COLIMITS FOR THE PRO CATEGORY OF TOWERS OF SIMPLICIAL SETS

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Abstract. We describe a certain category of Ind-towers into which the Pro category of towers of simplicial sets embeds, and in which all colimits (rather than just the finite ones) may be constructed explicitly.

1. Introduction

The Pro category of towers of spaces (and of other categories) has been studied in several contexts, and used for a variety of applications in homotopy theory, shape theory, geometric topology, and algebraic geometry – see for example [AM, BK, DF, EH, F, G, GV, H, HP, MS]. Our interest in it first arose, in [BT], in the study of v_n-periodicity in unstable homotopy theory (cf. [Bo, D, Md, MT]).

One problem in the usual version of the Pro category of towers is that, while *finite* limits and colimits exist, and may be constructed in a straightforward (levelwise) manner, the same does not hold for *infinite* colimits; and these were needed for the application we had in mind in [BT]. It is the purpose of the present note to improve on the rather ad hoc solution to this difficulty presented in [BT, §3] (in terms of of what were there called "virtual towers"), by enlarging the Pro category of towers in such a way as to allow a straightforward construction of arbitrary colimits. One object of this is to enable us to then provide a suitable framework for studying periodicity in unstable homotopy theory in terms of a Quillen model category structure for our version of the Pro category of towers (see [Bc]).

The construction we provide embeds (a suitable subcategory of) the Pro category $\mathcal{T}ow$ of towers of simplicial sets in a certain category $\mathcal{N}et$ of strict Ind-towers, in which we have explicit constructions for all colimits, as well as finite limits. This category $\mathcal{N}et$ can thus be thought of as a cocompletion of the Pro category of towers of spaces. We shall show in [Bc] how this construction can also provide a "homotopy theory of finite simplicial sets" (compare [Q, II, 4.10, remark 1]); it may be of use in other contexts, too.

There are other cocomplete categories in which $\mathcal{T}ow$ may be embedded – for example, the category $Pro\text{-}\mathcal{S}_\star$ of all pro-simplicial sets (cf. [AM, A.4.3 & A.4.4]), or the full category $Ind\text{-}\mathcal{T}ow$ of all inductive systems of towers (cf. [J, VI, Thm. 1.6] – this is actually "the" cocompletion of $\mathcal{T}ow$, in an appropriate sense: see [J, VI, §1] or [TT]. One advantage of the approach described here is that one obtains a smaller,

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and more mangeable, cocompletion, in this special case, and the construction of the colimits may be made quite explicitly.

A side effect of our approach is the elimination of certain "phantom phenomena" from the Pro category of towers (see §2.10(b) and §4.13 below).

1.1. conventions and notation. Let $\mathcal{T}_\star$ denote the category of pointed topological spaces, $\mathcal{S}$ the category of simplical sets, and $\mathcal{S}_\star$ that of pointed simplicial sets (see [My]). We shall refer to the objects of $\mathcal{S}_\star$ simply as *spaces*. A *finite* simplicial set $X_\bullet$ is one with only finitely many non-degenerate simplices (in all dimensions together).

(For technical convenience we prefer to work with simplicial sets, rather than topological spaces. This makes no difference for our purposes, since $\mathcal{T}_\star$ and $\mathcal{S}_\star$ have equivalent homotopy theories, in the sense of Quillen – see [Q, I, §4].)

The category (ordered set) of natural numbers will be denoted by $\mathbb{N}$, the category of abelian groups by $\mathcal{A}b\mathcal{G}p$, and the category of R-modules (for a commutative ring R) by $R\text{-}Mod$.

For any category $\mathcal{C}$ we shall denote by $Ind\text{-}\mathcal{C}$ the category of Ind-objects over $\mathcal{C}$ – that is, diagrams $F : J \to \mathcal{C}$, where J is a small filtered category (cf. [GV, Defs. 2.7 & 8.2.1]) – with the appropriate morphisms (see [GV, Def. 8.2.4]). Similarly, $Pro\text{-}\mathcal{C}$ denotes the category of Pro-objects over $\mathcal{C}$ (i.e., diagrams $F : J \to \mathcal{C}$ where J^{op} is filtered – cf. [GV, Def. 8.10.1]).

For any functor $F : I \to \mathcal{C}$ we denote the (inverse) limit of F simply by $\lim F$ or $\lim_I F$, (rather than $\lim_{\leftarrow}$), and the colimit (=direct limit) by $\operatorname{colim} F$. The (co)limit is *finite* if the category I is such (finitely many objects and morphisms between them). All limits and colimits in this paper are assumed to be small – i.e., $Ob(I)$ is a set.

A category $\mathcal{C}$ is called *pointed* if it has a zero object (i.e., one which is both initial and terminal). This object will be denoted by $*_\mathcal{C}$ (or simply $*$).

1.2. organization. In section 2 we give some background on towers of simplicial sets, their Pro category $\mathcal{T}ow$, and the finite (co)limits in $\mathcal{T}ow$. In section 3 we define "good" subcategories" – a concept which merely codifies those properties of $\mathcal{T}ow$ which are needed to construct its cocompletion. In section 4 we show that the category $\mathcal{N}et$, consisting of certain strict Ind-objects over such a good subcategory of $\mathcal{C}$, serves as a cocompletion for $\mathcal{C}$, in the sense of having all colimits (and all finite limits).

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2. The category of towers

In order to fix notation, we recall the definition of the usual Pro category of towers of spaces:

2.1. towers of spaces. The objects we shall be studying are *towers* in $\mathcal{S}_\star$ – i.e., sequences of pointed spaces and maps

$$\mathfrak{X} \;=\; \{\; \dots \boldsymbol{X}[n+1] \xrightarrow{p_n} \boldsymbol{X}[n] \xrightarrow{p_{n-1}} \boldsymbol{X}[n-1] \to \dots \xrightarrow{p_0} \boldsymbol{X}[0] \;\},$$

where the space $\boldsymbol{X}[n]$ is called the n-th *level* of $\mathfrak{X}$ $(n \geq 0)$, and the map p_n is called the n-th *level map* (or bonding map) of $\mathfrak{X}$. We denote such towers by Gothic letters: $\mathfrak{X}, \mathfrak{Y}, \ldots$.

For any $n > m$, the *iterated level map* $p^n_m : \boldsymbol{X}[n] \to \boldsymbol{X}[m]$ is defined to be the composite of

$$\boldsymbol{X}[n] \xrightarrow{p_{n-1}} \boldsymbol{X}[n-1] \to \ldots \to \boldsymbol{X}[m+1] \xrightarrow{p_m} \boldsymbol{X}[m]$$

(so $p^{n+1}_n = p_n$). We set $p^n_n = id_{\boldsymbol{X}[n]}$.

Definition 2.2. Such towers are simply objects in the functor category $\mathcal{S}_\star^{\mathbb{N}}$ of diagrams in $\mathcal{S}_\star$ indexed by the ordered set $(\mathbb{N}, \geq)$ of natural numbers. Thus a morphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ between two towers

$$\mathfrak{X} = \{\ldots \boldsymbol{X}[n] \xrightarrow{p_n} \boldsymbol{X}[n-1] \ldots\} \text{ and } \mathfrak{Y} = \{\ldots \boldsymbol{Y}[n] \xrightarrow{q_n} \boldsymbol{Y}[n-1] \to \ldots\}$$

is just a sequence $(f[k] : \boldsymbol{X}[k] \to \boldsymbol{Y}[k])_{k=0}^\infty$ of maps such that $q_k \circ f[k+1] = f[k] \circ p_k$ for $k \geq 0$.

The category $\mathcal{S}_\star^{\mathbb{N}}$ has all limits and colimits, of course. However, we are interested rather in the *Pro category* of towers of spaces:

Definition 2.3. Let $\mathcal{N} \subset \mathbb{N}^\omega$ denote the set of sequences $(n_s)_{s=0}^\infty$ of natural numbers, such that $n_s \geq \max\{n_{s-1}, s\}$ for all $s > 0$. We shall denote the elements of $\mathcal{N}$ by lower case Greek letters, with the convention that $\mu = (m_s)_{s=0}^\infty$, $\nu = (n_s)_{s=0}^\infty$, and so on. The set $\mathcal{N}$ is partially ordered by the relation $\mu \preceq \nu \Leftrightarrow m_s \leq n_s$ for all $s \geq 0$ – in fact, $\langle \mathcal{N}, \preceq \rangle$ is a lattice. $\mathcal{N}$ has a least element $\omega = (s)_{s=0}^\infty$ (though of course no maximal elements). Moreover, $\mathcal{N}$ is a monoid under composition (where $\nu = \lambda \circ \mu$ is defined by $n_s = \ell_{m_s}$), with ω as the unit and $\lambda, \mu \preceq \lambda \circ \mu$.

Definition 2.4. Given a tower $\mathfrak{X}$ and a sequence $\nu = (n_k)_{k=0}^\infty \in \mathcal{N}$, we define the *$\nu$-spaced tower* over $\mathfrak{X}$, denoted $\mathfrak{X}\langle \nu \rangle$, to be:

$$\ldots \boldsymbol{X}[n_{k+1}] \xrightarrow{p^{n_{k+1}}_{n_k}} \boldsymbol{X}[n_k] \to \ldots \to \boldsymbol{X}[n_1] \xrightarrow{p^{n_1}_{n_0}} \boldsymbol{X}[n_0].$$

In particular $\mathfrak{X}\langle \omega \rangle = \mathfrak{X}$. Note that $(-)\langle \nu \rangle$ is a functor on $\mathcal{S}_\star^{\mathbb{N}}$.

If $\mu \preceq \nu$ in $\mathcal{N}$, there is an $\mathcal{S}_\star^{\mathbb{N}}$-map $\mathfrak{p}^\nu_\mu : \mathfrak{X}\langle \nu \rangle \to \mathfrak{X}\langle \mu \rangle$ defined by $p^{n_k}_{m_k} : \boldsymbol{X}[n_k] \to \boldsymbol{X}[m_k]$ for all $k \geq 0$. Such a $\mathfrak{p}^\nu_\mu$ will be called a *self-tower map* (with respect to $\mathfrak{X}$). If $\mu = \omega$ we write simply $\mathfrak{p}^\nu$ for $\mathfrak{p}^\nu_\mu$, and call $\mathfrak{p}^\nu$ a *basic* self-tower map (for $\mathfrak{X}$). For $\mu = \nu$ we have $\mathfrak{p}^\nu_\mu = id$, and the composite of two self-tower maps (with respect to the same $\mathfrak{X}$), when defined, is a self-tower map.

Definition 2.5. We define now define the *category of towers*, denoted $\mathcal{T}ow$, in which the self-tower maps have been inverted: the objects of $\mathcal{T}ow$ are towers of spaces (as in §2.1) and its morphisms, called *tower maps*, are defined for any $\mathfrak{X}$, $\mathfrak{Z}$ by:

$$Hom_{\mathcal{T}ow}(\mathfrak{X}, \mathfrak{Y}) \overset{Def}{=} \operatorname*{colim}_{\nu \in \mathcal{N}} Hom_{\mathcal{S}_\star^{\mathbb{N}}}(\mathfrak{X}\langle \nu \rangle, \mathfrak{Y}) \tag{2.6}$$

(compare [GV, §8.2.4]). Equivalently, one may define Pro maps of towers (cf. [EH, 2.1]) by:

$$Hom_{\mathcal{T}ow}(\mathfrak{X}, \mathfrak{Y}) \cong \lim_{s \in \mathbb{N}} \operatorname*{colim}_{n \in \mathbb{N}} Hom_{\mathcal{S}_\star}(\boldsymbol{X}[n], \boldsymbol{Y}[s]).$$

(It is not hard to see this is equivalent to the above, since any Pro map of towers from $\mathfrak{X}$ to $\mathfrak{Y}$ is represented by a sequence of maps $f_s : \boldsymbol{X}[n_s] \to \boldsymbol{Y}[s]$ $(s \geq 0)$ in $\mathcal{S}_\star$, which are compatible in the sense that for each s there is an $m_s \geq n_s, n_{s+1}$ such that $q_s \circ f_{s+1} \circ p^{m_s}_{n_{s+1}} = f_s \circ p^{m_s}_{n_s} : \boldsymbol{X}[m_s] \to \boldsymbol{Y}[s]$. This defines an $\mathcal{S}_\star^{\mathbb{N}}$-map $\mathfrak{f} : \mathfrak{X}\langle\kappa\rangle \to \mathfrak{Y}$ for $\kappa \in \mathcal{N}$ defined by $k_0 = n_0$ and $k_{s+1} = \max\{m_s, k_s\}$. One readily verifies this correspondence between the two definitions of tower maps is bijective.)

Proposition 2.7. *The category $\mathcal{T}ow$ has all finite limits and colimits.*

Proof. This is well known; for completeness we recapitulate the proof:

To show that $\mathcal{T}ow$ has all finite limits, it suffices to show that $\mathcal{T}ow$ has a terminal object and pullbacks (cf. [Bx1, Prop. 2.8.2]). The tower $*$ (with $*[n]$ consisting of a single point for each n) is clearly a terminal object in $\mathcal{T}ow$. In order to define the pullback in $\mathcal{T}ow$ of two tower maps:

$$\mathfrak{X} \xrightarrow{\hat{\mathfrak{f}}} \mathfrak{Z} \xleftarrow{\hat{\mathfrak{g}}} \mathfrak{Y} \ , \tag{2.8}$$

choose any two $\mathcal{S}_\star^{\mathbb{N}}$-maps $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Z} \xleftarrow{\mathfrak{g}} \mathfrak{Y}$ representing (2.8); their pullback $\mathfrak{P}$ (in $\mathcal{S}_\star^{\mathbb{N}}$) is defined levelwise (i.e., $P[n]$ is the pullback of $\boldsymbol{X}[n] \xrightarrow{\mathfrak{f}[n]} \boldsymbol{Z}[n] \xleftarrow{\mathfrak{g}[n]} \boldsymbol{Y}[n]$ in $\mathcal{S}_\star$), and similarly for the structure maps $\mathfrak{i} : \mathfrak{P} \to \mathfrak{X}$ and $\mathfrak{j} : \mathfrak{P} \to \mathfrak{Y}$ such that $\mathfrak{f} \circ \mathfrak{i} = \mathfrak{g} \circ \mathfrak{j}$. The level maps of $\mathfrak{P}$ are induced from those of $\mathfrak{X}$ and $\mathfrak{Y}$ by the naturality of the pullback in $\mathcal{S}_\star$.

Now given a basic self-tower map $\mathfrak{p}^\nu : \mathfrak{X}\langle\nu\rangle \to \mathfrak{X}$, denote by $\tilde{\mathfrak{P}}$ the pullback (in $\mathcal{S}_\star^{\mathbb{N}}$) of $\mathfrak{X}\langle\nu\rangle \xrightarrow{\mathfrak{f}\circ\mathfrak{p}^\nu} \mathfrak{Z} \xleftarrow{\mathfrak{g}} \mathfrak{Y}$.

Again by the naturality of the pullback we can fit suitable spacings of $\mathfrak{P}$ and $\tilde{\mathfrak{P}}$ together as follows:

$$\begin{tikzcd}
\ldots \tilde{P}[n_k] \arrow[r] \arrow[rr, bend right, "\tilde{p}^{n_k}_k"'] & P[n_k] \arrow[r] \arrow[rr, bend left, "p^{n_k}_k"] & \tilde{P}[k] \arrow[r] & P[k]
\end{tikzcd}$$

implying that $\mathfrak{P}$ and $\tilde{\mathfrak{P}}$ are isomorphic; this shows that the pullback $\hat{\mathfrak{P}}$ of (2.8) in $\mathcal{T}ow$ is well-defined by taking the pullback $\mathfrak{P}$ in $\mathcal{S}_\star^{\mathbb{N}}$ of any representatives of $\hat{\mathfrak{f}}$, $\hat{\mathfrak{g}}$.

Next, given tower maps $\hat{\mathfrak{h}} : \mathfrak{W} \to \mathfrak{X}$ and $\hat{\mathfrak{k}} : \mathfrak{W} \to \mathfrak{Y}$ with $\hat{\mathfrak{g}} \circ \hat{\mathfrak{k}} = \hat{\mathfrak{f}} \circ \hat{\mathfrak{h}}$ (in $\mathcal{T}ow$), there are $\mathcal{S}_\star^{\mathbb{N}}$-representatives $\mathfrak{h} : \mathfrak{W} \to \mathfrak{X}$ and $\mathfrak{k} : \mathfrak{W} \to \mathfrak{Y}$ with $\mathfrak{g} \circ \mathfrak{k} = \mathfrak{f} \circ \mathfrak{h}$ in $\mathcal{S}_\star^{\mathbb{N}}$ (for suitable spacings of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{W}$). Thus by the universal property in $\mathcal{S}_\star$ the $\mathcal{S}_\star^{\mathbb{N}}$-maps $\mathfrak{h}$ and $\mathfrak{k}$ factor through the unique "universal" $\mathcal{S}_\star^{\mathbb{N}}$-map $\mathfrak{l} : \mathfrak{W} \to \mathfrak{P}$.

Conversely, given a tower map $\hat{\mathfrak{l}} \in Hom_{\mathcal{T}ow}(\mathfrak{W}, \mathfrak{P})$ such that

$$\hat{\mathfrak{i}} \circ \hat{\mathfrak{l}} = \hat{\mathfrak{h}} \qquad \text{and} \qquad \hat{\mathfrak{j}} \circ \hat{\mathfrak{l}} = \hat{\mathfrak{k}} \qquad \text{in } \mathcal{T}ow \ , \tag{2.9}$$

one can find $\mathcal{S}_\star^{\mathbb{N}}$-representatives for the maps in question such that $\mathfrak{i} \circ \mathfrak{l} = \mathfrak{h}$ and $\mathfrak{j} \circ \mathfrak{l} = \mathfrak{k}$ in $\mathcal{S}_\star^{\mathbb{N}}$, so that the $\mathcal{S}_\star^{\mathbb{N}}$-map $\mathfrak{l} : \mathfrak{W} \to \mathfrak{P}$ is the "universal map" as above. Thus it suffices to check that any two universal $\mathcal{S}_\star^{\mathbb{N}}$-maps $\mathfrak{l} : \mathfrak{W} \to \mathfrak{P}$ and $\mathfrak{l}' : \mathfrak{W} \to \mathfrak{P}\langle\nu\rangle$ represent the same tower map; but this follows readily from the uniqueness of the universal maps in $\mathcal{S}_\star^{\mathbb{N}}$.

To show that $\mathcal{T}ow$ has all finite colimits, we show analogously that it has an initial object and pushouts, again defined levelwise. $\square$

Remark 2.10. The category $\mathcal{T}ow$ may be embedded in a category with *all* colimits (and all filtered limits) – namely, the category $Pro\text{-}\mathcal{S}_\star$ of all pro-simplicial sets (see [AM, A.4.3 & A.4.4] or [GV, Props. 8.9.1 & 8.9.5]). The problem is that the limit or colimit of an infinite diagram of towers will not itself be a tower, and is rather difficult to construct explicitly.

Note that the naive (levelwise) construction of colimits in $\mathcal{T}ow$ can fail in two different ways:

(a) If $\{\mathfrak{X}_\alpha\}_{\alpha\in A}$ is some (infinite) collection of towers, and we define a tower $\mathfrak{Y}$ by $\boldsymbol{Y}[n] = \coprod_{\alpha\in A}\boldsymbol{X}_\alpha[n]$, then $\mathfrak{Y}$ is "too small" – in general, there will be maps $\hat{\mathfrak{f}}_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Z}$ in $\mathcal{T}ow$ such that for any choice of representatives $\mathfrak{f}_\alpha : \mathfrak{X}_\alpha\langle\nu^\alpha\rangle \to \mathfrak{Z}$ in $\mathcal{S}_\star^{\mathbb{N}}$, the set of numbers $\{n_0^\alpha\}_{\alpha\in A}$ is unbounded. Thus there will be no way to define a $\mathcal{S}_\star^{\mathbb{N}}$-representative of the putative corresponding map $\hat{\mathfrak{f}} : \mathfrak{Y} \to \mathfrak{Z}$ which restricts to $\hat{\mathfrak{f}}_\alpha$ on $\mathfrak{X}_\alpha$.

(b) On the other hand, let $(\mathcal{A}_i)_{i=0}^\infty$ be some sequence of non-trivial spaces, and define towers $(\mathfrak{X}_i)_{i=0}^\infty$ by letting $\boldsymbol{X}_i[n] = \mathcal{A}_i$, and $(p_i)_n = Id_{\mathcal{A}_i}$, if $n \leq i$, and $\boldsymbol{X}_i[n] = *$ otherwise. If again we set $\boldsymbol{Y}[n] = \coprod_{i=0}^\infty \boldsymbol{X}_i[n]$, we see $\mathfrak{Y}$ is now "too big":
For a given tower $\mathfrak{Z}$, any collection of maps $(f_i : \mathcal{A}_i \to \boldsymbol{Z}[i])_{i=0}^\infty$ yields a unique $\mathcal{S}_\star^{\mathbb{N}}$-map $\mathfrak{f} : \mathfrak{Y} \to \mathfrak{Z}$ in the obvious way, and two such choices $(f_i)_{i=0}^\infty$ and $(g_i)_{i=0}^\infty$ yield equivalent tower maps ($\hat{\mathfrak{f}} = \hat{\mathfrak{g}}$ in $\mathcal{T}ow$) if and only if there is an N such that $f_i = g_i$ for $i \geq N$ (at least for suitable $\mathfrak{Z}$ – e.g., if $\mathfrak{Z}$ is constant) . Thus there are many such tower maps $\hat{\mathfrak{f}} : \mathfrak{Y} \to \mathfrak{Z}$; but the corresponding maps $\hat{\mathfrak{f}}_i : \mathfrak{X}_i \to \mathfrak{Z}$ are all trivial in $\mathcal{T}ow$. (In some sense the maps $\hat{\mathfrak{f}}$ so defined may be thought of as "phantom tower maps" – compare [GM]).

3. Good subcategories

We now describe those properties of the category $\mathcal{T}ow$ which are needed to construct the extension. Since this construction is also needed for [Bc], we describe it in greater generality than required for our immediate purposes.

Definition 3.1. Let $\mathcal{C}$ be a pointed category, and $\mathcal{F}$ a small full subcategory. For each $A \in \mathcal{C}$, let $\mathcal{F}_A$ denote the subcategory of the over category $\mathcal{C}/A$ (cf. [Bx1, §1.2.7]), whose objects are monomorphisms $i : F \hookrightarrow A$ with $F \in \mathcal{F}$ and whose morphisms are (necessarily monic) maps $j : F \to F'$ such that $i' \circ j = i$. Similarly, let $\mathcal{F}^A$ denote the subcategory of the under category $A/\mathcal{C}$, whose objects are epimorphisms $q : A \twoheadrightarrow F$ with $F \in \mathcal{F}$, and whose morphisms are (epic) maps $p : F \to F'$ with $p \circ q = q'$.

We say that $\mathcal{F}$ is a *good* subcategory of $\mathcal{C}$ if:

(a) $\mathcal{F}$ is closed under taking subobjects and quotient objects.

(b) $\mathcal{F}$ is finite-complete and -cocomplete, and the inclusion $I : \mathcal{F} \to \mathcal{C}$ is $\aleph_0$-(co)continuous (i.e., any finite diagram $(F_\alpha)_{\alpha\in A}$ ($|A| < \infty$) over $\mathcal{F}$ has a limit L and a colimit C in $\mathcal{C}$, with $L, C \in \mathcal{F}$).

(c) For any $F \in \mathcal{F}$ the category $\mathcal{F}^F$ is *co-artinian* – that is, given a sequence of quotient maps

$$F \xrightarrow{q_0} G_0 \xrightarrow{q_1} G_1 \xrightarrow{q_2} \dots G_{n-1} \xrightarrow{q_n} G_n \xrightarrow{q_{n+1}} \dots ,$$

there is an N such that q_n is an isomorphism for $n \geq N$ (compare [GV, §8.12.6]).
(d) Any morphism $f : F \to C$, with $F \in \mathcal{F}$ and $C \in \mathcal{C}$, has an epimorphic image $Im(f)$ (see [Bx1, Def. 4.4.4]) – which is necessarily in $\mathcal{F}_C$.

The inclusions $i_G : G \to C$ thus induce a natural bijection:

$$\Phi_{F,C} : \operatorname*{colim}_{G \in \mathcal{F}_C} Hom_{\mathcal{F}}(F, G) \xrightarrow{=} Hom_{\mathcal{C}}(F, C). \tag{3.2}$$

Definition 3.3. Let $\mathcal{T}ow^{st}$ denote the category of (essentially) *strict* towers of simplicial sets (cf. [GV, §8.12.1]) – that is, the full subcategory of $\mathcal{T}ow$ whose objects are towers $\mathfrak{X}$ for which there is an N such that all level maps $p_n : \boldsymbol{X}[n+1] \to \boldsymbol{X}[n]$ are epimorphisms for $n \geq N$. (We think of these as being "good" towers, because they avoid the pathologies mentioned in 2.10(b)). Note that $\mathcal{T}ow^{st}$ has all finite colimits and products, but not all pullbacks.

Let $\mathcal{F} = \mathcal{F}^{\mathcal{T}ow}$ denote the full subcategory of $\mathcal{T}ow^{st}$ whose objects are towers $\mathfrak{X}$ such that each $\boldsymbol{X}[n]$ is a finite simplicial set (§1.1), and there is an N such that p_n is an isomorphism for $n \geq N$. We denote by $\boldsymbol{S}p(\mathfrak{X})$ the finite simplicial set $\lim_k \boldsymbol{X}[k]$ (which is naturally isomorphic to $\boldsymbol{X}[n]$ for $n \geq N$).

Proposition 3.4. *$\mathcal{F} = \mathcal{F}^{\mathcal{T}ow}$ is a good subcategory of $\mathcal{T}ow^{st}$.*

Proof. (1) Given $\mathfrak{Y} \in \mathcal{F}$, let $\hat{\mathfrak{f}} : \mathfrak{X} \to \mathfrak{Y}$ be a monomorphism in $\mathcal{T}ow^{st}$, with a representative $\mathfrak{f} : \mathfrak{X}\langle\nu\rangle \to \mathfrak{Y}$. For simplicity of notation let $\mathfrak{X} = \mathfrak{X}\langle\nu\rangle$. Now let $\mathfrak{Z} = \{\dots \boldsymbol{Z}[n] \xrightarrow{s_n} \boldsymbol{Z}[n-1] \to \dots\}$ be the (levelwise) pullback (in $\mathcal{S}_\star^{\mathbb{N}}$) of

$$\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y} \xleftarrow{\mathfrak{f}} \mathfrak{X} ,$$

with $\mathfrak{h}_1, \mathfrak{h}_2 : \mathfrak{Z} \to \mathfrak{X}$ the two projections.

Since $\mathfrak{f} \circ \mathfrak{h}_1 = \mathfrak{f} \circ \mathfrak{h}_2$ and $\hat{\mathfrak{f}}$ is a monomorphism in $\mathcal{T}ow$, there is a $\nu \in \mathcal{N}$ such that $\mathfrak{h}_1 \circ \mathfrak{s}^\nu = \mathfrak{h}_2 \circ \mathfrak{s}^\nu : \mathfrak{Z}\langle\nu\rangle \to \mathfrak{X}$ in $\mathcal{S}_\star^{\mathbb{N}}$.

Now let $(x_0, x_1) \in \boldsymbol{Z}[k]_t \subseteq \boldsymbol{X}[k]_t \times \boldsymbol{X}[k]_t$ be a pair of k-th level t-simplices of $\mathfrak{X}$. Since the level maps $p_n : \boldsymbol{X}[n+1] \to \boldsymbol{X}[n]$ of $\mathfrak{X}$ are epimorphisms, there are t-simplices $\bar{x}_0, \bar{x}_1 \in \boldsymbol{X}[n_k]_t$ such that $p_k^{n_k}(\bar{x}_i) = x_i$ $(i = 0, 1)$.

Thus $q_k^{n_k}(f(\bar{x}_0)) = f(p_k^{n_k}(\bar{x}_0)) = f(x_0) = f(x_1) = f(p_k^{n_k}(\bar{x}_1)) = q_k^{n_k}(f(\bar{x}_1))$, and since $\mathfrak{Y} \in \mathcal{F}$, each level map q_n of $\mathfrak{Y}$ is monic, and so $(f(\bar{x}_0)) = (f(\bar{x}_1))$, – i.e., $(\bar{x}_0, \bar{x}_1) \in \boldsymbol{Z}[n_k]_t$, with $s_k^{n_k}(\bar{x}_0, \bar{x}_1) \in \boldsymbol{Z}[n_k]_t = (x_0, x_1)$. But then $x_0 = x_1$, since $\mathfrak{v}_1 \circ \mathfrak{s}^\nu = \mathfrak{v}_2 \circ \mathfrak{s}^\nu$. Thus $\mathfrak{f}$ is levelwise monic.

But this implies that each $q_n \circ f[n+1] = f[n] \circ p_n$ is monic, so p_n is, too, and since $\mathfrak{Y} \in \mathcal{F}$ we see $\mathfrak{X} \in \mathcal{F}$, too.

(2) If $\hat{\mathfrak{f}} : \mathfrak{X} \to \mathfrak{Y}$ is any epimorphism in $\mathcal{T}ow^{st}$, we shall show more generally that $\hat{\mathfrak{f}}$ may be represented by a levelwise epimorphism: without loss of generality, $\hat{\mathfrak{f}}$ has an $\mathcal{S}_\star^{\mathbb{N}}$-representative $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$; by factoring $\mathfrak{f}$ via its (levelwise) image, $Im(\mathfrak{f})$, we may

assume that $\mathfrak{f}$ is levelwise *monic*. Now set $\mathfrak{Z} = \mathfrak{Y}/\mathfrak{X}$ to be the (levelwise) pushout (in $\mathcal{S}_\star^{\mathbb{N}}$) of

$$* \longleftarrow \mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y} ,$$

with two $\mathcal{S}_\star^{\mathbb{N}}$-maps: $\mathfrak{g} : \mathfrak{Y} \twoheadrightarrow \mathfrak{X}$ the quotient map, and $* = \mathfrak{h} : \mathfrak{Y} \twoheadrightarrow \mathfrak{X}$ the trivial map. Clearly $\mathfrak{g} \circ \mathfrak{f} = * = \mathfrak{h} \circ \mathfrak{h}$ (in $\mathcal{S}_\star^{\mathbb{N}}$), so there is a $\nu \in \mathcal{N}$ such that $\mathfrak{g} \circ \mathfrak{q}^\nu = \mathfrak{h} \circ \mathfrak{q}^\nu : \mathfrak{Y}\langle\nu\rangle \to \mathfrak{Z}$, since $\hat{\mathfrak{f}}$ is an epimorphism in $\mathcal{T}ow$. But then $\mathfrak{g} \circ \mathfrak{q}^\nu = *$, so $\mathfrak{q}^\nu$ factors as $\mathfrak{f} \circ \bar{\mathfrak{q}}^\nu$ (for $\bar{\mathfrak{q}}^\nu : \mathfrak{Y}\langle\nu\rangle \to \mathfrak{X}$), with $\bar{\mathfrak{q}}^\nu \circ \mathfrak{f}\langle\nu\rangle = \mathfrak{p}^\nu$, so that $\mathfrak{f}, \bar{\mathfrak{q}}^\nu$ are inverese to each other in $\mathcal{T}ow$, and thus $\mathfrak{f}$ is an isomorphism. As before we conclude that if $\mathfrak{X} \in \mathcal{F}$ then also $\mathfrak{Y} \in \mathcal{F}$.

(3) Given a finite diagram over $\mathcal{F}$, its limit and colimit in $\mathcal{T}ow$ may be defined levelwise by Proposition 2.7, so are in $\mathcal{F}$.

(4) Since epimorphisms in $\mathcal{T}ow^{st}$ are actually levelwise surjections, the category $\mathcal{F}^{\mathfrak{Y}}$ is equivalent to a finite category for any $\mathfrak{Y} \in \mathcal{F}$ – so in particular it is co-artinian.

(5) Given $\hat{\mathfrak{f}} : \mathfrak{X} \to \mathfrak{Y}$ in $\mathcal{T}ow^{st}$, with $\mathfrak{X} \in \mathcal{F}$, one can define $Im(\mathfrak{f}) = \mathfrak{Z}$ to be the levelwise image tower for some $\mathcal{S}_\star^{\mathbb{N}}$-representative $\mathfrak{f} : \mathfrak{X}\langle\nu\rangle \to \mathfrak{Y}$ of $\hat{\mathfrak{f}}$. This is independent of the representative $\mathfrak{f}$ chosen, since given another representative $\mathfrak{f}' : \mathfrak{X}\langle\mu\rangle \to \mathfrak{Y}$, there is a $\lambda \succeq \mu, \nu$ such that $\mathfrak{f}' \circ \mathfrak{s}^\lambda_\mu = \mathfrak{f} \circ \mathfrak{s}^\lambda_\nu$, and thus both maps have the same (levelwise) image, because $\mathfrak{Z}$ is in $\mathcal{F}$ by (2), and thus the level maps s_n of $\mathfrak{Z}$ are epimorphic. We write $\mathfrak{i} : \mathfrak{Z} \to \mathfrak{Y}$ for the inclusion, with $\bar{\mathfrak{f}} : \mathfrak{X} \twoheadrightarrow \mathfrak{Z}$ such that $\mathfrak{f} = \mathfrak{i} \circ \bar{\mathfrak{f}}$.

Now if $\hat{\mathfrak{j}} : \mathfrak{W} \to \mathfrak{Y}$ is another monomorphism in $\mathcal{T}ow^{st}$, equipped with a tower map $\hat{\mathfrak{g}} : \mathfrak{X} \to \mathfrak{W}$ such that $\hat{\mathfrak{j}} \circ \hat{\mathfrak{g}} = \hat{\mathfrak{f}}$ in $\mathcal{T}ow$, we may assume without loss of generality that $\hat{\mathfrak{g}}$ is represented by $\mathfrak{g} : \mathfrak{X}\langle\nu\rangle :\to \mathfrak{W}$ with $\mathfrak{j} \circ \mathfrak{g} = \mathfrak{f} = \mathfrak{i} \circ \bar{\mathfrak{f}}$, and moreover by factoring $\mathfrak{g}$ itself through its image we may assume $\mathfrak{g}$ is levelwise epimorphic, so $\mathfrak{W} \in \mathcal{F}$ by (2), being a quotient of $\mathfrak{X}$.

Finally, factoring our chosen $\mathcal{S}_\star^{\mathbb{N}}$-representative $\mathfrak{j} : \mathfrak{W} \to \mathfrak{Y}$ through $Im(\mathfrak{j})$ (which is in $\mathcal{F}$, by (2)), we find that the $\mathcal{S}_\star^{\mathbb{N}}$-map $\mathfrak{W} \twoheadrightarrow Im(\mathfrak{j})$ is an isomorphism, as in (a); but since $Im(\mathfrak{j}) \hookrightarrow \mathfrak{Y}$ is a levelwise monomorphism, by the universal property of Im in $\mathcal{S}_\star$ (and thus in $\mathcal{S}_\star^{\mathbb{N}}$) there is a (levelwise) monomorphism $\mathfrak{k} : \mathfrak{Z} \to \mathfrak{W}$ through which $\mathfrak{g}$ and $\mathfrak{i}$ factor, showing that $\mathfrak{f} = \mathfrak{i} \circ \bar{\mathfrak{f}}$ is indeed initial in $\mathcal{T}ow^{st}$ among the factorizations $\hat{\mathfrak{f}} = \hat{\mathfrak{j}} \circ \hat{\mathfrak{g}}$ of $\hat{\mathfrak{f}}$ with $\hat{\mathfrak{j}}$ monic. □

3.5. good generating subcategories. We shall in fact be interested in good subcategories $\mathcal{F} \subseteq \mathcal{C}$ which *generate* $\mathcal{C}$ (cf. [Bx1, Def. 4.5.1] or [Me, V, §7]) – that is, such that for any object $C \in \mathcal{C}$, $\{f : F \to C\}_{f:F \to C, F \in \mathcal{F}}$ is an epimorphic family ([GV, §10.3]).

Note that because of 3.1(d) and (3.2), this is equivalent to requiring that, for all $C, D \in \mathcal{C}$, there be a canonical natural inclusion of sets:

$$J_{C,D} : Hom_{\mathcal{C}}(C, D) \hookrightarrow \lim_{F \in \mathcal{F}_C} \operatorname{colim}_{G \in \mathcal{F}_D} Hom_{\mathcal{F}}(F, G), \tag{3.6}$$

induced by the restrictions $f|_F$ for any $f : C \to D$ and the correspondences $\Phi^{-1}_{F,D}$ of (3.2).

Example 3.7. (i) The category of pointed sets is generated by the good subcategory of finite pointed sets.

(ii) The category $\mathcal{S}_\star$ of pointed simplicial sets is generated by the good subcategory $\mathcal{S}_\star^f$ of finite pointed simplicial sets (§1.1).

(iii) The category of torsion groups is generated by the good subcategory of finite groups.

(iv) The category $\mathcal{A}b\mathcal{G}p$ is generated by the good subcategory of finitely generated abelian groups, and more generally $R\text{-}Mod$ is generated by the good subcategory of f.g. R-modules for any noetherian ring R.

In these cases the natural inclusion $J_{C,D}$ of (3.6) is actually bijective.

Proposition 3.8. *The category $\mathcal{T}ow^{st}$ is generated by the subcategory $\mathcal{F}^{\mathcal{T}ow}$.*

Proof. Let $\hat{\mathfrak{f}} \neq \hat{\mathfrak{g}} : \mathfrak{X} \to \mathfrak{Y}$ be two different tower maps, with $\mathfrak{X} \in \mathcal{T}ow^{st}$; without loss of generality we may assume they have $\mathcal{S}_\star^{\mathbb{N}}$-representatives $\mathfrak{f}, \mathfrak{g} : \mathfrak{X}\langle\nu\rangle \to \mathfrak{Y}$ respectively. By definition 2.5, there is a $k \geq 0$ such that for all $n \geq k$ we have $f[n] \circ p^n_k \neq g[n] \circ p^n_k$, so in particular for each $n \geq k$ there is a t-simplex $x_n \in \boldsymbol{X}[n]_t$ (t independent of n) such that $f[n](x_n) \neq g[n](x_n)$. Since $\mathfrak{X} \in \mathcal{T}ow^{st}$, the level maps p_n of $\mathfrak{X}$ are surjective, and we may evidently assume $p_n(x_{n+1}) = x_n$ for all $n \geq N$.

To each $x_n \in \boldsymbol{X}[n]_t$ there corresponds a map $\varphi_{x_n} : \boldsymbol{\Delta}[t] \to \boldsymbol{X}[n]_t$, and let $\boldsymbol{Z}[n] \in \mathcal{S}_\star$ denote the simplicial set $Im(\varphi_{x_n})$. Then $\mathfrak{Z} = \{\dots \boldsymbol{Z}[n] \xrightarrow{s_n} \boldsymbol{Z}[n-1] \to \dots\}$ is in fact a sub-tower of $\mathfrak{X}$, with $s_n = p_n|_{\boldsymbol{Z}[n+1]}$, and because each $\boldsymbol{Z}[n]$ is a quotient of both $\boldsymbol{\Delta}[t]$ and $\boldsymbol{Z}[n+1]$, for sufficiently large n the maps s_n must be isomorphisms (since $\boldsymbol{\Delta}[t]$ has only finitely many non-isomorphic quotients), so that $\mathfrak{Z} \in \mathcal{F}_{\mathfrak{X}}$.

Clearly $\mathfrak{f}|_{\mathfrak{Z}} \neq \mathfrak{g}|_{\mathfrak{Z}}$, and both have images in $\mathcal{F}$ by Definition 3.1(d) and Proposition 3.4 – which proves $J_{\mathfrak{X},\mathfrak{Y}}$ is indeed one-to-one. □

It may be useful to think of the finite subtowers $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ ($\mathfrak{Z} \in \mathcal{F}$) as the analogue of the stable cells of a CW-spectrum – compare [A, III, §3].

Remark 3.9. Note that in general our category $\mathcal{C}$ will not be *locally generated* by the subcategory $\mathcal{F}$, in the sense of [GU, §§7,9], because $\mathcal{C}$ need not be cocomplete – and we are interested precisely in such cases, because only then will the cocompletion of $\mathcal{C}$ be of interest. $\mathcal{C}$ need not even be $\aleph_0$-*accessible* in the sense of [Bx2, Def. 5.3.1], because we do not assume that $\mathcal{F}_C$ has all colimits for arbitrary $C \in \mathcal{C}$.

4. Nets and cocompletion

When $\mathcal{C}$ is generated by a good subcategory $\mathcal{F}$, it embeds in the category $Ind^{st}\text{-}\mathcal{F}$ of strict Ind-objects over $\mathcal{F}$; by constructing all colimits for $Ind^{st}\text{-}\mathcal{F}$, (or rather, for an equivalent subcategory $\mathcal{N}et$), we show that this can serve as a cocompletion for $\mathcal{C}$.

In analogy with the completion of a metric space, the objects of $\mathcal{N}et$ are themselves directed systems of suitable towers; one should think of these as representing their colimit (which may not exist in $\mathcal{T}ow$).

Definition 4.1. A *strict Ind* object over a category $\mathcal{G}$ is a diagram $X : I \to \mathcal{G}$ indexed by a small filtered partially ordered category I, such that all *bonding maps* $X(f) : X_\alpha \to X_\beta$ (for $f : \alpha \to \beta$ in A) are monomorphisms (cf. [GV, Def. 8.12.1]). The full subcategory of $Ind\text{-}\mathcal{G}$ whose objects are strict will be denoted by $Ind^{st}\text{-}\mathcal{G}$.

In order to simplify our constuctions, it is convenient to consider the subcategory $\mathcal{N}et \subseteq Ind^{st}\text{-}\mathcal{G}$ defined as follows (this is actually equivalent to $Ind^{st}\text{-}\mathcal{G}$, under suitable assumptions – see Fact 4.4 below):

Definition 4.2. If $\mathcal{G}$ is a pointed category, a *net* over $\mathcal{G}$ to be a strict Ind-object $(X_\alpha)_{\alpha\in A}$ indexed by a lattice $\langle A, \preceq, \vee, \wedge\rangle$ with least element 0, such $X_0 = *$, and for each $\alpha, \beta \in A$, the square:

$$\begin{tikzcd} X_{\alpha\wedge\beta} \arrow[r, "i_{\alpha\wedge\beta,\alpha}"] \arrow[d, "i_{\alpha\wedge\beta,\beta}"'] & X_\alpha \arrow[d, "i_{\alpha,\alpha\vee\beta}"] \\ X_\beta \arrow[r, "i_{\beta,\alpha\vee\beta}"] & X_{\alpha\vee\beta} \end{tikzcd}$$

FIGURE 1

is both cartesian and cocartesian. (Since we required the bonding maps of the net to be monic, this simply means that $X_{\alpha\wedge\beta}$ is the intersection $X_\alpha \cap X_\beta$ of X_α and X_β and $X_{\alpha\vee\beta}$ is their union $X_\alpha \cup X_\beta$ (cf. [Bx1, Def. 4.2.1 & Prop. 4.2.3])

Definition 4.3. If $(X_\alpha)_{\alpha\in A}$ and $(Y_\beta)_{\beta\in B}$ are two nets over $\mathcal{G}$, a *proper net map* between them is a pair $\langle\phi, (f_\alpha)_{\alpha\in A}\rangle$, where $\phi : A \to B$ is a order-preserving map with $\phi(0) = 0$, and for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_{\phi(\alpha)}$ is a morphism in $\mathcal{G}$. We require that for all $\alpha \preceq \beta$ in A, the diagram:

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "i_{\alpha,\beta}"'] & Y_{\phi(\alpha)} \arrow[d, "i_{\phi(\alpha),\phi(\beta)}"] \\ X_\beta \arrow[r, "f_\beta"] & Y_{\phi(\beta)} \end{tikzcd}$$

commutes. If $Y_{\phi(\alpha)} = Im(f_\alpha)$ for all $\alpha \in A$ – in other words, each f_α is epic – we say $\langle\phi, (f_\alpha)_{\alpha\in A}\rangle$ is a *minimal* proper net map.

Two proper net maps $\langle\phi, (f_\alpha)_{\alpha\in A}\rangle, \langle\psi, (g_\beta)_{\beta\in B}\rangle : (X_\alpha)_{\alpha\in A} \to (Y_\beta)_{\beta\in B}$ are *equivalent* – written $\langle\phi, (f_\alpha)_{\alpha\in A}\rangle \simeq \langle\psi, (g_\beta)_{\beta\in B}\rangle$ – if for each $\alpha \in A$ there is an $\rho(\alpha)$ such that $\phi(\alpha) \vee \psi(\alpha) \preceq \rho(\alpha)$ (in the lattice B) and the diagram

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "g_\alpha"'] & Y_{\phi(\alpha)} \arrow[d, "i_{\phi(\alpha),\rho(\alpha)}"] \\ Y_{\psi(\alpha)} \arrow[r, "i_{\psi(\alpha),\rho(\alpha)}"] & Y_{\rho(\alpha)} \end{tikzcd}$$

commutes. Note that if $\mathcal{G}$ is a category with images, then each equivalence class of proper net maps will have a unique minimal representative.

The category of nets over $\mathcal{G}$, with equivalence classes of proper net maps as morphisms, will be denoted $\mathcal{N}et_{\mathcal{G}}$. We shall sometimes use the notation $\mathfrak{f} : (X_\alpha)_{\alpha\in A} \to (Y_\beta)_{\beta\in B}$ to denote a morphism of nets (i.e., an equivalence class of proper maps) – cf. [GV, §8.2.4-5].

Fact 4.4. If $\mathcal{G}$ has finite unions and intersections, every object in Ind^{st}-$\mathcal{G}$ is isomorphic to one in $\mathcal{N}et_{\mathcal{G}}$.

Proof. By the dual of [MS, I, §1, Thm. 4] every (strict) Ind-object over $\mathcal{G}$ is Ind-isomorphic to a (strict) Ind-object $(X_\lambda)_{\lambda\in\Lambda}$ indexed by a directed ordered set $\langle\Lambda, \preceq\rangle$

which is *closure finite* – i.e., the set of predecessors of every $\lambda \in \Lambda$ is finite. Now let A be the free lattice generated by Λ, and set $X_{\alpha\wedge\beta} = \bigcup_{\gamma\preceq\alpha,\beta} X_\gamma$, with $X_{\alpha\vee\beta}$ defined by Figure 1. Since Λ is cofinal in A, we actually have an Ind-isomorphism $(X_\lambda)_{\lambda\in\Lambda} \hookrightarrow (X_\alpha)_{\alpha\in A}$. $\square$

This shows that we could assume, if we wish, that our nets are always indexed by closure finite lattices (and this will in fact be the case for $\mathcal{F}^{\mathcal{T}ow}$, of course, because in this case $\mathcal{F}_F$ will be a finite category for each $F \in \mathcal{F}$), but this is not needed for our constructions.

Proposition 4.5. *If $\mathcal{F} \subset \mathcal{C}$ is good (so in particular has all finite colimits), then $\mathcal{N}et_{\mathcal{F}}$ has all colimits.*

Proof. It suffices to show $\mathcal{N}et_{\mathcal{F}}$ has coproducts and pushouts (cf. [P, §2.6, Prop. 1 & 2]):

I. Given any collection $\{(X^i_\alpha)_{\alpha\in A_i}\}_{i\in I}$ of nets over $\mathcal{F}$ (indexed by an arbitrary set I), let $B = \coprod_{i\in I} A_i$ denote the coproduct lattice – so that the elements of B are of the form $\beta = \alpha_{i_1} \vee \ldots \vee \alpha_{i_n}$ for $\alpha_{i_j} \in A_{i_j}$ (and $i_j \neq i_k$ for $j \neq k$).

The coproduct net is then defined to be

$$(\bigvee_{j=1}^{n} X^{i_j}_{\alpha_{ij}})_{\alpha_{i_1}\vee\ldots\vee\alpha_{i_n}\in B} \ ,$$

and the universal property for the coproduct evidently holds.

II. Given two net maps with minimal proper representatives:

$$\begin{tikzcd}
(X_\alpha)_{\alpha\in A} \arrow[r, "{\langle \phi, (f_\alpha)_{\alpha\in A}\rangle}"] \arrow[d, "{\langle \psi, (g_\alpha)_{\alpha\in A}\rangle}"'] & (Y_\beta)_{\beta\in B} \\
(Z_\gamma)_{\gamma\in C} &
\end{tikzcd}$$

FIGURE 2

For each $\beta \in B$ and $\gamma \in C$, let $A_{(\beta,\gamma)} = \{(\alpha \in A \mid \phi(\alpha) \preceq \beta \ \& \ \psi(\alpha) \preceq \gamma\}$ (a sublattice of A), and for each $\alpha \in A_{(\beta,\gamma)}$, let $W^\alpha = W^\alpha_{\beta\vee\gamma}$ denote the pushout in:

$$\begin{tikzcd}
X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "g_\alpha"'] & Y_{\phi(\alpha)} \arrow[r, "i_{\phi(\alpha),\beta}"] & Y_\beta \arrow[dd, dotted] \\
Z_{\psi(\alpha)} \arrow[d, "i_{\psi(\alpha),\gamma}"'] & & \\
Z_\gamma \arrow[rr, dotted] & & W^\alpha_{\beta\vee\gamma}
\end{tikzcd}$$

Now for each $\beta_0 \vee \gamma_0$ in the coproduct lattice $B \amalg C$ let

$$\mathcal{L}_{\beta_0\vee\gamma_0} \ = \ \bigcup_{\beta_0\preceq\beta} \ \bigcup_{\gamma_0\preceq\gamma} \{(\alpha,\beta,\gamma) \in A\times B\times C \mid \alpha \in A_{(\beta,\gamma)}\}.$$

For any $(\alpha,\beta,\gamma)\in\mathcal{L}_{\beta_0\vee\gamma_0}$, the bonding maps $i_{\beta_0,\beta}$ and $i_{\beta_0,\beta}$ induce a map

$$q^{\beta_0\vee\gamma_0}_{(\alpha,\beta,\gamma)}:Y_{\beta_0}\amalg Z_{\gamma_0}\to W^{\alpha}_{\beta\vee\gamma}.$$

We let $U_{(\alpha,\beta,\gamma)}=U^{\beta_0\vee\gamma_0}_{(\alpha,\beta,\gamma)}$ denote $Im(q^{\beta_0\vee\gamma_0}_{(\alpha,\beta,\gamma)})\subseteq W^{\alpha}_{\beta\vee\gamma}$.

Note that, for fixed $\beta_0\vee\gamma_0\in B\amalg C$, the objects $U_{(\alpha,\beta,\gamma)}$ form a diagram in $\mathcal{F}$ indexed by the (possibly infinite) filtered set $\mathcal{L}_{\beta_0\vee\gamma_0}$, and set

$$W_{\beta_0\vee\gamma_0}\overset{Def}{=}\operatorname*{colim}_{(\alpha,\beta,\gamma)\in\mathcal{L}_{\beta_0\vee\gamma_0}}U^{\beta_0\vee\gamma_0}_{(\alpha,\beta,\gamma)}. \tag{4.6}$$

This limit exists in $\mathcal{F}$ – in fact, in $\mathcal{F}'=\mathcal{F}^{Y_{\beta_0}\amalg Z_{\gamma_0}}$, by [Bx1, Prop. 2.16.3] – since $\mathcal{F}'$ is co-artinian by Def. 3.1(c), and thus has all filtered colimits. The natural map

$$i_{\beta_0\vee\gamma_0,\beta_1\vee\gamma_1}:W_{\beta_0\vee\gamma_0}\to W_{\beta_1\vee\gamma_1}$$

(induced by the fact that each $q^{\beta_0\vee\gamma_0}_{(\alpha,\beta,\gamma)}$ factors through $q^{\beta_1\vee\gamma_1}_{(\alpha,\beta,\gamma)}$) is always a monomorphism. Thus we have defined as net $(W_{\beta\vee\gamma})_{\beta\vee\gamma\in B\amalg C})$ over $\mathcal{F}$. (Had we not required that our nets be *strict* Ind-objects, we could have defined $W_{\beta_0\vee\gamma_0}$ more simply as the colimit of the objects $W^{\alpha}_{\beta_0\vee\gamma_0}$ for $\alpha\in A_{(\beta_0,\gamma_0)}$).

We claim that this net is the pushout for the diagram in Figure 2: given a commutative diagram in $\mathcal{N}et_{\mathcal{F}}$

$$\begin{tikzcd}
(X_\alpha)_{\alpha\in A} \arrow[r, "{\langle\phi,(f_\alpha)_{\alpha\in A}\rangle}"] \arrow[d, "{\langle\psi,(g_\alpha)_{\alpha\in A}\rangle}"'] & (Y_\beta)_{\beta\in B} \arrow[d, "{\langle\rho,(h_\beta)_{\beta\in B}\rangle}"] \\
(Z_\gamma)_{\gamma\in C} \arrow[r, "{\langle\sigma,(k_\gamma)_{\gamma\in C}\rangle}"] & (V_\varepsilon)_{\varepsilon\in E}
\end{tikzcd}$$

(where we may assume the proper representatives indicated make it commute on the nose), we define a net map $\langle\tau,(\ell_{\beta\vee\gamma})_{\beta\vee\gamma\in B\amalg C}\rangle:(W_{\beta\vee\gamma})_{\beta\vee\gamma\in B\amalg C}\longrightarrow(V_\varepsilon)_{\varepsilon\in E}$ as follows: set

$$\tau(\beta\vee\gamma)\overset{Def}{=}\bigvee_{(\alpha',\beta',\gamma')\in\hat{\mathcal{L}}_{\beta\vee\gamma}}\rho(\beta')\vee\sigma(\gamma').$$

We then have $\bar{\ell}_{\beta\vee\gamma}:Y_\beta\amalg Z_\gamma\longrightarrow V_{\tau(\beta\vee\gamma)}$ induced by the appropriate bonding maps, and if $\phi(\alpha)\preceq\beta$ and $\psi(\alpha)\preceq\gamma$, the diagram

$$\begin{tikzcd}
X_\alpha \arrow[r, "f_\alpha"] \arrow[d, "g_\alpha"'] & Y_{\phi(\alpha)} \arrow[r, "i_{\phi(\alpha),\beta}"] \arrow[d] & Y_\beta \arrow[dd, "i\circ h_\beta"] \\
Z_{\psi(\alpha)} \arrow[r] \arrow[d, "i_{\psi(\alpha),\gamma}"'] & W^{\alpha}_{\beta\vee\gamma} \arrow[dr, dotted, "\ell^{\alpha}_{\beta\vee\gamma}"] & \\
Z_\gamma \arrow[rr, "i\circ k_\gamma"] & & V_{\tau(\beta\vee\gamma)}
\end{tikzcd}$$

commutes, so $\bar{\ell}_{\beta\vee\gamma}$ induces a map $\ell^{\alpha}_{\beta\vee\gamma}:W^{\alpha}_{\beta\vee\gamma}\longrightarrow V_{\tau(\beta\vee\gamma)}$, and thus $\ell_{\beta\vee\gamma}:W_{\beta\vee\gamma}\to V_{\tau(\beta\vee\gamma)}$. One may also verify that $\langle\tau,(\ell_{\beta\vee\gamma})_{\beta\vee\gamma\in B\amalg C}\rangle$ has the appropriate universal property. $\square$

Proposition 4.7. *If $\mathcal{F} \subset \mathcal{C}$ is good (so in particular has all finite limits), then $\mathcal{N}et_{\mathcal{F}}$ has all finite limits, too.*

Proof. It suffices to show that $\mathcal{N}et_{\mathcal{F}}$ has pullbacks (it clearly has a terminal object – namely, the zero net indexed by the zero lattice). Thus, given two net maps:

$$\begin{tikzcd} & (Y_\beta)_{\beta\in B} \arrow[d, "\langle\phi,(f_\beta)_{\beta\in B}\rangle"] \\ (Z_\gamma)_{\gamma\in C} \arrow[r, "\langle\psi,(g_\gamma)_{\gamma\in C}\rangle"] & (X_\alpha)_{\alpha\in A} \end{tikzcd}$$

where we assume the indicated representatives are minimal, for each $(\beta,\gamma) \in B \times C$, set $W_(\beta,\gamma)$ to be the pullback of

$$Z_\gamma \xrightarrow{g_\gamma} X_{\psi(\gamma)\vee\phi(\beta)} \xleftarrow{f_\beta} Y_\beta.$$

It is readily verified that this defines a pullback net with the required universal property. $\square$

Proposition 4.8. *If $\mathcal{C}$ is generated by a good subcategory $\mathcal{F} \subset \mathcal{C}$, then there is an embedding of categories $I : \mathcal{C} \to \mathcal{N}et_{\mathcal{F}}$, defined $I(C) = \mathcal{F}_C = (F)_{F\in\mathcal{F}_C}$.*

(Note that $\mathcal{F}_C$ is both the lattice indexing the net $I(C) \in \mathcal{N}et_{\mathcal{F}}$, and the net itself)

Proof. For any $f : C \to D$ in $\mathcal{C}$ and $F \in \mathcal{F}_C$ a subobject of C which is in $\mathcal{F}$, the image $Im(f|_F)$ is in $\mathcal{F}_D$ by Def. 3.1(e). Thus we may define a proper net map $\langle\phi_f,(f_F)_{F\in\mathcal{F}_C}\rangle : \mathcal{F}_C \to \mathcal{F}_D$ by $\phi_f(F) = Im(f|_F) \in \mathcal{F}_D$ and $f_F = f|_F : F \to Im(f|_F) \subset D$, for any $F \in \mathcal{F}_C$. This defines I on morphisms. Definition 3.1(e) also implies that $I : Hom_{\mathcal{C}}(C,D) \to Hom_{\mathcal{N}et_{\mathcal{F}}}(\mathcal{F}_C,\mathcal{F}_D)$ is monic, since if $f,g : C \to D$ satisfy $f|_F = g|_F$ for all $F \in \mathcal{F}_C$, then $f = g$. $\square$

We may summarize our results for the Pro category of towers of spaces in the following

Theorem 4.9. *The functor $I : \mathcal{T}ow \to \mathcal{N}et_{\mathcal{F}}$, defined by $I(\mathfrak{X}) = \mathcal{F}_{\mathfrak{X}}$ restricts to an embedding of $\mathcal{T}ow^{st}$ in the cocomplete and finite complete category of nets over $\mathcal{F}^{\mathcal{T}ow}$. I preserves all finite limits, and the functor $I|_{\mathcal{T}ow^{st}}$ preserves all colimits.*

Proof. If $\mathfrak{W}$ is the pullback in $\mathcal{T}ow$ of

$$\mathfrak{Z} \xrightarrow{\hat{\mathfrak{f}}} \mathfrak{X} \xleftarrow{\hat{\mathfrak{g}}} \mathfrak{Z}\ , \tag{4.10}$$

(which may not be in $\mathcal{T}ow^{st}$, even if (4.10) is), then (as in the proof of Proposition 2.7) $\mathfrak{W}$ may be constructed as the levelwise pullback of any $\mathcal{S}_\star^{\mathbb{N}}$-representatives of (4.10), so $\boldsymbol{W}[n]$ is a subobject of $\boldsymbol{Y}[n] \times \boldsymbol{Z}[n]$ (by the usual construction in $\mathcal{S}_\star$). Thus any finite subtower of $\mathfrak{W}$ is just a finite subobject of $\mathfrak{Y} \times \mathfrak{Z}$, satisfying the appropriate (levelwise) compatibility condition – so that $\mathcal{F}_{\mathfrak{W}}$ is isomorphic to the pullback net for

$$\mathcal{F}_{\mathfrak{Z}} \xrightarrow{\mathcal{F}_{\hat{\mathfrak{f}}}} \mathcal{F}_{\mathfrak{X}} \xleftarrow{\mathcal{F}_{\hat{\mathfrak{g}}}} \mathcal{F}_{\mathfrak{Z}}$$

constructed in the proof of Proposition 4.7.

Similarly, if $\mathfrak{W}$ is the pushout in $\mathcal{T}ow^{st}$ of

$$\mathfrak{Z} \xleftarrow{\hat{\mathfrak{f}}} \mathfrak{X} \xrightarrow{\hat{\mathfrak{g}}} \mathfrak{Z}\ , \tag{4.11}$$

then $\mathfrak{W}$ may be may be constructed as the levelwise pushout of any representatives of (4.11) and $\boldsymbol{W}[n]$ is thus a quotient of $\boldsymbol{Y}[n] \amalg \boldsymbol{Z}[n]$. Note that the structure maps of the pushout induce an epimorphism $\hat{\mathfrak{h}} : \mathfrak{Y} \amalg \mathfrak{Z} \twoheadrightarrow \mathfrak{W}$.

Now if $\mathfrak{U}$ is a finite subobject of $\mathfrak{W}$, then it is in fact a quotient of some finite subobject $\mathfrak{V}' \amalg \mathfrak{V}'' \hookrightarrow \mathfrak{Y} \amalg \mathfrak{Z}$, with $\mathfrak{V}' \in \mathcal{F}_{\mathfrak{Y}}$ and $\mathfrak{V}'' \in \mathcal{F}_{\mathfrak{Z}}$, as in the proof of Proposition 3.8. But since $\boldsymbol{W}[n] \cong (\boldsymbol{Y}[n] \amalg \boldsymbol{Z}[n])/\sim$, where the equivalence relation $\sim$ is generated by $f[n](x) \sim g[n](x)$, we see that any finite subspace $\boldsymbol{U}[n] \subseteq \boldsymbol{W}[n]$ (and thus $\mathfrak{U} \subseteq \mathfrak{W}$) is obtained form a finite subspace $\boldsymbol{V}' \amalg \boldsymbol{V}'' \subseteq \boldsymbol{Y}[n] \amalg \boldsymbol{Z}[n]$ by a finite colimit as in the proof of Proposition 4.5. This shows that $\mathcal{F}_{\mathfrak{W}}$ is isomorphic to the pushout net for

$$\mathcal{F}_{\mathfrak{Z}} \xleftarrow{\mathcal{F}_{\hat{\mathfrak{f}}}} \mathcal{F}_{\mathfrak{X}} \xrightarrow{\mathcal{F}_{\hat{\mathfrak{g}}}} \mathcal{F}_{\mathfrak{Z}}. \qquad \square$$

Remark 4.12. The fact that $\mathcal{N}et_{\mathcal{F}}$ serves as a cocompletion of $\mathcal{C}$, when $\mathcal{F}$ is a good subcategory generating $\mathcal{C}$, follows directly from more general results:

By [J, VI, Thms. 1.6 & 1.8] we know that $Ind\text{-}\mathcal{C}$ is the cocompletion of $\mathcal{C}$ (assuming $\mathcal{C}$ itself is finite-cocomplete), and it is easy to see that $\mathcal{C}$ embeds in $Ind\text{-}\mathcal{F}$ (as in the proof of Proposition 4.8), so that $Ind\text{-}\mathcal{C}$ embeds cocontinuously in $Ind\text{-}(Ind\text{-}\mathcal{F})$, which is equivalent to $Ind\text{-}\mathcal{F}$ (see [GV, Cor. 8.9.8]). Because $\mathcal{F}$ is co-artinian (Def. 3.1(c)), $Ind\text{-}\mathcal{F}$ is equivalent to $Ind^{st}\text{-}\mathcal{F}$ (see [GV, §8.12.6]), which is equivalent in turn to $\mathcal{N}et_{\mathcal{F}}$ by Fact 4.4 and Def. 3.1(b).

However, we believe that the explicit description fo the colimits in $\mathcal{N}et_{\mathcal{F}}$ given above may be more useful than that obtained form unwinding the above chain of equivalences.

The results relating specifically to towers of simplicial sets – Propositions 3.4 & 3.8 – may also be extended to other Pro categories of towers over categories $\mathcal{C}$ generated by a good subcategory $\mathcal{F}$, such as towers of sets (cf. §3.7).

Remark 4.13. The example in 2.10(b) shows that the functor $I : \mathcal{T}ow \to \mathcal{N}et_{\mathcal{F}}$ of the Theorem fails to be an embedding, since the tower $\mathfrak{Y}$ defined there has no non-trivial finite subobjects, and thus $I(\mathfrak{Y}) = \mathcal{F}_{\mathfrak{Y}} = *$, even though there are non-trivial maps $\mathfrak{Y} \to \mathfrak{Z}$ in $\mathcal{T}ow$.

This is not a serious flaw, since one often chooses to work with the “good” towers of $\mathcal{T}ow^{st}$ in applications. In fact, there is a certain advantage to this fact, from our point of view, since it yields a version of the Pro category of towers from which we have eliminated the phantom phenomena (as in the case of $\mathcal{T}ow^{st}$), but still have finite limits (and have actually added infinite colimits).

Question 4.14. Although only colimits were needed for our application in [BT], one can obviously ask the same question regarding the *completion* of $\mathcal{T}ow$ – that is, embedding the Pro category of towers in one where arbitrary limits (ideally: both limits and colimits) may be constructed. While the categorical part of our construction could presumably be dualized, it is not clear that the category $\mathcal{T}ow$, or any other version of the Pro category of towers, will indeed satisfy the required assumptions, since specific properties of $\mathcal{T}ow$ and $\mathcal{S}_\star$ were used in the proof of Proposition 3.8 and Theorem 4.9.

Note however that for any small finite-complete category $\mathcal{C}$, the category $Ind\text{-}\mathcal{C}$ has all limits (as well as all colimits), and the inclusion $\mathcal{C} \hookrightarrow Ind\text{-}\mathcal{C}$ preserves all limits which exist in $\mathcal{C}$, by [J, VI, Prop. 1.7].

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