MAPPING SPACES AND M-CW COMPLEXES

DAVID BLANC

Abstract. We describe a procedure for recovering $\mathbf{X}$ from the space of maps from $\mathbf{M}$ into $\mathbf{X}$, when $\mathbf{M}$ is constructed by cofibers of self-maps. This can be used to define an $\mathbf{M}$-CW approximation functor. The case when $\mathbf{M}$ is a Moore space is discussed in greater detail.

1. introduction

The concept of "homotopy groups with coefficients", in which spheres are replaced by a Moore spaces as the representing objects, were first studied by Peterson in [P], and in greater detail by Neisendorfer in his thesis ([N1]; see [N2]). Much of homotopy theory can be redone in this spirit, with an arbitrary but fixed space $\mathbf{M}$ and its suspensions replacing the spheres not only in the definition of homotopy groups, but also in that of a CW-complex, loop space, and so on. In particular, an $\mathbf{M}$-CW complex is a space constructed inductively by successively attaching $\mathbf{M}$-cells. This concept has been studied (under various names) by Bousfield, Dror-Farjoun, and others (cf. [Bo1, Bo2, BT1, Ch, D]. Some of the properties of ordinary CW complexes carry over *mutatis mutandis* to $\mathbf{M}$-CW complexes – e.g., the Whitehead theorem – but others do not. Compare [BT2].

In this note we address the question of recovering the space $\mathbf{X}$ from the mapping space $\mathbf{X}^M$, for a special class of "self-map resolvable" spaces $\mathbf{M}$ (see §2.1 below), a question analogous to the classical one of recovering $\mathbf{X}$ from $\Omega^n\mathbf{X}$ (§3.6). Just as for loop spaces, one needs some additional structure on $\mathbf{X}^M$ in order to do so. Our procedure for recovering $\mathbf{X}$ is given recursively by a sequence of homotopy colimits, in Theorem 2.13. We may also think of this procedure as another construction of an $\mathbf{M}$-CW approximation functor. Our approach can be made more explicit in the case of the mod p^r Moore space (see Theorem 3.8).

1.1. **notation and conventions.** $\mathcal{T}_*$ will denote the category of pointed CW complexes with base-point preserving maps, and by a *space* we shall always mean an object in $\mathcal{T}_*$, for which we shall use boldface letters: $\mathbf{X}$, $\mathbf{S}^n$. Then n-fold smash product will be denoted $\mathbf{X}^{\wedge n} \stackrel{Def}{=} \mathbf{X}\wedge\ldots\wedge\mathbf{X}$. The space of (pointed) maps between $\mathbf{X}$ and $\mathbf{Y}$ will be written $\mathbf{Y}^X$, rather than $\mathbf{map}_*(\mathbf{X},\mathbf{Y})$, and the map induced by $f:\mathbf{A}\to\mathbf{B}$ will be written $f^\#:\mathbf{X}^B\to\mathbf{X}^A$. When f is the degree k map between spheres, we write simply $k:\Omega^n\mathbf{X}\to\Omega^n\mathbf{X}$.

Let $\mathbf{M}\in\mathcal{T}_*$ be some d-dimensional "model space", which we shall assume to be a suspension. We adopt the stable convention that $\mathbf{M}^r$ denotes $\Sigma^{r-d}\mathbf{M}$ (so that $\mathbf{M}^r$ does not necessarily exist for $r<d$).

Date: July 7, 1996.

1991 *Mathematics Subject Classification.* Primary 55P99; secondary 55Q70, 55P60, 55S37.

Key words and phrases. mapping space, homotopy with coefficients, Moore space, cellular space, $\mathbf{M}$-CW complex.

I would like to thank the Institute of Mathematics at the Hebrew University of Jerusalem for its hospitality during the period when this paper was written.

Definition 1.2. The k-th *homotopy group with coefficients in* $\mathbf{M}$ of any space $\mathbf{X} \in \mathcal{T}_*$ is defined to be $\pi_k(\mathbf{X}; M) \stackrel{Def}{=} [\mathbf{M}^k, \mathbf{X}] \cong \pi_{k-d}\mathbf{X}^M$, for $k \geq d$. We say that a map $f : \mathbf{X} \to \mathbf{Y}$ in $\mathcal{T}_*$ is an $\mathbf{M}$-*equivalence* if $f_* : \pi_t(\mathbf{X}; M) \to \pi_t(\mathbf{Y}; M)$ is an isomorphism for all $t \geq d$ – that is, if $f^M : \mathbf{X}^M \to \mathbf{Y}^M$ is a weak homotopy equivalence (since $\mathbf{M}$ was assumed to be a suspension).

Definition 1.3. The class $\mathcal{C}_M$ is defined to be the smallest class of spaces in $\mathcal{T}_*$ containing $\mathbf{M}$ and closed under homotopy equivalences and arbitrary pointed homotopy colimits (cf. [D, Ch. 2, D.1]). If $\mathbf{X} \in \mathcal{C}_M$, we say $\mathbf{X}$ is an $\mathbf{M}$-*CW complex.*

Because $\mathbf{M}$ is a suspension, any $\mathbf{M}$-CW complex may be constructed inductively, starting with a discrete space, by taking cofibers of maps from suspensions of $\mathbf{M}$ (see [D, Ch. 2, E.3]).

Definition 1.4. An $\mathbf{M}$-CW complex $\hat{\mathbf{X}}$ equipped with an $\mathbf{M}$-equivalence $f : \hat{\mathbf{X}} \to \mathbf{X}$ is called an $\mathbf{M}$-CW *approximation* for $\mathbf{X}$. Any such $\hat{\mathbf{X}}$ will be denoted by $CW_M\mathbf{X}$; it is unique up to homotopy equivalence, by the analogue of the Whitehead Theorem ([W, V, Thm 3.8] and [D, Ch. 2, Thm E.1]).

There are a number of different constructions of $CW_M\mathbf{X}$ – see [Bo1], [CaPP], and [D, Ch. 2, B.1 & E.6].

1.5. **organization.** In section 2 we describe a recursive procedure for recovering $\mathbf{X}$ from $\mathbf{X}^M$ when $\mathbf{M}$ is "self-map resolvable" – that is, constructed by taking cofibers of self-maps. In section 3 we specialize to the case where $\mathbf{M}$ is a Moore space, and describe $\mathbf{M}$-CW complexes explicitly. In section 4 we describe the simplicial resolution of [Stv] in our context, and discuss its relevance to the construction of mapping spaces (Remark 4.14, as well as to $\mathbf{M}$-CW approximations (Proposition 4.15).

Acknowledgements. I would like to thank the referee for many useful comments and suggestions.

2. Mapping spaces

We first consider the following question: given a space $\mathbf{M}$ as in §1.1, and a homotopy equivalence $\mathbf{Y} \simeq \mathbf{X}^M$, what information regarding $\mathbf{Y}$ is needed in order to recover $\mathbf{X}$ from it? Note that, by Definition 1.2, we can only hope to recover $\mathbf{X}$ up to $\mathbf{M}$-equivalence.

The problem of recovering $\mathbf{X}$ from $\mathbf{X}^{\mathbf{M}}$ appears to be a hard one for arbitrary $\mathbf{M}$, so we restrict attention to the following special case:

Definition 2.1. We say that a space $\mathbf{V} \in \mathcal{T}_*$ is *self-map resolvable* (cf. [BT2, §2.1]) if there is a sequence of spaces $\{\mathbf{V}(m)\}_{m=-1}^{n}$, with $\mathbf{V} = \mathbf{V}(n)$, such that for each $m \geq 0$ there is a self-map $v_m : \Sigma^{d_m}\mathbf{V}(m-1) \to \mathbf{V}(m-1)$ with cofiber $\mathbf{V}(m)$. We always start with a (possibly localized) sphere $\mathbf{V}(-1) = \mathbf{S}^j$ (or $\mathbf{S}^j_{(p)}$), and assume $v_0 : \mathbf{V}(-1) \to \mathbf{V}(-1)$ is the degree k map $(k \geq 2)$, so $\mathbf{V}(0)$ is the $(j+1)$-dimensional mod k Moore space. For simplicity we assume that each v_m is a suspension, and all spaces are simply connected finite CW complexes.

Thus each $\mathbf{V}(m)$ satisfies Bousfield's n-supported J-torsion condtion (see [Bo2, §7.1]) for $J = \{p : p|k\}$ and some $n \geq 1$.

The most useful case is when each space $\mathbf{V}(m-1)$ is a p-local space of type m, and the map $v_m : \Sigma^{d_m}\mathbf{V}(m-1) \to \mathbf{V}(m-1)$ is a v_m-self-map (see [R, §1.5] or [HS, Thm. 5.12]). Such spaces and maps play a central role in the definition of v_n-periodicity.

2.2. **notation.** To simplify the notation, for the rest of this section we fix a self-map $v : \mathbf{M}^{t+d} \to \mathbf{M}^d$, and denote its homotopy cofiber by $\mathbf{V}$. By abuse of notation, any suspension of v will also be denoted by v; thus r-fold composites of suspensions of v will be written simply v^r (with dimensions understood from the context). The cofiber of $v^r : \mathbf{M}^{rt+s} \to \mathbf{M}^s$ will be denoted by $\mathbf{V}_r^{rt+s+1}$, and we shall write simply $\mathbf{V}_r$ for $\mathbf{V}_r^{rt+d+1}$ (so $\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_1^{t+d+1}$). Since we want $\mathbf{V}$ to be a co-H-space, we shall assume that $v : \mathbf{M}^{t+d} \to \mathbf{M}^d$ is a suspension.

Thus the problem at hand is that of recovering $\mathbf{X}$ from $\mathbf{X}^V$, where $\mathbf{V}$ is self-map resolvable. We assume inductively that we have a suitable procedure for recovering $\mathbf{X}$ (up to $\mathbf{M}$-equivalence, so in particular up to $\mathbf{V}$-equivalence) from $\mathbf{X}^M$, so in fact the inductive step, which we consider in this section, involves recovering $\mathbf{X}^M$ (up to $\mathbf{V}$-equivalence) from $\mathbf{X}^V$.

Remark 2.3. Just as in the case of loop spaces, one needs some structure on $\mathbf{X}^V$, beyond its bare homotopy type (in addition to the loop structure, which comes from the fact that $\mathbf{V}$ is a suspension), in order to recover $\mathbf{X}$. As in Stasheff's approach to delooping, our approach is iterative; so we shall not describe in advance the full structure on $\mathbf{X}^V$ which we postulate; the additional data needed at each stage in the process is given in §2.5. Of course, such data depends on the homotopy type of $\mathbf{X}$, not only on that of $\mathbf{X}^V$.

Again in analogy with the case of loop spaces, one really expects only to recover $CW_V\mathbf{X}$ from $\mathbf{X}^V$ (since by definition the augmentation $f : CW_V\mathbf{X} \to \mathbf{X}$ induces a homotopy equivalence $f^V : (CW_V\mathbf{X})^V \to \mathbf{X}^V$). Thus we may as well assume that $\mathbf{X}$ itself is a $\mathbf{V}$-CW complex to begin with. This may be restated as follows:

The additional structure on $\mathbf{Y} = \mathbf{X}^V$ which we need for our procedure consists of homotopy invariants of $\mathbf{X}$ (see §2.5 below). However, on the face of it these need not be invariant under $\mathbf{V}$-equivalences, and choosing a different space $\mathbf{X}'$ which is $\mathbf{V}$-equivalent to $\mathbf{X}$ – so that we may also view $\mathbf{Y}$ as $(\mathbf{X}')^V$ – may yield different invariants, and thus different constructions for recovering $\mathbf{X}'$. We do not claim that our procedure works for any such choice of $\mathbf{X}'$ (and thus of the invariants), but only for the "canonical" invariants – those associated to viewing $\mathbf{Y}$ as $(CW_V\mathbf{X})^V$.

2.4. **cofibration sequences.** The iterates of v fit into commutative diagrams, in which both the rows and the columns are cofibration sequences (up to homotopy):

$$\begin{tikzcd}
\mathbf{M}^{2rt+d} \arrow[r, "="] \arrow[d, "v^r"] & \mathbf{M}^{2rt+d} \arrow[r, "*"] \arrow[d, "v^{2r}"] & * \arrow[r, "*"] \arrow[d, "*"] & \mathbf{M}^{2rt+d+1} \arrow[d, "v^r"] \\
\mathbf{M}^{rt+d} \arrow[r, "v^r"] \arrow[d, "i_r^{rt+d}"] & \mathbf{M}^{d} \arrow[r, "i_r^d"] \arrow[d, "i_{2r}^d"] & \mathbf{V}_r^{rt+d+1} \arrow[r, "j_r^{rt+d+1}"] \arrow[d, "="] & \mathbf{M}^{rt+d+1} \arrow[d, "i_r^{rt+d+1}"] \\
\mathbf{V}_r^{2rt+d+1} \arrow[r, "\kappa_r^{2rt+d+1}"] \arrow[d, "j_r^{2rt+d+1}"] & \mathbf{V}_{2r}^{2rt+d+1} \arrow[r, "\lambda_r^{2rt+d+1}"] \arrow[d, "j_{2r}^{2rt+d+1}"] & \mathbf{V}_r^{rt+d+1} \arrow[r, "\beta_r^{rt+d+1}"] \arrow[d, "*"] & \mathbf{V}_r^{2rt+d+2} \arrow[d, "j_r^{rt+d+2}"] \\
\mathbf{M}^{2rt+d+1} \arrow[r, "="] & \mathbf{M}^{2rt+d+1} \arrow[r, "*"] & * \arrow[r, "*"] & \mathbf{M}^{2rt+d+2}
\end{tikzcd}$$

FIGURE 1

Thus we have:

(a) $\beta_r^s = i_r^{s+1} \circ j_r^s$ (so that $\beta_r^{s+1} \circ \beta_r^s \sim *$).
(b) $j_r^s = j_{2r}^s \circ \kappa_r^s$ and $i_r^s = \lambda_r^{s+2rt+1} \circ i_{2r}^s$ (so that $\beta_r^s = \lambda_{2r}^{s+2rt+1} \circ \beta_{2r}^s \circ \kappa_r^s$).

Remark 2.5. Mapping the third row of Figure 1 into $\mathbf{X}$ yields a fibration sequence: $\dots\mathbf{X}^{V_r} \xrightarrow{\lambda_r^\#} \mathbf{X}^{V_{2r}} \xrightarrow{\kappa_r^\#} \mathbf{X}^{V_r^{2rt+d+1}}$ (see [W, I, 7.8]), and thus a long exact sequence

$$\dots \to \pi_{2rt+d+2}(\mathbf{X};\mathbf{V}) \cong \pi_1\mathbf{X}^{V_r^{2rt+d+1}} \xrightarrow{\beta_r^*} \pi_0\mathbf{X}^{V_r} \xrightarrow{\lambda_r^*} \pi_0\mathbf{X}^{V_{2r}} \xrightarrow{\kappa_r^*} \pi_0\mathbf{X}^{V_r^{2rt+d+1}}.$$

Since $\mathbf{V}_r$, $\mathbf{V}_{2r}$, and all the maps in Figure 1 were assumed to be suspensions, this is actually an exact sequence of groups, which may be continued to the right as a sequence of pointed sets (cf. [Sp, 7,§2,Thm. 10]). In particular, one has a short exact sequence of groups:

$$0 \to Coker(\beta_r^*) \to \pi_0\mathbf{X}^{V_{2r}} \to Ker(\bar{\beta}_r^*) \to 0 \tag{2.6}$$

for $\beta_r^* : \pi_{2rt+d+2}(\mathbf{X};\mathbf{V}) \to \pi_{rt+d+1}(\mathbf{X};\mathbf{V}) = \pi_0\mathbf{X}^{V_r}$ and $\bar{\beta}_r^* : \pi_{2rt+d+1}(\mathbf{X};\mathbf{V}) \to \pi_{rt+d}(\mathbf{X};\mathbf{V}) = \pi_0\mathbf{X}^{\mathbf{V}_r^{rt+d}}$, where $\Sigma\mathbf{V}_r^{rt+d} = \mathbf{V}_r^{rt+d+1} = \mathbf{V}_r$ and $\Sigma\bar{\beta}_r = \beta_r$. Note that $\pi_0\mathbf{X}^{\mathbf{V}_r^{rt+d}}$ is just a pointed set, but $Ker(\bar{\beta}_r^*) = Im(\kappa_r^\#)$ is a group.

Thus $Coker(\beta_r^*)$ and $Ker(\bar{\beta}_r^*)$ determine $\pi_0\mathbf{X}^{V_{2r}} = [\mathbf{V}_{2r}^{2rt+d+1}, \mathbf{X}] = [\mathbf{V}_{2r}, \mathbf{X}]$ *as a set*, but since we shall also need its group structure, we must assume the extension in (2.6) is given, in addition to the homotopy class of the map $\beta_r^\# : \mathbf{X}^{V_{2rt+d+1}} \to \mathbf{X}^{V_{rt+d}}$, as part of the additional structure needed for our recovery procedure (see §2.3)

Remark 2.7. Moreover, again because Figure 1 desuspends, there is a topological group $\mathbf{H}$ and a closed normal subgroup $\mathbf{G} \overset{i}{\hookrightarrow} \mathbf{H}$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\cdots\ \Omega^{rt+d+2}\mathbf{X}^{V_r} \simeq \mathbf{X}^{V_r^{rt+d+2}} \arrow[r, "\beta_r^\#"] & \mathbf{X}^{V_r} \arrow[r, "\lambda_r^\#"] & \mathbf{X}^{V_{2r}} \arrow[r, "\kappa_r^\#"] & \mathbf{X}^{V_r} \\
\cdots\ \mathbf{G} \arrow[u, "\simeq"'] \arrow[r, "i"] & \mathbf{H} \arrow[u, "\simeq"'] \arrow[r, "q"] & \mathbf{H}/\mathbf{G} \arrow[u] \arrow[r, "Bi"] & B\mathbf{G} \arrow[u]
\end{tikzcd}$$

FIGURE 2

where the rows are fibration sequences, the two left vertical maps are homotopy equivalences, and the two right vertical maps are inclusions of components (up to homotopy), by [M3, Prop 7.9]. Since all the components of $\mathbf{X}^{V_{2r}}$ are homotopy equivalent, we have

$$\mathbf{X}^{V_{2r}} \simeq (\mathbf{H}/\mathbf{G})_{(0)} \times [\mathbf{V}_{2r}^{2rt+d+1}, \mathbf{X}] \simeq \coprod_{[\mathbf{V}_{2r},\mathbf{X}]} (\mathbf{H}/\mathbf{G})_{(0)} \tag{2.8}$$

(as a loop space), where $(\mathbf{H}/\mathbf{G})_{(0)}$ is the 0-component of $\mathbf{H}/\mathbf{G}$, say, and the group $[\mathbf{V}_{2r}^{2rt+d+1}, \mathbf{X}]$ is assumed to be known, since the extension (2.6) is given as part of our initial data.

Moreover, $\mathbf{H}/\mathbf{G}$ is homotopy equivalent to the realization $B(\mathbf{H},\mathbf{G},*)$ of the geometric bar construction $B_*(\mathbf{H},\mathbf{G},*)$, which is the simplicial space with $B_j(\mathbf{H},\mathbf{G},*) = \mathbf{H}\times\mathbf{G}^n\times\{*\}$ (and the obvious face and codegeneracy maps – see [M3, §7]). By [D, Ch. 2, D.16] we have thus exhibited $\mathbf{X}^{V_{2r}}$ as a homotopy colimit of a diagram defined in terms of $\mathbf{X}^{V_r}$, $\mathbf{X}^{\mathbf{V}_r^{rt+d+2}}$, the map $\beta_r^\#$ and the extension (2.6).

The cofibration sequences of Figure 1 fit into an inverse system:

$$\begin{tikzcd}
\cdots \arrow[r, "v^4"] & \mathbf{M}^{4t+d} \arrow[r, "v^2"] \arrow[d, "v^4"] & \mathbf{M}^{2t+d} \arrow[r, "v"] \arrow[d, "v^2"] & \mathbf{M}^{t+d} \arrow[d, "v"] \\
\cdots \arrow[r, "="] & \mathbf{M}^{d} \arrow[r, "="] \arrow[d, "i_4"] & \mathbf{M}^{d} \arrow[r, "="] \arrow[d, "i_2"] & \mathbf{M}^{d} \arrow[d, "i_1"] \\
\cdots \arrow[r, "\lambda_4"] & \mathbf{V}_4^{4t+d+1} \arrow[r, "\lambda_2"] \arrow[d, "j_4"] & \mathbf{V}_2^{2t+d+1} \arrow[r, "\lambda_1"] \arrow[d, "j_2"] & \mathbf{V}_1^{t+d+1} \arrow[d, "j_1"] \\
\cdots \arrow[r, "v^4"] & \mathbf{M}^{4t+d+1} \arrow[r, "v^2"] & \mathbf{M}^{2t+d+1} \arrow[r, "v"] & \mathbf{M}^{t+d+1}
\end{tikzcd}$$

FIGURE 3

and thus the fibration sequences obtained by mapping the above diagram into a fixed space $\mathbf{X}$ form a directed system as follows:

$$\begin{tikzcd}
\cdots & \mathbf{X}^{M^{4t+d}} \simeq \Omega^{4t}\mathbf{X}^{\mathbf{M}} \arrow[l, "(v^4)^{\#}"'] & \Omega^{2t}\mathbf{X}^{\mathbf{M}} \arrow[l, "(v^2)^{\#}"'] & \Omega^{t}\mathbf{X}^{\mathbf{M}} \arrow[l, "v^{\#}"'] \\
\cdots & \mathbf{X}^{\mathbf{M}} \arrow[l, "="'] \arrow[u, "(v^4)^{\#}"'] & \mathbf{X}^{\mathbf{M}} \arrow[l, "="'] \arrow[u, "(v^2)^{\#}"'] & \mathbf{X}^{\mathbf{M}} \arrow[l, "="'] \arrow[u, "v^{\#}"'] \\
\cdots & \mathbf{X}^{V_4} \arrow[l, "\lambda_4^{\#}"'] \arrow[u, "i_4^{\#}"'] & \mathbf{X}^{V_2} \arrow[l, "\lambda_2^{\#}"'] \arrow[u, "i_2^{\#}"'] & \mathbf{X}^{V} \arrow[l, "\lambda_1^{\#}"'] \arrow[u, "i_1^{\#}"'] \\
\cdots & \Omega^{4t+1}\mathbf{X}^{\mathbf{M}} \arrow[l, "(v^4)^{\#}"'] \arrow[u, "j_4^{\#}"'] & \Omega^{2t+1}\mathbf{X}^{\mathbf{M}} \arrow[l, "(v^2)^{\#}"'] \arrow[u, "j_2^{\#}"'] & \Omega^{t+1}\mathbf{X}^{\mathbf{M}} \arrow[l, "v^{\#}"'] \arrow[u, "j_1^{\#}"']
\end{tikzcd}$$

FIGURE 4

Remark 2.9. The diagram in Figure 4 can be changed up to homotopy so that all the maps $\lambda_{2^s}^{\#}$ are cofibrations, and $\Omega^{2^s t+1}\mathbf{X}^{\mathbf{M}} \xrightarrow{j_{2^s}^{\#}} \mathbf{X}^{\mathbf{V}_{2^s}} \xrightarrow{i_{2^s}^{\#}} \mathbf{X}^{\mathbf{M}} \xrightarrow{(v^{2^s})^{\#}} \Omega^{2^s t}\mathbf{X}^{\mathbf{M}}$ is a fibration sequence.

Definition 2.10. Recall from [Bo2, §1] that, given a fixed space $\mathbf{W}$, a space $\mathbf{X}$ is called $\mathbf{W}$-*local* (or $\mathbf{W}$-*periodic*) if $\mathbf{X}^{W} \simeq \star$. A map $f : \mathbf{A} \to \mathbf{B}$ is called a P_W-*equivalence* if $\mathbf{X}^{B} \xrightarrow{f^{\#}} \mathbf{X}^{A}$ is a (weak) homotopy equivalence for every $\mathbf{W}$-local space $\mathbf{X}$. Finally, a map $\varphi : \mathbf{X} \to \tilde{\mathbf{X}}$ is a $\mathbf{W}$-*localization* if $\tilde{\mathbf{X}}$ is $\mathbf{W}$-local and φ is a P_W-equivalence.

Such localizations exist for any $\mathbf{W}$. A functorial version of $\mathbf{W}$-localization is denoted by $P_W\mathbf{X}$; see [D, Ch. 1, §B] and [Bo2, §2].

Proposition 2.11. *If* $\mathbf{X}$ *is a* $\mathbf{V}$-*CW complex, and we set*

$$\mathbf{V}_\infty\mathbf{X} \stackrel{Def}{=} \operatorname{hocolim} \{\mathbf{X}^{\mathbf{V}} \xrightarrow{\lambda_1^{\#}} \mathbf{X}^{V_2} \xrightarrow{\lambda_2^{\#}} \mathbf{X}^{V_4} \ldots\},$$

then the map $i^{\#} : \mathbf{V}_\infty\mathbf{X} \to \mathbf{X}^{\mathbf{M}}$, *induced by the maps* $i_r : \mathbf{X}^{\mathbf{V}_r} \to \mathbf{X}^{\mathbf{M}}$, *is a weak homotopy equivalence.*

Proof. Set $\mathbf{M}_\infty\mathbf{X} \stackrel{Def}{=} \operatorname{hocolim}\{\Omega^t\mathbf{X}^{\mathbf{M}} \xrightarrow{v^\#} \Omega^{2t}\mathbf{X}^{\mathbf{M}} \xrightarrow{(v^2)^\#} \Omega^{4t}\mathbf{X}^{\mathbf{M}} \xrightarrow{(v^4)^\#} \dots\}$, and take the colimits of each row in Figure 4 to obtain a fibration sequence

$$\Omega\mathbf{M}_\infty\mathbf{X} \xrightarrow{j^\#} \mathbf{V}_\infty\mathbf{X} \xrightarrow{i^\#} \mathbf{X}^{\mathbf{M}} \xrightarrow{v^\#} \mathbf{M}_\infty\mathbf{X}.$$

(To see this is indeed a fibration sequence, consider the colimits as unions of simplicial groups; note that the homotopy colimits used to define $\mathbf{V}_\infty\mathbf{X}$ and $\mathbf{M}_\infty\mathbf{X}$ are just ordinary colimits, if we apply Remark 2.9). Now

$$\pi_i\mathbf{M}_\infty\mathbf{X} \;=\; \operatorname{colim}\,\{\pi_i(\mathbf{X};M) \xrightarrow{v^\#} \pi_{i+t}(\mathbf{X};M) \xrightarrow{(v^2)^\#} \pi_{i+2t}(\mathbf{X};M)\dots\} \qquad \text{for } i \geq 0$$

(cf. [Gr, Prop. 15.9]), and the right hand side is by definition the i-th v-periodic homotopy group of $\mathbf{X}$, usually written $v^{-1}\pi_i(\mathbf{X};M)$.

Taking $\mathbf{V}_{n-1} = \mathbf{M}$, $\omega = v$ and thus $\mathbf{V}_n = \mathbf{V}$ in [Bo2, §11.3], by [Bo2, Def. 9.1 & Thm. 9.12], we have $\langle V\rangle \leq \langle\Sigma\mathbf{W}_n\rangle$ for a suitable space $\mathbf{W}_n$ (see [Bo2, 10.1]), so the periodization map $\mathbf{X} \to P_V\mathbf{X}$, which is a P_V-equivalence, is also a P_{v_n}-equivalence (see [Bo2, §10.2]), and thus induces an isomorphism $v^{-1}\pi_*(\mathbf{X};M) \cong v^{-1}\pi_*(P_V\mathbf{X};M) = 0$ by [Bo2, Thm 11.5].

However, since $\mathbf{X}$ is a $\mathbf{V}$-CW complex, $P_V\mathbf{X}$ is contractible (cf. [D, Ch. 3, Prop. B.1]). We deduce that $\mathbf{M}_\infty\mathbf{X}$ is (weakly) contractible, and so $i_\infty : \mathbf{V}_\infty\mathbf{X} \simeq \mathbf{X}^M$. □

Remark 2.12. If $\phi : \mathbf{X}^V \to \mathbf{V}_\infty\mathbf{X}$ is the obvious map, then $i_\infty \circ \phi \simeq i_1^\# : \mathbf{X}^V \to \mathbf{X}^M$; thus ϕ is up to homotopy the inclusion of a closed normal subgroup, with quotient map $\psi : \mathbf{V}_\infty\mathbf{X} \to \mathbf{X}^M/\mathbf{X}^V$ (as in Remark 2.7 above). Now let $\omega : \mathbf{V}_\infty\mathbf{X}/\mathbf{X}^V \to B\mathbf{X}^V$ denote the classifying map of ψ (which, up to homotopy, is just j_1). Then the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathbf{X}^V \arrow[r, "\phi"] \arrow[d, "="] & \mathbf{V}_\infty\mathbf{X} \arrow[r, "\psi"] \arrow[d, "i_\infty"', "\simeq"] & \mathbf{V}_\infty\mathbf{X}/\mathbf{X}^V \arrow[r, "\omega"] \arrow[d, "\simeq"] & B\mathbf{X}^V \arrow[d, "="] \\
\mathbf{X}^V \arrow[r, "i_1^\#"] & \mathbf{X}^{\mathbf{M}} \arrow[r, "v^\#"] & \Omega^t\mathbf{X}^{\mathbf{M}} \arrow[r, "B(j_1^\#)"] & B\mathbf{X}^V
\end{tikzcd}$$

By 2.4(a) we have $B(\beta_1^\#) \simeq \Omega^t\phi \circ \omega$. This implies we can recover the columns of Figure 3 from the spaces $\{\mathbf{X}^{\mathbf{V}_{2^s}}\}_{s=0}^\infty$ and maps between them.

We may summarize the results of this section in the following

Theorem 2.13. *Suppose $v : \Sigma^t\mathbf{M} \to \mathbf{M}$ is a suspended self-map with cofiber $\mathbf{V}$, and $\mathbf{Y} = \mathbf{X}^V$ is a mapping space; then $\mathbf{X}^{\mathbf{M}}$ is $\mathbf{V}$-equivalent to the homotopy colimit of $\mathbf{X}^V \to \mathbf{X}^{V_2} \to \dots$ as in Proposition 2.11, and each $\mathbf{X}^{V_{2^{s+1}r}}$ is a disjoint union as in* (2.8)*, where each component is given by the homotopy colimit $B(\mathbf{X}^{V_{2^s r}}, \Omega^{rt}\mathbf{X}^{V_{2^s r}}, *)_{(0)}$ as in §2.7. At the s-th stage, the data needed to determine the diagrams over which we take these homotopy colimits consists of the map $\beta_{2^s r}^\#$ and the extension* (2.6).

Corollary 2.14. *If $\mathbf{V} = \mathbf{V}(n)$ is self-map resolvable, and $\mathbf{Y} \simeq \mathbf{X}^V$, $CW_V\mathbf{X}$ may be recovered from $\mathbf{Y}$ by a countable sequence of homotopy colimits.*

Proof. Since we assumed that $\mathbf{X}$ was a $\mathbf{V}$-CW complex (§2.3), Proposition 2.11 implies that $\mathbf{V}_\infty\mathbf{X} \simeq \mathbf{X}^M$. Since any $\mathbf{V}$-CW complex is in particular an $\mathbf{M}$-CW complex, the same holds throughout the inductive application of Theorem 2.13. □

Remark 2.15. The procedure we defined above is one for *recovering* $\mathbf{X}$ (up to $\mathbf{V}$-equivalence) from $\mathbf{X}^V$, rather than *recognizing* when $\mathbf{Y} \simeq \mathbf{X}^V$. Of course this yields an implicit method for recognizing spaces of maps from a self-map resolvable space $\mathbf{V}$: apply the procedure to all possible candidates for the maps $\beta_r^{\#}:\Omega^{rt+d+2}\mathbf{Y}(\simeq \mathbf{X}^{V_r^{rt+d+2}})\to\mathbf{Y}(\simeq\mathbf{X}^{V_r})$, etc., and verify that the resulting space $\bar{\mathbf{X}}$ satisfies $\bar{X}^V\simeq\mathbf{Y}$. In fact, one can say something about which maps $b:\Omega^{rt+d+2}\mathbf{Y}\to\mathbf{Y}$ *could be* of the form $\beta_r^{\#}$ (assuming $\mathbf{Y}\simeq\mathbf{X}^V$), by checking

$$b_{\#}:\pi_{i+rt+d+2}(\mathbf{Y};\mathbf{V})\to\pi_i(\mathbf{Y};\mathbf{V})=\pi_{i-(rt+d+1)}\mathbf{Y}^V=\pi_{i-(rt+d+1)}(\mathbf{X}^{V\wedge V}).$$

However, this is still far from an explicit recognition principle for spaces of maps from $\mathbf{V}$, comparable to those of May [M2], Cobb [Co], Smith [Sm], and others.

3. Moore CW complexes

We now specialize to the simplest example of a self-map resolvable space, the d-dimensional mod p^r Moore space $\mathbf{M}=\mathbf{M}^d(p^r)=\mathbf{S}^{d-1}\cup_{p^r}\mathbf{e}^d$. Throughout this section we shall write $\mathbf{X}^{M(r)}$ for the mapping space $\mathbf{X}^{\mathbf{M}^d(p^r)}$, and $\pi_t(\mathbf{X};p^r)$ for $\pi_t(\mathbf{X};M)=[\mathbf{M}^t(p^r),\mathbf{X}]$.

In this case we can say explicitly when a space $\mathbf{X}\in\mathcal{T}_*$ is of the homotopy type of an $\mathbf{M}$-CW complex:

Proposition 3.1. *A space $\mathbf{X}\in\mathcal{T}_*$ is an $\mathbf{M}^d(p^r)$-CW complex if and only if $\mathbf{X}$ is $(d-2)$-connected, $\pi_*\mathbf{X}$ is p-torsion, and $p^r\cdot\pi_{d-1}\mathbf{X}=0$.*

Proof. (I) If we assume that $\mathbf{X}$ is an $\mathbf{M}$-CW complex, then since $\mathbf{M}$ is a suspension, we have $\mathbf{X}=\operatorname{hocolim}_n\mathbf{X}_n$, where each $\mathbf{X}_{n+1}$ is obtained as the cofiber of a map $f_n:\mathbf{M}^{k_n}\to\mathbf{X}_n$, starting with $\mathbf{X}_0\simeq\bigvee_{\gamma\in\Gamma}\mathbf{M}_{k_\gamma}$ (see [D, Ch. 2, E.3]). Since $\tilde{H}_i(\mathbf{X}_n,\mathbb{Z})=0$ for each $i\le d-2$ and $\tilde{H}_i(\mathbf{X}_n,\mathbb{Z})$ is a p-group for all i, the same holds for $\pi_i\mathbf{X}_n$ by [Sp, IX, §6, Thm 20], and thus for $\pi_i\mathbf{X}$ by [BoK, XII, 5.7]. Moreover, by the Blakers-Massey Theorem (cf. [W, VII, Thm 7.12]) $\pi_{d-1}\mathbf{X}_{n+1}$ is a quotient of $\pi_{d-1}\mathbf{X}_n$, so it has exponent p^r by induction, and thus $p^r\cdot\pi_{d-1}\mathbf{X}=0$ by [Gr, Prop. 15.9].

(II) Now let $\mathbf{X}\in\mathcal{T}_*$ be arbitrary: the cofibration sequence $\mathbf{S}^{d-1}\xrightarrow{p^r}\mathbf{S}^{d-1}\to\mathbf{M}^d(p^r)$ then yields a long exact sequence

$$\ldots\to\pi_t\mathbf{X}\xrightarrow{p^r}\pi_t\mathbf{X}\to\pi_t(\mathbf{X};p^r)\to\pi_{t-1}\mathbf{X}\xrightarrow{p^r}\pi_{t-1}\mathbf{X}\to\ldots \tag{3.2}$$

and thus a short exact sequence

$$0\to\pi_t\mathbf{X}\otimes\mathbb{Z}/p^r\to\pi_t(\mathbf{X};p^r)\to Tor(\pi_{t-1}\mathbf{X},\mathbb{Z}/p^r)\to 0\qquad\text{for } t>d. \tag{3.3}$$

(See [N2, §1] for the case $t=d=2$). This short exact sequence implies that, if $f:\mathbf{X}\langle d-1\rangle\to\mathbf{X}$ is the $(d-2)$-connected cover of $\mathbf{X}$, then f is an $\mathbf{M}$-equivalence, and thus $CW_M(f):CW_M\mathbf{X}\langle d-1\rangle\to CW_M\mathbf{X}$ is a homotopy equivalence. Similarly, the p-localization $\mathbf{X}\to\mathbf{X}_{(p)}$ is an $\mathbf{M}$-equivalence. Thus we may assume without loss of generality that $\mathbf{X}$ is p-local and $(d-2)$-connected.

In [Bo2, Thm 5.2] Bousfield shows that, for $\mathbf{W}=\mathbf{M}^{d+1}(p^r)$ we have

$$\pi_iP_W\mathbf{X}=\begin{cases}\pi_i\mathbf{X} & \text{if } i<d\\ \pi_d\mathbf{X}/(p\text{-torsion}) & \text{if } i=d\\ \pi_i\mathbf{X}\otimes\mathbb{Z}[1/p] & \text{if } i>d,\end{cases} \tag{3.4}$$

Thus if $\mathbf{X}$ is $(d-1)$-connected and $\pi_*\mathbf{X}$ is p-torsion (i.e., all elements are of orders which are powers of p) then $P_W\mathbf{X}\simeq *$. In this case [D, Ch. 3, Prop. B.3] implies that $CW_M\mathbf{X}\simeq\mathbf{X}$.

Now let π be an (abelian) p-group; then

$$\pi_{d+s}(\mathbf{K}(\pi,d-1);M)\cong\tilde{H}^{d-1}(\mathbf{M}^{d+s}(p^r);\pi)=Tor(\mathbb{Z}/p^r,\pi)$$

for $s=0$, and $\pi_{d+s}(\mathbf{K}(\pi,d-1);M)=0$ for $s>0$. Thus the inclusion induces an $\mathbf{M}$-equivalence $\mathbf{K}(Tor(\mathbb{Z}/p^r,\pi),d-1)\to\mathbf{K}(\pi,d-1)$.

By (I) we know that $\pi_*\hat{\mathbf{X}}$ is p-torsion for any $\mathbf{M}$-CW complex $\hat{\mathbf{X}}$. On the other hand, $CW_M\mathbf{K}(\pi,d-1)$ is a GEM (generalized Eilenberg-Mac Lane space) by [D, Ch. 5, Thm E.1], since $\mathbf{M}$ is a suspension and $\mathbf{Y}^{\Sigma M}\simeq *$. This implies that $CW_M\mathbf{K}(\pi,d-1)\simeq\mathbf{K}(Tor(\mathbb{Z}/p^r,\pi),d-1)$, by (3.3).

Now if $\mathbf{X}$ is any $(d-2)$-connected space with $\pi_*\mathbf{X}$ p-torsion, such that $p^r\cdot\pi=0$ for $\pi=\pi_{d-1}\mathbf{X}$, and $\mathbf{X}\langle d\rangle$ is the $(d-1)$-connected cover of $\mathbf{X}$, then we have a map of fibration sequences

$$\begin{tikzcd}
 & Fib\,(CW_M f) \arrow[r, "i"] \arrow[d, "h"] \arrow[dl, dotted] & CW_M\mathbf{X} \arrow[r, "CW_M f"] \arrow[d, "k"] & CW_M\mathbf{K}(\pi,d-1) \arrow[d, "\ell", "\simeq"'] \\
CW_M(\mathbf{X}\langle d\rangle) \arrow[r, "\simeq"] & \mathbf{X}\langle d\rangle \arrow[r, "g"] & \mathbf{X} \arrow[r, "f"] & \mathbf{K}(\pi,d-1)
\end{tikzcd}$$

in which ℓ is a homotopy equivalence, by the above. If $\mathbf{F}=Fib\,(h)$ is the homotopy fiber of h, then $Fib\,(k)\simeq\mathbf{F}$, and since k (and ℓ) are $\mathbf{M}$-equivalences, so is h, so $\mathbf{F}$ is $\mathbf{M}$-local – that is, $\pi_*(\mathbf{F};p^r)=0$ for $i\geq d$.

Moreover, by [D, Ch. 5, §E.7], $\mathbf{F}$ is a 2-stage Postnikov system; but since $\mathbf{X}$ and $CW_M\mathbf{X}$ are $(d-2)$-connected, $\mathbf{F}$ is $(d-3)$-connected, and thus by (3.3) we have $\pi_i\mathbf{F}=0$ for $i\neq d-1,d-2$, and $\pi_{d-1}\mathbf{F}$ has no p-torsion. But since $\pi_*\mathbf{X}$ is p-torsion by assumption, and $\pi_*CW_M\mathbf{X}$ is p-torsion by (I), the same is true of $\mathbf{F}$, so that in fact $\mathbf{F}=\mathbf{K}(\pi',d-2)$ for some p-torsion group π', which fits into the short exact sequence $0\to\pi_{d-1}CW_M\mathbf{X}\to\pi_{d-1}\mathbf{X}\to\pi'\to 0$.

By comparing the long exact sequences (3.2) for $CW_M\mathbf{X}$ and $\mathbf{X}$, we see that $\pi'\cong\pi_{d-1}\mathbf{X}/Tor(\pi_{d-1}\mathbf{X},\mathbb{Z}/p^r)$. This completes the proof. □

An alternative proof of the Proposition may be obtained by using [Ch, Thm. 20.9]. Note that the Proposition does not provide us with any obvious *construction* of a $\mathbf{M}^d(p^r)$-CW approximation functor (but compare Bousfield's p-cocompletion functor in [Bo2, §14.1]).

Corollary 3.5. *If* $(p,q)=1$, *and we let* $\mathbf{M}=\mathbf{M}^d(p^rq^s)$, $\mathbf{M}'=\mathbf{M}^d(p^r)$, *and* $\mathbf{M}''=\mathbf{M}^d(q^s)$, *then* $\mathbf{X}^M\simeq\mathbf{X}^{M'}\times\mathbf{X}^{M''}$ *and* $CW_M\mathbf{X}\simeq CW_{M'}\mathbf{X}\vee CW_{M''}\mathbf{X}$ *for any* $\mathbf{X}\in\mathcal{T}_*$.

Proof. By [N2, Prop. 1.5] we have $\mathbf{M}\simeq\mathbf{M}'\vee\mathbf{M}''$, so a map is $\mathbf{M}$-equivalence if and only if it is both an $\mathbf{M}'$-equivalence and an $\mathbf{M}''$-equivalence. The Corollary then follows from the Proposition. □

3.6. **n-fold loop spaces.** To start the inductive sequence of procedures described in section 2 for recovering $\mathbf{X}$ from $\mathbf{X}^V$, we need to consider the initial case, when $\mathbf{V}=\mathbf{V}(-1)=\mathbf{S}^n$:

Recall from [M2, §4] or [BV] the *little n-cubes operad* $\mathcal{C}_n$, which operates on any n-fold loop space $\mathbf{Y}\simeq\Omega^n\mathbf{X}$ (cf. [M2, §5]); conversely, any (connected) space on which $\mathcal{C}_n$ operates is weakly equivalent to an n-fold loop space: in fact, May defines an "n-fold classifying space" functor $B(\mathbf{S}^n, \mathcal{C}_n, -)$ which recovers $\mathbf{X}$ from $\mathbf{Y} \simeq \Omega^n\mathbf{X}$ (if $\mathbf{Y}$ is connected, of course – cf. [M2, Thm 13.1], and see also [Be]).

Lemma 3.7. *For $\mathbf{M} = \mathbf{S}^{d-1}$ and $v = p^r$ (and thus $\mathbf{V} = \mathbf{M}^d(p^r)$, the mod p^r Moore space), the classifying space $B(\mathbf{S}^{d-1}, \mathcal{C}_{d-1}, \mathbf{V}_\infty\mathbf{X}/\mathbf{X}^{V_r})$ (see §2.12) is a $\mathbf{V}$-CW complex.*

Proof. Since $\pi_i\mathbf{X}^{V_r} \cong \pi_{d+i}(\mathbf{X}; p^r)$, we see by (3.3) that

(a) $\pi_i\mathbf{X}^{V_r}$ is p-torsion for all $i \geq 0$, and
(b) $p^r \cdot \pi_0\mathbf{X}^{V_r} = 0$

(In fact, $\pi_*\mathbf{X}^{V_r}$ has exponent p^r for $p > 2$, by [N2, Prop. 7.1]).
By considering the homtopy colimits used to define them (see §2.7, Proposition 2.11, §2.12), we conclude that each of the spaces $\mathbf{X}^{V_{2r}}, \mathbf{X}^{V_{4r}}, \ldots,$ as well as $\mathbf{V}_\infty$ (which is the colimit of $\mathbf{X}^{V_r} \to \mathbf{X}^{V_{2r}} \to \ldots$), and $\mathbf{V}_\infty\mathbf{X}/(\mathbf{X}^{V_r})$, satisfies properties (a) and (b) above. The Lemma then follows from Proposition 3.1, since $B(\mathbf{S}^{d-1}, \mathcal{C}_{d-1}, -)$ takes values in $(d-2)$-connected spaces. □

When $\mathbf{M}$ is a Moore space, we thus obtain from Theorem 2.13 a more explicit description of the procedure for recovering $\mathbf{X}$ from $\mathbf{X}^M$:

Theorem 3.8. *Let p be a prime, $\mathbf{V} = \mathbf{M}^d(p^r)$ the d-dimensional mod p^r Moore space $(d \geq 3)$, and $\mathbf{Y} = \mathbf{X}^{V_r}$, and assume that $[\mathbf{V}, \mathbf{X}] = 0$; then $CW_V\mathbf{X} \simeq B(\mathbf{S}^{d-1}, \mathcal{C}_{d-1}, \mathbf{V}_\infty\mathbf{X})$, where $\mathbf{V}_\infty\mathbf{X}$ is the sequential homotopy colimit of $\mathbf{X}^{V_r} \to \mathbf{X}^{V_{2r}} \to \ldots$ as in Proposition 2.11, and each $\mathbf{X}^{V_{2^{s+1}r}}$ is the homotopy colimit $B(\mathbf{X}^{V_{2^s r}}, \Omega\mathbf{X}^{V_{2^s r}}, *)$, in the notation of §2.7.*

If we do not assume that $[\mathbf{V}, \mathbf{X}] = 0$, we must include the extensions (2.6) in the construction of $\mathbf{X}^{V_{2^{s+1}r}}$ from $\mathbf{X}^{V_{2^s r}}$. Corollary 3.5 allows one to generalize the Theorem to the mapping space from any mod k Moore space $(k \geq 2)$.

4. $\mathbf{M}$-CW APPROXIMATIONS

In a sense, Corollary 2.14 provides a way of constructing an $\mathbf{M}$-CW approximation for any space $\mathbf{X}$, when $\mathbf{M}$ is self-map resolvable. However, this procedure is somewhat unsatisfactory, because it requires the full mapping space $\mathbf{X}^M$ as part of the initial data. We now show how one may construct $CW_M\mathbf{X}$ out of simpler building blocks:

4.1. **the mapping cotriple.** One obvious candidate for such a functorial and relatively efficient (i.e., countable) construction for the $\mathbf{M}$-CW approximation of a space is the $\mathbf{M}$-analogue of Stover's construction of "simplicial resolutions by spheres": for any space $\mathbf{M} \in \mathcal{T}_*$, one can define a functor $J : \mathcal{T}_* \to \mathcal{T}_*$, as in [Stv, §2], by

$$J\mathbf{X} = \Big(\bigvee_{i=0}^{\infty} \bigvee_{f \in Hom_{\mathcal{T}_*}(\Sigma^i M, X)} \Sigma^i\mathbf{M}_f \;\vee\; \bigvee_{i=0}^{\infty} \bigvee_{F \in Hom_{\mathcal{T}_*}(C\Sigma^i\mathbf{M}, X)} C\Sigma^i\mathbf{M}_F\Big)/\sim \tag{4.2}$$

where for each $\mathbf{A} = \Sigma^i\mathbf{M}$, the subspace $\partial(C\mathbf{A}_F) \cong \mathbf{A}$ of $C\mathbf{A}_F$ (which is the copy of the cone on $\mathbf{A}$ indexed by $F : C\mathbf{A} \to \mathbf{X}$) is identified under $\sim$ with $\mathbf{A}_f$, the copy of $\mathbf{A}$ indexed by $f = F|_{\partial C\mathbf{A}}$. Note that $J\mathbf{X}$ is homotopy equivalent to a wedge of copies of $\mathbf{M}$ and its suspensions.

J is clearly a comonad (cf. [M, VI, §1]) on $\mathcal{T}_*$, with the obvious counit $\varepsilon : J\mathbf{X} \to \mathbf{X}$ – namely, "evaluation" – and comultiplication $\mu : J(\mathbf{X}) \to J^2\mathbf{X}$ – where $\mu|_{\Sigma^i\mathbf{M}_f}$ is an isomorphism onto the copy of $\Sigma^i\mathbf{M}$ in $L^2\mathbf{X}$ indexed by the inclusion $\Sigma^i\mathbf{M}_f \hookrightarrow J\mathbf{X}$, for any $f:\Sigma^i\mathbf{M}\to\mathbf{X}$; and similarly for $C\Sigma^i\mathbf{M}_F$.

Now given $\mathbf{X}\in\mathcal{T}_*$, one may define a functorial simplicial space $\mathbf{J}_\bullet=\mathbf{J}_\bullet(\mathbf{X})$ by setting $\mathbf{J}_n = J^{n+1}\mathbf{X}$, with face and degeneracy maps induced by the counit and comultiplication respectively (cf. [Go, Appendix, §3]). The counit also induces an augmentation $\varepsilon:\mathbf{J}_\bullet\to\mathbf{X}$. Moreover, if $\mathbf{M}$ is a suspension, then one has:

Proposition 4.3. *For any $t\geq d$, the augmented simplicial group $\pi_t(\mathbf{J}_\bullet;M)\xrightarrow{\varepsilon_\#}\pi_t(\mathbf{X};M)$ is acyclic – that is, $\pi_s(\pi_t(\mathbf{J}_\bullet;M))=0$ for $s\geq 1$, and $\pi_0(\pi_t(\mathbf{J}_\bullet;M))\cong\pi_t(\mathbf{X};M)$.*

Proof. Same as that of [Stv, Prop. 2.6]. □

The *realization* of a simplicial space $\mathbf{Y}_\bullet$ (cf. [M2, §11.1]) is its homotopy colimit, denoted by $\|\mathbf{Y}_\bullet\|$, and constructed analogously to the geometric realization of a simplicial set. In particular, for $\mathbf{J}_\bullet=\mathbf{J}_\bullet(\mathbf{X})$ as above we see that $\|\mathbf{J}_\bullet\|$ is an $\mathbf{M}$-CW complex. Recall from [BoF, Thm B.5] & [BrL, App.] that for any suitable simplicial space $\mathbf{Y}_\bullet$ there is a first quadrant spectral sequence with

$$E^2_{s,t}=\pi_s(\pi_t\mathbf{Y}_\bullet)\Rightarrow\pi_{s+t}\|\mathbf{Y}_\bullet\|. \tag{4.4}$$

("Suitable" includes the cases where $\mathbf{Y}_\bullet$ is a simplicial loop space, or where each $\mathbf{Y}_n$ is connected). Applying the mapping space functor $(-)^M$ to $\mathbf{Y}_\bullet$ yields a simplicial space $\mathbf{Y}_\bullet^M$, and we have:

Corollary 4.5. *If $\mathbf{M}=\Sigma\mathbf{M}'$ is a suspension, then for any space $\mathbf{X}\in\mathcal{T}_*$ we have $\|(\mathbf{J}_\bullet(\mathbf{X})^{\mathbf{M}}\|\simeq\mathbf{X}^{\mathbf{M}}$.*

4.6. **the map ε.** What one would really like to conclude from Proposition 4.3 is that the augmentation $\varepsilon:\|\mathbf{J}_\bullet\|\to\mathbf{X}$ is an $\mathbf{M}$-equivalence, and thus $\|\mathbf{J}_\bullet\|\simeq CW_M\mathbf{X}$. Unfortunately, this is not true in general:

To see why, let $\mathbf{M}=\mathbf{M}^d(p)$ be the d-dimensional mod p Moore space, (so that each $\mathbf{J}_n$ is $(d-2)$-connected). Let $\mathbf{K}_\bullet$ be the simplical Eilenberg-Mac Lane space with $\mathbf{K}_n=\mathbf{K}(\pi_{d-1}\mathbf{J}_n,d-1)$, and $g_\bullet:\mathbf{J}_\bullet\to\mathbf{K}_\bullet$ the obvious map of simplicial spaces made into a fibration, with (dimensionwise) fiber $\tilde{\mathbf{J}}_\bullet=\mathbf{J}_\bullet\langle d-1\rangle$, the $(d-1)$-connected cover of $\mathbf{J}_\bullet$.

By [A] one then has a fibration sequence $\|\tilde{\mathbf{J}}_\bullet\|\to\|\mathbf{J}_\bullet\|\xrightarrow{\|g_\bullet\|}\|\mathbf{K}_\bullet\|$, and thus a fibration sequence

$$\|\tilde{\mathbf{J}}_\bullet\|^{\mathbf{M}}\to\|\mathbf{J}_\bullet\|^{\mathbf{M}}\xrightarrow{g}\|\mathbf{K}_\bullet\|^{\mathbf{M}}. \tag{4.7}$$

Recall from §2.4(a) that $\beta^s_r=i^{s+1}_r\circ j^s_r$, so from the long exact sequence (3.2) we see that $\Gamma_n\overset{Def}{=}\pi_{d-1}\mathbf{K}_n=\pi_{d-1}(\mathbf{J}_n)\cong\pi_d(\mathbf{J}_n;M)/Im(\beta^d_r)^\#$. Since the spectral sequence (4.4) for $\mathbf{K}_\bullet$ collapses at E^2, if we set $G_n=\pi_n(\pi_{d-1}\mathbf{K}_\bullet)$ we see that $\|\mathbf{K}_\bullet\|\simeq\prod_{n=d-1}^\infty\mathbf{K}(G_{n+d-1},n)$ (since $\mathbf{K}_\bullet$ is a simplicial GEM, $\|\mathbf{K}_\bullet\|$ is a GEM, too). Moreover, for any G we have $\pi_t\mathbf{K}(G,n)^M\cong\tilde{H}^{n-t}(\mathbf{M},G)$, and $\mathbf{K}(G,n)^M$ is again a GEM (cf. [T, Thm. 3]), so

$$\|\mathbf{K}_\bullet\|^M\simeq\prod_{n=0}^\infty(\mathbf{K}(G_{n+d-1},n)\times\mathbf{K}(G_{n+d},n)) \tag{4.8}$$

since $p^r \cdot G_n = 0$ for all i (as in the proof of 3.7), and thus $\tilde{H}^{d-1}(\mathbf{M}, G_n) \cong Hom(\mathbb{Z}/p^r, G_n) \cong G_n$, and $\tilde{H}^d(\mathbf{M}, G_n) \cong Ext(\mathbb{Z}/p^r, G_n) \cong G_n$ by [Sp, V, §5, Thm. 3].

One similarly sees that $\mathbf{K}_n^M \simeq \mathbf{K}(\Gamma_n, 0)$, so that again the spectral sequence for the simplical space $\mathbf{K}_\bullet^M$ collapses, and

$$\|\mathbf{K}_\bullet^M\| \simeq \prod_{i=d-1}^{\infty} \mathbf{K}(\pi_n \Gamma_\bullet, n-d+1). \tag{4.9}$$

On the other hand, applying realization to the fibration sequence of simplicial spaces $\tilde{\mathbf{J}}_\bullet^M \to \mathbf{J}_\bullet^M \to \mathbf{K}_\bullet^M$ yields a fibration sequence (by [A] again), which maps into that of (4.7) by naturality:

$$\begin{tikzcd}
\|\tilde{\mathbf{J}}_\bullet^{\mathbf{M}}\| \arrow[r] \arrow[d, "\gamma"', "\simeq"] & \|\mathbf{J}_\bullet^{\mathbf{M}}\| \arrow[r] \arrow[d, "\delta"] & \|\mathbf{K}_\bullet^{\mathbf{M}}\| \arrow[d, "\eta"] \\
\|\tilde{\mathbf{J}}_\bullet\|^{\mathbf{M}} \arrow[r] & \|\mathbf{J}_\bullet\|^{\mathbf{M}} \arrow[r] & \|\mathbf{K}_\bullet\|^{\mathbf{M}}
\end{tikzcd}$$

FIGURE 5

The map $\gamma : \|\tilde{\mathbf{J}}_\bullet\|^{\mathbf{M}} \to \|\tilde{\mathbf{J}}_\bullet\|^{\mathbf{M}}$ is a weak equivalence, by [BT1, Lemma 6.1]. Now Proposition 4.3 implies that the spectral sequence for $\mathbf{J}_\bullet^M$ collapses, and thus

$$\pi_t \|\mathbf{J}_\bullet^M\| \cong \pi_0 \pi_t \mathbf{J}_\bullet^M \cong \pi_{t+d}(\mathbf{X}; M). \tag{4.10}$$

4.11. **M-Π-algebras.** We now show how to interpret the groups appearing in the right hand side of (4.8) and (4.9) as derived functors, which allows us to show they often do not vanish, and thus to conclude that $\varepsilon : \|\mathbf{J}_\bullet\| \to \mathbf{X}$ is *not* in general an $\mathbf{M}$-equivalence.

Definition 4.12. Recall that a Π-*algebra* is an algebraic object modeled on the homotopy groups of a space, together with the action of the primary homotopy operations on them. If we replace the spheres representing ordinary homotopy groups by a model space $\mathbf{M} = \mathbf{M}^d$ (as in §1.1), we get $\mathbf{M}$-*homotopy operations* corresponding to each homotopy class $\alpha \in \pi_r(\mathbf{M}^{n_1} \vee \ldots \vee \mathbf{M}^{n_k}; M)$ (subject to the universal relations among such operations, corresponding to compositions of maps).

We then define an $\mathbf{M}$-Π-*algebra* to be a graded set $\{X_i\}_{i=d}^{\infty}$, together with an action of the $\mathbf{M}$-homotopy operations on them. As usual, the *free* $\mathbf{M}$-Π-algebras are those isomorphic to $\pi_\star(\bigvee_{\alpha \in A} \mathbf{M}^{r_\alpha}; M)$ for some (possibly infinite) wedge of model spaces. (In the case of Moore spaces, one can be more explicit – cf. [Bl2, §5.6]).

The category of $\mathbf{M}$-Π-algebras will be denoted by $\mathbf{M}$-Π-$\mathcal{A}lg$; since it is a category of universal (graded) algebras, one has a concept of free simplicial resolutions, and thus of left derived functors $L_n T : \mathbf{M}\text{-}\Pi\text{-}\mathcal{A}lg \to \mathcal{A}b\mathcal{G}p$ for any functor $T : \mathbf{M}\text{-}\Pi\text{-}\mathcal{A}lg \to \mathcal{A}b\mathcal{G}p$ (see [Q, I,§4] or [BS, §2.2.4] for more details).

In particular, given an $\mathbf{M}$-Π-algebra $X_* = \{X_i\}_{i=d}^{\infty}$, one has a functor $T : \mathbf{M}\text{-}\Pi\text{-}\mathcal{A}lg \to \mathcal{A}b\mathcal{G}p$ defined $T(X_*) = \mathbf{X}_d / Im\{(\beta_r^d)^\# : X_{d+1} \to X_d\}$. Now Proposition 4.3 shows that $\pi_*(\mathbf{J}_\bullet; M) \to \pi_*(\mathbf{X}; M)$ is a free simplicial resolution, so that $\pi_{d-1}(\mathbf{J}_n) \cong \pi_d(\mathbf{J}_n / Im(\beta_r^d)^\#; M) = T(\pi_*(\mathbf{J}_n; M))$, and thus $G_n \overset{Def}{=} \pi_n(\pi_{d-1}\mathbf{J}_\bullet) = (L_n T)\pi_*(\mathbf{X}; M)$. Of course, we may use *any* resolution of $\pi_*(\mathbf{X}; M)$ to calculate these derived functors.

Lemma 4.13. *Let* $\mathbf{M} = \mathbf{M}^d(p^r)$ *with* $d \geq 4$, $p > 2$, *and* $\mathbf{X} = \mathbf{K}(\mathbb{Z}, n)$ *with* $n < \min\{2p-4, 2d-4\}$*; then* $\varepsilon : \|\mathbf{J}_\bullet\| \to \mathbf{X}$ *is not an* $\mathbf{M}$*-equivalence.*

Proof. By (3.3), $\pi_i(\mathbf{X}; M) = \mathbb{Z}/p^r$ for $i = n$, and 0 otherwise. Since the only $\mathbf{M}$-homotopy operations in dimensions $\leq n$ are the Bocksteins $(\beta^i_r)^\#$ for $i = d, \dots n-1$ (cf. [Y]), we have $\pi_k(\mathbf{M}^k; M) = \mathbb{Z}/p^r\langle \iota^k \rangle$, $\pi_{k-1}(\mathbf{M}^k; M) = \mathbb{Z}/p^r\langle (\beta^k_r)^\# \iota^k \rangle$, and $\pi_t(\mathbf{M}^k; M) = 0$ for $k < t < 2(p-2)$.

In the stable range it suffices to find an ordinary chain-complex resolution $A_* \to \pi_*(\mathbf{X}; M)$ (cf. [Bl1, Lemma 6.10]), so we may choose $A_i \cong \pi_*(\mathbf{M}^{n-i}; M)$ in dimensions $\leq n$ for $i \leq n-d$, with $d_i \iota^{n-i} = (\beta^{n-i+1}_r)^\# \iota^{n-i+1}$. Thus $G_i = (L_i T)\pi_*(\mathbf{X}; M) \cong \pi_i(TA_*) = \mathbb{Z}/p^r$ for $i = n-d$ (and $G_i = 0$ for $0 \leq i < n-d$). Thus the map η in Figure 5 cannot be an equivalence, so the fiber of η, and thus of δ, is not trivial. The Lemma follows. □

Remark 4.14. We have shown that the simplicial space $\mathbf{J}_\bullet$ does not in general provide an $\mathbf{M}$-CW approximation functor. Nevertheless, as long as $\mathbf{M}$ is a suspension, it does give us a (relatively) explicit construction of $\mathbf{X}^{\mathbf{M}}$ as the homotopy colimit of a diagram of copies of spaces $\Sigma^i \mathbf{M}^{\wedge j})^M$ $(i, j = 0, 1, \dots)$. In particular, if $\mathbf{M}$ is self-map resolvable, combined with Corollary 2.14 this gives a construction of $CW_M \mathbf{X}$ from the spaces $(\mathbf{M}^{\wedge j})^M$ by a countable sequence of homotopy colimits.

This is because the Hilton-Milnor Theorem (cf. [W, XI, Thm. 6.7]) shows that $\mathbf{J}^M_n \simeq \prod_\beta (\bigwedge_{i=1}^{n_\beta} \Sigma^{n_i} \mathbf{M})^M$ (since $\mathbf{J}_n \simeq \bigvee_\alpha \Sigma^{n_\alpha} \mathbf{M}$). As this is in fact a weak product, and all finite products may be expressed as pointed homotopy colimits by [D, Ch. 2, Thm D.16], the statement follows by induction.

We observe that for most Moore spaces this result can be slightly improved:

Proposition 4.15. *Let* $\mathbf{M} = \mathbf{M}^d(k)$ *be the d-dimensional mod k Moore space, where k is odd or* $4|k$. *Then* $CW_M \mathbf{X}$ *may be constructed by homotopy colimits as in Theorem 2.13 from the simplicial space* $\mathbf{J}_\bullet(\mathbf{X})^{\mathbf{M}}$ *of §4.1, where each* $\mathbf{J}^M_n$ *is homotopy equivalent to a product of mapping spaces of the form* $(\Sigma^i \mathbf{M})^M$.

Proof. By [N2, Cor. 6.6] we have $\mathbf{M}^r(k) \wedge \mathbf{M}^s(k) \simeq \mathbf{M}^{r+s-1}(k) \vee \mathbf{M}^{r+s}(k)$ if $r, s \geq 3$ and k is odd or $4|k$. Thus the Proposition follows from the above Remark, again by induction on the dimensions. □

References

[A] D.W. Anderson, "Fibrations and geometric realizations", *Bull. AMS* **84** (1978) No. 5, pp. 765-788.

[Be] J. Beck, "Classifying spaces for homotopy-everything H-spaces", in F. Sigrist, ed., *H-Spaces - Neuchâtel 1970*, Springer-Verlag *Lec. Notes Math.* **196**, Berlin-New York 1971, pp. 54-62.

[Bl1] D. Blanc, "Higher homotopy operations and the realizability of homotopy groups", *Proc. Lond. Math. Soc. (3)* **70** (1995), pp. 214-140.

[Bl2] D. Blanc, "Homotopy operations and the obstructions to being an H-space", *Manus. Math.* **88** (1995) No. 4, pp. 497-515.

[BS] D. Blanc & C.R. Stover, "A generalized Grothendieck spectral sequence", in N. Ray & G. Walker, eds., *Adams Memorial Symposium on Algebraic Topology, Vol. 1*, Lond. Math. Soc. Lec. Notes Ser. **175**, Cambridge U. Press, Cambridge, 1992, pp. 145-161.

[BT1] D. Blanc & R.D. Thompson, "A suspension spectral sequence for v_n-periodic homotopy groups", *Math. Zeit.* **220** (1995) No. 1, pp. 11-35.

[BT2] D. Blanc & R.D. Thompson, "M-equivalences and homotopy colimits", in B. Cenkl & H.R. Miller. ed., *The Čech Centennial*, Cont. Math. **81**, AMS Providence, RI, 1995, pp. 27-33.

[BV] J.M. Boardman & R.M. Vogt, Homotopy-everything H-spaces. *Bull. AMS* **74** (1968) pp. 1117-1122.

[Bo1] A.K. Bousfield, "Construction of factorization systems in categories", *J. Pure & Appl. Alg.* **9** (1977) pp. 207-220.

[Bo2] A.K. Bousfield, "Localization and periodicity in unstable homotopy theory", *J. AMS* **7** (1994) No. 4, pp. 831-873.

[BoF] A.K. Bousfield & E.M. Friedlander, "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets", in M.G. Barratt & M.E. Mahowald, eds., *Geometric Applications of Homotopy Theory, II*, Springer-Verlag *Lec. Notes Math.* **658**, Berlin-New York, 1978, pp. 80-130.

[BoK] A.K. Bousfield & D.M. Kan, *Homotopy limits, Completions, and Localizations*, Springer--Verlag *Lec. Notes Math.* **304**, Berlin-New York, 1972.

[BrL] R. Brown & J.-L. Loday, "Van Kampen theorems for diagrams of spaces", *Topology* **26** (1987) No. 3, pp. 311-335.

[CaPP] C. Casacuberta, G. Peschke & M. Pfenniger, "On orthogonal pairs in categories and localization", In N. Ray and G. Walker, eds., *Adams Memorial Symposium on Algebraic Topology, I*, Lond. Math. Soc. Lec. Notes Ser. **175**, Cambridge U. Press, Cambridge, UK, 1992, pp. 211-223.

[Ch] W. Chachólski, "On the functors CW_A and P_A", Preprint, 1995.

[Co] P.V.Z. Cobb, "P_n-spaces and n-fold loop spaces", *Bull. AMS* **80** (1974) No. 5, pp. 910-914.

[D] E. Dror-Farjoun, "Cellular spaces, null spaces, and homotopy localization", Springer-Verlag *Lec. Notes Math.* **1622**, Berlin-New York, 1995.

[Go] R. Godement, *Topologie algébrique et théorie des faisceaux*, Act. Sci. & Ind. No. 1252, Publ. Inst. Math. Univ. Strasbourg **XIII**, Hermann, Paris 1964.

[Gr] B. Gray, *Homotopy Theory: An Introduction to Algebraic Topology*, Academic Press, New York-San Francisco-London, 1975.

[HS] M.J. Hopkins & J.H. Smith, "Nilpotence and stable homotopy theory, II", Preprint, May 1992.

[M] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag *Grad. Texts in Math.* **5**, Berlin-New York, 1971.

[M1] J.P. May, *Simplicial Objects in Algebraic Topology*, U. Chicago Press, Chicago-London, 1967.

[M2] J.P. May, *The Geometry of Iterated Loop Spaces*, Springer-Verlag *Lec. Notes Math.* **271**, Berlin-New York, 1972.

[M3] J.P. May, *Classifying spaces and fibrations*, Mem. AMS **155**, AMS, Providence, RI, 1975.

[N1] J.A. Neisendorfer, "Homotopy theory modulo an odd prime", Ph.D. Thesis, Princeton Univ., 1972.

[N2] J.A. Neisendorfer, *Primary homotopy theory*, Mem. AMS **25**, AMS, Providence, RI, 1980.

[P] F.P. Peterson, "Generalized cohomotopy groups", *Amer. J. Math.* **78** (1956), pp. 259-281.

[Q] D.G. Quillen, *Homotopical Algebra*, Springer-Verlag *Lec. Notes Math.* **20**, Berlin-New York, 1963.

[R] D.C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Ann. Math. Studies **128**, Princeton U. Press, Princeton, NJ, 1992.

[T] R. Thom, "L'homologie de espaces fonctionnels", in *Colloque de Topologie Algébrique (Louvain 1956)*, Centre Belge de Recherches Math., G. Thone, Liège Masson & Cie., Paris, 1957, pp. 29-39.

[Sm] J.H. Smith, "Simplicial group models for $\Omega^n S^n X$", *Isr. J. Math.* **66** (1989) No. 1-3, pp. 330-350.

[Sp] E.H. Spanier, *Algebraic Topology*, Springer-Verlag, Berlin-New York, 1966.

[St1] J.D. Stasheff, "Homotopy associativity of H-spaces, I", *Trans. AMS* **108** (1963) pp. 275-292.

[St2] J.D. Stasheff, "Homotopy associativity of H-spaces, II", *Trans. AMS* **108** (1963) pp. 293-312.

[Stv] C.R. Stover, "A Van Kampen spectral sequence for higher homotopy groups", *Topology* **29** (1990) No. 1, pp. 9-26.

[W] G.W. Whitehead, *Elements of homotopy theory*, Springer-Verlag *Grad. Texts Math.* **61**, Berlin-New York 1971.

[Y] N. Yamamoto, "Algebra of stable homotopy of Moore spaces", *J. Math. Osaka City U.* **14** (1963), pp. 45-67.

Dept. of Mathematics and Computer Science, Haifa University, Haifa 31905, Israel
E-mail address: `blanc@mathcs2.haifa.ac.il`