NON-SIMPLY CONNECTED H-SPACES WITH FINITENESS CONDITIONS

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Abstract. This article is concerned with homotopy properties of H-spaces X that are reflected in the module of indecomposables $QH^*(X;\mathbb{F}_p)$. It is shown that mod p H-spaces X of finite type with finite transcendence degree mod p cohomology and locally finite $QH^*(X;\mathbb{F}_p)$ are $B\mathbb{Z}/p$-null spaces, Eilenberg-Mac Lane spaces $K(\hat{\mathbb{Z}}_p, 2)$, $K(\mathbb{Z}/p^r, 1)$, and extensions of those. If we restrict attention to H-spaces with noetherian mod p cohomology algebra, then we are left with finite mod p H-spaces and Eilenberg-Mac Lane spaces.

1. Introduction

The mod p cohomology of an H-space X is a Hopf algebra $H^*(X;\mathbb{F}_p)$ that carries a compatible action of the Steenrod algebra $\mathcal{A}$. The Steenrod algebra action is inherited by the module of the indecomposables $QH^*(X;\mathbb{F}_p)$ and we are interested in homotopy properties of X that are reflected in this $\mathcal{A}$-action, specially in the non-simply connected case. In this article we will restrict our attention to the class of H-spaces X that satisfy the following finiteness conditions,

$(F1)$ $H^*(X;\mathbb{F}_p)$ is of finite type.
$(F2)$ $H^*(X;\mathbb{F}_p)$ has finite transcendence degree.
$(F3)$ $QH^*(X;\mathbb{F}_p)$ is locally finite as module over the Steenrod algebra.

Recall that a module over the Steenrod algebra is called locally finite provided any $\mathcal{A}$-submodule generated by a single element is finite (cf. [11]). Since H-spaces are simple they are p-good in the sense of Bousfield-Kan [2] so, there will be no loss of generality if we assume all H-spaces to be p-completed.

If we further restrict to the case where the transcendence degree of $H^*(X;\mathbb{F}_p)$ is actually zero, we end up with the class of $B\mathbb{Z}/p$-null H-spaces of finite type. Recall

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that $B\mathbb{Z}/p$-null spaces are the local spaces for the nullification functor $P_{B\mathbb{Z}/p}$ of Bousfield and Dror Farjoun [6]. We will denote by $F(-) = P_{B\mathbb{Z}/p}(-)^\wedge_p$ the composition of the nullification functor and p-completion.

Proposition 1.1. *A connected H-space X of finite type is $B\mathbb{Z}/p$-null if and only if it satisfies the equivalent conditions*

1. *The evaluation map is a homotopy equivalence,* $\mathrm{Map}(B\mathbb{Z}/p, X) \simeq X$.
2. *$H^*(X;\mathbb{F}_p)$ is a locally finite $\mathcal{A}$-module.*
3. *$QH^*(X;\mathbb{F}_p)$ is a locally finite $\mathcal{A}$-module and $H^*(X;\mathbb{F}_p)$ has transcendence degree zero.*

Proof. Condition (1) follows by definition (see [6]). This is equivalent to condition (2) by [7, 6.3.1]. The fact that the transcendence degree is zero in $H^*(X;\mathbb{F}_p)$ implies that there is just one component in $\mathrm{Map}(B\mathbb{Z}/p, X)$, that of the constant map. Then (1) and (2) are equivalent to condition (3) by [11, 3.9.7 and 6.4.5]. ∎

Immediate examples of H-spaces that satisfy $(F1)$, $(F2)$ and $(F3)$ but are not $B\mathbb{Z}/p$-null are products of Eilenberg-Mac Lane spaces $K(\hat{\mathbb{Z}}_p, 2)$ and $K(\mathbb{Z}/p^k, 1)$. A product of a finite number of copies of these Eilenberg-Mac Lane spaces is the p-completed classifying space of a compact abelian Lie group and it is usually called an abelian p-toral group. Our main result generalizes [3, 4] to the non-simply connected case and establishes the precise sense in which these examples are the only difference between these classes of H-spaces.

Theorem 1.2. *Let X be a connected H-space. If X satisfies $(F1)$, $(F2)$ and $(F3)$, then there exists a principal H-fibration*

$$BP \longrightarrow X \longrightarrow F(X) \longrightarrow B^2P$$

where $F(X)$ is a $B\mathbb{Z}/p$-null H-space of finite type and P is an abelian p-toral group.

Condition $(F2)$ is the matter of discussion. We first observe that if a connected H-space X satisfies $(F1)$ and $(F3)$, then $H^*(X;\mathbb{F}_p)$ has transcendence degree zero if and only if it is locally finite as $\mathcal{A}$-module. Theorem 1.2 is thus interpreted as a reduction of the case of finite transcendence degree to the case of transcendence degree zero.

On the other hand we are not aware of any example of an H-space satisfying $(F1)$ and $(F3)$ but not $(F2)$.

It is worth mentioning that H-spaces with noetherian mod p cohomology algebra fit in the conditions of the above Theorem. Next Theorem improves the conclusion in such cases.

Theorem 1.3. *Let X be a connected H-space. If the mod p cohomology ring $H^*(X, \mathbb{F}_p)$ is noetherian, then $F(X)$ is a mod p finite H-space.*

Structure theorems for simply connected H-spaces with noetherian mod p cohomology already appeared in [3] for the 2-local version and in [4] for the odd primary case. Notice that in the non-simply connected situation and according to theorems 1.2, 1.3 the possible extensions of mod p finite H-spaces that still have noetherian mod p cohomology can only be obtained by killing part of the three dimensional homotopy. More generally, we are allowed to kill two and three dimensional homotopy classes of $B\mathbb{Z}/p$-null spaces if we want to end up with H-spaces satisfying $(F1)$, $(F2)$, and $(F3)$.

While dealing with non-simply connected spaces a new interesting sort of extension comes into the picture. Rather than killing homotopy classes we can enlarge the fundamental group. The extreme case is that of H-spaces having $B\mathbb{Z}/p$-null universal cover, or even mod p finite universal cover.

Example 1.4. The fundamental group of the compact Lie group $SO(3)$ is $\mathbb{Z}/2$. Now for any $n > 1$ consider the projection $\mathbb{Z}/2^n \longrightarrow \mathbb{Z}/2$ and define the H-space F by the pull-back diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\mathbb{Z}/2^n \arrow[d] \\ SO(3) \arrow[r] & B\mathbb{Z}/2 \end{tikzcd}$$

so that we have modified the fundamental group of $SO(3)$ to $\pi_1(F) = \mathbb{Z}/2^n$, while the universal cover is still the three sphere S^3. One computes

$$H^*(F; \mathbb{F}_2) \cong H^*(S^3; \mathbb{F}_2) \otimes H^*(B\mathbb{Z}/2^n; \mathbb{F}_2) = E[x_1, x_3] \otimes P[\beta_{(n)}(x_1)]$$

but F does not split as $S^3 \times B\mathbb{Z}/2^n$. In fact, using Miller's theorem [9] one shows that a section of $F \longrightarrow B\mathbb{Z}/2^n$ would factors through $B\mathbb{Z}/2^{n-1}$, the homotopy fibre of $F \longrightarrow SO(3)$. And this is not possible.

Theorem 1.5. *Let X be a connected mod p H-space of finite type, then its universal cover is $B\mathbb{Z}/p$-null if and only if it fits in a homotopy pull-back diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\pi_1 \arrow[d, "B\rho"] \\ F(X) \arrow[r] & B\pi_1(F(X)) \end{tikzcd} \tag{1}$$

where $F(X)$ is a $B\mathbb{Z}/p$-null H-space and $\rho : \pi_1 \twoheadrightarrow \pi_1(F(X))$ is a group epimorphism for some finitely generated $\hat{\mathbb{Z}}_p$-module π_1, with kernel a finite abelian p-group Q.

Theorem 1.6. *Let X be a connected H-space. Its universal cover $\tilde{X}$ is mod p finite if and only if there is a diagram as 1 with $F(X)$ mod p finite.*

Example 1.7. Let p be an odd prime number. Let $S^3\{p^k\}$ denote the homotopy fibre of the degree p^k self map of S^3. It is a loop space with mod p cohomology algebra

$$H^*(S^3\{p^k\}; \mathbb{F}_p) \cong \Gamma[x_2] \otimes E[x_3]$$

with the relation $\beta_{(k)}(x_2) = x_3$, that is $H^2(S^3\{p^k\}; \mathbb{Z}) \cong \mathbb{Z}/p^k$.

Consider a map $S^3\{p^k\} \xrightarrow{p^s} B^2\mathbb{Z}/p^k$ with $0 < s < k$ and let X be the homotopy fibre of this map. We have an H-fibration sequence

$$B\mathbb{Z}/p^k \longrightarrow X \longrightarrow S^3\{p^k\} \xrightarrow{p^s} B^2\mathbb{Z}/p^k$$

from which it follows $H^1(X; \mathbb{Z}) \cong \mathbb{Z}/p^s$ and $H^2(X; \mathbb{Z}) \cong \mathbb{Z}/p^s$. Hence we have non-trivial classes $x_1 \in H^1(X; \mathbb{F}_p)$ and $x_2 \in H^2(X; \mathbb{F}_p)$ linked via $\beta_{(s)}$ and also $y_2 \in H^2(X; \mathbb{F}_p)$ and $y_3 = \beta_{(s)}(y_2) \in H^3(X; \mathbb{F}_p)$. The Serre spectral sequence with coefficients in $\mathbb{Z}/p$ collapses at the E_2 term, and we obtain

$$H^*(X; \mathbb{F}_p) \cong E[x_1] \otimes \Gamma[x_2] \otimes P[y_2] \otimes E[y_3] \, .$$

Notice that in this example we combine the two facts mentioned above, on the one hand the fundamental group has been enlarged and on the other hand we have killed some classes in $\pi_2(X)$.

Spaces Y with mod p cohomology $P[x_2] \otimes E[\beta_{(s)}(x_2)]$ are shown in [1]. Moreover a straightforward computation gives that $H^*(\Omega Y; \mathbb{F}_p) \cong E[y_1] \otimes \Gamma[\beta_{(s)}(y_1)]$, hence $H^*(X; \mathbb{F}_p) \cong H^*(Y \times \Omega Y; \mathbb{F}_p)$. One may ask if the space X constructed above is homotopy equivalent to the product $Y \times \Omega Y$. Observe that the answer is no. On the one hand we know that $F(X) = S^3\{p^r\}$ which is simply connected. On the other hand, since the functor F commutes with products $F(Y \times \Omega Y) = F(Y) \times F(\Omega Y)$, but ΩY is already $B\mathbb{Z}/p$-null, so that $F(\Omega Y) = \Omega Y$ and this is not simply connected.

Recall that the functor F preserves many of the structures that one likes to attach to an H-space and therefore the above theorems apply directly to H-spaces with additional structure. One outstanding example is that of loop structures. We have therefore obtained structure theorems for mod p loop spaces under conditions $(F1)$, $(F2)$, and $(F3)$, or with noetherian mod p cohomology.

The paper is organized as follows. Section §2 is devoted to the general theory of BP-principal fibrations of H-spaces. In section §3 we describe the relationship between the mod p cohomology of an H-space of finite type and that of its universal cover and derive the proof of theorems 1.5 and 1.6. In Section 4 we prove theorems 1.2 and 1.3. All spaces will be considered p-complete in the sense of Bousfield-Kan. $H^*(-)$ will stand for mod p cohomology.

2. BP-principal H-fibrations

In this section we look at the question whether or not a fibration with fibre the classifying space of an abelian p-toral group BP

$$BP \xrightarrow{f} X \xrightarrow{g} Y$$

and X and Y mod p H-spaces, is a principal fibration and if this is the case, we are also interested in whether or not this is a principal H-fibration. By this we mean that the classifying map $Y \xrightarrow{h} B^2P$ is an H-map. The results here generalize that of [3] stated for elementary abelian p-groups.

Proposition 2.1. *Let X be a connected mod p H-space with finite type mod p cohomology $(F1)$. The following conditions are equivalent,*

1. *The module of the indecomposables $QH^*(X)$ is locally finite as module over the Steenrod algebra $(F3)$.*
2. *For any p-toral group P and any map $f\colon BP \longrightarrow X$, the evaluation*

$$\mathrm{Map}(BP, X)_f \longrightarrow X$$

is a homotopy equivalence.

Proof. We first prove that 1 implies 2. Since X is a connected H-space it admits a homotopy inverse that is inherited by the mapping space $\mathrm{Map}(BP, X)$, hence all of its connected components are homotopy equivalent, and it will be enough to show the equivalence of condition 2 for the case of the constant map $f = c\colon BP \longrightarrow X$.

If P is elementary abelian, the result is already stated in [3] and follows from [5] or [11, 3.9.7]. For finite p-groups P we argue by induction on the order of P. If the order is small enough P is elementary abelian. Otherwise we can write a central extension $\mathbb{Z}/p \longrightarrow P \longrightarrow P/(\mathbb{Z}/p)$ and the Zabrodsky's lemma [13, 9] applies, so that

$$\mathrm{Map}(BP, X)_c \simeq \mathrm{Map}(B(P/(\mathbb{Z}/p)), X)_c \simeq X\,,$$

this last by induction hypothesis.

Finally, if P is a p-toral group, we choose a discrete approximation; that is, a sequence $\{P_k\}$ of finite groups ordered by inclusion such that the inclusion $\bigcup_k P_k = P_\infty \longrightarrow P$ induces a mod p cohomology equivalence in the level of classifying spaces. That is to say $\mathrm{hocolim}_k BP_k \longrightarrow BP$ is a homotopy equivalence after completion and then

$$\mathrm{Map}(BP, X) \simeq \operatorname*{holim}_k \mathrm{Map}(BP_k, X)\,.$$

Since every P_k is a finite p-group, we have a homotopy equivalence, induced by evaluation

$$\operatorname*{holim}_k \mathrm{Map}(BP_k, X)_c \simeq \operatorname*{hocolim}_k X \simeq X$$

thus $\mathrm{holim}_k \mathrm{Map}(BP_k, X)_c$ contains just one component and

$$\mathrm{Map}(BP, X)_c \simeq \operatorname*{holim}_k \mathrm{Map}(BP_k, X)_c \simeq X\,.$$

That condition 2 implies 1, follows immediately from [11, 6.4.5]. ∎

Lemma 2.2. *Let X be an H-space and P an abelian p-toral group. For an H-map $f\colon BP \longrightarrow X$ the following conditions are equivalent*

1. *$H^*(BP)$ becomes a finitely generated $H^*(X)$-module induced by f^*.*
2. *For any other abelian p-toral group P' and homomorphism $\rho\colon P' \longrightarrow P$, the composition $BP' \xrightarrow{B\rho} BP \xrightarrow{f} X$ is null-homotopic if and only if ρ is trivial.*
3. *For any other abelian p-toral group P', the induced homomorphism*
$$f_\sharp\colon [BP', BP] \longrightarrow [BP', X]$$
is injective.

Proof. Assume first condition 1. Let $\rho\colon P' \longrightarrow P$ be a non-trivial homomorphism of abelian p-toral groups. Notice that we can restrict to the case where ρ is injective, otherwise we can factor ρ through its image. If ρ is injective, then $H^*(BP')$ is finitely generated over $H^*(BP)$ and hence it is also a finitely generated $H^*(X)$-module induced by the composition $B\rho^* \circ f^*$. Hence $f \circ B\rho$ cannot be null-homotopic.

Conversely, assume that condition 2 is satisfied. Let V be the maximal elementary abelian subgroup of P. If we choose an isomorphism $(\mathbb{Z}/p)^r \cong V$, each restriction $B\mathbb{Z}/p \longrightarrow BP \longrightarrow X$ is not null-homotopic, hence $H^*(B\mathbb{Z}/p)$ becomes finitely generated over $H^*(X)$. Then $H^*(BV)$ is itself finitely generated over $H^*(X)$, in particular, the image of $H^*(X) \longrightarrow H^*(BV)$ is a sub Hopf algebra of $H^*(BV)$ of maximal transcendence degree, and then the image of $H^*(X) \longrightarrow H^*(BP)$ is also a sub Hopf algebra of maximal transcendence degree of $H^*(BP)$. Therefore $H^*(BP)$ is finitely generated as $H^*(X)$-module.

We have shown that conditions 1 and 2 are equivalent. That 2 and 3 are equivalent follows immediately from the fact that $[BP', BP]$ is a group, $[BP', X]$ also inherits from X a memorialisation and a zero element so that $f_\sharp$ preserves multiplication and the zero element. Hence to check injectivity we only need to look at the kernel of $f_\sharp$. ∎

Definition 2.3. An H-map $BP \longrightarrow X$ is called mod p homotopy injective if it satisfies the equivalent conditions of Lemma 2.2.

Remark 2.4. There is also a way to define a concept of homotopy kernel for an H-map $BP \xrightarrow{f} X$. If $\check{P}$ is a discrete approximation of P, we set $\check{K} = \{\, x \in \check{P} \mid B\langle x\rangle \longrightarrow BP \longrightarrow X$ is null-homotopic $\}$. The map $BK = (B\check{K})^\wedge_p \longrightarrow BP$ induced by the inclusion, might be called the homotopy kernel of $BP \longrightarrow X$.

It turns out that if X satisfies $(F1)$ and $(F3)$, then any map $g\colon BP' \longrightarrow BP$ for which the composition is null-homotopic, factors through the homotopy kernel $BK \longrightarrow BP$. Also, f itself factors uniquely as $BP \longrightarrow B(P/K) \longrightarrow X$ with $B(P/K) \longrightarrow X$ mod p homotopy injective.

Proposition 2.5. *Let P be an abelian p-toral group and $BP \xrightarrow{f} X \longrightarrow Y$ a fibration with Y connected. Assume that X is an H-space and f and H-map. If f is mod p homotopy injective, then for any abelian p-toral group P',* $\mathrm{Map}(BP', X)_c \simeq X$ *if and only if* $\mathrm{Map}(BP', Y)_c \simeq Y$.

Proof. Apply $\mathrm{Map}(BP', -)$ to the fibration $BP \xrightarrow{f} X \longrightarrow Y$. One obtains another fibration:

$$\mathrm{Map}(BP', BP)_\Phi \longrightarrow \mathrm{Map}(BP', X)_c \longrightarrow \mathrm{Map}(BP', Y)_c$$

where c stands for the constant map and Φ denotes the set of maps $h \in [BP', BP]$ such that $BP' \xrightarrow{h} BP \longrightarrow X$ is null-homotopic. Since $f : BP \longrightarrow X$ is mod p homotopy injective then $\Phi = \{c\}$. Evaluation provides a map of fibrations

$$\begin{tikzcd}
\mathrm{Map}(BP', BP)_c \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}(BP', X)_c \arrow[r] \arrow[d] & \mathrm{Map}(BP', Y)_c \arrow[d] \\
BP \arrow[r, "f"] & X \arrow[r] & Y
\end{tikzcd}$$

where the left vertical arrow is known to be a homotopy equivalence by Proposition 2.1, and the result follows. ∎

Assume now that we have a connected mod p H-space X that satisfies $(F1)$ and $(F3)$ and an H-map

$$f\colon BP \longrightarrow X$$

from the classifying space of an abelian p-toral group. According to Proposition 2.1

$$ev\colon\ \mathrm{Map}(BP, X)_f \longrightarrow X$$

is a homotopy equivalence, thus we have obtained a different model for the spaces X that supports an action of the topological group BP. We define the homotopy quotient as

$$X_{hBP} = \mathrm{Map}(BP, X)_f \times_{BP} EBP$$

and we have a sequence of fibrations

$$BP \xrightarrow{f} X \xrightarrow{g} X_{hBP} \xrightarrow{h} B^2P\,. \tag{2}$$

In what follows we will show that X_{hBP} is an H-space and the maps g and h are H-maps. Furthermore, the sequence (2) is the only way to complete $f\colon BP \longrightarrow X$ to a sequence of fibrations.

Proposition 2.6. *Let X be a connected mod p H-space that satisfies conditions $(F1)$ and $(F3)$. If P is an abelian p-toral group and $f\colon BP \longrightarrow X$ a mod p homotopy injective H-map, then*

1. *X_{hBP} is an H-space that satisfies $(F1)$ and $(F3)$,*
2. *the quotient map $g\colon X \longrightarrow X_{hBP}$ is an H-map, and*
3. *if Y is another H-space satisfying $(F1)$ and $(F3)$, a map $k\colon X_{hBP} \longrightarrow Y$ is an H-map if and only if the composition $X \xrightarrow{g} X_{hBP} \xrightarrow{k} Y$ is an H-map.*

Proof. We use an argument similar to that of [3, 2.5]. Proposition 2.5 provides a homotopy equivalence $\mathrm{Map}(BP \times BP, X_{hBP})_c \simeq X_{hBP}$. Then, the diagram

$$\begin{tikzcd}
BP \times BP \arrow[r, "m_{BP}"] \arrow[d, "f \times f"'] & BP \arrow[d, "f"] \\
X \times X \arrow[r, "m_X"] \arrow[d, "g \times g"'] & X \arrow[d, "g"] \\
X_{hBP} \times X_{hBP} \arrow[r, dashed, "m_{X_{hBP}}"] & X_{hBP}
\end{tikzcd}$$

is completed by the map $m_{X_{hBP}}$ due to Zabrodsky's lemma. Hence X_{hBP} becomes an H-space and g an H-map.

That X_{hBP} satisfies $(F1)$ follows from the Serre spectral sequence. Propositions 2.1 and 2.5 imply that X_{hBP} satisfies condition $(F3)$.

Finally we prove point 3. If k is an H-map then $k \circ g$ is clearly an H-map. Conversely, if $k \circ g$ is an H-map again an argument using Zabrodsky's lemma shows that the map $X \times X \longrightarrow Y \times Y \longrightarrow Y$ factors uniquely through $X_{hBP} \times X_{hBP}$. Now, the compositions $X_{hBP} \times X_{hBP} \xrightarrow{m_{hBP}} X_{hBP} \xrightarrow{k} Y$ and $X_{hBP} \times X_{hBP} \xrightarrow{k \times k} Y \times Y \xrightarrow{m_Y} Y$ are two possible factorizations, hence they are homotopic; that is, k is an H-map. ∎

Proposition 2.7. *Let X be a connected mod p H-space that satisfies conditions $(F1)$ and $(F3)$, P is an abelian p-toral group and $f\colon BP \longrightarrow X$ a mod p homotopy injective H-map.*

If Y is a connected space and $BP \xrightarrow{f} X \xrightarrow{g'} Y$ a fibration, there is a homotopy equivalence $X_{hBP} \simeq Y$ that completes the commutative diagram

$$\begin{tikzcd} BP \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d, equal] & X_{hBP} \arrow[d, "\simeq"] \\ BP \arrow[r, "f"] & X \arrow[r, "g'"] & Y \end{tikzcd}$$

Furthermore if Y is an H-space and g' an H-map then all arrows in the above diagram are H-maps.

Proof. Proposition 2.1 implies that $\mathrm{Map}(BP, X)_c \simeq X$ and this together with Proposition 2.5, that $\mathrm{Map}(BP, Y)_c \simeq Y$. So, Zabrodsky's lemma applies to the principal fibration

$$BP \xrightarrow{f} X \xrightarrow{g} X_{hBP}$$

and maps with target Y. Since $g'\colon X \longrightarrow Y$ restricts trivially to BP, it factors as $X \xrightarrow{g} X_{hBP} \xrightarrow{k} Y$, making the above diagram homotopy commutative. It is then clear that k is a homotopy equivalence.

Finally, if Y is an H-space Proposition 2.6 applies and hence k is an H-map. ∎

Theorem 2.8. *Let X be a connected mod p H-space that satisfies conditions $(F1)$ and $(F3)$. If P is an abelian p-toral group and $f\colon BP \longrightarrow X$ a mod p homotopy injective H-map, the extension*

$$BP \xrightarrow{f} X \xrightarrow{g} X_{hBP} \xrightarrow{h} B^2P \tag{3}$$

provides an H-fibration sequence.

Proof. We have just to check that the map h is an H-map. An abelian p-toral group P is the product of a p-completed torus T and a finite abelian p-group Q. Thus, we can write two components $h_T\colon Y \longrightarrow B^2T$ and $h_Q\colon Y \longrightarrow B^2Q$ where $Y = X_{hBP}$.

Since g is an H-map, $g \circ h_Q \simeq *$ and the induced map $\pi_1(X) \longrightarrow \pi_1(Y)$ is an epimorphism, we can apply proposition 2.3.1 in [12] and conclude that h_Q is an H-map.

Let us turn our attention to h_T. Notice that if B^2p_T is the projection from B^2P to B^2T then h_T factors as the composition $B^2p_T \circ h$ and this extends to a diagram of fibrations

$$\begin{tikzcd}
BQ \arrow[r, "f_Q"] \arrow[d, "B_{i_Q}"'] & X \arrow[r, "\hat{g}"] \arrow[d, equal] & F \arrow[r, "\hat{h}"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^2Q \arrow[d, "B^2 i_Q"] \\
BP \arrow[r, "f"] & X \arrow[r, "g"] & Y \arrow[r, "h"] \arrow[d, "h_T"'] & B^2P \arrow[d, "B^2 p_T"] \\
 & & B^2T \arrow[r, equal] & B^2T
\end{tikzcd}$$

Since the upper row fibration fits in the conditions of propositions 2.6 and 2.7, F is an H-space, $\hat{g}$ is an H-map and furthermore j is an H-map.

Finally, $j\colon F \longrightarrow Y$ and $h_T\colon Y \longrightarrow B^2T$ satisfy the conditions of proposition 2.3.1 in [12], namely the composition is null-homotopic and by inspection of the Serre spectral sequence, j induces in homology an isomorphism in degree one and an epimorphism in degree two, hence h_T is an H-map. ∎

3. Universal covers

Let X denote a connected mod p H-space with finite type mod p cohomology. The fundamental group will be a finitely generated $\hat{\mathbb{Z}}_p$-module

$$\pi_1(X) \cong \mathbb{Z}/p^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{r_k}\mathbb{Z} \times (\hat{\mathbb{Z}}_p)^l \,.$$

Let $\tilde{X}$ denote the universal cover of X. The fibration $\tilde{X} \xrightarrow{p} X \xrightarrow{j} B\pi_1(X)$ is an H-fibration and its Serre spectral sequence shows that $j^*\colon H^1(B\pi_1(X)) \longrightarrow H^1(X)$ is an isomorphism and $j^*\colon H^2(B\pi_1(X)) \longrightarrow H^2(X)$ is a monomorphism. The image of j^* is a sub Hopf algebra of $H^*(X)$ isomorphic to $H^*(B\pi_1(X))/\operatorname{Ker} j^*$ and then the argument in the proof of [10, thm. 7.11] proves in our case that there is a complementary Hopf algebra A of finite type such that

$$H^*(X) \cong \frac{H^*(B\pi_1(X))}{\operatorname{Ker} j^*} \otimes A$$

where A is 1-connected. Up to a change of generators we can write

$$H^*(B\pi_1(X)) \cong E[u_1, \dots, u_{k+l}] \otimes P[v_1, \dots, v_k]$$

with $\deg u_i = 1$, $\deg v_i = 2$, and $\beta_{(r_i)}(u_i) = v_i$ for $i = 1, \dots, k$ in such a way that $\operatorname{Ker} j^*$, being 2-connected, can be written as $(v_1^{p^{\alpha_1}}, \dots, v_s^{p^{\alpha_s}})$, where $\alpha_i \in \mathbb{N}$ and $s \leq k$, so that,

$$H^*(B\pi_1(X))/\operatorname{Ker} j^* \cong E[u_1, \dots, u_{k+l}] \otimes \frac{P[v_1, \dots, v_s]}{(v_1^{p^{\alpha_1}}, \dots, v_s^{p^{\alpha_s}})} \otimes P[v_{s+1}, \dots v_k], \tag{4}$$

and therefore

$$H^*(X) \cong E[u_1, \dots, u_{k+l}] \otimes \frac{P[v_1, \dots, v_s]}{(v_1^{p^{\alpha_1}}, \dots, v_s^{p^{\alpha_s}})} \otimes P[v_{s+1}, \dots v_k] \otimes A. \tag{5}$$

The Eilenberg-Moore spectral sequence

$$E_2^{*,*} \cong \operatorname{Tor}_{*,*}^{H^*(B\pi_1(X))}(H^*(X), \mathbb{F}_p) \Longrightarrow H^*(\tilde{X}).$$

will now make clear the relation between $H^*(X)$ and $H^*(\tilde{X})$. It is not hard to obtain that $\operatorname{Tor}_{*,*}^{P[v]}(\frac{P[v]}{(v^{p^\alpha})}, \mathbb{F}_p) \cong E[z]$ with $\operatorname{bideg} z = (-1, 2p^\alpha)$ and then

$$\begin{aligned}\operatorname{Tor}_{*,*}^{H^*(B\pi_1(X))}(H^*(X), \mathbb{F}_p) &\cong \operatorname{Tor}_{*,*}^{H^*(B\pi_1(X))}\left(\frac{H^*(B\pi_1(X))}{\operatorname{Ker} j^*} \otimes A, \mathbb{F}_p\right) \cong \\ &\cong \operatorname{Tor}_{*,*}^{H^*(B\pi_1(X))}\left(\frac{H^*(B\pi_1(X))}{\operatorname{Ker} j^*}, \mathbb{F}_p\right) \otimes A \cong E[z_1, \dots, z_s] \otimes A\end{aligned}$$

where $\operatorname{bideg} z_i = (-1, 2p^{\alpha_i})$ and A remains in bidegrees $(0, *)$. Notice then that the Eilenberg-Moore spectral sequence collapses at the E_2 term and we obtain a exact sequence of $\mathcal{A}$-Hopf algebras

$$1 \longrightarrow A \longrightarrow H^*(\tilde{X}) \longrightarrow H^*(\tilde{X})//A \longrightarrow 1 \tag{6}$$

where $H^*(\tilde{X})//A$ is finitely generated by elements represented by $\bar{z}_1, \dots, \bar{z}_s \in H^*(\tilde{X})$, $\deg \bar{z}_i = 2p^{\alpha_i} - 1 \geq 3$, and A is the image of $p^*\colon H^*(X) \longrightarrow H^*(\tilde{X})$. Since the elements $\bar{z}_i$ appear in odd degrees, if we are working at an odd prime p, ${\bar{z}_i}^2 = 0$ and then the above sequence splits to give $H^*(\tilde{X}) \cong A \otimes E[\bar{z}_1, \ldots, \bar{z}_s]$, but this is not clear for $p = 2$. In any case $H^*\tilde{X}$ turns out to be generated by $z_1, \ldots, z_s$ as an A-module.

Proposition 3.1. *The mod p cohomology of a connected mod p H-space of finite type X is described as*

$$H^*(X) \cong \frac{H^*(B\pi_1(X); \mathbb{F}_p)}{\operatorname{Ker} j^*} \otimes A$$

where

1. *A is a finite 1-connected Hopf algebra if $\tilde{X}$ is mod p finite, or*
2. *A is a locally finite 1-connected $\mathcal{A}$-Hopf algebra if $\tilde{X}$ is $B\mathbb{Z}/p$-null.*

Proof. Follows from (6) ∎

Proposition 3.2. *Let X be a connected mod p H-space of finite type. Then, its universal cover $\tilde{X}$ is also a mod p H-space of finite type and*

1. *$H^*(X)$ is noetherian if and only if $H^*(\tilde{X})$ is noetherian.*
2. *$H^*(X)$ has finite transcendence degree if and only if $H^*(\tilde{X})$ has finite transcendence degree.*
3. *$QH^*(X)$ is locally finite if and only if $QH^*(\tilde{X})$ is locally finite.*

Proof. (1) and (2) follow from the description of $H^*(X)$ and $H^*(\tilde{X})$ in (5) and (6). The proof of (3) is same involved. Although it could be obtained in a purely algebraic setting it seems to us more immediate a topological argument. From (5) and (6) and the fact that maps out from BV are controlled by mod p cohomology, we obtain that the sequence

$$0 \longrightarrow [BV, \tilde{X}] \longrightarrow [BV, X] \longrightarrow [BV, B\pi_1(X)]$$

is exact for every elementary abelian p-group V. Hence we have a fibration

$$\operatorname{Map}(BV, \tilde{X})_c \longrightarrow \operatorname{Map}(BV, X)_c \longrightarrow \operatorname{Map}(BV, B\pi_1(X))_c$$

Evaluation gives a homotopy equivalence $\operatorname{Map}(BV, B\pi_1(X))_c \simeq B\pi_1(X)$ and then $\operatorname{Map}(BV, \tilde{X})_c \simeq \tilde{X}$ if and only if $\operatorname{Map}(BV, X)_c \simeq X$. This is the geometric reason of the statement (3) by Proposition 1.1. ∎

Let X be a connected $B\mathbb{Z}/p$-null H-space, then in equation (5) $s = k$ and A is a locally finite $\mathcal{A}$-module. Hence $\tilde{X}$ is also $B\mathbb{Z}/p$-null. The universal cover might be $B\mathbb{Z}/p$-null in more general situations. Those are determined by Theorem 1.5.

Proof of Theorem 1.5. If X fits in a pull-back diagram like (1), its universal cover is homotopy equivalent to that of $F(X)$ and this last is $B\mathbb{Z}/p$-null by hypothesis.

Assume now that $\tilde{X}$ is $B\mathbb{Z}/p$-null. In particular $H^*(\tilde{X})$ is a locally finite $\mathcal{A}$-module and according to equation (6) $A = Im\{p^*\colon H^*(X) \longrightarrow H^*(\tilde{X})\}$ is also a locally finite $\mathcal{A}$-module and $H^*(X)$ is described according to (5)

$$H^*(X) \cong E[u_1, \dots, u_{k+l}] \otimes \frac{P[v_1, \dots, v_s]}{(v_1^{p^{\alpha_1}}, \dots, v_s^{p^{\alpha_s}})} \otimes P[v_{s+1}, \dots v_k] \otimes A$$

with A a locally finite 1-connected $\mathcal{A}$-Hopf algebra.

Notice that the factor $P[v_{s+1}, \dots v_k]$ is what prevents the mod p cohomology of X from being locally finite and this factor is inherited from $H^*(B\pi_1(X))$, more precisely from the cohomology of the torsion part of $\pi_1(X)$.

Our objective is to eliminate such polynomial generators. Fix v to be one of those. Nil-localization provides a map of $\mathcal{A}$-Hopf algebras

$$f^*\colon H^*(X) \longrightarrow H^*(B\mathbb{Z}/p)$$

that detects precisely our polynomial generator v [4, thm. 2.10] .

By Lannes theory [8] we can realize f^* as an H-map $f\colon B\mathbb{Z}/p \longrightarrow X$ that turns out to be mod p homotopy injective. Now by Theorem 2.8 f fits in an H-fibration sequence

$$B\mathbb{Z}/p \xrightarrow{f} X \xrightarrow{g} E \xrightarrow{h} B^2\mathbb{Z}/p$$

where E is the Borel construction $\mathrm{Map}(B\mathbb{Z}/p, X)_f \times_{B\mathbb{Z}/p} EB\mathbb{Z}/p$ (see [3] and [4] for the details) that in turn extends to a pull-back diagram

$$\begin{tikzcd}
 & B\mathbb{Z}/p \arrow[r, equal] \arrow[d, "f"'] & B\mathbb{Z}/p \arrow[d, "j_X \circ f"] \\
\tilde{X} \arrow[r, "p_X"] \arrow[d, equal] & X \arrow[r, "j_X"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & B\pi_1(X) \arrow[d] \\
\tilde{X} \arrow[r, "p_E"] & E \arrow[r, "j_E"] & B\pi_1(E)
\end{tikzcd} \tag{7}$$

where $j_X \circ f$ is non trivial and $\pi_1(E) \cong \pi_1(X)/\mathbb{Z}/p$.

Notice that $j_X \circ f$ is non-trivial because the generator $v \in H^*(X)$ detected by f was inherited from $H^*(B\pi_1(X))$. Now, the same construction applies to $j_X \circ f \colon B\mathbb{Z}/p \longrightarrow B\pi_1(X)$ and gives the fibration in the right column of the diagram 7. Finally the two fibration fit together by naturality of the construction.

Now observe that the H-space E satisfies the same conditions as X namely, it is connected, mod p H-space of finite type and its universal cover $\tilde{E} \simeq \tilde{X}$ is $B\mathbb{Z}/p$-null. Hence, in case it is not yet $B\mathbb{Z}/p$-null we can iterate the same construction. We obtain, in this way, a sequence of H-spaces and H-maps

$$X = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \ldots$$

where E_k and E_{k+1} are related by a pull-back diagram

$$\begin{tikzcd}
 & B\mathbb{Z}/p \arrow[r, equal] \arrow[d, "f_k"'] & B\mathbb{Z}/p \arrow[d, "j_k \circ f_k"] \\
\tilde{X} \arrow[r, "p_k"] \arrow[d, equal] & E_k \arrow[r, "j_k"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B\pi_1(E_k) \arrow[d] \\
\tilde{X} \arrow[r, "p_{k+1}"] & E_{k+1} \arrow[r, "j_{k+1}"] & B\pi_1(E_{k+1})
\end{tikzcd}$$

where again $\pi_1(E_{k+1}) \cong \pi_1(E_k)/\mathbb{Z}/p$.

Since the torsion part of $\pi_1(X)$ is finite, and at each step we reduce the torsion part of the fundamental group this process will finish at a finite stage E_m that will be a $B\mathbb{Z}/p$-null connected mod p H-space.

Gluing together all the sequence of pull-backs we obtain

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B\pi_1(X) \arrow[d] \\
E_m \arrow[r] & B\pi_1(E_m)
\end{tikzcd}$$

where $B\pi_1(X) \longrightarrow B\pi_1(E_m)$ is induced by an epimorphism $\rho \colon \pi_1(X) \longrightarrow \pi_1(E_m)$ with kernel a finite abelian p-group.

In order to finish the proof of Theorem 1.5 we only need to show that E_m is obtained functorially from X as $F(X)$. This follows from basic properties of the nullification functor that can be found in [6]. Since $P_{B\mathbb{Z}/p}$ annihilates $B\mathbb{Z}/p$, each fibration $B\mathbb{Z}/p \xrightarrow{f_k} E_k \longrightarrow E_{k-1}$ induces a homotopy equivalence $P_{B\mathbb{Z}/p}(E_k) \simeq P_{B\mathbb{Z}/p}(E_{k+1})$. Hence $P_{B\mathbb{Z}/p}(X) \simeq P_{B\mathbb{Z}/p}(E_m)$. But E_m is already $B\mathbb{Z}/p$-null, and then $P_{B\mathbb{Z}/p}(X) \simeq E_m$, thus also $F(X) \simeq E_m$. ∎

Proof of Theorem 1.6. If $F(X)$ is mod p finite, then its universal cover $\tilde{F}(X)$ is mod p finite too. If we assume in addition that there is a pull-back diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B\pi_1 \arrow[d, "B\rho"] \\ F(X) \arrow[r] & B\pi_1(F(X)) \end{tikzcd}$$

then we have that $\tilde{F}(X)$ coincides with the universal cover of X.

Conversely, assume that $\tilde{X}$ is mod p finite, in particular $B\mathbb{Z}/p$-null. Hence Theorem 1.5 applies and we obtain a diagram like the one above, thus proving that the universal cover of $F(X)$ is homotopy equivalent to $\tilde{X}$. Since $F(X)$ is a $B\mathbb{Z}/p$-null H-space of finite type, Proposition 1.1 implies that $H^*(F(X))$ has transcendence degree zero and then Proposition 3.1 together with equation 5 gives

$$H^*(F(X)) \cong E[u_1, \dots, u_{k+l}] \otimes \frac{P[v_1, \dots, v_s]}{(v_1^{p^{\alpha_1}}, \dots, v_s^{p^{\alpha_s}})} \otimes A$$

with A finite. In particular $F(X)$ is mod p finite. ∎

4. Proof of the Main Theorem

This section is devoted to the proof of our structural results for connected H-spaces with the prescribed finiteness conditions $(F1)$, $(F2)$, and $(F3)$, that is, Theorems 1.2 and 1.3. We first need to establish the statement that fibrewise localization [6] preserves H-fibrations.

Lemma 4.1. *If $X \to E \to B$ is an H-fibration, then its fibrewise localization $F(X) \to \bar{E} \to B$ is also an H-fibration. Furthermore, the induced map $E \longrightarrow \bar{E}$ is also an H-map.*

Proof. By the naturality of the fibrewise localization and by using the homotopy equivalence $F(X \times Y) \simeq F(X) \times F(Y)$. ∎

Proof of Theorem 1.2. Assume X is a connected H-space such that its mod p cohomology satisfies conditions $(F1)$, $(F2)$ and $(F3)$. We can write its universal cover and its fundamental group in an H-fibration sequence

$$\tilde{X} \to X \to B\pi_1(X)\,. \tag{8}$$

According to Proposition 3.2 the mod p cohomology of $\tilde{X}$ satisfies also the conditions $(F1)$, $(F2)$ and $(F3)$ and then the results of [3, 4] apply. There is then an H-fibration sequence

$$BP \longrightarrow \tilde{X} \longrightarrow F(\tilde{X}) \longrightarrow B^2P \tag{9}$$

where P is an abelian p-toral group, and $F(\tilde{X})$ is a simply connected $B\mathbb{Z}/p$-null H-space.

Fibrewise localization of the fibration (8) is again an H-fibration. Together with fibration (9) this fits in a diagram

$$\begin{tikzcd}
BP \arrow[r] \arrow[d, equal] & \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\tilde{X}) \arrow[r] \arrow[d] & B^2P \arrow[d, equal] \\
BP \arrow[r] & X \arrow[r] \arrow[d] & \bar{X} \arrow[r] \arrow[d] & B^2P \\
 & B\pi_1(X) \arrow[r, equal] & B\pi_1(X) &
\end{tikzcd} \tag{10}$$

Notice that the existence of the H-fibration sequence in the middle row is not immediate but rather follows from Theorem 2.8. Indeed, $\bar{X} \simeq X_{hBP}$.

Observe also that it follows from the above diagram that $F(\tilde{X})$ is homotopy equivalent to the universal cover of $\bar{X}$, and so, therefore Theorem 1.5 applies. We obtain a pullback diagram

$$\begin{tikzcd}
BQ \arrow[r, equal] \arrow[d] & BQ \arrow[d] \\
\bar{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\pi_1(\bar{X}) \arrow[d, "B\rho"] \\
F(\bar{X}) \arrow[r] & B\pi_1(F(\bar{X}))
\end{tikzcd} \tag{11}$$

where BQ is the classifying space of a finite abelian p-group Q.

Now (10) and (11) combine to a diagram of fibrations

$$\begin{tikzcd}
BP \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & BQ \arrow[d] \\
BP \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \bar{X} \arrow[d] \\
* \arrow[r] & F(\bar{X}) \arrow[r, equal] & F(\bar{X})
\end{tikzcd}$$

where W is the homotopy fibre of the map $X \to F(\bar{X})$. From the fibration

$$BP \longrightarrow W \longrightarrow BQ$$

we deduce that $W \simeq BP'$ is the classifying space of a p-toral group and since W is the fibre of an H-map it is itself an H-space, and therefore P' is an abelian p-toral group.

Finally $F(X) \simeq F(\bar{X})$ and by Theorem 2.8 we obtain the required H-fibration sequence

$$BP' \longrightarrow X \longrightarrow F(X) \longrightarrow B^2P' \qquad \blacksquare$$

Proof of Theorem 1.3. Assume now that $H^*(X)$ is noetherian. In particular X satisfies conditions $(F1)$, $(F2)$, $(F3)$, and so, in particular, the arguments in the proof of Theorem 1.2 above apply. Now, by Proposition 3.2 one has that $H^*(\tilde{X})$ is also noetherian. Hence following [3, 4], $F(\tilde{X})$ is a finite H-space. But $F(\tilde{X})$ is homotopy equivalent to the universal cover of $\bar{X}$, hence Theorem 1.6 applies and we get that $F(\bar{X}) \simeq F(X)$ is mod p finite. $\blacksquare$

References

1. J. Aguadé, C. Broto, and D. Notbohm, *Homotopy classification of spaces with interesting cohomology and a conjecture of Cooke, part I*, Topology **33** (1994), 455–492.
2. A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, 1972.
3. C Broto and J.A. Crespo, *H-spaces with noetherian mod two cohomology algebra*, Topology **38** (1999), 353–386.
4. J. A. Crespo, *Structure of mod p H-spaces with finiteness conditions*, Proceedings of the 1998 BCAT, To appear.
5. W.G. Dwyer and C.W. Wilkerson, *Spaces of nulhomotopic maps*, Asterisque **191** (1990), 97–108.

6. E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Springer-Verlag, Berlin, 1996.
7. J. Lannes and L. Schwartz, *Sur la structure des $\mathcal{A}$-modules instables injectifs*, Topology **28** (1989), no. 2, 153–169.
8. J. Lannes, *Sur les espaces fonctionels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Publ. Math. IHES **75** (1992), 135–244.
9. H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120** (1984), no. 1, 39–87.
10. J.W. Milnor and J.C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.
11. L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, University of Chicago Press, Chicago, IL, 1994.
12. A. Zabrodsky, *Hopf spaces*, North-Holland Publishing Co., Amsterdam, 1976, North-Holland Mathematics Studies, Vol. 22, Notas de Matemática, No. 59.
13. ______, *On phantom maps and a theorem of H. Miller*, Israel J. Math. **58** (1987), no. 2, 129–143.

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