(0)}$, which we can use to push forward $\lambda'_1 : \bar{A}_2 \to \pi_{\mathcal{F}} Z_1 Q'_\bullet$ to $\lambda''_1 : \bar{A}_2 \to \pi_{\mathcal{F}} Z_1 Q_\bullet$ so it is meaningful to consider $\delta_{\lambda_1,\lambda'_1} := \delta_{\lambda_1,\lambda''_1} \in H^2(J_*, \Omega J_*)$. Proceeding in this way we obtain the following

4.21. Theorem. *Assume given a Π-algebra J_*, a CW resolution $A_\bullet \in s\Pi\text{-}\mathcal{A}lg$ of J_*, and two topological spaces $\mathbf{X}, \mathbf{X}' \in \mathcal{T}_*$ realizing J_*, corresponding to $X, X' \in \mathcal{G}$ under (4.13). Let $Q_\bullet, Q'_\bullet \in s\mathcal{G}$ be CW resolutions of X, X' respectively, determined as in §4.16 by successive choices of lifts $\lambda_{n+1} : \bar{A}_{n+1} \to \pi_{\mathcal{F}} Z_n Q_\bullet$ and $\lambda'_{n+1} : \bar{A}_{n+1} \to \pi_{\mathcal{F}} Z_n Q'_\bullet$. If the difference obstructions $\delta_{\lambda_{n+1},\lambda'_{n+1}} \in H^{n+2}(J_*, \Omega^{n+1} J_*)$ vanish for all $n \geq 0$, then $\mathbf{X}$ and $\mathbf{X}'$ are weakly equivalent.*

Again, these classes satsify certain naturality conditions, which are more easily stated for simplicial $\Pi_{\mathcal{F}}$-algebras: see [BG].

4.22. Remark. Theorem 4.21 provides a collection of algebraic invariants – starting with the homotopy Π-algebra $\pi_* \mathbf{X}$ – for distinguishing between (weak) homotopy types of spaces. As with the ordinary Postnikov systems and their k-invariants, these are not actually invariant, in the sense that distinct values (i.e., non-vanishing difference obstructions) do not guarantee distinct homotopy types. Thus we are still far from a full algebraization of homotopy theory – even if we disregard the fact that Π-algebras, not too mention their cohomology groups, are rather mysterious objects, and no non-trivial naturally occurring examples are fully known to date.

Note, however, that we have a considerable simplification of the theory in the case of the rational homotopy type of simply-connected spaces: in this case the $\Pi_{\mathcal{F}}$-algebras in question are just connected graded Lie algebras over $\mathbb{Q}$, and the cohomology theory reduces to the usual Cartan-Eilenberg cohomology of Lie algebras. The obstruction theory we define appears to be the Lie algebra version of the theory for graded algebras due to Halperin and Stasheff in [HS]. See also [O, §III] and [F].

Another such simplification occurs when we consider only the stable homotopy type: in this case $\Pi_{\mathcal{F}}$-algebras are just graded modules over the stable homotopy ring $\pi := \pi_*^S S^0$, and the cohomology groups in question are $\mathrm{Ext}^*_\pi(J_*, \Sigma^{-n} J_*)$. Here we have no action of the fundamental group to worry about.

Furthermore, the spectral sequence of (4.14) implies that if $Q_\bullet^{(n)} \cong (Q'_\bullet)^{(n)}$, then also $(\mathrm{diag}\, Q_\bullet)^{(n)} \cong (\mathrm{diag}\, Q'_\bullet)^{(n)}$, so one can also use the theory described above "within a range".

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