COMPARING HOMOTOPY CATEGORIES

DAVID BLANC

Abstract. Given a suitable functor $T : \mathcal{C} \to \mathcal{D}$ between model categories, we define a long exact sequence relating the homotopy groups of any $X \in \mathcal{C}$ with those of TX, and use this to describe an obstruction theory for lifting an object $G \in \mathcal{D}$ to $\mathcal{C}$. Examples include finding spaces with given homology or homotopy groups.

0. Introduction

A number of fundamental problems in algebraic topology can be described as measuring the extent to which a given functor $T : \mathcal{C} \to \mathcal{D}$ between model categories induces an equivalence of homotopy categories: more specifically, which objects (or maps) from $\mathcal{D}$ are in the image of T, and in how many different ways. For example:

a) How does one distinguish between different topological spaces with the same homology groups, or with chain-homotopy equivalent chain complexes? How can one realize a given map of chain complexes up to homotopy?

b) When do two simply-connected topological spaces have the same rational homotopy type?

c) When is a given topological space a suspension, up to homotopy? Dually, how many distinct loop space structures, if any, can a given topological space carry?

d) Is a given Π-algebra (that is, a graded group with an action of the primary homotopy operations) realizable as the homotopy groups of a topological space, and if so, in how many ways?

Our goal is to describe a unified approach to such problems that works for functors between *spherical* model categories, for which several familiar concepts and constructions are available. These include a set $\mathcal{A}$ of *models* (to play the role of spheres, in particular determining the corresponding homotopy groups $\pi^{\mathcal{C}}_{*}$), Postnikov systems, and k-invariants. If a functor $T : \mathcal{C} \to \mathcal{D}$ respects this additional structure, we obtain a natural long exact sequence of the form:

$$\ldots \to \Gamma_n X \xrightarrow{s} \pi^{\mathcal{C}}_n X \xrightarrow{h} \pi^{\mathcal{D}}_n TX \xrightarrow{\partial} \Gamma_{n-1} X \ldots , \tag{0.1}$$

which generalizes the EHP sequence, J.H.C. Whitehead's "certain exact sequence", and the spiral exact sequence of Dwyer, Kan, and Stover. See (4.4) below.

Under these hypotheses, given an object G in $\mathcal{D}$, we want to find an object X in $\mathcal{C}$ with $TX \simeq G$. The key step is to choose $\pi^{\mathcal{C}}_{*} X$ which fits into (0.1). We describe

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an inductive procedure for doing this, using the Postnikov systems in both categories, together with an obstruction theory for lifting G to $\mathcal{C}$, along the following lines:

Theorem. *Given* $T:\mathcal{C}\to\mathcal{D}$ *and* $G\in\mathcal{D}$ *as above, for each* $X\in\mathcal{C}$ *with* $TX\simeq G$, *there is a tower of fibrations in* $\mathcal{C}$*:*

$$\dots \xrightarrow{p^{(n+1)}} \hat{X}\langle n+1\rangle \xrightarrow{p^{(n)}} \hat{X}\langle n\rangle \xrightarrow{p^{(n-1)}} \dots \xrightarrow{p^{(0)}} \hat{X}\langle 0\rangle \ ,$$

called the modified Postikov tower for X *(Def. 5.21), with G mapping compatibly to* $T\hat{X}\langle n\rangle$ *for each n, and* $X\simeq \operatorname{holim}_n \hat{X}\langle n\rangle$.

Conversely, given such a tower up to level n, the obstruction to extending it to level $n+1$ *lies in* $H^{n+3}_{\Lambda}(G;\,\Gamma_{n+1}\hat{X}\langle n\rangle)$, *and the choices for* $\hat{X}\langle n+1\rangle$ *are classified by:*

- *a class in* $H^{n+2}_{\Lambda}(G;\,\Gamma_{n+1}\hat{X}\langle n\rangle)$*;*
- *a class in* $H^{n+2}_{\Lambda}(\hat{X}\langle n\rangle;\,K_{n+1})$, *where* $K_{n+1} := \operatorname{Coker}\,\pi_{n+2}\rho^{(n)}$, *for* $\rho^{(n)}: P_{n+2}G\to P_{n+2}T\hat{X}\langle n\rangle$.

See Theorem 6.8.

0.2. **Related work.** The comparison problems discussed above are familiar ones in algebraic topology:

a) The question of the realizability of a graded algebra as a cohomology ring was first raised explicitly by Steenrod in [Ste], but it goes back to Hopf (in [Ho]) in the rational case. The "Steenrod problem" of realizing a given π_1-action in homology has been studied, for example, in [T, Sm].

b) The comparison between integral and rational homotopy type was implicit in the notion of a Serre class (cf. [Se, AC]), although an explicit formulation was only possible after the construction of the rationalization functors of Quillen and Sullivan in [Q2, Sul].

c) Possible loop space structures on a given H-space were analyzed extensively, starting with the work of Sugawara and Stasheff (cf. [Sug, Sta]). The dual question on identifying suspensions has also been studied (see, e.g., [BH]).

d) The question of the realizability of homotopy groups goes back to J.H.C. Whitehead, in [W2] (see also [W5]), and has reappeared in recent years in the context of Π-algebras (cf. [DKS1, DKS2]). The relationship between homology and homotopy groups, which is relevant to the realization problem for both, was studied in [W3, W4] (in which the "certain exact sequence" was introduced).

In [Ba4], H.-J. Baues gave what appears to be the first general theory covering a wide spectrum of such realization problems. This was an outgrowth of his earlier work on classifying homotopy types of finite dimensional CW complexes in [Ba2, Ba3] (which in turn builds on [W1]).

His initial setting consists of a homological cofibration category $\mathcal{C}$ (corresponding to, and extending, the notion of a resolution model category) under a theory of coactions $\mathbb{T}$ (corresponding to the category $\Pi_{\mathcal{A}}$ of §1.2). Baues then constructs a generalized "certain exact sequence" similar to (0.1), and provides an inductive obstruction theory for realizing a chain complex (or a chain map) by a $\mathbb{T}$-complex (corresponding to a CW complex, or more generally a cofibrant object in $\mathcal{C}$) – see [Ba4, VI, (2.2-2.3)]).

These results apply inter alia to the problem of realizing a chain complex by a topological space (the motivating example for Baues's approach), as well as to the realization of a Π-algebra (cf. [Ba4, D, (7.9)]). However, here we consider functors between two different model categories that are not covered by [Ba4]. In particular, our original motivating example – the realization of a *simplicial* Π-algebra (by a simplicial space) – shows that in the relative context a more refined obstruction theory may be necessary: compare Theorem (2.3) of [Ba4, VI] with Theorem 6.8 below.

0.3. *Remark.* Another set of closely related questions – which do not quite fit into the framework described here, though they can also be stated as realization problems – arise in categories of structured ring spectra; see for example [R] and [GH, Cor. 5.9].

0.4. **Notation and conventions.** $\mathcal{T}_*$ denotes the category of pointed connected topological spaces; $\mathcal{S}et_*$ that of pointed sets, and $\mathcal{G}p$ that of groups. For any category $\mathcal{C}$, $\operatorname{gr}\mathcal{C}$ denotes the category of non-negatively graded objects over $\mathcal{C}$, and $s\mathcal{C}$ the category of simplicial objects over $\mathcal{C}$. $s\mathcal{S}et$ is denoted by $\mathcal{S}$, $s\mathcal{S}et_*$ by $\mathcal{S}_*$, and $s\mathcal{G}p$ by $\mathcal{G}$. The constant simplicial object an an object $X \in \mathcal{C}$ is written $c(X) \in s\mathcal{C}$.

If $\mathcal{C}$ has all coproducts, then given $A \in \mathcal{S}$ and $X \in \mathcal{C}$, we define $X\hat{\otimes}A \in s\mathcal{C}$ by $(X\hat{\otimes}A)_n := \coprod_{a\in A_n} X$, with face and degeneracy maps induced from those of A. For $Y \in s\mathcal{C}$, define $Y \otimes A \in s\mathcal{C}$ by $(Y \otimes A)_n := \coprod_{a\in A_n} Y_n$ (the diagonal of the bisimplicial object $Y\hat{\otimes}A$) – so that for $X \in \mathcal{C}$ we have $X\hat{\otimes}A = c(X) \otimes A$.

The category of chain complexes of R-modules is denoted by $\mathcal{C}hain_R$ (or simply $\mathcal{C}hain$, for $R = \mathbb{Z}$).

0.5. **Organization:** In Section 1 we define *spherical* model categories, having the additional structure mentioned above. Most examples of such categories are in particular *resolution* model categories, which are described in Section 2; we explain how to produce the needed structure for them in Section 3. We define *spherical functors* between such categories, and construct the comparison exact sequence for them, in Section 4. This is applied in Section 5 to study the effect of a spherical functor on Postnikov systems. Finally, in Section 6 we construct an obstruction theory as above for the fiber of a spherical functor. In Section 7 we indicate how the theory works for the above examples.

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1. Spherical model categories

Before defining the additional structure we shall need, we briefly recapitulate the relevant homotopical algebra:

1.1. **Model categories.** Recall that a *model category* is a bicomplete category $\mathcal{C}$ equipped with three classes of maps: weak equivalences, fibrations, and cofibrations, related by appropriate lifting properties. By inverting the weak equivalences we obtain the associated homotopy category $\operatorname{ho}\mathcal{C}$, with morphism set $[X, Y] = [X, Y]_{\mathcal{C}}$. We shall concentrate on *pointed* model categories (with null object $*$). See [Q1] or [Hi].

1.2. **The set of models.** The additional initial data that we shall require for our model category consists of a set $\mathcal{A}$ of cofibrant homotopy cogroup objects in $\mathcal{C}$, called *models* (playing the role of the spheres in $\mathcal{T}_*$). Given such a set $\mathcal{A}$, let $\Pi_{\mathcal{A}}$ denote the smallest subcategory of $\mathcal{C}$ containing $\mathcal{A}$ and closed under weak equivalences, arbitrary coproducts, and suspensions. Note that every object in $\Pi_{\mathcal{A}}$ is a homotopy cogroup object, too.

1.3. **Example.** Let $\mathcal{C} = \mathcal{G}$ be the category of simplicial groups, $S^k = \Delta[k]/\partial\Delta[k]$ the standard simplicial k-sphere in $\mathcal{S}_*$, $G : \mathcal{S}_* \to \mathcal{G}$ the Kan's loop functor (cf. [May, §26.3]), and $F : \mathcal{S}_* \to \mathcal{G}$ the free group functor. For each $n \geq 1$, $\mathbb{S}^n := GS^n \in \mathcal{G} \cong FS^{n-1}$ will be called the *n-dimensional $\mathcal{G}$-sphere*, with $\Sigma^k \mathbb{S}^n \simeq \mathbb{S}^{n+k}$. These, and their coproducts, are cofibrant strict cogroup objects for $\mathcal{G}$. Here $\mathcal{A} := \{\mathbb{S}^1 = c(\mathbb{Z})\}$; in fact, throughout this paper $\mathcal{A}$ will be either a singleton, or countable.

1.4. *Remark.* The adjoint pairs of functors:

$$\mathcal{T}_* \underset{\|-\|}{\overset{S}{\rightleftarrows}} \mathcal{S}_* \underset{\bar{W}}{\overset{G}{\rightleftarrows}} \mathcal{G}$$

induce equivalences of the corresponding homotopy categories – where $\bar{W} : \mathcal{G} \to \mathcal{S}_*$ is the Eilenberg-Mac Lane classifying space functor, $S : \mathcal{T}_* \to \mathcal{S}_*$ is the singular set functor, and $\| - \| : \mathcal{S}_* \to \mathcal{T}_*$ is the geometric realization functor (cf. [May, §14,23]). Thus to study the usual homotopy category of (pointed connected) topological spaces, we can work in $\mathcal{G}$ (or $\mathcal{S}_*$), rather than $\mathcal{T}_*$.

1.5. **Definition.** If $\mathcal{A}$ is a set of models for $\mathcal{C}$, then given $X \in \mathcal{C}$, for each $A \in \mathcal{A}$ let $\pi^{\mathcal{C}}_{A,k}(X) := [\Sigma^k A, X']_{\mathcal{C}}$, where $X' \to X$ is a (functorial) fibrant replacement. We write $\pi^{\mathcal{C}}_k X$ for $(\pi^{\mathcal{C}}_{A,k} X)_{A \in \mathcal{A}}$, and $\pi^{\mathcal{C}}_* X := (\pi^{\mathcal{C}}_k X)^{\infty}_{k=0}$.

1.6. **Theories and algebras.** Recall that a *theory* is a small category Θ with finite products (so in particular, an FP-sketch – cf. [Bor, §5.6]), and a Θ-*algebra* (or *model*) is a product-preserving functor $\Theta \to \mathcal{S}et$. Think of Θ as encoding the operations and relations for a "variety of universal algebras", the category Θ-$\mathcal{A}lg$ of Θ-algebras (which is *sketched* by Θ).

For example, the obvious category $\mathfrak{G}$, which sketches groups, is equivalent to the opposite of the homotopy category of (finite) wedges of circles. An $\mathfrak{G}$-*theory* Θ (cf. [BP, §2]) is one equipped with a map of theories $\coprod_S \mathfrak{G} \to \Theta$ (coproduct taken in the category of theories, over some index set S) which is bijective on objects. This implies that each Θ-algebra has the underlying structure of an S-graded group, so that Θ-$\mathcal{A}lg$ can be thought of as a "variety of (graded) groups with operators" (cf. [Ba4, I, (2.5)]).

1.7. *Remark.* We will assume that all the functors $\pi^{\mathcal{C}}_n$ $(n \geq 0)$ take value in a category $\Pi_{\mathcal{C}}$-$\mathcal{A}lg$ sketched by a $\mathfrak{G}$-theory Θ, and thus equipped with a faithful forgetful functor $U_{\mathcal{C}} : \Pi_{\mathcal{C}}\text{-}\mathcal{A}lg \to \mathcal{G}p^{\mathcal{A}}$ into the category of $\mathcal{A}$-graded groups. The objects of $\Pi_{\mathcal{C}}$-$\mathcal{A}lg$ are called $\Pi_{\mathcal{C}}$-*algebras*.

For topological spaces, with $\mathcal{A} = \{\mathbf{S}^1\}$, the $\Pi_{\mathcal{C}}$-algebras are simply groups. If we use rational spheres as the models, then $\Pi_{\mathcal{C}}$-$\mathcal{A}lg$ is the category of $\mathbb{Q}$-vector spaces. A more interesting example appears in §2.9 below.

1.8. **Constructions based on models.** There are a number of familiar constructions for topological spaces which we require for our purposes. We can *define* them once we are given a set of models $\mathcal{A}$ as above, although they do not always exist (see §3.10 below).

1.9. **Definition.** A *Postnikov tower* (with respect to $\mathcal{A}$) is a functor that assigns to each $Y \in \mathcal{C}$ a tower of fibrations:

$$\ldots \to P^{\mathcal{A}}_n Y \xrightarrow{p^{(n)}} P^{\mathcal{A}}_{n-1} Y \xrightarrow{p^{(n-1)}} \cdots \to P^{\mathcal{A}}_0 Y \ ,$$

as well as a weak equivalence $r : Y \to P^{\mathcal{A}}_\infty Y := \lim_n P^{\mathcal{A}}_n Y$ and fibrations $P^{\mathcal{A}}_\infty Y \xrightarrow{r^{(n)}} P^{\mathcal{A}}_n Y$ such that $r^{(n-1)} = p^{(n)} \circ r^{(n)}$ for all n. Finally, $(r^{(n)} \circ r)_\# : \pi^{\mathcal{C}}_k Y \to \pi^{\mathcal{C}}_k (P^{\mathcal{A}}_n Y)$ is an isomorphism for $k \leq n$, and $\pi^{\mathcal{C}}_k (P^{\mathcal{A}}_n Y)$ is zero for $k > n$.

When $\mathcal{A}$ is clear from the context, we denote $P^{\mathcal{A}}_n$ simply by P_n.

1.10. **Example.** For a free chain complex $C_* \in \mathcal{C}hain_R$ of modules over a ring R, we may take $C'_* := P_n C_*$ where $C'_i = C_i$ for $i \leq n+1$, $C'_{n+2} = Z_{n+1} C_*)$, and $C'_i = 0$ for $i \geq n+3$. The map $r^{(n)} : C_* \to C'_*$ is defined by $r^{(n)}_{n+2} := \partial_{n+2} : C_{n+2} \to Z_{n+1} C_*$.

1.11. **Definition.** Given an $\Pi_{\mathcal{C}}$-algebra Λ, a *classifying object* $B_{\mathcal{C}}\Lambda$ (or simply $B\Lambda$) for Λ is any $B \in s\mathcal{C}$ such that $B \simeq P_0 K$ and $\pi^{\mathcal{C}}_0 B \cong \Lambda$.

The name is used by analogy with the classifying space of a group, which classifies G-bundles. One can interpret $B_{\mathcal{C}}\Lambda$ similarly, though perhaps less naturally (see, e.g., [BJT, §4.6]).

1.12. **Definition.** A *module over a* $\Pi_{\mathcal{C}}$*-algebra* Λ is an abelian group object in $\Pi_{\mathcal{C}}$-$\mathcal{A}lg/\Lambda$ (cf. [Q3, §2]), and the category of such is denoted by Λ-$\mathcal{M}od$.

1.13. *Remark.* Since any $\Pi_{\mathcal{C}}$-algebra is in particular a (graded) group, if $p : Y \to \Lambda$ is a module, then $Y = K \times \Lambda$ (as sets!) for $K := \mathrm{Ker}\,(p)$, with an appropriate $\Pi_{\mathcal{C}}$-algebra structure (cf. [Bl3, §3]). We may call K itself a Λ-*module* (which corresponds to the traditional description of an R-module, for a ring R).

1.14. **Example.** For any object $X \in \mathcal{C}$ as above, the $\mathcal{A} \times \mathbb{N}$-graded group $\pi^{\mathcal{C}}_* X$ has an action of the $\mathcal{A}$-primary homotopy operations, corepresented by the maps in $\mathrm{ho}\,\Pi_{\mathcal{A}}$ (see §2.9 below). In particular, one of these operations, corresponding to the action of the fundamental group on the higher homotopy groups, makes each $\pi^{\mathcal{C}}_n X$ $(n \geq 1)$ into a module over $\pi^{\mathcal{C}}_0 X$ (see Fact 3.6 below).

1.15. **Definition.** Given an abelian $\Pi_{\mathcal{C}}$-algebra M and an integer $n \geq 1$, an *n-dimensional M-Eilenberg-Mac Lane object* $E_{\mathcal{C}}(M,n)$ (or simply $E(M,n)$) is any $E \in s\mathcal{C}$ such that $\pi^{\mathcal{C}}_n E \cong M$ and $\pi^{\mathcal{C}}_k E = 0$ for $k \neq n$.

1.16. **Definition.** Given a $\Pi_{\mathcal{C}}$-algebra Λ, a module M over Λ, and an integer $n \geq 1$, an *n-dimensional extended M-Eilenberg-Mac Lane object* $E^{\Lambda}_{\mathcal{C}}(M,n)$ (or simply $E^{\Lambda}(M,n)$) is any homotopy abelian group object $E \in s\mathcal{C}/\Lambda$, equipped with a section s for $p^{(0)} : E \to P_0 E \simeq B\Lambda$, such that $\pi^{\mathcal{C}}_n E \cong M$ as modules over Λ; and $\pi^{\mathcal{C}}_k E = 0$ for $k \neq 0, n$.

1.17. **Definition.** Given a Postnikov tower functor as in §1.9, an n-th *k-invariant square* (with respect to $\mathcal{A}$) is a functor that assigns to each $Y \in \mathcal{C}$ a homotopy pull-back square:

$$\begin{tikzcd}
P^{\mathcal{A}}_{n+1} Y \arrow[r, "p^{(n+1)}"] \arrow[d] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & P^{\mathcal{A}}_n Y \arrow[d, "k_n"] \\
B\Lambda \arrow[r] & E^{\Lambda}(M, n+2)
\end{tikzcd} \tag{1.18}$$

for $\Lambda := \pi_0^{\mathcal{C}} Y$ and $M := \pi_{n+1}^{\mathcal{C}} Y$, where $p^{(n+1)} : P_{n+1}Y \to P_nY$ is the given fibration of the Postnikov tower.

The map $k_n : P_nY \to E^{\Lambda}(M, n+2)$ is the n-th (functorial) *k-invariant* for Y.

1.19**. Example.** If C_* is a chain complex of R-modules, and $P_nC_* = C'_*$ as in §1.10, we may take $E(H_{n+1}C_*, n+2) = E_*$, where $E_i = 0$ for $i < n+2$, $E_{n+2} = Z_{n+1}C_*$, and $E_{n+3} = B_{n+1}C_*$. Then $k_n : C'_* \to E_*$ is defined by $\mathrm{Id} : C'_{n+1} \to E_{n+1}$.

Of course, if R is a principle ideal domain (or a hereditary ring), such as $\mathbb{Z}$, then the k-invariants for C_* are trivial, since in that case any two free (or projective) chain complexes with the same homology are homotopy equivalent, by [D, Prop. 3.5]. But this need not hold for an arbitrary ring R.

1.20. **Spherical models.** A set of objects $\mathcal{A} := \{A\}_{A \in \mathcal{A}}$ in a model category $\mathcal{C}$ is called a collection of *spherical models* if the following axioms hold:

Ax 1. Each $\Sigma^n A$ $(A \in \mathcal{A},\ n \in \mathbb{N})$ is a cofibrant homotopy cogroup object in $\mathcal{C}$.

Ax 2. For any $X \in \mathcal{C}$ and $n \geq 1$, $\pi_n^{\mathcal{C}} X$ has a natural structure of a module over $\pi_0^{\mathcal{C}} X$.

Ax 3. A map $f : X \to Y$ is a weak equivalence if and only if $\pi_{A,n}^{\mathcal{C}} f$ is a weak equivalence for each $A \in \mathcal{A}$ and $n \in \mathbb{N}$.

Ax 4. $\mathcal{C}$ has Postnikov towers with respect to $\mathcal{A}$.

Ax 5. For every $\Pi_{\mathcal{C}}$-algebra Λ and module M over Λ, the classifying object $B\Lambda$ and extended M-Eilenberg-Mac Lane object $E^{\Lambda}(M, n)$ exist (and are unique up to homotopy) for each $n \geq 1$.

Ax 6. $\mathcal{C}$ has k-invariant squares with respect to $\mathcal{A}$ for each $n \geq 0$.

If each model $\Sigma^k A$ $(A \in \mathcal{A},\ k \in \mathbb{N})$ is a cofibrant *strict* cogroup object – which implies that every object in $\Pi_{\mathcal{A}}$ is such, up to weak equivalence – we call $\mathcal{A}$ a collection of *strict* spherical models.

A pointed simplicial model category $\mathcal{C}$ equipped with a collection $\mathcal{A} := \{A\}_{A \in \mathcal{A}}$ of spherical models is called a *spherical model category*, and we denote it by $\langle \mathcal{C}; \mathcal{A} \rangle$. Such a category is *stratified* in the sense of Spaliński (cf. [Sp]).

1.21**. Example.** The category $\mathcal{S}_*$ of pointed simplicial sets, as well as the category $\mathcal{T}_*$ of pointed connected topological spaces, have spherical model category structures with $\mathcal{A} = \{\mathbf{S}^1\}$. (Functorial k-invariants in these categories are provided by the construction of [BDG, §5]; in both cases $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg \approx \mathcal{G}p$). Similarly for the category $\mathcal{C}hain_R$ of chain complexes over R, with the constructions indicated in §1.10 and §1.19.

In the examples we have in mind, our model categories enjoy additional useful properties, which we can summarize in the following:

1.22**. Definition.** A spherical model category $\langle \mathcal{C}; \mathcal{A} \rangle$ as above is called *strict* if the following axioms hold:

Ax 1. $\mathcal{C}$ is a pointed right-proper cofibrantly generated simplicial model category (cf. [Hi, 11.1, 13.1]), in which every object is fibrant.

Ax 2. $\mathcal{C}$ is equipped with a faithful forgetful functor $\hat{U} : \mathcal{C} \to \mathcal{D}$, with left adjoint $\hat{F}$ – where $\mathcal{D}$ is one of the "categories of groups" $\mathcal{D} = \mathcal{G}p$, $gr\mathcal{G}p$, $\mathcal{G}$, $R\text{-}\mathcal{M}od$, or $sR\text{-}\mathcal{M}od$, for some ring R.

Ax 3. The adjoint pair $(\hat{U}, \hat{F})$ *create* the model category structure on $\mathcal{C}$ in the sense of [Bl1, §4.13] – so in particular $\hat{U}$ creates all limits in $\mathcal{C}$.

Ax 4. $\mathcal{A}$ is a collection of strict spherical models, each of which lies in the image of the composite $\hat{F} \circ F' : \mathcal{S} \to \mathcal{C}$, where $F' : \mathcal{S} \to \mathcal{D}$ is adjoint to the forgetful functor $U' : \mathcal{D} \to \mathcal{S}$, with the group structure on $\mathrm{Hom}_{\mathcal{C}}(A, X)$ induced from that of $\hat{U}(X)$.

2. Resolution model categories

Many examples of spherical model categories fit into the framework originally conceived by Dwyer, Kan and Stover in [DKS2] under the name of "E^2 model categories," and later generalized by Bousfield (see [Bou, J]. A slightly different generalization is given by Baues in [Ba4, Ch. D, §2] under the name of *spiral* model categories.

First, some preliminary concepts:

2.1. **Definition.** The n-th *matching object* for a simplicial object X over $\mathcal{C}$ is defined by

$$M_n X = \{(x_0, \dots, x_n) \in (X_{n-1})^{n+1} \mid d_i x_j = d_{j-1} x_i \ \text{ for all } 0 \leq i < j \leq n\}$$

(see [BK, X,§4.5]). Note that each face map $d_k : X_n \to X_{n-1}$ factors through the obvious map $\delta_n : X_n \to M_n X$.

2.2. **Definition.** The n-th *latching object* of a simplicial object X over $\mathcal{C}$ is defined $L_n X := \coprod_{0 \leq i \leq n-1} X_{n-1}/\sim$, where for any $x \in X_{n-k-1}$ and $0 \leq i \leq j \leq n-1$ we set $s_{j_1} s_{j_2} \dots s_{j_k} x$ in the i-th copy of X_{n-1} equivalent to $s_{i_1} s_{i_2} \dots s_{i_k} x$ in the j-th copy of X_{n-1} whenever the simplicial identity $s_i s_{j_1} s_{j_2} \dots s_{j_k} = s_j s_{i_1} s_{i_2} \dots s_{i_k}$ holds. The map $\sigma_n : L_n X \to X_n$ is defined $\sigma_n(x)_i = s_i x$, where $(x)_i \in (X_{n-1})_i$.

There are two canonical ways to extend a given model category structure on $\hat{\mathcal{C}}$ to $\mathcal{C} := s\hat{\mathcal{C}}$:

2.3. **The Reedy model structure.** This is defined by letting a simplicial map $f : X \to Y$ in $\mathcal{C} := s\hat{\mathcal{C}}$ be:

(i) a weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence in $\hat{\mathcal{C}}$ for each $n \geq 0$;

(ii) a (trivial) cofibration if $f_n \amalg \sigma_n : X_n \amalg_{L_n X} L_n Y \to Y_n$ is a (trivial) cofibration in $\hat{\mathcal{C}}$ for each $n \geq 0$;

(iii) a (trivial) fibration if $f_n \times \delta_n : X_n \to Y_n \times_{M_n Y} M_n X$ is a (trivial) fibration in $\hat{\mathcal{C}}$ for each $n \geq 0$.

See [Hi, 15.3].

2.4. **The resolution model category.** Let $\hat{\mathcal{C}}$ be a pointed cofibrantly generated right proper model category (in our cases, every object will be fibrant, though this is not needed in general – cf. [J]). Given a set $\hat{\mathcal{A}}$ of models for $\hat{\mathcal{C}}$ (§1.2), we let $\mathcal{A} := \{c(\Sigma^k \hat{A})\}_{k \in \mathbb{N}, \hat{A} \in \hat{\mathcal{A}}}$ (the constant simplicial objects on $\Sigma^k \hat{A} \in \hat{\mathcal{C}}$) be the set of models for $\mathcal{C}$. Note that $\Sigma^n c(\Sigma^k \hat{A}) := c(\Sigma^k \hat{A}) \hat{\otimes} S^n$ (§1.2), so we shall generally reserve the notation Σ^k for (internal) suspension in $\hat{\mathcal{C}}$, and $-\otimes S^n$ for the (simplicial) suspension in $\mathcal{C} = s\hat{\mathcal{C}}$.

2.5. *Remark.* If we do not assume that each $\hat{A} \in \hat{\mathcal{A}}$ is a homotopy cogroup object in $\hat{\mathcal{C}}$, we take $\mathcal{A} := \{(\Sigma^k \hat{A}) \otimes S^1\}_{A \in \mathcal{A}}$ as our collection of models for $\mathcal{C}$.

2.6. **Definition.** A map $f : V \to Y$ in $\mathcal{C} = s\hat{\mathcal{C}}$ is called *homotopically $\hat{\mathcal{A}}$-free* if for each $n \geq 0$, there is

a) a cofibrant object W_n in $\Pi_{\hat{\mathcal{A}}} \subset \hat{\mathcal{C}}$, and
b) a map $\varphi_n : W_n \to Y_n$ in $\mathcal{C}$ inducing a trivial cofibration $(V_n \amalg_{L_n V} L_n Y) \amalg W_n \to Y_n$.

We define the *resolution model category structure* on $s\hat{\mathcal{C}}$ determined by $\hat{\mathcal{A}}$, by letting a simplicial map $f : X \to Y$ be:

(i) a *weak equivalence* if $\pi^{\mathcal{C}}_{A,n} f$ is a weak equivalence of simplicial groups for each $A \in \mathcal{A}$ and $n \geq 0$.
(ii) a *cofibration* if it is a retract of a homotopically $\hat{\mathcal{A}}$-free map;
(iii) a *fibration* if it is a Reedy fibration (§2.3(iii)) and $\pi^{\mathcal{C}}_{A,n} f$ is a fibration of simplicial groups for each $A \in \mathcal{A}$ and $n \geq 0$

2.7. **Definition.** Given a fibrant $X \in s\hat{\mathcal{C}}$, define its *n-cycles* object $Z_n X$ to be $\{x \in X_n \mid d_i x = 0 \text{ for } i = 0, \ldots, n\}$ (the fiber of $\delta_n : X_n \to M_n X$ of §2.1). Similarly, the *n-chains* object for X is $C_n X = \{x \in X_n \mid d_i x = 0 \text{ for } i = 1, \ldots, n\}$.

If X is fibrant, the map $\mathbf{d}_0 = \mathbf{d}_0^n := d_0|_{C_n X} \colon C_n X \to Z_{n-1} X$ fits into a fibration sequence:

$$\cdots \Omega Z_n X \to Z_{n+1} X \xrightarrow{j^X_{n+1}} C_{n+1} X \xrightarrow{\mathbf{d}_0^{n+1}} Z_n X \tag{2.8}$$

(see [DKS2, Prop. 5.7]).

2.9. **Definition.** A *$\Pi_{\mathcal{A}}$-algebra* is a product-preserving functor from $(\mathrm{ho}\,\Pi_{\mathcal{A}})^{\mathrm{op}}$ to sets. The category of $\Pi_{\mathcal{A}}$-algebras is denoted by $\Pi_{\mathcal{A}}\text{-}\mathcal{A}lg$.

Equivalently, we can think of an $\Pi_{\mathcal{A}}$-algebra Λ as an $\mathbb{N} \times \mathcal{A}$-graded group equipped with an action of the $\mathcal{A}$-primary homotopy operations (corepresented by the maps in $\mathrm{ho}\,\Pi_{\mathcal{A}}$).

Thus we can think of the functor $\pi^{\mathcal{C}}_*$ as taking value in $\Pi_{\mathcal{A}}\text{-}\mathcal{A}lg$. This explains the additional $\Pi_{\mathcal{C}}$-algebra structure on the $\mathcal{A}$-graded groups $\pi^{\mathcal{C}}_n X$, mentioned in §1.7: when $\mathcal{C} = s\hat{\mathcal{C}}$, we have $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg := \Pi_{\hat{\mathcal{A}}}\text{-}\mathcal{A}lg$.

2.10. **Example.** When $\mathcal{C} = \mathcal{G}$, and $\mathcal{A} = \{\mathbb{S}^1\}$ – so $\Pi_{\mathcal{A}}$ is the category of wedges of $\mathcal{G}$-spheres (§1.3) – then (up to indexing) $\Pi_{\mathcal{A}}\text{-}\mathcal{A}lg$ is the usual category of Π-algebras (see [Sto, §2]): graded groups equipped with an action of the primary homotopy operations (Whitehead products and compositions).

2.11. **Examples of resolution model categories.** In this paper we shall be interested mainly in the following instances of resolution model categories:

(a) Let $\hat{\mathcal{C}} = \mathcal{G}p$ with the *trivial* model category structure: i.e., only isomorphisms are weak equivalences, and every map is both a fibration and a cofibration. Let $\hat{\mathcal{A}} = \{\mathbb{Z}\}$ consist of the free cyclic group (whose coproducts are the cogroup objects in $\mathcal{G}p$). The resulting resolution model category structure on $\mathcal{G} := s\hat{\mathcal{C}}$ is the usual one (cf. [Q1, II,§3]. Here $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg \approx \mathcal{G}p$ – there is no extra structure on the individual homotopy groups of a simplicial group.

Note that if we tried to do the same for $\hat{\mathcal{C}} = \mathcal{S}et$, there are no nontrivial cogroup objects, while in $\mathcal{S}$ not all objects are fibrant. Note also that the category $\mathcal{T}_*$ of pointed topological spaces, which is one of the main examples we have in mind, has a spherical model category structure which is not strict (§1.22). This explains the significance of Remark 1.4 in our context.

(b) The previous example extends to any category $\hat{\mathcal{C}}$ of (possibly graded) universal algebras with an underlying group structure – such as rings, R-modules, associative algebras, Lie algebras, and so on – so that $\mathcal{C}$ is corepresented by a $\mathfrak{G}$-theory Θ, in the language of [BP, §4]. Here $\mathcal{A}$ consists of free monogenic algebras (one for each isomorphism class), and thus once more $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg \approx \mathcal{C}$.

(c) We can iterate the process by taking $\mathcal{G}$ for $\hat{\mathcal{C}}$, and letting $\hat{\mathcal{A}} := \{\mathbb{S}^n\}_{n=1}^{\infty}$ (§1.3). We thus obtain a resolution model category structure on $s\mathcal{G}$ (or equivalently, on the category of simplicial spaces).

In this case the homotopy groups $\pi^{s\mathcal{G}}_{k,n}X$, denoted briefly by $\pi^{\natural}_{n}X$, are the "bigraded groups" of [DKS2], and Proposition 5.8 there shows that, for a fibrant simplicial space $X \in s\mathcal{G}$, we have $\pi^{\mathcal{C}}_{A,n}X \cong \pi_0\ \mathrm{map}(A \otimes S^n, X)$.

(d) If $\mathcal{C}$ is a resolution model category and I is some small category, the category $\mathcal{C}^I$ of I-diagrams in $\mathcal{C}$ also has a resolution model category structure, in which the models consists of all free I-diagrams $F[A,i]$ for $i \in \mathrm{Obj}\, I$ and $A \in \mathcal{A}$, where $F[A,i](j) := \coprod_{\mathrm{Hom}_I(i,j)} A$. See [BJT, §1]).

2.12. *Remark.* In all these examples, if $Y \in \mathcal{C} = s\hat{\mathcal{C}}$, is fibrant, then for each $n \geq 0$ we have an exact sequence:

$$\pi^{\hat{\mathcal{C}}}_{*}C_{n+1}X \xrightarrow{(\mathbf{d}_0^{n+1})_{\#}} \pi^{\hat{\mathcal{C}}}_{*}Z_nX \xrightarrow{\hat{\vartheta}_n} \pi^{\mathcal{C}}_{n}Y \to 0. \tag{2.13}$$

3. Constructions in resolution model categories

Not all spherical model categories are resolution model categories (see §1.21), but all known examples appear to be Quillen equivalent to such. Conversely, the examples of resolution model categories $\langle \mathcal{C} = s\hat{\mathcal{C}}; \mathcal{A}\rangle$ we are interested in are spherical (though this does not hold in general – see §3.10 below). We briefly indicate why this is so.

3.1. **Postnikov sections.** Given $Y \in s\hat{\mathcal{C}}$, for each $n \geq 0$ define $Y^{(n)} \in s\hat{\mathcal{C}}$ by setting $Y^{(n)}_k := Y_k$ for $k \leq n+1$ and $Y^{(n)}_k := M_k(Y^{(n)})$ (§2.1) for $k \geq n+2$. Note that for any $X \in s\mathcal{C}$, M_kX depends only on X through dimension $(k-1)$, so this definition is valid inductively. Denote the obvious maps by $r^{(n)} : Y \to Y^{(n)}$ and $p^{(n)} : Y^{(n+1)} \to Y^{(n)}$ (see [DK2, §1.2]).

Now for any $X \in s\hat{\mathcal{C}}$, choose a functorial fibrant replacement Y, and set $P_nX := Y^{(n)}$, with $\varphi^{(n)} : X \to P_nX$ defined to be the composite of $r^{(n)}$ with the trivial cofibration $i : X \to Y$, and $p^{(n)} : P_{n+1}X \to P_nX$ defined as above.

3.2. *Remark.* The functor $-^{(n)} : \mathcal{C} \to \mathcal{C}$ is right adjoint to the $(n+1)$-skeleton functor sk_{n+1}, so P_nX depends only on $\mathrm{sk}_{n+1}X$, even if X is not fibrant. If X is fibrant, we can find $Y \simeq P_nX$ with $\mathrm{sk}_{n+1}Y = \mathrm{sk}_{n+1}X$.

3.3. **Fact.** In each of the examples of §2.11(a-d), the tower:

$$X \to \ldots \to P_{n+1}X \xrightarrow{p^{(n)}} P_nX \to \ldots \to P_0X$$

is a functorial Postnikov tower for $\mathcal{C} = s\hat{\mathcal{C}}$ with respect to $\mathcal{A}$ (§1.9).

Proof. From §2.3 and §3.1 it follows that if $Y \in s\hat{\mathcal{C}}$ is fibrant, then so is each $Y^{(n)}$, and for each n, $Y^{(n+1)} \to Y^{(n)}$ is a fibration, $Z_kY^{(n)} = 0$ and $C_kY^{(n)} \xrightarrow{\mathbf{d}_0} Z_{k-1}Y$ is an isomorphism for $k \geq n+2$. The claim then follows from (2.13). $\square$

3.4. **Fact.** In each of the examples of §2.11(a-d), there is a classifying object $B\Lambda$ for any $\Pi_{\mathcal{A}}$-algebra Λ, and it is unique up to homotopy.

Proof. In the algebraic cases of §2.11(a-b), we may take $B\Lambda$ to be (a cofibrant model for) the constant simplicial object on Λ. For simplicial spaces, $B\Lambda$ may be constructed as for topological spaces, using generators and relations (see [BDG, §8.9]). The extension to the diagram case of §2.11(d) is objectwise. □

3.5. **Fact.** In each of the examples of §2.11(a-d), for each $n \geq 1$ there is an n-dimensional M-Eilenberg-Mac Lane object $E(M,n)$ for any abelian $\Pi_{\mathcal{A}}$-algebra M, and there is an n-dimensional extended M-Eilenberg-Mac Lane object $E^{\Lambda}(M,n)$ for any $\Pi_{\mathcal{A}}$-algebra Λ and module M over Λ. Each of these is unique up to homotopy.

Proof. In the algebraic cases of §2.11(a-b), we may take $E(M,n)$ to be the iterated Eilenberg-Mac Lane construction $\bar{W}$ on BM, while $E^{\Lambda}(M,n)$ is a semi-direct product $E(M,n) \ltimes B\Lambda$ (see [BDG, Prop. 2.2]). For simplicial spaces, use the explicit construction of [BDG, §8.9] The extension to the diagram case is again objectwise. □

3.6. **Fact.** In each of the examples of §2.11(a-d), for each $n \geq 1$ and $X \in \mathcal{C}$, $\pi_n^{\mathcal{C}} X$ has a natural structure of a module over $\pi_0^{\mathcal{C}} X$.

Proof. Note that by [Q1, II,1,(6)] we have $\operatorname{map}(A \otimes S^n, X) \cong \operatorname{map}_{\mathcal{S}}(S^n, \operatorname{map}(A,X))$ (unpointed maps), so $\pi_n^{\mathcal{C}} X \to \pi_0^{\mathcal{C}} X$ associates to each $f : A \otimes S^n \to X$ its component in $\operatorname{map}(A,X)$. This defines an *abelian* algebra over $\pi_0^{\mathcal{C}} X$ by [BP, Prop. 6.26]). □

3.7. **Fact.** In each of the examples of §2.11(a-d), for each $X \in s\mathcal{C}$, $\Lambda := \pi_0^{\mathcal{C}} X$ and $n \geq 1$, the commutative square obtained by applying the functor P_{n+2} to the pushout diagram:

$$\begin{tikzcd}
P_{n+1}X \arrow[r, "p^{(n+1)}"] \arrow[d] & P_n X \arrow[d, "k_n"] \\
B\Lambda \arrow[r] \arrow[ur, phantom, "\boxed{\mathrm{PO}}"] & Y
\end{tikzcd}$$

is an n-th k-invariant square (Def. 1.17) – that is, $P_{n+2}Y \simeq E^{\Lambda}(\pi_{A,n+1}^{\mathcal{C}} X, n+2)$.

Proof. See [BDG, §5]. □

We may summarize these facts in the following:

3.8. **Theorem.** *The following resolution model categories (cf. §2.11) are strict spherical model categories:*

i. *The category* $\mathcal{C} = s\Theta\text{-}\mathcal{S}et_*$ *of simplicial* Θ*-algebras for any* $\mathfrak{G}$*-theory* Θ*, with* $\hat{\mathcal{A}}$ *consisting of monogenic free* Θ*-algebras;*
ii. *In particular, the category* $\mathcal{C} = \mathcal{G}$ *of simplicial groups, with* $\hat{\mathcal{A}} = \{\mathbb{Z}\}$*;*
iii. *The category* $s\mathcal{G}$ *of bisimplicial groups ("simplicial spaces"), with* $\hat{\mathcal{A}} = \{\mathbb{S}^1 \otimes S^k\}_{k=0}^{\infty}$.
iv. *The category* $\mathcal{C}^I$ *of* I*-diagrams in a strict spherical model category* $\mathcal{C}$.

3.9. **Theorem.** *The following are spherical model categories (which are not strict):*

i. *The category* $\mathcal{S}_*$ *of pointed simplicial sets, with* $\mathcal{A} = \{S^1\}$*;*
ii. *The category* $\mathcal{T}_*$ *of pointed topological spaces, with* $\mathcal{A} = \{\mathbf{S}^1\}$*;*
iii. *The category* $s\mathcal{T}_*$ *of simplicial pointed topological spaces, with* $\hat{\mathcal{A}} = \{\mathbf{S}^1 \otimes S^k\}_{k=1}^{\infty}$.

3.10. **Non-spherical model categories.** Consider the trivial model category structure on $\hat{\mathcal{C}} = \mathcal{G}p$, with $\hat{\mathcal{A}} := \{A = \mathbb{Z}/p\}$ (for p an odd prime). This defines a resolution model category structure on $\mathcal{G}$ – or equivalently, on $\mathcal{T}_*$ (see Remark 2.5). Note that $-\otimes S^n$ corresponds to suspension of simplicial sets, not simplicial abelian group, so the model $A \otimes S^n \in \mathcal{G}$ corresponds to the n-dimensional mod p Moore space $\mathbf{S}^{n-1} \cup_p \mathbf{e}^n$.

Thus $\pi^{\mathcal{C}}_{A,k} X := [A \otimes S^k, X]$ is by definition the k-th *mod p homotopy group* of X – denoted by $\pi_k(X;\mathbb{Z}/p)$ in [Ne, Def. 1.2] – which fits into a short exact sequence:

$$0 \to \pi_k X \otimes \mathbb{Z}/p \to \pi_k(X;\mathbb{Z}/p) \to \operatorname{Tor}^{\mathbb{Z}}_1(\pi_{k-1}X, \mathbb{Z}/p) \to 0 \tag{3.11}$$

for $k \geq 2$ (see [Ne, Prop. 1.4]). In particular, for $Y := A \otimes S^n$ $(n \geq 4)$ we have

$$\pi_i(Y;\mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & \text{for } \ i = n-1, n, \\ 0 & \text{for } \ 2 \leq i < n-1 \text{ or } \ i = n+1, \end{cases}$$

with the two non-trivial groups connected by a Bockstein (cf. [Ne, §1]).

However, the resolution model category structure on $\mathcal{G}$ determined by $\mathcal{A}$ is not spherical: if it were, in particular there would be Postnikov functors $P_k = P^{\mathcal{A}}_k$ for all $k \geq 1$ (Def. 1.9). From (3.11) we see that, disregarding torsion prime to p, because of the Bockstein we must have $P_{n-1}Y \simeq E(\mathbb{Z}, n-1)$ and $P_n Y \simeq E(\mathbb{Z}/p, n-1)$ (for $Y = S^{n-1} \cup_p e^n$). But then there is no non-trivial map $P_n Y \to P_{n-1} Y$.

3.12. **Cohomology in spherical model categories.** Note that the k-invariants of a simplicial object actually take value in cohomology groups, as expected:

3.13. **Proposition.** *For each* $\Pi_{\mathcal{A}}$*-algebra* Λ *and module* M *over* Λ*, the functors* $D^n : \mathcal{C}/B\Lambda \to \mathcal{A}b\mathcal{G}p$ $(n > 0)$*, defined* $D^n(X) := [X, E^{\Lambda}(M,n)]_{B\Lambda}$*, are* cohomology functors *on* $\mathcal{C}$ *– that is, they are homotopy invariant, take arbitrary coproducts to products, vanish on the spherical models* $\Sigma^n A$*, except in degree n, and have Mayer-Vietoris sequences for homotopy pushouts.*

We therefore denote $[X, E^{\Lambda}(M,n)]_{B\Lambda}$ by $H^n_{\Lambda}(X; M)$.

Proof. See [BP, Thm. 7.14]. □

Fact 3.5 then follows from Brown Representability, since $E^{\Lambda}(M,n)$ represents the n-th André-Quillen cohomology group in $\mathcal{C}$; see [BDG, §6.7] and [Bl3, §4].

4. Spherical functors

Our objective is to study functors between model categories, and investigate the extent to which they induce an equivalence of homotopy categories. Our methods work only for functors between spherical model categories which take models to models, in the following sense:

4.1. **Definition.** Let $\langle \mathcal{C}; \mathcal{A} \rangle$ and $\langle \mathcal{D}; \mathcal{B} \rangle$ be two spherical model categories. A functor $T : \mathcal{C} \to \mathcal{D}$ is called *spherical* if

i. T defines a bijection $\mathcal{A} \to \mathcal{B}$;
ii. $T|_{\Pi_{\mathcal{A}}}$ preserves coproducts and suspensions;
iii. T induces an equivalence of categories $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg \approx \Pi_{\mathcal{D}}\text{-}\mathcal{A}lg$ (in fact, it suffices that $\Pi_{\mathcal{D}}\text{-}\mathcal{A}lg$ be a full subcategory of $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg$).

4.2. **Examples of spherical functors.** In the cases we shall be considering (those mentioned in the introduction), $\mathcal{C}$ and $\mathcal{D}$ will be strict spherical resolution model categories, with $\mathcal{C} = s\hat{\mathcal{C}}$ and $\mathcal{D} = s\hat{\mathcal{D}}$, and T will be prolonged from a functor $\hat{T} : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$.

The four examples:

(a) For $\langle \hat{\mathcal{C}}; \hat{\mathcal{A}} \rangle = \langle \mathcal{G}p; \{\mathbb{Z}\} \rangle$ and $\langle \hat{\mathcal{D}}, \hat{\mathcal{B}} \rangle = \langle \mathcal{A}b\mathcal{G}p; \{\mathbb{Z}\} \rangle$, let $\hat{T} = \mathcal{A}b : \mathcal{G}p \to \mathcal{A}b\mathcal{G}p$ be the abelianization functor.

Here $\mathcal{C} = s\hat{\mathcal{C}} = \mathcal{G}$, so $\operatorname{ho}\mathcal{C}$ is equivalent to the homotopy category of pointed connected topological spaces (§1.4), while $\mathcal{D} = s\hat{\mathcal{D}}$, the category of simplicial abelian groups, is equivalent to the category of chain complexes under the Dold-Kan correspondence (see [D, §1]). Thus $T : \mathcal{C} \to \mathcal{D}$ represents the singular chain complex functor $C_* : \mathcal{T}_* \to \mathcal{C}hain$.

Note that $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg = \mathcal{G}p$, while $\Pi_{\mathcal{D}}\text{-}\mathcal{A}lg = \mathcal{A}b\mathcal{G}p$, in this case, so strictly speaking T does not induce an equivalence of categories. But since $\mathcal{A}b\mathcal{G}p$ is a full subcategory of $\mathcal{G}p$, we can in fact think of $\pi^\natural$ as taking values in groups.

(b) For $\langle \hat{\mathcal{C}}; \hat{\mathcal{A}} \rangle = \langle \mathcal{G}p; \{\mathbb{Z}\} \rangle$ and $\langle \hat{\mathcal{D}}, \hat{\mathcal{B}} \rangle = \langle \mathcal{H}opf; \{H\} \rangle$, where $\mathcal{H}opf$ is the category of complete Hopf algebras over $\mathbb{Q}$, H is the monogenic free object in this category, let $\hat{Q} : \mathcal{G}p \to \mathcal{H}opf$ be the functor which associates to a group G the completion of the group ring $\mathbb{Q}[G]$ by powers of the augmentation ideal.

Again, $\mathcal{C} = s\hat{\mathcal{C}}$ is a model category for connected topological spaces, while $\mathcal{D} = s\hat{\mathcal{D}}$ is a model category for the rational simply-connected spaces (see [Q2]); Q (when restricted to connected simplicial groups) represents the rationalization functor. Once more, $\Pi_{\mathcal{C}}\text{-}\mathcal{A}lg = \mathcal{G}p$, while $\Pi_{\mathcal{D}}\text{-}\mathcal{A}lg$ is the subcategory of vector spaces over $\mathbb{Q}$.

(c) For $\langle \hat{\mathcal{C}}, \hat{\mathcal{A}} \rangle = \langle \mathcal{S}et_*; \{S^0\} \rangle$ (so that $\langle \mathcal{C}, \mathcal{A} \rangle = \langle \mathcal{S}; \{S^1\} \rangle$, by Remark 2.5), and $\langle \hat{\mathcal{C}}; \hat{\mathcal{A}} \rangle = \langle \mathcal{G}p; \{\mathbb{Z}\} \rangle$, let $\hat{F} : \mathcal{S}et_* \to \mathcal{G}p$ be the free group functor.

Again, we think of both $\mathcal{C} = s\hat{\mathcal{C}} = \mathcal{G}$ and $\mathcal{D} = s\hat{\mathcal{D}} = \mathcal{S}_*$ as model categories for pointed topological spaces, (under the respective equivalences of §1.4) – so F here represents the suspension functor $\Sigma : \mathcal{T}_* \to \mathcal{T}_*$ (rather than $\Omega\Sigma$, as one might think at first glance).

(d) For $\langle \hat{\mathcal{C}}; \hat{\mathcal{A}} \rangle = \langle \mathcal{G}; \{\mathcal{S}^k\}_{k=0}^\infty \rangle$ and $\langle \hat{\mathcal{D}}, \hat{\mathcal{B}} \rangle = \langle \Pi\text{-}\mathcal{A}lg; \{\pi_*\mathcal{S}^k\}_{k=0}^\infty \rangle$, let $\widehat{\pi_*} : \mathcal{G} \to \Pi\text{-}\mathcal{A}lg$ be the graded homotopy group functor $X \mapsto \pi_* X$. Here $\mathcal{C} = s\mathcal{G}$ is a model category for simplicial spaces.

4.3. **Theorem.** *Let $\langle \mathcal{C}; \mathcal{A} \rangle$ and $\langle \mathcal{D}; \mathcal{B} \rangle$ be spherical model categories, and let $T : \mathcal{C} \to \mathcal{D}$ be a spherical functor. Then for each $X \in \mathcal{C}$ and $A \in \mathcal{A}$ there is a natural long exact sequence of $\Pi_{\mathcal{C}}$-algebras:*

$$\ldots \to \Gamma^X_{\alpha,n} X \xrightarrow{s^X_n} \pi^{\mathcal{C}}_{A,n} X \xrightarrow{h^X_n} \pi^{\mathcal{C}}_{T_*(\alpha),n} TX \xrightarrow{\partial^X_n} \Gamma^T_{\alpha,n-1} X \ldots \tag{4.4}$$

We call (4.4) the *comparison exact sequence* for T. Compare [Ba4, V, (5.4)].

Proof. If $\tilde{X} \to X$ is a functorial fibrant replacement, the functor T induces a natural transformation $\tau : \operatorname{map}_{\mathcal{C}}(A, \tilde{X}) \to \operatorname{map}_{\mathcal{D}}(TA, \widetilde{T\tilde{X}})$, which we may functorially change to a fibration of simplicial sets, with fiber $F(X)$. Setting $\Gamma^T_{\alpha,n} := \pi_n F(X)$, the corresponding long exact sequence in homotopy is (4.4).

Note that the map $h^X_n = h^X$ is also natural in the variable A, so the graded map $h^X_* : \pi^{\mathcal{C}}_n X \to \pi^{\mathcal{D}}_n TX$ is a morphism of $\Pi_{\mathcal{C}}$-algebras (i.e., $\Pi_{\hat{\mathcal{A}}}$-algebras). □

4.5. **Applications of Theorem 4.3.** The Theorem is not very useful in this generality. However, in all the examples of §4.2, we obtain interesting (though mostly known) exact sequences:

(a) For $\hat{T} = \mathcal{A}b : \mathcal{G}p \to \mathcal{A}b\mathcal{G}p$ the abelianization functor, where $T : \mathcal{G} \to s\mathcal{A}b\mathcal{G}p$ represents the singular chain complex functor $C_* : \mathcal{T}_* \to \mathcal{C}hain$ (cf. §4.2(a)), the sequence (4.4) is the "certain exact sequence" of J.H.C. Whitehead:

$$\ldots \to \Gamma_n X \to \pi_n X \xrightarrow{h_n} H_n(X;\mathbb{Z}) \to \Gamma_{n-1} X \ldots \tag{4.6}$$

(See [W4]). In particular, the third term in this sequence, $\Gamma^{\mathcal{A}}_n(X)$, is simply the n-th homotopy group of the commutator subgroup of GX.

(b) For $Q : \mathcal{G} \to s\mathcal{H}opf$ of §4.2(b), representing the rationalization functor, we obtain a long exact sequence relating the integral and rational homotopy groups of a simply-connected space X. The third term in (4.4) may be described in terms of the torsion subgroup of $\pi_* X$ together with $\pi_* X \otimes \mathbb{Q}/\mathbb{Z}$.

(c) The free group functor $\hat{F} : \mathcal{S}et_* \to \mathcal{G}p$ of §4.2(c) represents the suspension $\Sigma : \mathcal{T}_* \to \mathcal{T}_*$, and indeed for $K \in \mathcal{S}_*$ the map h^K, which is the composite:

$$\begin{aligned} \pi_n K \;=\; \pi_0 \operatorname{map}_{\mathcal{S}_*}(S^n, K) \;&\longrightarrow\; \pi_0 \operatorname{map}_{\mathcal{G}}(FS^n, FK) \\ &\xrightarrow{\cong} \pi_0 \operatorname{map}_{\mathcal{S}_*}(\Sigma S^n, \Sigma K) \;=\; \pi_{n+1}\Sigma K , \end{aligned}$$

is the suspension homomorphism, so (4.4) is a generalized EHP sequence (cf. [Ba1, G, No]).

(d) For $\pi_* : s\mathcal{G} \to s\Pi\text{-}\mathcal{A}lg$ as in §4.2(d), it turns out that for any simplicial space $X \in s\mathcal{G}$, the induced map h^X_n is the "Hurewicz homomorphism" $h_n : \pi^\natural_n X \to \pi_n \pi_* X$ of [DKS2, 7.1], while $\Gamma^T_n X$ is just $\Omega \pi^\natural_{n-1} X$ – that is, $\Gamma^T_{i,n} X = \pi^\natural_{i+1,n-1} X$ for each i. Thus (4.4) is the *spiral long exact sequence*:

$$\ldots \pi_{n+1}\pi_* X \xrightarrow{\partial^\star_{n+1}} \Omega \pi^\natural_{n-1} X \xrightarrow{s_n} \pi^\natural_n X \xrightarrow{h_n} \pi_n \pi_* X \to \cdots \pi^\natural_0 X \xrightarrow{h_0} \pi_0 \pi_* X \to 0 \tag{4.7}$$

of [DKS2, 8.1]. Of course, $\pi^\natural_{-1} X = 0$, so h_0 is an isomorphism.

Note that for $T : \mathcal{C} \to \mathcal{D}$ as above, the homotopy groups $\pi^{\mathcal{D}}_n TX$ for any $X \in \mathcal{C} = s\hat{\mathcal{C}}$ may be computed using the Moore chains $C_* TX$ as in §2.7; each $\pi^{\mathcal{D}}_n TX$ is a $\Pi_{\mathcal{D}}$-algebra, abelian for $n \geq 1$.

4.8. **Explicit construction of the spiral exact sequence.** It may be helpful to inspect in detail the construction of last long exact sequence, since it is perhaps the least familar of the four. Specificializing to $\hat{\mathcal{C}} = \mathcal{G}$ and $T = \pi_*$, we have:

4.9. **Lemma.** *For fibrant $X \in \mathcal{C}$, the inclusion $\iota : C_n X \hookrightarrow X_n$ induces an isomorphism $\iota_\star : \pi_* C_n X \cong C_n(\pi_* X)$ for each $n \geq 0$.*

Proof. See [Bl2, Prop. 2.11]. □

Together with (2.13), this yields a commuting diagram:

$$
\tag{4.10}
\begin{tikzcd}
\pi_*C_{n+1}X \arrow[r, "(\mathbf{d}_0)_\#"] \arrow[d, "\iota_\star"', "\cong"] & \pi_*Z_nX \arrow[r, two heads, "\hat{\vartheta}_n"] \arrow[d, "\hat{\iota}_\star"] & \pi^\natural_n X \arrow[d, dotted, "h_n"] \\
C_{n+1}(\pi_*X) \arrow[r, "d_0^{\pi_*X}"] & Z_n(\pi_*X) \arrow[r, two heads, "\vartheta_n"] & \pi^\natural_n\pi_*X
\end{tikzcd}
$$

which defines the dotted morphism of Π-algebras $h_n : \pi^\natural_n X \to \pi_n(\pi_*X)$. Note that for $n = 0$ the map $\hat{\iota}_\star$ is an isomorphism, so h is, too.

If $X \in s\mathcal{G}$ is fibrant, applying π_* to the fibration sequence (2.8) yields a long exact sequence, with connecting homomorphism $\partial_n : \Omega\pi_*Z_nX = \pi_*\Omega Z_nX \to \pi_*Z_{n+1}X$; (2.13) then implies that

$$
\tag{4.11}
\Omega\pi^\natural_n X = \Omega\,\mathrm{Coker}\,(\mathbf{d}_0^{n+1})_\# \cong \mathrm{Im}\ \partial_n \cong \mathrm{Ker}\,(j^X_{n+1})_\# \subseteq \pi_*Z_{n+1}X,
$$

and the map $s_{n+1} : \Omega\pi^\natural_n X \to \pi^\natural_{n+1}X$ in (3.11) is then obtained by composing the inclusion $\mathrm{Ker}\,(j^X_{n+1})_\# \hookrightarrow \pi_*Z_{n+1}X$ with the quotient map $\hat{\vartheta}_{n+1} : \pi_*Z_{n+1}X \to \pi^\natural_{n+1}X$ of (2.13).

Similarly, $h_n : \pi^\natural_n X \to \pi^\natural_n\pi_*X$ is induced by the inclusion $(j^X_n)_\# : \pi_*Z_nX \to Z_n\pi_*X \subseteq C_n\pi_*X$, and $\partial^\star_{n+2} : \pi^\natural_{n+2}\pi_*X \to \Omega\pi^\natural_n X$ is induced by the composite

$$
Z_{n+2}\pi_*X \subseteq C_{n+2}\pi_*X \cong \pi_*C_{n+2}X \xrightarrow{(\mathbf{d}_0^{n+2})_\#} Z_{n+1}\pi_*X,
$$

which actually lands in $\mathrm{Ker}\,(j^X_{n+1})_\# \cong \Omega\pi^\natural_n X$ by the exactness of the long exact sequence for the fibration.

Moreover, for each $n \geq 0$, (4.10) may be extended (after rotating by $90°$) to a commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}\ s_n \arrow[r, tail] \arrow[d] & B_{n+1}X \arrow[r, two heads, "(j_n)_*"] \arrow[d] & B_{n+1}\pi_*X_{n+2} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega\pi^\natural_{n-1}X \arrow[r, tail] \arrow[d] & \pi_*Z_nX \arrow[r, "(j_n)_*"] \arrow[d, "\hat{\vartheta}_n"] & Z_n\pi_*X \arrow[r, two heads] \arrow[d, "\vartheta_n"] & \mathrm{Coker}\ h_n \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{Ker}\ h_n \arrow[r, tail] \arrow[d] & \pi^\natural_n X \arrow[r, "h_n"] \arrow[d] & \pi_n\pi_*X \arrow[r, two heads] \arrow[d] & \mathrm{Coker}\ h_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

in which $B_{n+1}X := \mathrm{Im}\,(\mathbf{d}_0^{X_{n+2}})_\# \subseteq \pi_*Z_nX$ and $B_{n+1}\pi_*X_{n+2} := \mathrm{Im}\ \mathbf{d}_0^{\pi_*X_{n+2}}$ are the respective boundary objects.

The maps $\partial^\star_{n+1}$, s_n, and h_n, as defined above, form the spiral long exact sequence.

4.12. **Inverse spherical functors.** We may sometimes be interested in functors between spherical model categories which are not quite spherical. Thus, if $T : \langle\mathcal{C};\mathcal{A}\rangle \to \langle\mathcal{D};\mathcal{B}\rangle$ is a spherical functor as in §4.1, a functor $V : \mathcal{D} \to \mathcal{C}$ equipped with a natural transformation $\vartheta : \mathrm{Id}_\mathcal{C} \to VT$ is called an *inverse spherical functor* to T.

4.13. **Example.** For the free group functor $F : \mathcal{S}et_* \to \mathcal{G}p$ of §4.2(c), the forgetful functor $\hat{U} : \mathcal{G}p \to \mathcal{S}et_*$ (right adjoint to F) with the adjunction counit $\eta : \mathrm{Id} \to UF$ as the natural transformation ϑ, yields the inverse spherical functor $U : \mathcal{G} \to \mathcal{S}_*$. Here we do not think of $\mathcal{G}$ as a model for $\mathcal{T}_*$ – rather, U represents the forgetful functor from loop spaces (topological groups) to spaces.

Similarly, the adjoint to the abelianization functor $\mathcal{A}b : \mathcal{G}p \to \mathcal{A}b\mathcal{G}p$ is the inclusion $\hat{I} : \mathcal{A}b\mathcal{G}p \to \mathcal{G}p$, and the corresponding functor $I : s\mathcal{A}b\mathcal{G}p \to \mathcal{G}$ represents the factorization of the Dold-Thom infinite symmetric product functor $SP^\infty : \mathcal{T}_* \to \mathcal{T}_*$ through $\mathcal{C}hain$.

4.14. **Proposition.** *If $V : \mathcal{D} \to \mathcal{C}$ is an inverse spherical functor to T, then for each $Y \in \mathcal{D}$ and $B \in \mathcal{B}$ there is a natural long exact sequence:*

$$\ldots \to \Delta^V_{B,n} Y \to \pi^{\mathcal{D}}_{B,n} Y \xrightarrow{V_\#} \pi^{\mathcal{C}}_{V_*(B),n} VY \to \Delta^V_{B,n-1} Y \ldots \tag{4.15}$$

Proof. If V is an inverse spherical functor, because $T|_{\mathcal{A}}$ is a bijection onto $\mathcal{B}$, there is an $A \in \mathcal{A}$ such that $B = TA$. As before, V induces a natural transformation $\nu : \mathrm{map}_{\mathcal{D}}(B, \tilde{Y}) \to \mathrm{map}_{\mathcal{D}}(VB, \widehat{V\tilde{Y}})$ and the natural transformation $\vartheta : A \to VTA$ yields $\vartheta^\# : \mathrm{map}_{\mathcal{D}}(VTA, \widehat{V\tilde{Y}}) \to \mathrm{map}_{\mathcal{D}}(A, \widehat{V\tilde{Y}})$ so we get a composite map $\mathrm{map}_{\mathcal{D}}(B, \tilde{Y}) \to \mathrm{map}_{\mathcal{D}}(A, \widehat{V\tilde{Y}})$, with homotopy fiber $E(Y)$. If we let $\Delta^V_{\beta,n} Y := \pi_n E(Y)$, the fibration long exact sequence is (4.15). □

4.16. *Remark.* Note that in contradistinction to Theorem 4.3, $V_\#$ of (4.15) need not respect any operations, since we only have a bijection $T|_{\mathcal{A}}: \mathcal{A} \to \mathcal{B}$, not a functor.

For $U : \mathcal{G} \to \mathcal{S}_*$ as in §4.13, we may assume $X \in \mathcal{G}$ is of the form $X \simeq GK$ for $K \in \mathcal{S}_*$, and then $V_\#$ is the identity:

$$\begin{aligned} \pi_n K \;=\; \pi^\natural_n X = \;& \pi_0\, \mathrm{map}_{\mathcal{G}}(FS^{n-1}, GK) \;\to\; \pi_0\, \mathrm{map}_{\mathcal{S}_*}(UFS^{n-1}, UGK) \\ & \xrightarrow{\eta^\#} \;\pi_0\, \mathrm{map}_{\mathcal{S}_*}(S^{n-1}, UGK) \;=\; \pi_n K\;, \end{aligned} \tag{4.17}$$

so (4.15) is not interesting in this case.

5. Comparing Postnikov systems

The basic problem under consideration in this paper may be formulated as follows:

Question. Given a spherical functor $T : \langle \mathcal{C}; \mathcal{A} \rangle \to \langle \mathcal{D}; \mathcal{B} \rangle$ and an object $G \in \mathcal{D}$, what are the different objects $X \in \mathcal{C}$ (up to homotopy) such that $TX \simeq G$?

As shown in the previous section, such a pair $\langle X, G \rangle$ must be connected by a comparison exact sequence. Thus, in order to reconstruct X from G, we first try to determine $\pi^{\mathcal{C}}_* X$, and its relation to $\pi^{\mathcal{D}}_* G$.

In order to proceed further, we must make an additional assumption on T, contained in the following:

5.1. **Definition.** A spherical (or inverse spherical) functor $T : \mathcal{C} \to \mathcal{D}$ is called *special* if:

i. $\mathcal{C} = s\hat{\mathcal{C}}$ and $\mathcal{D} = s\hat{\mathcal{D}}$ are spherical resolution model categories, and T is prolonged from a functor $\hat{T} : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$.
ii. For any $\Pi_{\mathcal{A}}$-algebra Λ and module M over Λ, T induces a homomorphism of (graded) groups $\phi_T : \Lambda \to \pi^{\mathcal{D}}_0 TB_{\mathcal{C}}\Lambda$.

iii. This ϕ_T induces a functor $\hat{T}:\Lambda\text{-}\mathcal{M}od \to \phi_T\Lambda\text{-}\mathcal{M}od$ which is an isomorphism on Λ-modules (see Remark 1.13).

iv. For each $n \geq 1$ and n-dimensional extended M-Eilenberg-Mac Lane object $E = E^{\Lambda}_{\mathcal{C}}(M,n)$, there is a natural isomorphism $\pi^{\mathcal{D}}_n TE \cong M$ which respects $\hat{T}$ in the obvious sense.

v. The natural map

$$[X, E^{\Lambda}_{\mathcal{C}}(M,n)]_{B_{\mathcal{C}}\Lambda} \to [TX, E^{\hat{T}\Lambda}_{\mathcal{D}}(M,n)]_{B_{\mathcal{D}}\hat{T}L}\ , \tag{5.2}$$

defined by composition with the projection

$$\rho : TE^{\Lambda}_{\mathcal{C}}(M,n) \to P_n TE^{\Lambda}_{\mathcal{C}}(M,n) \ = \ E^{\hat{T}\Lambda}_{\mathcal{D}}(M,n)\ ,$$

is an isomorphism.

5.3. **Example.** All the functors we have considered hitherto, except for the rationalization functor $Q : \mathcal{G} \to s\mathcal{H}opf$ of §4.2(b), are special:

(a) For the singular chain functor $T : \mathcal{G} \to s\mathcal{A}b\mathcal{G}p$, induced by abelianization, this follows from the Hurewicz Theorem (recall that $\pi^{\mathcal{C}}_0 X$ is the fundamental group, in our indexing for $X \in \mathcal{G}$).

(b) For the suspension $\Sigma : \mathcal{T}_* \to \mathcal{T}_*$, induced by the free group functor $F : \mathcal{S}et_* \to \mathcal{G}p$, this follows (in the simply connected case) from the Freudenthal Suspension Theorem.

(c) For the homotopy groups functor $\pi_* : s\mathcal{G} \to s\Pi\text{-}\mathcal{A}lg$, (i)-(iii) follow by inspecting the spiral long exact sequence (4.7), while (iv) is [BDG, Prop. 8.7].

(d) For the inverse spherical functor $U : \mathcal{G} \to \mathcal{S}_*$ of §4.13, induced by the forgetful functor $\hat{U} : \mathcal{G}p \to \mathcal{S}et_*$, this is immediate from (4.17).

5.4. **Lemma.** *Any special spherical functor $T : \mathcal{C} \to \mathcal{D}$ as above* respects Postnikov systems – *that is, for any $X \in \mathcal{C}$ and $n \geq 0$ we have:*

$$P^{\mathcal{D}}_n T P^{\mathcal{C}}_n X \cong P^{\mathcal{D}}_n TX\ - \tag{5.5}$$

so that $\pi^{\mathcal{C}}_k TX \cong \pi^{\mathcal{D}}_k TP_n X$ and $\Gamma_k X \cong \Gamma_k P_n X$ for $k \leq n$.

Proof. This follows from the constructions in §3.1 and the proof of Theorem 4.3. □

5.6. **Postnikov systems and spherical functors.** From now on, assume $T : \mathcal{C} \to \mathcal{D}$ is a special spherical functor. Ultimately, for each object $G \in \mathcal{D}$, we would like find any and all $X \in \mathcal{C}$ such that $TX \simeq G$. First, however, we try to discover what can be said about TX and its Postnikov systems for a given $X \in \mathcal{C}$. Using the comparison exact sequence for T and Lemma 5.4, we see that:

$$\pi^{\mathcal{D}}_k TP_n X \cong \begin{cases} \pi^{\mathcal{D}}_k TX & \text{for } k \leq n, \\ \operatorname{Coker}\{h^X_{n+1} : \pi^{\mathcal{C}}_{n+1} X \to \pi^{\mathcal{D}}_{n+1} TX\} & \text{for } k = n+1, \\ \Gamma_{k-1} P_n X & \text{for } k \geq n+2\ . \end{cases} \tag{5.7}$$

5.8. **Fact.** If $T : \mathcal{C} \to \mathcal{D}$ is a special spherical functor, applying $\pi^{\mathcal{C}}_{n+2}$ to the n-th k-invariant $k_n : P_n X \to E^{\Lambda}_{\mathcal{C}}(\pi^{\mathcal{C}}_{n+1} X, n+2)$ yields the homomorphism $s^X_{n+1} : \Gamma_{n+1} X \to \pi^{\mathcal{C}}_{n+1} X$.

Proof. Since T is special, $\pi^{\mathcal{D}}_{n+2} T E^{\Lambda}_{\mathcal{C}}(\pi^{\natural}_{n+1}X, n+2) \cong \pi^{\mathcal{C}}_{n+1}X$, and $\pi^{\mathcal{D}}_{n+2} T P_n X \cong \Gamma_{n+1}X$ from (5.7), so this follows from the naturality of the comparison exact sequence, applied to the maps in (1.18). $\square$

5.9. **Lemma.** *If $T : \mathcal{C} \to \mathcal{D}$ is a special spherical functor, for any $X \in \mathcal{C}$,*

$$\begin{tikzcd} P_{n+1} T P_n X \arrow[r] \arrow[d] & P_{n+1} T P_{n-1} X \arrow[d] \\ P_{n+1} T B_{\mathcal{C}}\Lambda \arrow[r, "Tk_n"] & P_{n+1} T E^{\Lambda}_{\mathcal{C}}(\pi^{\mathcal{C}}_n X, n+2) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{D}/TB_{\mathcal{C}}\Lambda$, where $\Lambda := \pi^{\mathcal{C}}_0 X$.

Proof. Set $E := TE^{\Lambda}_{\mathcal{C}}(J, n+1)$, $M^{n-1} := TP_{n-1}X$, and $M^n := TP_n X$. The naturality of the comparison exact sequence, applied to the maps in (1.18), combined with Fact 5.8, imply that the vertical maps in the following commutative diagram are isomorphisms:

$$\begin{tikzcd} \pi^{\mathcal{D}}_{n+2} E \arrow[r] \arrow[d, "\cong"] & \pi^{\mathcal{D}}_{n+1} M^n \arrow[r] \arrow[d, "\cong"] & \pi^{\mathcal{D}}_{n+1} M^{n-1} \arrow[r, "Tk_{n-1}"] \arrow[d, "\cong"] & \pi^{\mathcal{D}}_{n+1} E \arrow[r] \arrow[d, "\cong"] & \pi^{\mathcal{D}}_n M^n \arrow[r] \arrow[d, "\cong"] & \pi^{\mathcal{D}}_n M^{n-1} \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{Coker}\ h^X_{n+1} \arrow[r, hook] & \Gamma_n X \arrow[r, "s^X_n"] & \pi^{\mathcal{C}}_n X \arrow[r, "h^X_n"] & \pi^{\mathcal{D}}_n TX \arrow[r, two heads] & \mathrm{Coker}\ h^T_n \end{tikzcd}$$

and since the bottom row is part of the comparison long exact sequence, and the rest of the top sequence to the right is exact for by (5.5), the k-invariant square (1.18) induces a long exact sequence after applying $\pi^{\natural}$ (except in the bottom dimensions). The obvious map from M^n to the fiber of Tk_{n-1} is thus a weak equivalence in $\mathcal{D}/TB_{\mathcal{C}}\Lambda$ through dimension $n+1$. $\square$

5.10. **Corollary.** *For $T : \mathcal{C} \to \mathcal{D}$ as above, for any $X \in \mathcal{C}$ and $n \geq 1$ the natural map $r^{(n)} : X \to P_n X$ of §3.1 induces an isomorphism $\Gamma_k X \cong \Gamma_k P_n X$ for $k \leq n+1$.*

Proof. For each $A \in \mathcal{A}$, take fibers vertically and horizontally of the commutative square:

$$\begin{tikzcd} \mathrm{map}_{B_{\mathcal{C}}\Lambda}(A, P_n X) \arrow[r, "h^{P_n X}"] \arrow[d, "(k_n)_*"] & \mathrm{map}_{B_{\mathcal{D}}\Lambda}(TA, TP_n X) \arrow[d, "(Tk_n)_*"] \\ \mathrm{map}_{B_{\mathcal{C}}\Lambda}(A, E^{\Lambda}_{\mathcal{C}}(\pi^{\mathcal{C}}_{n+1}X, n+2)) \arrow[r, "h^E"] & \mathrm{map}_{B_{\mathcal{D}}\Lambda}(TA, TE^{\Lambda}_{\mathcal{C}}(\pi^{\mathcal{C}}_{n+1}X, n+2)) \end{tikzcd} ,$$

and use Lemma 5.9 and §5.1(iv). $\square$

5.11. *Remark.* For $\mathcal{C} = s\mathcal{G}$ this follows from the fact that $\Gamma_n X \cong \Omega\pi^{\natural}_{n-1}X$, while for the algebraic cases of §2.11(i-ii), this follows from the fact that $H_{n+1}(K(\pi, n); \mathbb{Z}) = 0$ for $n \geq 1$.

5.12. **The extension.** The map $r^{(n)} : X \to P_nX$ induces a map of comparison exact sequences:

(5.13)
$$\begin{tikzcd}
\pi^{\mathcal{C}}_{n+2}X \arrow[r, "h^X_{n+2}"] \arrow[d] & \pi^{\mathcal{C}}_{n+2}TX \arrow[r, "\partial^\star_{n+2}"] \arrow[d, "\pi^{\mathcal{D}}_{n+2}Tr^{(n)}"] & \Gamma_{n+1}X \arrow[r, "s_{n+1}"] \arrow[d, "="] & \pi^{\mathcal{C}}_{n+1}X \arrow[r, "h^X_{n+1}"] \arrow[d] & \pi^{\mathcal{D}}_{n+1}TX \arrow[r, "\partial^\star_{n+1}"] \arrow[d, "\pi^{\mathcal{D}}_{n+1}Tr^{(n)}"] & \Gamma_nX \arrow[d, "="] \\
0 \arrow[r] & \pi^{\mathcal{D}}_{n+2}M^n \arrow[r, "\cong"] & \Gamma_{n+1}P_nX \arrow[r] & 0 \arrow[r] & \pi^{\mathcal{D}}_{n+1}M^n \arrow[r] & \Gamma_nP_nX
\end{tikzcd}$$

so that $\pi^{\mathcal{C}}_{n+1}X$ fits into a short exact sequence of $\Pi_{\mathcal{A}}$-algebras:

$$0 \to \mathrm{Coker}\ \pi^{\mathcal{D}}_{n+2}Tr^{(n)} \to \pi^{\mathcal{C}}_{n+1}X \to \mathrm{Ker}\ \pi^{\mathcal{D}}_{n+1}Tr^{(n)} \to 0, \tag{5.14}$$

where

$$\mathrm{Coker}\ \pi^{\mathcal{D}}_{n+2}Tr^{(n)} \cong \mathrm{Ker}\ h^X_{n+1} \quad \text{and} \quad \mathrm{Ker}\ \pi^{\mathcal{D}}_{n+1}Tr^{(n)} \cong \mathrm{Im}\ h^X_{n+1}. \tag{5.15}$$

Since h^X_{n+1} is a map of modules over $\Lambda := \pi^{\mathcal{C}}_0X$, by Theorem 3.8, (5.14) is actually a short exact sequence of modules over Λ, and we can classify the possible values of $J \in \Lambda\text{-}\mathcal{M}od$ (the candidates for $\pi^{\mathcal{C}}_{n+1}X$) using the following:

5.16**. Proposition.** *Given $Tr^{(n)} : TX \to TP_nX$, a choice for the isomorphism class of $\pi^{\mathcal{C}}_{n+1}X$ uniquely determines an element of*

$$\mathrm{Ext}_{\Lambda\text{-}\mathcal{M}od}(\mathrm{Ker}\ (Tr^{(n)})_{n+1}, \mathrm{Coker}\ (Tr^{(n)})_{n+2}).$$

Proof. Since $\Lambda\text{-}\mathcal{M}od$ is an abelian category, with a set $\{A_{\mathrm{ab}} \otimes S^n \amalg B_{\mathcal{D}}\Lambda\}_{A\in\mathcal{A},n\in\mathbb{N}}$ of projective generators, the argument of [Mc, III] carries over to our setting. $\square$

5.17. *Remark.* Observe that given P_nX, we know the comparison exact sequence (4.4) for X only from $s_n : \Gamma_{n-1}X \to \pi^{\mathcal{C}}_nX$ down. However, if $\pi^{\mathcal{D}}_iTr^{(n)} : \pi^{\mathcal{D}}_iTX \to \pi^{\mathcal{D}}_iM^n$ (for $i \geq 0$) and the extension (5.14) are also known, all we need in order to determine (4.4) for X from $\partial^\star_{n+3} : \pi^{\mathcal{D}}_{n+3}TX \to \Gamma_{n+1}X$ down is the homomorphism

$$\pi^{\mathcal{D}}_{n+3}Tr^{(n+1)} : \pi^{\mathcal{D}}_{n+3}TX \to \pi^{\mathcal{D}}_{n+3}TP_{n+1}X\ ,$$

which is just $\partial^\star_{n+3}$, as one can see from (5.13).

5.18**. Proposition.** *For any $\Lambda \in \mathcal{D}$, $J', J'' \in \Lambda\text{-}\mathcal{M}od$, and $n \geq 2$, there is a natural isomorphism $\mathrm{Ext}_{\Lambda\text{-}\mathcal{M}od}(J'', J') \cong H^{n+1}_{\Lambda}(E^{\Lambda}_{\mathcal{D}}(J'', n);\ J')$.*

In particular, this implies that $H^{n+1}_{\Lambda}(E^{\Lambda}_{\mathcal{D}}(-, n);\ -)$ is *stable* – i.e., independent of n.

Proof. By Proposition 3.13*ff.* there is a natural isomorphism

$$H^{n+1}_{\Lambda}(E^{\Lambda}_{\mathcal{D}}(J'', n);\ J') \cong [E^{\Lambda}_{\mathcal{D}}(J'', n), E^{\Lambda}_{\mathcal{D}}(J', n+1)]_{s\mathcal{D}/B_{\mathcal{D}}\Lambda},$$

and given a map $\psi : E^{\Lambda}_{\mathcal{D}}(J'', n) \to E^{\Lambda}_{\mathcal{D}}(J', n+1)$, we can form the fibration sequence over $B_{\mathcal{D}}\Lambda$ (that is, pullback square as in (1.18)):

$$\Omega E^{\Lambda}_{\mathcal{D}}(J'', n) \xrightarrow{\Omega\psi} \Omega E^{\Lambda}_{\mathcal{D}}(J', n+1) \simeq E^{\Lambda}_{\mathcal{D}}(J', n) \to F \to E^{\Lambda}_{\mathcal{D}}(J'', n) \xrightarrow{\psi} E^{\Lambda}_{\mathcal{D}}(J', n+1).$$

From the corresponding long exact sequence in homotopy for this sequence in $\mathcal{D}$, we obtain a short exact sequence of modules over Λ:

$$0 \to J' \to J \to J'' \to 0. \tag{5.19}$$

On the other hand, given a short exact sequence (5.19) in $\Lambda\text{-}\mathcal{M}od$, we can construct a map $\psi : E^{\Lambda}_{\mathcal{D}}(J'', n) \to E^{\Lambda}_{\mathcal{D}}(J', n+1)$ over $B_{\mathcal{D}}\Lambda$ as follows:

Assume $E := E^{\Lambda}_{\mathcal{D}}(J'', n)$ is constructed starting with $\mathrm{sk}_{n-1} E^{\Lambda}_{\mathcal{D}}(J'', :) = \mathrm{sk}_{n-1} B_{\mathcal{D}}\Lambda$, and $E_n \simeq W \amalg L_n B_{\mathcal{D}}\Lambda$ (cf. §2.2), where W is free, equipped with a surjection $\phi : W \to J''$. Because $J \twoheadrightarrow J''$ is a surjection, and W is free, we can lift ϕ to $\phi' : W \to J$, defining a map $\tilde{\phi}' : Z_n E^{\Lambda}_{\mathcal{D}}(J'', n) \to J$. Since $\pi^{\mathcal{D}}_n E^{\Lambda}_{\mathcal{D}}(J'', n) = J''$, the restriction of $\tilde{\phi}'$ to $B_n E^{\Lambda}_{\mathcal{D}}(J'', n) = \mathrm{Ker}\,\{Z_n E^{\Lambda}_{\mathcal{D}}(J'', n) \to J''\}$ factors through $\psi : B_n E^{\Lambda}_{\mathcal{D}}(J'', n) \to J' = \mathrm{Ker}\,\{J \to J''\}$. Precomposing with $\mathbf{d}_0 : C_{n+1} E^{\Lambda}_{\mathcal{D}}(J'', n) \to B_n E^{\Lambda}_{\mathcal{D}}(J'', n)$ defines $\psi : E^{\Lambda}_{\mathcal{D}}(J'', n) \to E^{\Lambda}_{\mathcal{D}}(J', n+1)$, which classifies (5.19) as before. ☐

5.20. **Corollary.** *For Λ, J', and J'' as above, there is a natural isomorphism:*

$$\mathrm{Ext}_{\Lambda\text{-}\mathcal{M}od}(J'', J') \cong H^{n+1}_{\Lambda}(E^{\Lambda}_{\mathcal{C}}(J'', n);\, J').$$

Proof. This follows from (5.2)-(5.7) and the naturality of $P^{\mathcal{D}}_{n+1}$. ☐

5.21. **Definition.** Given $X \in \mathcal{C}$, its *n-th modified Postnikov section*, denoted by $\hat{P}_n X$, is defined as follows:

Let $\mathcal{K} := \{f : A \otimes S^{n+1} \to X \mid A \in \mathcal{A},\ [f] \in \mathrm{Ker}\ h^T_{n+1} \subset \pi^{\mathcal{C}}_{n+1} X\}$, and let C be the cofiber of the obvious map $\Phi : \bigvee_{f \in \mathcal{K}} A \otimes S^{n+1} \to X$ (so that $\pi^{\mathcal{C}}_{n+1} C \cong \mathrm{Coker}\ \Phi$), with $\hat{P}_n X := P_{n+1} C$. There are then natural maps $\hat{p}^{(n+1)} : P_{n+1} X \to \hat{P}_n X$ (induced by $X \to C$), as well as $\check{p}^{(n)} : \hat{P}_n X \to P_n X$ (which is just $p^{(n)}_C : P_{n+1} C \to P_n C \cong P_n X$), with $\check{p}^{(n)} \circ \hat{p}^{(n)} = p^{(n)}_X : P_{n+1} X \to P_n X$. Note that $\pi^{\mathcal{C}}_{n+1} \hat{P}_n X \cong \mathrm{Im}\ h^X_{n+1}$, and $P_n \hat{P}_n X \cong P_n X$.

The map $\hat{r}^{(n)} := \hat{p}^{(n)} \circ r^{(n)} : X \to \hat{P}_n X$ induces a map of comparison exact sequences:

$$\begin{array}{ccccccccccc}
\pi^{\mathcal{C}}_{n+2} X & \xrightarrow{h^X_{n+2}} & \pi^{\mathcal{D}}_{n+2} T X & \xrightarrow{\partial^{\star}_{n+2}} & \Gamma_{n+1} X & \xrightarrow{s_{n+1}} & \pi^{\mathcal{C}}_{n+1} X & \xrightarrow{h^X_{n+1}} & \pi^{\mathcal{D}}_{n+1} T X & \xrightarrow{\partial_{n+1}} & \Gamma_n X \\
\downarrow & & \downarrow{\scriptstyle \pi^{\mathcal{D}}_{n+2} T \hat{r}^{(n)}} & & \downarrow{\scriptstyle =} & & \downarrow & & {\scriptstyle =}\downarrow{\scriptstyle \pi^{\mathcal{D}}_{n+1} T \hat{r}^{(n)}} & & \downarrow{\scriptstyle =} \\
0 & \longrightarrow & \pi^{\mathcal{D}}_{n+2} T \hat{P}_n X & \xrightarrow{\cong} & \Gamma_{n+1} \hat{P}_n X & \xrightarrow{0} & \pi^{\mathcal{C}}_{n+1} \hat{P}_n X & \hookrightarrow & \pi^{\mathcal{D}}_{n+1} T \hat{P}_n X & \longrightarrow & \Gamma_n \hat{P}_n X
\end{array}$$

so that:

$$\pi^{\mathcal{D}}_k T \hat{P}_n X \cong \begin{cases} \pi^{\mathcal{D}}_k T X & \text{for } k \leq n+1, \\ \Gamma_{n+1} X & \text{for } k = n+2, \\ \Gamma_{k-1} \hat{P}_n X & \text{for } k \geq n+3\ . \end{cases} \tag{5.22}$$

Thus $\hat{r}^{(n)}$ induces a weak equivalence $P_{n+1} T X \simeq P_{n+1} T \hat{P}_n X$, which, together with the existence of the appropriate maps $P_{n+1} X \xrightarrow{\hat{p}^{(n)}} \hat{P}_n X \xrightarrow{\check{p}^{(n)}} P_n X$, determines $\hat{P}_n X$ up to homotopy. In fact we have:

5.23. **Proposition.** *$\hat{P}_n X$ is determined uniquely (up to weak equivalence) by $P_n X$ and the map $\rho := P_{n+1} T r^{(n)} : P_{n+1} T X \to P_{n+1} T P_n X$.*

Proof. Note that $I_{n+1} := \mathrm{Ker}\ \pi^{\mathcal{D}}_{n+1} \rho$ is isomorphic to $\mathrm{Im}\ h^X_{n+1}$ and $C_{n+1} := \mathrm{Im}\ \pi^{\mathcal{D}}_{n+1} \rho$ is isomorphic to $\mathrm{Coker}\ h^X_{n+1}$ by (5.15).

We construct $Y \simeq \hat{P}_n X$ as follows, starting with $\mathrm{sk}_{n+1} Y := \mathrm{sk}_{n+1} P_n X$; by Remark 3.2, we may assume $\mathrm{sk}_{n+1} T X = \mathrm{sk}_{n+1} T P_n X$, so that $P_n T X \cong P_n T P_n X$.

By Fact 3.7), the lower right hand square in Figure 5.24 commutes in $\mathcal{D}$, thus inducing the rest of the diagram, in which the rows and columns are fibration sequences over $B_{\mathcal{D}}\Lambda$.

$$\begin{tikzcd}
F \arrow[rr, dotted, "\hat{\rho}"] \arrow[d, "\simeq" left, "\lambda"] & & P_{n+1}TP_nX \arrow[r, dotted, "\hat{k}_n"] \arrow[d, "p^{(n)}_{TP_nX}"] & E^{\Lambda}_{\mathcal{D}}(I_{n+1}, n+2) \arrow[d, "i_*"] \\
P_{n+1}TX \arrow[rr, "p^{(n)}_{TX}"] \arrow[d] & & P_nTX \cong P_nTP_nX \arrow[r, "k^{TX}_n"] \arrow[d, "k^{TP_nX}_n"] & E^{\Lambda}_{\mathcal{D}}(\pi^{\mathcal{D}}_{n+1}TX, n+2) \arrow[d, "q_*"] \\
B_{\mathcal{D}}\Lambda \arrow[rr] & & E^{\Lambda}_{\mathcal{D}}(C_{n+1}, n+2) \arrow[r, "="] & E^{\Lambda}_{\mathcal{D}}(C_{n+1}, n+2)
\end{tikzcd}$$

FIGURE 5.24

In particular, the induced map $\hat{k}_n : P_{n+1}TP_nX \to E^{\Lambda}(I_{n+1}, n+2)$ provides a canonical lifting of:

$$k^{TX}_n \circ p^{(n)}_{TP_nX} : P_{n+1}TP_nX \to E^{\Lambda}_{\mathcal{D}}(\pi^{\mathcal{D}}_{n+1}TX, n+2)$$

to $E^{\Lambda}_{\mathcal{D}}(I_{n+1}, n+2)$. Composing it with the natural map $r^{(n+1)} : TP_nX \to P_{n+1}TP_nX$ defines an element in:

$$[TP_nX, E^{\Lambda}_{\mathcal{D}}(I_{n+1}, n+2)] \cong H^{n+2}_{\Lambda}(P_nX;\, I_{n+1})\ ,$$

which we call the *n-th modified k-invariant* for X.

If $\hat{\mathbf{k}}_n : P_nX \to E^{\Lambda}_{\mathcal{C}}(I_{n+1}, n+2)$ is the map corresponding to $\hat{k}_n$ under (5.2)), then its homotopy fiber Y is (weakly equivalent to) $\hat{P}_nX$, as one can verify by calculating $\pi^{\mathcal{C}}_*Y$. Note that Lemma 5.9 implies that $F \simeq P_{n+1}TP_nX$, so that λ is the homotopy inverse of the weak equivalence $P_{n+1}\rho : TX \to TP_nX$, which completes the construction. □

5.25. *Remark.* Note that there is a certain indeterminacy in our description of $\hat{k}_n$, and thus of $\hat{\mathbf{k}}_n$, since we must make the lower right corner of Figure 5.24 into a strict commuting diagram of fibrations, rather than one which commutes only up to homotopy. However,

5.26. **Fact.** The indeterminacy for $\hat{k}_n$ as an induced map is contained in the indeterminacy for $\hat{k}_n$ as a k-invariant for $P_{n+1}TX = P_{n+1}TY$.

Proof. Let $M := TP_nX$. Making the lower right corner of Figure 5.24 commute on the nose (assuming q_* is already a fibration) requires the choice of a homotopy

$$H : P_nTX \to \Omega E^{\Lambda}_{\mathcal{D}}(C_{n+1}, n+2) = E^{\Lambda}_{\mathcal{D}}(C_{n+1}, n+1)\ ,$$

so the indeterminacy for $\hat{k}_n$ as defined above is $\psi_*p^*[P_nTX, E^{\Lambda}_{\mathcal{D}}(C_{n+1}, n+1)]$, where $\psi : E^{\Lambda}_{\mathcal{D}}(C_{n+1}, n+1) \to E^{\Lambda}_{\mathcal{D}}(I_{n+1}, n+2)$ classifies the extension

$$0 \to I_{n+1} \to \pi^{\mathcal{D}}_{n+1}TX \to C_{n+1} \to 0$$

(Proposition 5.18), and $p = p^{(n)}_M : P_{n+1}M \to P_nM = P_nTX$.

On the other hand, the k-invariant $\hat{k}^M_n : P_{n+1}M \to E^{\Lambda}_{\mathcal{D}}(I_{n+1}, n+2)$ for $P_{n+1}TP_nX$ (which is $P_{n+1}TX$) is determined only up to the actions of the group $\mathrm{haut}_{\Lambda}(P_{n+1}M)$

of homotopy self-equivalences of $P_{n+1}M$ over $B_{\mathcal{D}}\Lambda$, and of $\mathrm{Aut}_\Lambda(I_{n+1})$, the group of automorphisms of modules over Λ of I_{n+1}, in $[P_{n+1}M, E^\Lambda_{\mathcal{D}}(I_{n+1}, n+2)]$. Thus given a map $f : P_nM \to E^\Lambda_{\mathcal{D}}(C_{n+1}, n+1)$, we obtain a self-map $g : P_{n+1}M \to P_{n+1}M$ such that $P_ng = \mathrm{Id}_{P_nM}$ and $\pi^{\mathcal{D}}_{n+1}g = \mathrm{Id}$, by letting $g = \mathrm{Id} + i_*p^*(f)$, for $i : E^\Lambda_{\mathcal{D}}(C_{n+1}, n+1) \to P_{n+1}M$ the inclusion of the fiber. It is readily verified that g induces the identity on $\pi^{\mathcal{D}}_* P_{n+1}M$, so $[g] \in \mathrm{haut}_\Lambda(P_{n+1}M)$, and that $\hat{k}_n + \psi_*p^*(f)$ is obtained from $\hat{k}_n$ under the action of $[g]$ on $H^{n+2}_\Lambda(P_{n+1}M; I_{n+1})$. □

5.27. *Notation.* Given $W \simeq P_nX$ and $\rho : P_{n+1}TX \to P_{n+1}TW$, Proposition 5.23 allows us to write $\hat{P}_n(W, \rho)$, or simply $\hat{P}_nW$ for $\hat{P}_nX \in \mathcal{C}$, which they determine up to homotopy. This comes equipped with a weak equivalence $\rho : P_{n+1}TX \to P_{n+1}T\hat{P}_nW$ lifting ρ.

5.28. **Corollary.** *The weak equivalence $\rho : P_{n+1}TX \to P_{n+1}T\hat{P}_nW$ is well-defined up to homotopy.*

Proof. The map ρ is inverse to λ in Figure 5.24, which is induced by the upper right hand square, which is determined by $\hat{k}_n$ and thus up to a self-equivalence $g : P_{n+1}TW \to P_{n+1}TW$, according to Fact 5.26. But such a g induces a canonical self-equivalence $g' : F' \to F$, where $F' := \mathrm{Fib}\,(\hat{k}_n \circ g)$, and the resulting $\lambda' : F' \simeq P_{n+1}TX$ satisfies $\lambda \circ g' \simeq \lambda'$. □

5.29. **Definition.** For $W \simeq P_nX$ and $\rho : P_{n+1}TX \to P_{n+1}TW$ as above, an extension

$$0 \to \mathrm{Coker}\ \pi^{\mathcal{D}}_{n+2}\rho \hookrightarrow J \twoheadrightarrow \mathrm{Ker}\ \pi^{\mathcal{D}}_{n+1}\rho \to 0 \tag{5.30}$$

is called *allowable* if its classifying cohomology class

$$[\psi] \in H^{n+3}_\Lambda(E^\Lambda_{\mathcal{D}}(\mathrm{Coker}\ \pi_{n+2}\rho, n+2); \mathrm{Ker}\ \pi_{n+1}\rho)$$

(cf. Proposition 5.18) satisfies $[\psi] \circ \hat{\mathbf{k}}_n = 0$.

5.31. **Proposition.** *For any $X \in \mathcal{C}$, the extension* (5.14) *is allowable.*

Proof. Writing $V \simeq P_{n+1}X$ and $Y \simeq \hat{P}_nX$, by naturality we have a commutative square:

$$\begin{tikzcd}
P_nV \arrow[r, "k_n"] \arrow[d, "="] & E^\Lambda_{\mathcal{C}}(\pi^{\mathcal{C}}_{n+1}V, n+2) \arrow[d, "q_*"] \\
P_nY \arrow[r, "k_n"] & E^\Lambda_{\mathcal{C}}(\mathrm{Ker}\ \pi_{n+1}Tr^{(n)}, n+2).
\end{tikzcd}$$

Lemma 5.9 and (5.2) then yield the following commuting diagram in $\mathcal{D}$ in which the rows and columns are all fibration sequences over $B_{\mathcal{D}}\Lambda$:

$$\begin{tikzcd}
E^{\Lambda}_{\mathcal{D}}(\text{Ker } h_n, n+1) \arrow[r] \arrow[d] & B_{\mathcal{D}}\Lambda \arrow[r] \arrow[d] & E^{\Lambda}_{\mathcal{D}}(\text{Ker } h_{n+1}, n+2) \arrow[d] \\
TP_{n+1}X \arrow[r, "Tr^{(n)}"] \arrow[d, "r^{(n+2)}"] & TP_nX \arrow[r, "k"] \arrow[d, "="] & E^{\Lambda}_{\mathcal{D}}(\pi^{\mathcal{C}}_{n+1}X, n+2) \arrow[d, "q_*"'] \\
TY \arrow[r] & TP_nX \arrow[r, "\hat{k}"] & E^{\Lambda}_{\mathcal{D}}(\text{Im } h_{n+1}, n+2) \arrow[d, "\psi"'] \\
 & & E^{\Lambda}_{\mathcal{D}}(\text{Ker } h_{n+1}, n+3)
\end{tikzcd}$$

The map k is induced by k_n, and $\hat{k}$ is induced by $\hat{\mathbf{k}}_n$. The claim then follows from the universal property for fibrations. $\square$

6. The fiber of a special spherical functor

Let $T : \mathcal{C} \to \mathcal{D}$ be a special spherical functor. We would like to use the results of Section 5 in order to determine whether a given $G \in \mathcal{D}$ is (up to homotopy) of the form TX for some $X \in \mathcal{C}$ – and if so, how we can distinguish between such *realizations*, or liftings.

6.1. **Lifting objects of $\mathcal{D}$.** Let us assume for simplicity that $\Lambda := \pi^{\mathcal{D}}_0 G$ is a $\Pi_{\mathcal{C}}$-algebra, and that the map $\phi_T : \Lambda \to \pi^{\mathcal{D}}_0 TB_{\mathcal{C}}\Lambda$ of §5.1(i) is an isomorphism. [In the general case, we are faced with an additional, purely algebraic, problem of determining the fiber of the functor $T_* : \Pi_{\mathcal{C}}\text{-}\mathcal{A}lg \to \Pi_{\mathcal{D}}\text{-}\mathcal{A}lg$ (compare [BP]); we bypassed this question in §4.1(iv).

We want a map $\varphi : TX \to G$ inducing isomorphisms $\pi^{\mathcal{D}}_i TX \to \pi^{\mathcal{D}}_i G$ for $i \geq 0$. Our approach is inductive: we are trying to define a tower in $\mathcal{C}$:

$$\cdots \xrightarrow{p^{(n+1)}} \hat{X}\langle n+1\rangle \xrightarrow{p^{(n)}} \hat{X}\langle n\rangle \xrightarrow{p^{(n-1)}} \cdots \xrightarrow{p^{(0)}} \hat{X}\langle 0\rangle \simeq B_{\mathcal{C}}\Lambda \tag{6.2}$$

which are to serve as the modified Postnikov tower of the (putative) $X \in \mathcal{C}$ – so that in the end we will have $X := \text{holim}_n \hat{X}\langle n\rangle$.

At the n-th stage $(n \geq 0)$, we have constructed $\hat{X}\langle n\rangle$ as our candidate for $\hat{P}_n X$ – so in particular if we let $X\langle n\rangle := P_n\hat{X}\langle n\rangle$, (our candidate for the ordinary n-th Postnikov section of X), then $TX\langle n\rangle$ satisfies (5.7), $T\hat{X}\langle n\rangle$ satisfies (5.22), and of course $\hat{X}\langle n\rangle = P_{n+1}\hat{X}\langle n\rangle$.

Assume also, as part of our inductive hypothesis, a given weak equivalence:

$$\hat{\rho}^{(n)} : P_{n+1}G \xrightarrow{\simeq} P_{n+1}T\hat{X}\langle n\rangle. \tag{6.3}$$

We start the induction with $X\langle 0\rangle := B_{\mathcal{C}}\Lambda$. The natural map $r^{(1)} : G \to P_1TB_{\mathcal{C}}\Lambda = B_{\mathcal{D}}\Lambda$ allows us to define $\hat{X}\langle 0\rangle$, together with $\hat{\rho}^{(0)} : P_1G \xrightarrow{\simeq} P_1T\hat{X}\langle 0\rangle$, as in Definition 5.21 (see §5.25).

6.4. **Lifting $\rho^{(n)}$.** The first stage in the inductive step occurs in $\mathcal{D}$: we must lift $\hat{\rho}^{(n)}$ to $\rho^{(n)} : P_{n+2}G \to P_{n+2}T\hat{X}\langle n\rangle$. Note that by Remark 5.17 and Fact 5.8, we already know the comparison exact sequence (4.4) for the putative X from h_{n+1} down; the lifting $\rho := \rho^{(n)}$ will determine $\partial_{n+2} : \pi^{\mathcal{D}}_{n+2}G \to \Gamma_{n+1}\hat{X}\langle n\rangle$ in addition, since this is just $\pi_{n+2}\rho$, so that $C_{n+2} := \operatorname{Im}\, \pi^{\mathcal{D}}_{n+2}\rho$ is our candidate for $\operatorname{Coker}\, h^X_{n+2}$, while $K_{n+1} := \operatorname{Coker}\, \pi^{\mathcal{D}}_{n+2}\rho$ is our candidate for $\operatorname{Ker}\, h^X_{n+1}$.

From (5.22) we see that the obstruction is the class:

$$\chi_n := k^{T\hat{X}\langle n\rangle}_{n+1} \circ \rho^{(n)} \in H^{n+3}_{\Lambda}(G;\, \Gamma_{n+1}\hat{X}\langle n\rangle)\ , \tag{6.5}$$

and the different liftings are classified by $H^{n+2}_{\Lambda}(G;\, \Gamma_{n+1}\hat{X}\langle n\rangle)$.

6.6. **Constructing $X\langle n+1\rangle$.** The next step is to choose a cohomology class $\hat{\mathbf{k}}_n$ in $H^{n+2}_{\Lambda}(\hat{X}\langle n\rangle;\, K_{n+1})$. This fits into a commutative diagram with rows and fibers all fibration sequences over $B_{\mathcal{C}}\Lambda$:

$$\begin{tikzcd}
B_{\mathcal{C}}\Lambda \arrow[r] \arrow[d] & E^{\Lambda}_{\mathcal{C}}(I_{n+1}, n+1) \arrow[r, "="] \arrow[d, "i"] & E^{\Lambda}_{\mathcal{C}}(I_{n+1}, n+1) \arrow[d, "\psi"] \\
X\langle n+1\rangle \arrow[r, "\hat{\rho}^{(n)}"] \arrow[d] & \hat{X}\langle n\rangle \arrow[r, "\hat{\mathbf{k}}_n"] \arrow[d] & E^{\Lambda}_{\mathcal{C}}(K_{n+1}, n+2) \arrow[d, "j_*"] \\
X\langle n+1\rangle \arrow[r] & X\langle n\rangle \arrow[r, dotted, "k_n"] & E^{\Lambda}_{\mathcal{C}}(J, n+2)
\end{tikzcd}$$

for the bottom fibration sequence $X\langle n+1\rangle \to X\langle n\rangle \to E^{\Lambda}_{\mathcal{C}}(J, n+2)$ as indicated (though we shall not need this).

Note that J, our candidate for $\pi^{\mathcal{C}}_{n+1}X$, fits into the short exact sequence of modules over Λ:

$$0 \to K_{n+1} \hookrightarrow J \twoheadrightarrow I_{n+1} \to 0,$$

as in (5.14), and is classified by $\psi := \hat{\mathbf{k}}_n \circ i \in H^{n+2}_{\Lambda}(E^{\Lambda}_{\mathcal{C}}(I_{n+1}, n+1);\, K_{n+1})$, as in Corollary 5.20. Moreover, this extension is obviously allowable in the sense of §5.29.

6.7. **Lifting ρ.** To complete the induction on (6.3), we must lift $\rho : G \to P_{n+2}T\hat{X}\langle n\rangle$. This will be done in two steps:

First, note that we obtain a commuting diagram:

$$\begin{tikzcd}
P_{n+2}G \arrow[rr, "\rho"] \arrow[dr, dotted, "\rho"'] \arrow[dd, "p^{(n+1)}_G"'] & & P_{n+2}T\hat{X}\langle n\rangle \arrow[dd, "p^{(n+1)}_{T\hat{X}\langle n\rangle}"] \\
& P_{n+2}TX\langle n+1\rangle \arrow[ur, "\tilde{i}_*"'] \arrow[d, "p^{(n+1)}_{TX\langle n+1\rangle}"] & \\
P_{n+1}G \arrow[r, "f", "\simeq"'] \arrow[dd, "k^G_{n+1}"'] & P_{n+1}TX\langle n+1\rangle \arrow[r, "g", "\simeq"'] \arrow[d, "k^{TX\langle n+1\rangle}_{n+1}"] & P_{n+1}T\hat{X}\langle n\rangle \arrow[dd, "k^{T\hat{X}\langle n\rangle}_{n+1}"] \\
& E^{\Lambda}_{\mathcal{D}}(C_{n+2}, n+3) \arrow[dr, "i_*"] & \\
E^{\Lambda}_{\mathcal{D}}(\pi^{\natural}_{n+2}G, n+3) \arrow[ur, dotted, "q_*"] \arrow[rr, "(\pi^{\natural}_{n+2}\rho)_*"] & & E^{\Lambda}_{\mathcal{D}}(\pi^{\mathcal{D}}_{n+1}TX, n+3)
\end{tikzcd}$$

in which the columns are fibration sequences over $B_{\mathcal{C}}\Lambda$, since by definition

$$\pi^{\mathcal{D}}_{n+2}\rho : \pi^{\mathcal{D}}_{n+2}G \to \pi^{\mathcal{D}}_{n+1}T\hat{X}\langle n\rangle = \pi^{\mathcal{D}}_{n+1}TX$$

factors through $C_{n+2} := \operatorname{Im}\,\pi^{\mathcal{D}}_{n+2}\rho$, so that the bottom triangle commutes.

Since the natural K-invariant k^{G}_{n+1} is given, the other two k-invariants in the diagram above are determined by inverting the given homotopy equivalences $f : P_{n+1}G \to P_{n+1}TX\langle n+1\rangle$ and $g : P_{n+1}G \to P_{n+1}T\hat{X}\langle n\rangle$ (assuming all objects in $\mathcal{D}$ are fibrant and cofibrant), and letting $k^{TX\langle n+1\rangle}_{n+1} := q_{*}\circ k^{G}_{n+1}\circ f^{-1}$ and $k^{T\hat{X}\langle n\rangle}_{n+1} := i_{*}\circ k^{G}_{n+1}\circ g^{-1}$, using Fact 3.7.

Therefore, the map $\rho : G \to P_{n+2}T\hat{X}\langle n\rangle$ lifts to $\rho : P_{n+2}G \to P_{n+2}TX\langle n+1\rangle$ (which is induced by q_{*}). In fact, the lifting ρ is unique up to homotopy. Moreover, from the proof of Proposition 5.23 we see that this suffices to define $\hat{X}\langle n+1\rangle$, as well as determining a lifting of ρ to a weak equivalence $\hat{\rho}^{(n+1)} : P_{n+2}G \to P_{n+2}T\hat{X}\langle n+1\rangle$.

We may summarize our results in:

6.8. **Theorem.** *Given $G \in \mathcal{D}$, there is an object $X \in \mathcal{C}$ such that $TX \simeq G$ if and only if there is a tower as in* (6.2)*, serving as the modified Postnikov tower for X. If we have constructed $\hat{X}\langle n\rangle$ satisfying* (6.3) *for n, a necessary and sufficient condition for the existence of an $\hat{X}\langle n+1\rangle$ satisfying* (6.3) *for $n+1$ is the vanishing of $\chi_{n} \in H^{n+3}_{\Lambda}(G;\,\Gamma_{n+1}\hat{X}\langle n\rangle)$. The choices are classified by:*

- $H^{n+2}_{\Lambda}(G;\,\Gamma_{n+1}\hat{X}\langle n\rangle)$ *(distinguishing the liftings of $\hat{\rho}^{(n)}$ to $P_{n+2}T\hat{X}\langle n\rangle$); and*
- $\hat{\mathbf{k}}_{n} \in H^{n+2}_{\Lambda}(\hat{X}\langle n\rangle;\,K_{n+1})$*, where* $K_{n+1} := \operatorname{Coker}\,\pi_{n+2}\rho^{(n)}$*, up to self-homotopy equivalences of $\hat{X}\langle n\rangle$ over $B_{\mathcal{C}}\Lambda$ and* $\operatorname{Aut}_{\Lambda}(K_{n+1})$*. In particular, this distinguishes the class of $\pi^{\mathcal{C}}_{n+1}X$ in* $\operatorname{Ext}_{\Lambda\text{-}\mathcal{M}od}(\operatorname{Ker}(Tr^{(n)})_{n+1}, \operatorname{Coker}(Tr^{(n)})_{n+2})$.

Note that $\Gamma_{n+1}\hat{X}\langle n\rangle = \Gamma_{n+1}\hat{X}\langle n+1\rangle = \Gamma_{n+1}X$, by Corollary 5.10.

6.9. **Moduli spaces.** It is possible to refine the statement of our fundamental problem of lifting $G \in \mathcal{D}$ to $\mathcal{C}$ in terms of *moduli* spaces:

Given a model category $\mathcal{C}$, let $\mathfrak{W}$ be a homotopically small subcategory of $\mathcal{C}$, such that all maps in $\mathfrak{W}$ are weak equivalences, and if $f : X \to Y$ is a weak equivalence in $\mathcal{C}$ with either X or Y in $\mathfrak{W}$, then $f \in \mathfrak{W}$. Recall from [DK1, §2.1] that the nerve $B\mathfrak{W}$ of such a category is called a *classification complex.* Its components are in one-to-one correspondence with the weak homotopy types (in $\mathcal{C}$) of the objects of $\mathfrak{W}$, and the component containing $X \in \mathcal{C}$ is weakly equivalent to the classifying space $B\,\mathrm{haut}\,X$ of the monoid of self-weak equivalences of X.

6.10. **Definition.** Given a spherical functor $T : \mathcal{C} \to \mathcal{D}$ and $G \in \mathcal{D}$, we denote by $\mathcal{M}(G)$ the category of objects in $\mathcal{D}$ weakly equivalent to G (with weak equivalences as morphisms), and by $\mathcal{T}M(G)$ the category of objects $X \in \mathcal{C}$ such that $TX \in \mathcal{M}(G)$ (again, with weak equivalences in $\mathcal{C}$ as morphisms). The "pointed" version is denoted by $\mathcal{R}(G)$ – the category of pairs (X, ρ), where $X \in \mathcal{C}$ and $\rho : G \to TX$ is a specified weak equivalence.

In all our examples the obvious functors $\mathcal{R}(G) \xrightarrow{F} \mathcal{T}M(G) \xrightarrow{T} \mathcal{M}(G)$ preserve fibrant and cofibrant objects, and thus induce a homotopy pullback diagram:

$$\begin{tikzcd} B\mathcal{R}(G) \arrow[r, "BF"] \arrow[d] & B\mathcal{T}M(G) \arrow[d, "BT"] \\ \{\mathrm{Id}_G\} \arrow[r] & B\mathcal{M}(G) \end{tikzcd}$$

and there are weak equivalences $B\mathcal{T}M(G) \simeq \coprod_{X\in\pi_0\mathcal{T}M(G)} B\,\mathrm{haut}\,X$, where $B\mathcal{M}(G) \simeq B\,\mathrm{Aut}(G)$ for $\mathrm{Aut}(G)$ the monoid of self weak equivalences of G.

6.11. **Towers of moduli spaces.** Although $B\mathcal{T}M(G)$ is the more natural object of interest in our context, it is more convenient to study $B\mathcal{R}(G)$ by means of a tower of fibrations, corresponding to the Postnikov system of $X \in \mathcal{R}(G)$:

Let $\mathcal{R}_n(G)$ denote the category whose objects are pairs $(\hat{X}\langle n\rangle, \rho')$, where $\hat{X}\langle n\rangle \in \mathcal{C}$ has $P_{n+1}\hat{X}\langle n\rangle \simeq \hat{X}\langle n\rangle$ and $\rho' : P_{n+1}G \to P_{n+1}T\hat{X}\langle n\rangle$ is a weak equivalence. The maps of $\mathcal{R}_n(G)$ are weak equivalences compatible with the maps $p^{(n)}$.

As in [BDG, Thm. 9.4], one can show that $B\mathcal{R}(G) \simeq \mathrm{holim}_n\, B\mathcal{R}_n(G)$, so we may try to obtain information about the moduli space $\mathcal{T}M(G)$ by studying the successive stages in the tower:

$$\ldots B\mathcal{R}_{n+1}(G) \xrightarrow{BF_n} B\mathcal{R}_n(G) \xrightarrow{BF_{n-1}} \ldots \to B\mathcal{R}_1(G). \tag{6.12}$$

However, from the discussion above we see that we need several intermediate steps in the study of $B\mathcal{R}_{n+1}(G) \to B\mathcal{R}_n(G)$, corresponding to the additional choices made in obtaining $\hat{P}_{n+1}X$ and $p^{(n+1)} : P_{n+2}G \xrightarrow{\simeq} P_{n+2}T\hat{P}_{n+1}X$ from $\hat{P}_nX$ and $p^{(n)} : P_{n+1}G \xrightarrow{\simeq} P_{n+1}T\hat{P}_nX$. As a result one obtains a refinement of the tower (6.12), where the successive fibers F are either empty, or else generalized Eilenerg-Mac Lane spaces, whose homotopy groups may be described in terms of appropriate Quillen cohomology groups. We leave the details to the reader; compare [BDG, Thm. 9.6].

7. Applying the theory

The approach to the lifting problem for a spherical functor $T : \mathcal{C} \to \mathcal{D}$ described in the previous section is somewhat unwieldy. However, in specific applications it may simplify in various ways. We illustrate this by a number of examples:

7.1. **Singular chains.** Consider the singular chain functor $C_* : \mathcal{T}_* \to \mathcal{C}hain$, which in the form $T : \mathcal{G} \to s\mathcal{A}b\mathcal{G}p$ is induced by abelianization (see §4.2(a)). Thus, given a chain complex G_*, we would like to find all topological spaces X (if any) with $C_*X \simeq G_*$. Over $\mathbb{Z}$, this is equivalent to the question of realizing a given sequence of homology groups.

Our approach uses Whitehead's exact sequence (4.6) to relate the (trivial) Postnikov system for the chain complex G_* to the modified Postnikov system for the space X, in which we attach at each stage not a single new homotopy group, but a pair of groups in adjacent dimensions, corresponding to the image and kernel respectively of the Hurewicz homomorphism.

It should be observed that the functor T involves only "algebraic" categories $\mathcal{C} = s\hat{\mathcal{C}}$, where $\hat{\mathcal{C}}$ – in our case, $\mathcal{G}p$ or $\mathcal{A}b\mathcal{G}p$ – has a trivial model category structure, as in §2.11(a-b). The analysis in Section 6 then simplifies considerably, in as much as the categories of $\Pi_{\mathcal{C}}$-algebras and $\Pi_{\mathcal{D}}$-algebras are simply $\mathcal{G}p$ and $\mathcal{A}b\mathcal{G}p$, respectively.

As noted in the Introduction, Baues's [Ba4, VI, (2.3)] is actually a generalization the obstruction theory described here for this case. His earlier approach in [Ba3] (as well as that of Benkhalifa in [Be] is parallel to this, though not framed in the same cohomological language. See [Man] for another viewpoint.

7.2. **Rationalization.** On the other hand, the rationalization functor $(-)_{\mathbb{Q}} : \mathcal{T} \to \mathcal{T}_{\mathbb{Q}}$, induced by the completed group ring functor $\hat{Q} : \mathcal{G}p \to \mathcal{H}opf$ (cf. §4.2(b)), is spherical but not special (Def. 5.1), and so the theory described here does not apply as is. In fact, one can see why if one considers the comparison exact sequence for $\hat{Q}$ (§4.5(b)): given a (simply-connected) rational space $G \in \mathcal{T}_{\mathbb{Q}}$, for each $\mathbb{Q}$-vector space $\pi_n G$, we need an abelian group $A = \pi_n X$ such that $A \otimes \mathbb{Q} \cong \pi_n G$, and then lift the rational k-invariants for X to integral ones. Thus, much of the indeterminacy for X is algebraic.

7.3. **Suspension.** The suspension functor $\Sigma : \mathcal{T}_* \to \mathcal{T}_*$, induced by the free group functor $\hat{F} : \mathcal{S}et_* \to \mathcal{G}p$ as in §4.2(c), is similar to singular chains, with the generalized EHP sequence replacing the "certain long exact sequence", and the modified Postniov systems involve the kernel and image of the suspension homomorphism $E : \pi_n X \to \pi_{n+1}\Sigma X$.

7.4. **Homotopy groups.** The motivating example for the treatment in this paper – and the only one which requires the full force of Section 6 – is the functor $\pi_* : \mathcal{T}_* \to \Pi\text{-}\mathcal{A}lg$, prolonged to simplicial spaces (as in as in §4.2(d)). However, even this case simplifies greatly if we want to realize a single Π-algebra Λ – that is, we take $G \in s\Pi\text{-}\mathcal{A}lg$ to be the constant simplicial Π-algebra $B\Lambda$.

Indeed, given a simplicial space X with $\pi_* X \simeq B\Lambda$ (which implies that $\pi_*\|X\| \cong G$), from the spiral exact sequence (4.7) we find that $\pi^\natural_n X \cong \Omega^n\Lambda$ for all $n \geq 0$, so that $h_n : \pi^\natural_n X \to \pi^\natural_n \pi_* X$ is trivial for $n > 0$. We do not need the modified Postnikov system in this case: the obstructions to realizing Λ (or G) are just the classes $\chi_n \in H^{n+3}(\Lambda; \Omega^{n+1}\Lambda)$, and the difference obstructions distinguishing between the different realizations are $\delta_n \in H^{n+2}(\Lambda; \Omega^{n+1}\Lambda)$ $(n \geq 1)$. See [BDG] and [BJT, §5] for two descriptions of this case.

7.5. *Remark.* Our obstruction theory is irrelevant, of course, for the inverse spherical functor $U : \mathcal{G} \to \mathcal{S}_*$ (see §4.13) – that is, in determining loop structures on a given topological space. Nevertheless, from (4.17) we can easily recover the well-known fact that $X \simeq \Omega Y$ is a loop space if and only if its k-invariants are suspensions of those of Y (cf. [AHK]).

7.6. **Lifting morphisms.** In all of the above examples, one can ask the analogous question regarding the lifting of *maps*, or more complicated diagrams, from $\mathcal{D}$ to $\mathcal{C}$. This can be addresses via Theorem 6.8 by transfering the spherical structure from $\mathcal{C}$ and $\mathcal{D}$ to the diagram categories $\mathcal{C}^I$ and $\mathcal{D}^I$ (cf. §2.11(d)). See [BJT, §8] for a detailed example.

Note that the k-invariants for a map of chain complexes are not trivial (cf. [D, (3.8)]), so the theory for realizing chain maps in $\mathcal{T}_*$ is correspondingly more complicated.

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Department of Mathematics, University of Haifa, 31905 Haifa, Israel

E-mail address: `blanc@math.haifa.ac.il`