H-SPACES WITH NOETHERIAN MOD TWO COHOMOLOGY ALGEBRA

CARLOS BROTO AND JUAN A. CRESPO

Abstract. The object of this paper is to analyse the structure of connected H-spaces with noetherian mod two cohomology algebra. We will show that, up to 2-completion, they are, essentially, finite mod 2 H-spaces and their 3-connected covers, $\mathbb{C}P^\infty$, $B\mathbb{Z}/2^r$ and certain extensions of these.

1. Introduction

A subject of great interest in algebraic topology is the understanding of the homotopy theoretic generalizations of the concept of compact Lie group. Among those, finite H-spaces or loop spaces and the localized versions, mod p finite H-spaces and the newer concept of p-compact group ([11]).

A finite H-space is an H-space whose underlying space is homotopy equivalent to a CW-complex with a finite number of cells. In the localized version, a mod p finite H-space stands for an H-space which is finite up to p-completion or equivalently for an H-space which mod p cohomology ring is finite dimensional. By p-completion we understand Bousfield-Kan p-completion [5]. An H-space, being simple, it is p-good in the sense of Bousfield-Kan, so we will assume without loose of generality that our mod p H-spaces are p-complete.

Other related spaces that play an important role are the three connected covers of compact Lie groups, finite H-spaces or mod p finite H-spaces. In fact, those connected covers carry most of the homotopy theoretic structure of the original finite H-spaces as, for instance, higher homotopy groups. Furthermore, recently it has been discovered that most of the times they even *recall* the p-completed homotopy type of the original finite H-space. More precisely, in [9] Dror Farjoun gives a nice construction of localization functors with respect to maps. Concretely the nullification functor for $B\mathbb{Z}/p$, $L_{B\mathbb{Z}/p}$, was investigated by Neisendorfer [23] who defined F as the composition of the functor $L_{B\mathbb{Z}/p}$ and the Bousfield-Kan p-completion and proved that for a 1-connected finite complex X with finite $\pi_2(X)$, if $X\langle n\rangle$ denotes the n-connected cover of X, then $F(X\langle n\rangle) \simeq X^\wedge_p$ for any positive integer n.

The three connected cover of a finite H-space is not homotopy equivalent to a finite CW-complex. In fact the mod p cohomology ring is no longer finite. It is however a noetherian ring. The universal cover of a mod p finite H-space X is again a mod p finite H-space $\tilde{X} \simeq X\langle 1\rangle$. $\tilde{X}$ is thus 2-connected and its third homotopy group is torsion free ([8]). In order to construct the 3-connected cover of X, we choose a map

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$\tilde{X} \to K(\hat{\mathbb{Z}}_p{}^m, 3)$ that induces an isomorphism between the three dimensional homotopy groups and the 3-connected cover $X\langle 3\rangle$ is defined as the homotopy fibre of that map, thus it fits in a principal fibration

$$((\mathbb{C}P^\infty)^\wedge_p)^m \to X\langle 3\rangle \to \tilde{X} .$$

A spectral sequence argument shows now that the mod p cohomology ring of $X\langle 3\rangle$ is not finite but finitely generated; that is, noetherian. Other H-spaces with noetherian mod p cohomology ring are $(\mathbb{C}P^\infty)^\wedge_p$ and $B\mathbb{Z}/p^r$ for any positive integer r.

Our aim is to prove that those H-spaces are essentially all mod p H-spaces with noetherian mod p cohomology ring. In this paper, both for clarity and simplicity we concentrate in the case $p = 2$. The necessary changes for the odd prime case will be considered in a forthcoming paper. So, at prime two, we obtain

Theorem 1.1. *Let X be a 1-connected mod 2 H-space with noetherian mod 2 cohomology algebra, then there exists a mod 2 finite H-space $F = F(X)$ and a principal H-fibration*

$$((\mathbb{C}P^\infty)^\wedge_2)^n \to X \to F(X) . \tag{1}$$

Thus we easily obtain a cohomological characterization of three connected covers of mod 2 finite H-spaces

Corollary 1.2. *A mod 2 H-space is the three connected cover of a mod 2 finite H-space if and only if its mod 2 cohomology ring is three connected and noetherian.* □

Corollary 1.3. *All 1-connected mod 2 H-spaces with noetherian mod 2 cohomology ring are finite mod 2 H-spaces, $(\mathbb{C}P^\infty)^\wedge_2$, products of those and extensions of the form (1).* □

This answers a question of Lin [18, Question 2.4], at prime 2.

The general case is reduced to the simply connected case by taking the universal cover of our H-space X:

$$\tilde{X} \to X \to B\pi_1(X) \tag{2}$$

Corollary 1.4. *All connected mod 2 H-spaces with noetherian mod 2 cohomology ring are finite mod 2 H-spaces, $(\mathbb{C}P^\infty)^\wedge_2$, $B\mathbb{Z}/2^r$ for any positive integer r, products of those and extensions of the form (1) or (2).* □

A connected mod p loop space X is a triple (X, BX, e) where X is connected and p-complete space, BX a 1-connected and p-complete space and e is a homotopy equivalence $e\colon X \to \Omega BX$. A connected mod p loop space X is a mod p finite loop space or p-compact group if $H^*(X; \mathbb{F}_p)$ is finite dimensional ([11]). Theorem 1.1 and corollaries 1.2, 1.3 and 1.4 remain true for mod 2 loop spaces instead of mod 2 H-spaces because according to [9] the nullification functor $L_{B\mathbb{Z}/2}$ and hence F preserves the loop structure.

We can also use Theorem 1.1 in order to reduce questions about H-spaces or loop spaces with noetherian mod 2 cohomology to finite ones. As an example we can easily obtain the classification of homotopy commutative mod 2 H-spaces with noetherian mod 2 cohomology, based in the corresponding result for mod 2 finite H-spaces ([12], [17]),

Corollary 1.5 (Slack [25], Lin-Williams [20]). *Let X be a homotopy commutative connected mod 2 H-space with noetherian mod 2 cohomology. Then X is the direct product of a finite number of Eilenberg-MacLane spaces*

$$K(\hat{\mathbb{Z}}_2, 2), \quad K(\hat{\mathbb{Z}}_2, 1), \quad K(\mathbb{Z}/2^r, 1)$$

for $r \geq 1$.

Proof. Let $\tilde{X}$ be the universal cover of X. It is as well a homotopy commutative H-space with noetherian mod 2 cohomology and now Theorem 1.1 applies. $\tilde{X}$ is the total space of a principal fibration $((\mathbb{C}P^\infty)\hat{_2})^n \to \tilde{X} \to F(\tilde{X})$. where $F(\tilde{X})$ is a 1-connected mod 2 finite H-space and, using the properties of F, also homotopy commutative. Hence the classical torus theorem of Hubbuck ([12], [17]) implies that $F(\tilde{X})$ is contractible and therefore $\tilde{X} \simeq ((\mathbb{C}P^\infty)\hat{_2})^n$.

We have obtained a covering $((\mathbb{C}P^\infty)\hat{_2})^n \to X \to B\pi_1(X)$ which is a simple fibration an then it should be classified by an H-map $B\pi_1(X) \to K(\hat{\mathbb{Z}}_2^n, 3)$. Moreover, this map should represent a primitive class in the cohomology of $B\pi_1(X)$ (cf. [26]).

But $\pi_1(X) \cong \hat{\mathbb{Z}}_2^r \times \mathbb{Z}/2^{k_1} \times \ldots \times Z/2^{k_s}$, a finitely generated $\hat{\mathbb{Z}}_2$-module, has no non trivial primitives in its 2-adic three dimensional cohomology, hence this classifying map is trivial and then,

$$X \simeq ((\mathbb{C}P^\infty)\hat{_2})^n \times B\hat{\mathbb{Z}}_2^r \times B\mathbb{Z}/2^{k_1} \times \ldots \times BZ/2^{k_s}\,. \quad \square$$

Our starting point is the classification by Aguadé, Broto and Notbohm [1] of the possible p-completed homotopy types of spaces having a mod p cohomology ring like that of the three connected covering of S^3,

$$P[x_{2p}] \otimes E[\beta x_{2p}] \tag{3}$$

where subscripts denote degree of the generators. Also, they proved in the final section that, among those spaces, the only one that admits an H-space structure is p-completion of the true three connected cover of S^3: $S^3\langle 3\rangle\hat{_p}$.

Later, in [2], it was considered the case of spaces with mod p cohomology isomorphic to the algebra

$$P[x_{2p^2}] \otimes E[y_{2p+1}, z_{2p^2+1}] \tag{4}$$

This is the cohomology of $SU(3)\langle 3\rangle$ for $p = 2$, $Sp(2)\langle 3\rangle$ for $p = 3$ and $G_2\langle 3\rangle$ if $p = 5$. A homotopy uniqueness result is obtained in that case for every prime, even with no H-structure assumption.

Our observation is the relevance in the proof of the above results of the fact that the considered algebras, (3) and (4), are noetherian.

There is one further observation about the Steenrod algebra action on the algebras (3) and (4) and other cohomology algebras of three connected covers of H-spaces (cf [13]). As the degree of a polynomial generator increases there are more and more nilpotent generators attached to that polynomial generator by means on Steenrod operations. For example, in (4) at prime 2 we have $Sq^1x_8 = z_9$ and $Sq^4y_5 = z_9$. Actually, this observation goes back to [3] where it is shown that the three connected cover of $Sp(k)$ has one polynomial generator of degree $2p^i$, for $k = (p^{i-1} + 1)/2$, together with a number of exterior generators. In [1] it is shown that an algebra like (3) would not be the cohomology of an H-space if the degree of the polynomial generator was larger than $2p$ (or rather, of any space if it was larger than $2p(p-1)$).

Within the proof Theorem 1.1 we find an explanation for this fact, actually, for a more general class of H-spaces. Namely, H-spaces X satisfying the finiteness conditions:

(F1) $H^*(X;\mathbb{F}_2)$ is of finite type.
(F2) $H^*(X;\mathbb{F}_2)$ has a finite number of polynomial generators.
(F3) The module of the indecomposables $QH^*(X;\mathbb{F}_2)$ is locally finite as module over the Steenrod algebra.

Recall that a module over the Steenrod algebra is called locally finite provided any submodule generated by a single element is finite (cf. [24]). We will denote by $\mathcal{A}$the mod two Steenrod algebra. It is generated by the Steenrod squares Sq^i, $i \geq 0$, subject to the Adem relations. The squares Sq^{2^n} for a system of algebra generators. We will denote

$$Sq^{\Delta_n} = Sq2^n Sq^{2^{n-1}} \dots Sq^2 Sq^1$$

for $n \geq 0$ and formally $Sq^{\Delta_r} = 0$ if $r < 0$. With this notation we can express the mod 2 cohomology of $B^2\mathbb{Z}/2$ as the polynomial algebra

$$H^*(B^2\mathbb{Z}/2;\mathbb{F}_2) \cong \mathbb{F}_2[\iota, Sq^1\iota, \dots, Sq^{\Delta_n}\iota, \dots]$$

where $\iota \in H^2(B^2\mathbb{Z}/2;\mathbb{F}_2)$ is the fundamental class.

Theorem 1.6. *For any mod 2 H-space X satisfying the conditions* F1, F2 and F3 *and any polynomial generator $x \in H^*(X;\mathbb{F}_2)$ of degree* $\deg x > 1$, *there exists a finite subquotient of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ of the form, either*

$$M_n^{\mathrm{I}} = \left\langle Sq^{\Delta_n}\iota, (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, (Sq^{\Delta_{n-2}}\iota)^{2^{m_2}}, \dots, (Sq^1\iota)^{2^{m_n}} \right\rangle_{\mathbb{F}_2}$$

with $n \geq 0$, $m_0 = 0$, $m_1 = 1$ and $m_{k-1} \leq m_k \leq m_{k-1}+1$ or

$$M_n^{\mathrm{II}} = \left\langle Sq^{\Delta_n}\iota, (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, (Sq^{\Delta_{n-2}}\iota)^{2^{m_2}}, \dots, (Sq^1\iota)^{2^{m_n}}, \iota^{2^m} \right\rangle_{\mathbb{F}_2}$$

with $n \geq 0$, $m_0 = 0$, $m_1 = 1$, $m_{k-1} \leq m_k \leq m_{k-1}+1$ and $m \geq m_n$, and an epimorphism of unstable $\mathcal{A}$-modules:

$$\tilde{\tau} : \Sigma QH^*X \twoheadrightarrow M_n^{\bullet}$$

with $\tilde{\tau}(x) = Sq^{\Delta_n}\iota$, where $M_n^{\bullet}$ denotes either M_n^{I} or M_n^{II}. Moreover, x might be completed to a system of generators where $\tilde{\tau}(y) = 0$ for other polynomial generators $y \neq x$.

In case that $H^*(X;\mathbb{F}_2)$ is noetherian and 1-connected it can only happen M_n^0.

This shows that a polynomial generator cannot happen in a large dimension unless is linked by Steenrod operations to other nilpotent generators in a way codified by the unstable $\mathcal{A}$-modules M_n.

Also Theorem 1.1 can be stated in a more general form, for 1-connected mod 2 H-spaces satisfying conditions F1,F2 and F3 (see theorem 8.4).

Example 1.7.

$n = 1$: $M_1^{\mathrm{I}} = \{(Sq^{\Delta_1}\iota), (Sq^1\iota)^2\}$. We can represent it in a diagram

$$\circ \xrightarrow{Sq^1} \bullet$$

where $\circ$ represents the class $Sq^{\Delta_n}\iota$ which will correspond to the polynomial generator and $\bullet$ represent the other classes that will correspond to nilpotent generators in the cohomology of X. This case clearly corresponds to $S^3\langle 3\rangle$.

$n = 2$: Now M_2^{I} is either $\{Sq^{\Delta_2}\iota, (Sq^{\Delta_1}\iota)^2, (Sq^1\iota)^2\}$ or $\{Sq^{\Delta_2}\iota, (Sq^{\Delta_1}\iota)^2, (Sq^1\iota)^4\}$; that is, one of the two diagrams

$$\circ \xrightarrow{Sq^1} \bullet \xleftarrow{Sq^4} \bullet$$

$$\circ \xrightarrow{Sq^1} \bullet \xrightarrow{Sq^2} \bullet$$

So, a polynomial generator in dimension 8, x_8 comes always together with two more generators linked by Steenrod operations according to one of the above two diagrams.

These cases are realized by $SU(3)\langle 3\rangle$ and $G_2\langle 3\rangle$.

$n \geq 1$: If $n \geq 1$, $M_n^{\bullet}$ contains the classes of $Sq^{\Delta_n}\iota$ and $(Sq^{\Delta_{n-1}}\iota)^2 = Sq^1(Sq^{\Delta_n}\iota)$:

$$\circ \xrightarrow{Sq^1} \bullet \ \ldots \ \bullet \ \ldots$$

thus every polynomial generator in degree bigger than or equal to two has degree a power of two and non trivial Sq^1. This has appeared with some restrictions in [19].

Our final corollary was suggested to us by R. Kane.

Corollary 1.8. *Let X be a connected H-space of finite integral type. If $H^*(X;\mathbb{F}_2)$ is noetherian, then al rational polynomial generators appear in degree two.*

Proof. Being X of finite integral type, the Bockstein spectral sequence applies. It starts with $H^*(X;\mathbb{F}_2)$ and, according to Theorem 1.6, all polynomial generators have non trivial Sq^1, unless possibly some generators in degree two. Thus the spectral sequence converges to a finitely generated Hopf algebra with all polynomial generators in degree two. Thus the same should happen rationally. □

The paper is organized as follows. In section 2 we construct H-fibrations $B\mathbb{Z}/2 \to X \to E \to B^2\mathbb{Z}/2$ for a given mod 2 H-space satisfying F1, F2 and F3 where $B\mathbb{Z}/2 \to X$ detects a prescribed polynomial generator. Then we study the Serre spectral sequence for $X \to E \to B^2\mathbb{Z}/2$. Section 3 contains information about the structure of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ and section 4 about differential Hopf algebras. The spectral sequence itself is analysed in sections 5, 6 and 7. Finally in section 8 we iterate the construction of section 2 and obtain the proofs of Theorems 1.1 and 1.6.

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2. Detecting polynomial generators with central elements

Let X denote a connected mod 2 H-space that satisfies the conditions F1, F2 and F3. according to conditions F1 and F2 and the Borel classification of finite type Hopf algebras, there is an algebra isomorphism

$$H^*(X;\mathbb{F}_2) \cong P[x_1,\dots,x_r] \otimes \frac{P[y_1,\dots,y_s,\dots]}{(y_1^{2^{\alpha_1}},\dots,y_s^{2^{\alpha_s}},\dots)} \tag{5}$$

We call r the depth of X.

The objective of this section is to show the existence of central elements in X detecting the polynomial generators; that is, maps

$$f\colon B\mathbb{Z}/2 \to X$$

for which a certain polynomial generator in $H^*(X;\mathbb{F}_2)$ has non trivial restriction to $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$ and such that $\mathrm{map}(B\mathbb{Z}/2, X)_f \simeq X$. The main tool here is Lannes theory on elementary abelian groups [15]. The proof of that last homotopy equivalence will be based on work of Dwyer-Wilkerson [10] and uses strongly the condition F3. Finally, we take the homotopy quotient by the central element thus obtaining a sequence of fibrations

$$B\mathbb{Z}/2 \to X \to E \to B^2\mathbb{Z}/2$$

that turn out to be H-fibrations. We use Zabrodsky's Lemma in order to prove this fact. Let us recall it here

Lemma 2.1 ([27, 22])**.** *Let G be a topological group and $G \to E \to B$ a principal fibration. If, for a space X, $\mathrm{map}(G,X)_c \simeq X$, where c is the constant map, then*

$$\mathrm{map}(B,X) \simeq \mathrm{map}(E,X)_{f|_G \simeq c}\ .$$

The mod 2 cohomology of an H-space is both, an unstable algebra over the Steenrod algebra and a Hopf algebra in a compatible way. We will say that it is an unstable $\mathcal{A}$-Hopf algebra.

Let ℓ denote the localization functor of unstable modules or algebras over the Steenrod algebra away from nilpotence (cf. [24],[6]). From the proof of proposition 1.23 in [7] or applying proposition 8.2.1 in [16] to the construction of ℓ, it follows that for a tensor product of unstable $\mathcal{A}$-algebras $R \otimes S$ one have a natural isomophism $\ell(R) \otimes \ell(S) \xrightarrow{\cong} \ell(R \otimes S)$. And therefore, the localization of an unstable $\mathcal{A}$-Hopf algebra is again an unstable $\mathcal{A}$-Hopf algebra and the coaugmentation is as well a map of unstable $\mathcal{A}$-Hopf algebras.

Theorem 2.2. *Let X be a connected H-space that satisfies conditions F1 and F2. The localization of $H^*(X;\mathbb{F}_2)$ gives a map of unstable $\mathcal{A}$-Hopf algebras*

$$\mu_X\colon H^*(X;\mathbb{F}_2) \longrightarrow H^*(BV;\mathbb{F}_2)$$

where V is an elementary abelian 2-group of rank the depth of X.

Furthermore, if $H^*(X;\mathbb{F}_2)$ is described as in (5), then there is a basis $u_1,\dots,u_r$ of $H^1(BV;\mathbb{F}_2)$ such that $\mu_X(x_i) = u_i^{2^{\beta_i}}$, for $\beta_i \geq 0$ and $i = 1,\dots,r$.

Proof. since X satisfies conditions F1 and F2 we can describe its mod 2 cohomology algebra as in (5). Let $\mu_X\colon H^*(X;\mathbb{F}_2) \to \ell\big(H^*(X;\mathbb{F}_2)\big)$ be the coaugmentation of the localization of $H^*(X;\mathbb{F}_2)$. The kernel of μ_X consists of the maximal nilpotent ideal of $H^*(X;\mathbb{F}_2)$. It should be, therefore, the ideal generated by the nilpotent generators $y_1,\dots,y_s,\dots$. Thus, we obtain that the image of μ_x is isomorphic to the polynomial algebra $P[x_1,\dots,x_r]$ and then, $\ell\big(H^*(X;\mathbb{F}_2)\big) \cong \ell\big(P[x_1,\dots,x_r]\big)$. This last is clearly given by the Adams-Wilkerson embedding into the mod 2 cohomology of an elementary abelian group of rank r, the depth of X, hence we obtain the desired map μ_X.

Finally, image of the localization should be sub Hopf algebra of $H^*(BV,\mathbb{F}_2)$ and then the precise description of μ_X is a consequence of the Borel classification of Hopf algebras (see [4]). □

What makes this result interesting is that using results of Lannes [15] we can realize the algebraic map of the theorem by a geometric map

$$f\colon BV \to X$$

with $f^* = \mu_X$. Moreover, since μ_X is a map of Hopf algebras, it commutes with the diagonal and geometrically this means that f is an H-map.

Assume that V' is any other elementary abelian group and $f'\colon BV' \to X$ a map. By universality of the coaugmentation μ_X, the induced map $f'^*\colon H^*(X;\mathbb{F}_2) \to H^*(BV';\mathbb{F}_2)$ factors as a composition $H^*(X;\mathbb{F}_2) \xrightarrow{\mu_X} H^*(BV;\mathbb{F}_2) \longrightarrow H^*(BV';\mathbb{F}_2)$ and then f' itself factors as $BV' \longrightarrow BV \xrightarrow{f} X$ for a certain homomorphism $V' \to V$ and so f' is as well an H-map.

Lemma 2.3. *In the above conditions,* $\mathrm{map}(BV',X)_{f'} \simeq X$*, provided X satisfies as well the condition F3.*

Proof. We have a map

$$g\colon BV' \times X \longrightarrow X$$

obtained as the composition of $f' \times id\colon BV \times X \to X \times X$ and the multiplication of X.

The map g induces in cohomology a map $g^*\colon H^*(X;\mathbb{F}_2) \to H^*(BV';\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2)$ and this in turn induces an adjoint map $T_{V'}(H^*(X;\mathbb{F}_2);f^*) \to H^*(X;\mathbb{F}_2)$, where $T_{V'}$ is the Lannes' T functor ([15]).

The computation of $T_{V'}(H^*(X;\mathbb{F}_2);f^*)$ follows from [10, 3.2,4.5] and in fact $T_{V'}(H^*(X;\mathbb{F}_2);f^*) \to H^*(X;\mathbb{F}_2)$ becomes an isomorphism if and only if condition F3 is satisfied. Then, the lemma follows from [15, 3.3.2]. □

Thus, we have proved that if we fix a connected mod 2 H-space X satisfying conditions F1, F2 and F3, any $BV' \to X$ is central. Next, we observe that BV' acts on $\mathrm{map}(BV',X)_f \simeq X$ and define

$$E = \mathrm{map}(BV',X)_f \times_{BV'} EBV' \tag{6}$$

so obtaining a sequence of fibrations

$$BV' \xrightarrow{f} X \xrightarrow{g} E \xrightarrow{h} B^2V'. \tag{7}$$

It remains to prove that the constructed space E is actually an H-space in such a way that the fibrations (7) are H-fibrations.

The argument is a variation of an argument in [1] in which we use the fibration (7) itself in order to compute the mapping spaces $\mathrm{map}(BW, E)_c$, for W any elementary abelian 2-group.

Lemma 2.4. *Let V be any elementary abelian 2-group and $BV \xrightarrow{f} X \longrightarrow Y$ a fibration. Assume that $H^*(BV;\mathbb{F}_2)$ becomes finitely generated as $H^*(X;\mathbb{F}_2)$-module induced by f^*. Then, for any W elementary abelian 2-group, $\mathrm{map}(BW, X)_c \simeq X$ if and only if $\mathrm{map}(BW, Y)_c \simeq Y$.*

Proof. The proof follows from the diagram of fibrations

$$\begin{array}{ccc} \mathrm{map}(BW, BV)_S & \xrightarrow{ev} & BV \\ \downarrow & & \downarrow \\ \mathrm{map}(BW, X)_c & \xrightarrow{ev} & X \\ \downarrow & & \downarrow \\ \mathrm{map}(BW, Y)_c & \xrightarrow{ev} & Y \end{array}$$

where S is the set of components of maps from BW to BV that become null-homotopic when composed with f.

Now, suppose that $g\colon BW \to BV$ represents one of the components of S. That is, $f \circ g\colon BW \to X$ is null-homotopic. Maps out of classifying spaces of elementary abelian groups are controlled by cohomology ([15]) and then the condition that $H^*(BV;\mathbb{F}_2)$ is finitely generated as $H^*(X;\mathbb{F}_2)$-module induced by f^* implies that $f \circ g$ is null-homotopic if and only if g itself is null-homotopic. This proves that in our case S consists of just one component, that of the constant map, and therefore the evaluation $\mathrm{map}(BW, BV)_S \xrightarrow{ev} BV$ is a homotopy equivalence and the lemma follows from the diagram. □

Proposition 2.5. *Let X be a connected mod 2 H-space that satisfies F1, F2 and F3. Assume that*

$$BV \xrightarrow{f} X \xrightarrow{g} E$$

is a principal fibration where V is an elementary abelian 2-group, f is an H-map and $H^*(BV;\mathbb{F}_2)$ becomes finitely generated as $H^*(X;\mathbb{F}_2)$-module induced by f^*. Then, E is an H-space and g an H-map.

Proof. The argument here is the same used in [1, 9.17]. Look at the diagram

$$\begin{array}{ccc} BV \times BV & \xrightarrow{m_{BV}} & BV \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ X \times X & \xrightarrow{m_X} & X \\ {\scriptstyle g\times g}\downarrow & & \downarrow{\scriptstyle g} \\ E \times E & \overset{m_E}{\dashrightarrow} & E \end{array}$$

where the columns are principal fibrations and the top square is homotopy commutative because f is an H-map. By 2.4 $\mathrm{map}(BV \times BV, E)_c \simeq E$ and then the lemma 2.1 implies the existence of m_E making the bottom square homotopy commutative.

A similar argument shows that this multiplication admits a two sided unit element up to homotopy and therefore E becomes an H-space and g an H-map. □

It follows from these results that the space E of (6) is an H-space and that the fibrations (7) are H-fibrations. Actually, we will restrict our detection result to one single polynomial generator, e.g. x_1 in which case we have obtained

Theorem 2.6. *Let X be a connected mod 2 H-space satisfying conditions F1, F2 and F3 and let x a polynomial generator of $H^*(X;\mathbb{F}_2)$. Then, there exists a map*

$$f\colon B\mathbb{Z}/2 \longrightarrow X$$

with $f^*(x) = u^{2^{n+1}}$, u the one dimensional generator of $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$ and a sequence of H-fibrations

$$B\mathbb{Z}/2 \xrightarrow{f} X \xrightarrow{g} E \xrightarrow{h} B^2\mathbb{Z}/2\,. \tag{8}$$

Proof. The map f is obtained as the composition $B\mathbb{Z}/2 \to BV \to X$ for a suitable chosen inclusion $\mathbb{Z}/2 \subset V$. □

3. PGBA-ideals of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$.

The motivation for this section is the study of certain ideals in the mod 2 cohomology of $B^2\mathbb{Z}/2$. Those among which we will find the possible kernels in mod 2 cohomology of the projection map p of an H-fibration $F \xrightarrow{j} E \xrightarrow{p} B^2\mathbb{Z}/2$.

Definition 3.1. *Let R be an $\mathcal{A}$-Hopf algebra. An ideal of R is called a Primitively Generated Borel $\mathcal{A}$-ideal (PGBA-ideal for short) if it is an $\mathcal{A}$-ideal generated by a regular sequence of primitive elements.*

The examples of interest to us will appear as $\ker p^*$ where p is a projection as above. The trivial ones correspond to the fibrations $* \to B^2\mathbb{Z}/2 \to B^2\mathbb{Z}/2$ where $\ker p^* = \{0\}$ and the universal principal bundle $B\mathbb{Z}/2 \to * \to B^2\mathbb{Z}/2$ where $\ker p^*$ is the ideal of all positively graded elements of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$. Our aim is the classification of all possible PGBA-ideals of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$.

Recall that

$$H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)\cong \mathbb{F}_2[\iota, Sq^1\iota,\dots,Sq^{\Delta_n}\iota,\dots]$$

is a primitively generated polynomial algebra. Thus, the primitives are the elements ι^{2^s} of degree 2^{s+1} and $(Sq^{\Delta_n}\iota)^{2^s}$ of degree $2^s(2^n+1)$ for all $s\geq 0$ and $n\geq 0$. It will be important in next sections the observation that in each degree there is at most one primitive element.

Lemma 3.2. *For $n\geq 0$ and $s\geq 0$,*

$$Sq^{2^s}Sq^{\Delta_n}\iota=\begin{cases}0 & s>n+1,\\ Sq^{\Delta_{n+1}}\iota & s=n+1,\\ 0 & 0<s\leq n,\\ (Sq^{\Delta_{n-1}}\iota)^2 & s=0, n\neq 0,\\ 0 & s=0,n=0.\end{cases}$$

Proof. The case $s>n+1$ follows by unstability and the case $s=n+1$ is just the recursive definition of the Sq^{Δ_n}. Now, the case $s=0$, also by unstability, we have $Sq^1Sq^{\Delta_n}\iota=Sq^{2^n+1}(Sq^{\Delta_{n-1}}\iota)=(Sq^{\Delta_{n-1}}\iota)^2$ if $n\geq 1$ or $Sq^1Sq^1\iota=0$ for $n=0$.

It remains to look at the case $0<s\leq n$. From the Adem relations applied to $Sq^{2^s}Sq^{\Delta_n}\iota=Sq^{2^s}Sq^{2^n}(Sq^{\Delta_{n-1}}\iota)$ we obtain that

$$Sq^{2^s}Sq^{\Delta_n}\iota=Sq^{2^s+2^n-2^{s-1}}Sq^{2^{s-1}}(Sq^{\Delta_{n-1}}\iota)=Sq^{2^n+2^{s-1}}Sq^{2^{s-1}}(Sq^{\Delta_{n-1}}\iota).$$

Iterating this formula we obtain

$$Sq^{2^s}Sq^{\Delta_n}\iota=Sq^{2^n+2^{s-1}}Sq^{2^n+2^{s-2}}\dots Sq^{2^n+1}Sq^1(Sq^{\Delta_{n-s}}\iota)$$

but $Sq^1(Sq^{\Delta_{n-s}}\iota)=(Sq^{\Delta_{n-s}}\iota)^2$ and then $Sq^{2^n+1}((Sq^{\Delta_{n-s}}\iota)^2)=0$ by the Cartan formula so $Sq^{2^s}Sq^{\Delta_n}\iota=0$ ☐

Lemma 3.3. *The minimal PGBA-ideal of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ containing $(Sq^{\Delta_n}\iota)^{2^s}$, $n\geq 0$, $s\geq 0$, is*

$$J(n,s)=\big((Sq^1\iota)^{2^{s+n}},\dots,(Sq^{\Delta_{n-1}}\iota)^{2^{s+1}},(Sq^{\Delta_n}\iota)^{2^s},\dots,(Sq^{\Delta_{n+r}}\iota)^{2^s},\dots\big).$$

Proof. We have seen in lemma 3.2 how the Steenrod algebra operates on $Sq^{\Delta_n}\iota$. Moreover, from the Cartan formula, we know: $Sq^{2^{s+r}}(x^{2^r})=(Sq^{2^s}x)^{2^r}$. So, $Sq^{2^{n+i+s}}(Sq^{\Delta_{n+i-1}}\iota)^{2^s}=(Sq^{\Delta_{n+i}}\iota)^{2^s}$ and then we obtain that these elements are required in an $\mathcal{A}$-ideal containing $(Sq^{\Delta_n}\iota)^{2^s}$.

On the other hand, we know $Sq^1Sq^{\Delta_n}\iota=(Sq^{\Delta_{n-1}})^2$ and so:

$$Sq^{2^s}(Sq^{\Delta_n}\iota)^{2^s}=(Sq^{\Delta_{n-1}}\iota)^{2^{s+1}}$$

Iterating this result we obtain:

$$Sq^{2^{s+i}}(Sq^{\Delta_{n-i}}\iota)^{2^{s+i}}=(Sq^{\Delta_{n-i-1}}\iota)^{2^{s+i+1}}$$

Then, all these elements must appear in our ideal, too. Finally one just check that the ideal generated by all those elements is already an $\mathcal{A}$-ideal and in fact a PGBA-ideal. ☐

Proposition 3.4. *The PGBA-ideals of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ are either 0 or one of the following types for $n \geq 0$, $s \geq 0$,*

Type Is:

$$J = \big((Sq^1\iota)^{2^{m_n}}, \dots , (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, (Sq^{\Delta_n}\iota)^{2^s}, \dots , (Sq^{\Delta_{n+r}}\iota)^{2^s}, \dots\big) ,$$

where $m_0 = s$, $m_1 = s + 1$ and $m_k = m_{k-1} + \epsilon$, $\epsilon = 0, 1$.

Type IIs:

$$J = \big(\iota^{2^m}, (Sq^1\iota)^{2^{m_n}}, \dots , (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, (Sq^{\Delta_n}\iota)^{2^s}, \dots , (Sq^{\Delta_{n+r}}\iota)^{2^s}, \dots\big) ,$$

where $m_0 = s$, $m_1 = s + 1$, $m_k = m_{k-1} + \epsilon$, $\epsilon = 0, 1$ and $m \geq m_n$.

Observe that the trivial PGBA-ideal $\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2) = (\iota, Sq^1\iota, \dots)$ is of type II0 with $n = 0$, $m = m_0 = 0$.

Proof. Being the ideal J non trivial, it contains a primitive of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ and therefore one of the minimal ideals described in lemma 3.3, but it might be bigger; that is, it might contain generators that are roots of the generators in the minimal ideal.

Assume that J contains the minimal ideal $J(n, s)$ and also that this is the largest $J(n, s)$ that it contains; that is, $(Sq^{\Delta_{n-1}}\iota)^{2^s} \notin J$. Now our ideal might contain other primitive generators like $(Sq^{\Delta_{n-k}})^{2^{m_k}}$ with $m_k \leq s + k$ so it would be

$$J = \big((Sq^1\iota)^{2^{m_n}}, \dots , (Sq^{\Delta_{n-k}}\iota)^{2^{m_k}}, \dots , (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, (Sq^{\Delta_n}\iota)^{2^s}, \dots , (Sq^{\Delta_{n+r}}\iota)^{2^s}, \dots\big),$$

with $m_1 = s + 1$. However, since it should be an $\mathcal{A}$-ideal and as we have observed before $Sq^{2^{m_k+n-k+1}}((Sq^{\Delta_{n-k}}\iota)^{2^{m_k}}) = (Sq^{\Delta_{n-k+1}}\iota)^{2^{m_k}}$ and $Sq^{2^{m_{k-1}}}((Sq^{\Delta_{n-k+1}}\iota)^{2^{m_{k-1}}}) = (Sq^{\Delta_{n-k}}\iota)^{m_{k-1}+1}$ for $k = 1, \dots , n$ with $m_0 = s$, it follows that the integers m_k should satisfy the condition specified in the Proposition.

In case J contains a power of ι, the equality $Sq^{2^m}(\iota^{2^m}) = (Sq^1\iota)^{2^m}$ implies that m has to be an integer larger than or equal to m_n. □

Notice that those ideals are determined by a sequence of integers: m, m_n, m_{n-1},..., m_1, s that determines the powers of the indecomposable primitives contained in the ideal.

We will now derive some properties of the systems of generators for the PGBA-ideals of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$, and will define some important quotients.

Definition 3.5 (cf. [24])**.** *An unstable $\mathcal{A}$-module M is said to be nilpotent of class l or l-nilpotent if for every homogeneous element of degree m and $0 \leq k < l$*

$$Sq^{2^r(m-k)} \dots Sq^{2(m-k)} Sq^{m-k} x = 0$$

for a large enough r.

An unstable $\mathcal{A}$-module M is nilpotent if it is 1-nilpotent and it is reduced if it does not contain a non-trivial nilpotent submodule.

For example l-fold suspensions of unstable $\mathcal{A}$-modules are l-nilpotent.

Let J be a PGBA-ideal of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$. The quotient

$$\mathbb{F}_2 \otimes_{H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)} J \cong J/J \cdot \tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$$

is generated as a vector space by the regular sequence that generates J as an ideal. Using the formulae in Lemma 3.2, we observe that the module $J/J\cdot\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ is nilpotent of class one. It is, indeed, a suspension. However it is not always 2-nilpotent. In fact, it is nilpotent of class two if and only if $s>0$. Let N_2 be the maximal 2-nilpotent submodule of $J/J\cdot\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ and ΣL the quotient, so we have a short exact sequence of unstable $\mathcal{A}$-modules

$$0\to N_2\to J/J\cdot\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)\to\Sigma L\to 0, \tag{9}$$

where, for $s=0$ and $J\neq\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)=(\iota,Sq^1\iota,\dots)$:

- $N_2=\langle(Sq^1\iota)^{2^{m_n}},\dots,(Sq^{\Delta_{n-1}}\iota)^2\rangle_{\mathbb{F}_2}$ and
- $\Sigma L=\langle Sq^{\Delta_{n+r}}\iota\,|\,r\geq 0\rangle_{\mathbb{F}_2}$ is the suspension of a reduced module, L,

while, in case $s>0$:

- $N_2=J/J\cdot\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ and
- $\Sigma L=0$

and for $J=\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$:

- $N_2=0$ and
- $\Sigma L=\langle\iota,Sq^1\iota,\dots\rangle_{\mathbb{F}_2}\cong QH^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$.

For J of type either I0 or II0, as described in Proposition 3.4, and $J\neq\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$, we define the sub $\mathcal{A}$-module

$$\Sigma\hat{L}=\langle Sq^{\Delta_{n+1}}\iota,\dots,Sq^{\Delta_{n+s}}\iota,\dots\rangle_{\mathbb{F}_2} \tag{10}$$

and for $J=\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$, $\Sigma\hat{L}=\langle Sq^1\iota,Sq^{\Delta_1},\dots\rangle_{\mathbb{F}_2}$, what amounts to set $n=-1$ in (10).

The corresponding quotients will be important in understanding the transgression map (15) in section 6.

Definition 3.6. *We define the unstable $\mathcal{A}$-modules $M_n^\bullet=J/\Sigma\hat{L}$, where J are PGBA-ideals of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ of type either I0 or II0. We will write M_n^{I} if J was of type I0 and M_n^{II} if J was of type II0 with $J\neq\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$. We will simply set $M_{-1}\cong\iota\mathbb{F}_2$ if $J=\tilde{H}^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$.*

Alternatively, we can describe the unstable $\mathcal{A}$-modules $M_n^\bullet$, $n\geq 0$, as the subquotients of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ described as vector spaces by

$$M_n^{\mathrm{I}}=\langle Sq^{\Delta_n}\iota,(Sq^{\Delta_{n-1}}\iota)^{2^{m_1}},(Sq^{\Delta_{n-2}}\iota)^{2^{m_2}},\dots,(Sq^1\iota)^{2^{m_n}}\rangle_{\mathbb{F}_2}$$

with $m_0=0$, $m_1=1$ and $m_k=m_{k-1}+\epsilon$, $\epsilon=0,1$, or

$$M_n^{\mathrm{II}}=\langle Sq^{\Delta_n}\iota,(Sq^{\Delta_{n-1}}\iota)^{2^{m_1}},(Sq^{\Delta_{n-2}}\iota)^{2^{m_2}},\dots,(Sq^1\iota)^{2^{m_n}},\iota^{2^m}\rangle_{\mathbb{F}_2}$$

with $m_0=0$, $m_1=1$, $m_k=m_{k-1}+\epsilon$, $\epsilon=0,1$, and $m\geq m_n$.

In the example 1.7 are described the modules possible modules M_n^{I} for $n=1,2$.

4. Differential Hopf algebras.

In this section we study the structure of some differential Hopf algebras related to the Serre spectral sequence of an H-fibration $F \to E \to B^2\mathbb{Z}/2$.

The model for a page of our spectral sequences is a bigraded Hopf algebra $E = \oplus_{s,t} E^{s,t}$, $0 \leq s, t$, which is isomorphic to a tensor product of two connected graded Hopf algebras A and B:

$$E^{s,t} \cong A^s \otimes B^t$$

and equipped with a differential d of degree $(n, 1-n)$. We will identify A with $E^{*,0}$ and B with $E^{0,*}$.

Then we have the following variant of the DHA lemma (see [14]):

Lemma 4.1. *Let $(E \cong A \otimes B, d)$ be a bigraded differential Hopf algebra as above, then:*

1. $d(B^m) \subset P^n(A) \otimes B^{m-n+1}$.
2. *Furthermore, if the* transgression $d\colon B^{n-1} \to P^n(A)$ *is trivial, then* $d \equiv 0$.

Proof. For an element $x \in B^m$, we can write

$$d(x) = \sum_i a_i b_i \in E^{n,m-n+1} \cong A^n \otimes B^{m-n+1}$$

with $\{b_i\}$ linearly independent.

The diagonal applied to this element can be written in terms of the diagonal of a_i and b_i:

$$\Delta(d(x)) = \sum_i \Delta(a_i)\Delta(b_i) = \tag{11}$$
$$= \sum_i \Big(a_i \otimes 1 + 1 \otimes a_i + \sum_j a'_{ij} \otimes a''_{ij}\Big)\Big(b_i \otimes 1 + 1 \otimes b_i + \sum_k b'_{ik} \otimes b''_{ik}\Big)$$

where $\deg(a'_{ij}), \deg(a''_{ij}) < n$ and $\deg(b'_{ij}), \deg(b''_{ij}) < m-n+1$.

On the other hand, counting degrees, one obtains that

$$\Delta(d(x)) = d(\Delta(x)) \in \bigoplus_{p+q=m} (E^{n,1-n+p} \otimes E^{0,q}) \oplus (E^{0,p} \otimes E^{n,1-n+j})$$

and this implies that many homogeneous summands in equation (11) must vanish. In particular:

$$\sum_{i,j} a'_{ij} b_i \otimes a''_{ij} = 0$$

in $E^{<n,m-n+1} \otimes E^{<n,0}$, and since $\{b_i\}$ are linearly independent,

$$\sum_j a'_{ij} \otimes a''_{ij} = 0$$

for all i, that is, the elements $a_i \in A^n$ are primitives. This proves part (1). (2) follows like in [14] since $P(E) = P(A) + P(B)$. □

Next we assume that B is of finite type and therefore isomorphic, as algebra, to a tensor product of monogenic Hopf algebras:

$$B \cong \frac{P[x_1, \dots, x_r, \dots]}{(x_1^{2^{\alpha_1}}, \dots, x_r^{2^{\alpha_r}}, \dots)},$$

where $\alpha_i \in \mathbb{N} \cup \{\infty\}$, and $\alpha_i = \infty$ means that there is no relation at all and x_i is a polynomial generator. With this notation we want to obtain a precise hold of the homology $H(A \otimes B, d)$, for a non trivial d. According to the previous lemma, the differential d is trivial unless the transgression $d\colon B^{n-1} \to P^n(A)$ is non trivial. And this means that it has a non trivial value in one of the generators of B. We can assume, without loose of generality, that $d(x_1) = a \in P^n(A)$ is the non trivial transgression and also that x_1 is of minimal height among elements x with $d(x) = a$. Then we have

Lemma 4.2. *Assume as above that $(E \cong A \otimes B, d)$ is a bigraded differential Hopf algebra with B of finite type, thus $B \cong \dfrac{P[x_1, \dots, x_r, \dots]}{(x_1^{2^{\alpha_1}}, \dots, x_r^{2^{\alpha_r}}, \dots)}$ as algebras. Assume also that there is a non trivial transgression $d(x_1) = a \in P^n(A)$, with x_1 of minimal possible height in $d^{-1}(a)$ and*

(i) *$P^n(A)$ is one dimensional.*
(ii) *a is not a zero divisor in A.*

Then, there is a new system of generators $x_1, \tilde{x}_2, \dots \tilde{x}_r, \dots$, such that

1. *for $j > 1$, $\tilde{x}_j = x_j + \sum_k b_k x_1^{2s_k+1}$ and $d(\tilde{x}_j) = 0$, where each b_k is a polynomial on the generators x_i, different of x_1 and with degrees smaller than $m = \deg(x_j)$,*
2. *$\tilde{x}_j^{2^{\alpha_j}} = 0$, so that B is equally expressed as $B \cong \dfrac{P[x_1, \tilde{x}_2, \dots, \tilde{x}_r, \dots]}{(x_1^{2^{\alpha_1}}, \tilde{x}_2^{2^{\alpha_2}}, \dots, \tilde{x}_r^{2^{\alpha_r}}, \dots)}$ and*
3. *$H(E, d) \cong \dfrac{P[{x_1}^2, \tilde{x}_2, \dots, \tilde{x}_r, \dots]}{(x_1^{2^{\alpha_1}}, \tilde{x}_2^{2^{\alpha_2}}, \dots, \tilde{x}_r^{2^{\alpha_r}}, \dots)} \otimes \dfrac{A}{(a)}$, as algebras.*

Proof. We need to study how the differential acts on the generators x_i. We have assumed that $d(x_1) = a \neq 0$. Let x_j the next generator of minimal degree such that $d(x_j) \neq 0$. We shall see that we can modify x_j and obtain a different generator $\tilde{x}_j$ with trivial differential.

By lemma 4.1 the degree of x_j, m, is bigger than or equal to $\deg(x_1) = n - 1$ and $d(x_j) \in P^n(A) \otimes B^{m-n+1} \subset E^{n,m-n+1}$. Actually, we can prove:

Claim 4.3. *$d(x_j) = a \sum_k b_k x_1^{2s_k}$, where each b_k is a polynomial on the generators x_i, different of x_1 and with degrees smaller than $m = \deg(x_j)$.*

Proof. According to lemma 4.1 $d(x_j) \in P^n(A) \otimes B^{m-n+1}$, so it can be written as $d(x_j) = a \sum_k b_k x_1^{s_k}$, where b_k are polynomials on generators x_i different than x_1 and of degree less than $m = \deg(x_j)$. Then $0 = d^2(x_j) = a^2 \sum_k b_k s_k x_1^{s_k-1}$ and b_k should be zero whenever s_k is odd. That is $d(x_j)$ might be written as above. □

Now, (1) is an easy computation. In order to prove the relation (2) we use the diagonal map: $\Delta(x_j) = x_j \otimes 1 + 1 \otimes x_j + \dots$. We are particularly interested in the component in $E^{0,n-1} \otimes E^{0,m-n+1}$, that might be written as $x_1 \otimes y + b \otimes y'$, where b is a polynomial on generators different of x_1 and y, y' are any elements in B^{m-n+1}. Thus we can write:

$$\Delta(x_j) = x_j \otimes 1 + 1 \otimes x_j + x_1 \otimes y + b \otimes y' + \text{ terms in different degrees.}$$

Now we compute $d(\Delta(x_j))$ and look particularly at the component in $E^{n,0} \otimes E^{0,m-n+1}$:

$$d(\Delta(x_j)) = d(x_j) \otimes 1 + 1 \otimes d(x_j) + a \otimes y + \text{ terms in different degrees.}$$

On the other hand,

$$\Delta(d(x_j)) = \Delta(a \sum_k b_k x_1^{2s_k}) = a \sum_k b_k x_1^{2s_k} \otimes 1 + 1 \otimes a \sum_k b_k x_1^{2s_k} + \\ + a \otimes \sum_k b_k x_1^{2s_k} + \text{ terms in different degrees.}$$

Hence, the equation $d\Delta(x_j) = \Delta d(x_j)$ implies $y = \sum_k b_k x_1^{2s_k}$ and then

$$\Delta(x_j) = x_j \otimes 1 + 1 \otimes x_j + x_1 \otimes \sum_k b_k x_1^{2s_k} + b \otimes y' + \text{ terms in different degrees.}$$

From this equation it follows that the relation $x_j^{2^{\alpha_j}} = 0$ implies that $\left(\sum_k b_k x_1^{2s_k}\right)^{2^{\alpha_j}} = 0$, and therefore that $\tilde{x}_j^{2^{\alpha_j}} = x_j + \left(\sum_k b_k x_1^{2s_k+1}\right)^{2^{\alpha_j}} = 0$ or $x_1^{2^{\alpha_j}}$ =0, but this second option gives us to the previous one.

Finally, we prove (3). We can use inductively the result of this claim and obtain a new system of generators $x_1, \tilde{x}_2, \dots \tilde{x}_r, \dots$ with

$$B \cong \frac{P[x_1, \dots, x_r, \dots]}{(x_1^{2^{\alpha_1}}, \dots, x_r^{2^{\alpha_r}}, \dots)} \cong \frac{P[x_1, \tilde{x}_2, \dots, \tilde{x}_r, \dots]}{(x_1^{2^{\alpha_1}}, \tilde{x}_2^{2^{\alpha_2}}, \dots, \tilde{x}_r^{2^{\alpha_r}}, \dots)}$$

as algebras, and such that $d(x_1) = a$ and $d(\tilde{x}_i) = 0$ for all $i > 1$. Hence, we can split E as differential algebra

$$E \cong \Big(\frac{P[x_1]}{(x_1^{2^{\alpha_1}})} \otimes A\Big) \otimes \frac{P[\tilde{x}_2, \dots, \tilde{x}_r, \dots]}{(\tilde{x}_2^{2^{\alpha_2}}, \dots, \tilde{x}_r^{2^{\alpha_r}}, \dots)}$$

where the differential on the right term is trivial. The homology of the left term is easily computed using that $d(x_1) = a$ is not a zero divisor and then the lemma follows. □

5. Serre Spectral sequence for H-fibrations over $B^2\mathbb{Z}/2$

We are interested in the behavior of the Serre spectral sequence of an H-fibration $F \to E \to B^2\mathbb{Z}/2$:

$$E_2^{*,*} \cong H^*(B^2\mathbb{Z}/2; \mathbb{F}_2) \otimes H^*(F; \mathbb{F}_2) \Longrightarrow H^*(E; \mathbb{F}_2) \tag{12}$$

Proposition 5.1. *Let $F \to E \to B^2\mathbb{Z}/2$ be an H-fibration, where $H^*(F;\mathbb{F}_2)$ is of finite type. Then each stage of the corresponding Serre spectral sequence is a bigraded differential Hopf algebra of the form:*

$$E_n \cong A_n \otimes B_n, \quad A_n = \frac{H^*(B^2\mathbb{Z}/2; \mathbb{F}_2)}{(\theta_1, \dots, \theta_r)}, \tag{13}$$

where $\theta_1, \dots, \theta_r$ is a regular sequence of primitive elements of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ and B_n is a sub Hopf algebra of $H^*(F;\mathbb{F}_2)$.

Moreover, the elements θ_i are the targets of the transgression homomorphisms of the previous stages of the spectral sequence.

Proof. We will proof by induction on n that each term E_n of the spectral sequence has the form (13) and the elements θ_i are targets of previous transgressions.

For $n = 2$, the E_2 term of the spectral sequence is as described in (12) and it clearly satisfies the above conditions. See section 3 for a description of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$.

Assume by induction that this is true for E_{n-1}. If d_{n-1} is trivial $E_n \cong E_{n-1}$ and there is nothing to prove. Thus, we suppose that d_{n-1} is non trivial. According to lemma 4.1 there should be a non trivial transgression which target is a primitive element of A_{n-1} in degree $n-1$.

Recall that the primitive elements of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2) \cong \mathbb{F}_2[\iota, Sq^1\iota, \dots Sq^{\Delta_n}\iota, \dots]$ are of the form $\theta_i = (Sq^{\Delta_{m_i}}\iota)^{2^{s_i}}$, hence by the induction hypothesis A_{n-1} is written as

$$A_{n-1} \cong \frac{H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)}{(\theta_1, \dots, \theta_{r-1})} \cong \frac{\mathbb{F}_2[\iota, Sq^1\iota, \dots Sq^{\Delta_n}\iota, \dots]}{\left((Sq^{\Delta_{m_1}}\iota)^{2^{s_1}}, \dots, (Sq^{\Delta_{m_{r-1}}}\iota)^{2^{s_{r-1}}}\right)}$$

where the primitives $\theta_i = (Sq^{\Delta_{m_i}}\iota)^{2^{s_i}}$ have degrees $2^{s_i}(2^{m_i}+1) < n-1$ because they appear as images of previous transgressions. It follows that the remaining primitives in A_{n-1} in degrees bigger than or equal to $n-1$ are the classes of $(Sq^{\Delta_m}\iota)^{2^s}$, with degree $2^s(2^m+1) \geq n-1$ and $m \neq m_i$ for $1 \leq i \leq r-1$. Such elements are still non zero divisors in A_n and there is at most one in each degree.

Hence the transgression in E_{n-1} should hit one of those primitives and we can apply Lemma 4.2(3) in order to compute E_n and check that it has the form (13). This finishes the induction step and then the proof of the Proposition. □

Proposition 5.2. *Let $F \to E \to B^2\mathbb{Z}/2$ be an H-fibration, with $H^*(F;\mathbb{F}_2)$ of finite type, thus we can write*

$$H^*(F;\mathbb{F}_2) \cong \frac{P[x_1, \dots, x_r, \dots]}{(x_1^{2^{\alpha_1}}, \dots, x_r^{2^{\alpha_r}}, \dots)}. \tag{14}$$

where the generators are ordered by degree and by height in case of coincidence of degree. Then, there exists a system of transgressive generators for $H^*(F;\mathbb{F}_2)$, $\tilde{x}_1, \dots, \tilde{x}_r, \dots$ such that

1. *for each i, $\tilde{x}_i = x_i + p_i$ where $p_i = p_i(x_1, \dots, x_{i-1})$ is a polynomial on the previous generators,*
2. *$\tilde{x}_i$ has the same height as x_i, so that*
$$H^*(F;\mathbb{F}_2) \cong \frac{P[\tilde{x}_1, \dots, \tilde{x}_r, \dots]}{(\tilde{x}_1^{2^{\alpha_1}}, \dots, \tilde{x}_r^{2^{\alpha_r}}, \dots)}.$$
3. *In each degree, there is at most one monomial on the $\tilde{x}_i$'s with non trivial transgression.*
4. *If $y \in H^*(F;\mathbb{F}_2)$ is a monomial on the $\tilde{x}_i$'s with non trivial transgression, then there exists i and $k \geq 0$ such that $y = (\tilde{x}_i)^{2^k}$.*
5. *If, for a given i, $\tilde{x}_i$ is a nilpotent generator, then $\tilde{x}_i^{2^k}$ has trivial transgression for all $k \geq 1$.*

Proof. We proceed by induction. For this, we assume that we have a system of ordered generators $x_1, \dots, x_r, \dots$ such $x_1, \dots, x_{q-1}$ are transgressive and satisfy (1), (2) and (3). Then we first prove:

Claim 5.3. *(4) and (5) applies to the generators $x_1, \dots, x_{q-1}$.*

Proof. Pick a monomial $y = x_1^{m_1 2^{k_1}} \cdots x_{q-1}^{m_{q-1} 2^{k_{q-1}}}$, with $m_1, \dots, m_{q-1}$ odd integers. A differential of y is computed in terms of that of $x_1^{2^{k_1}}, \dots x_{q-1}^{2^{k_{q-1}}}$. Not all of that elements can transgress trivially for if they did, y would transgress trivially as well. Assume that $x_i^{2^{k_i}}$ is an element of minimal degree in the decomposition of y that transgresses non trivially. Then, this same differential kills y and then y would not be transgressive unless it is exactly $x_i^{2^{k_i}}$. This proves that (4) applies to the generators $x_1, \dots, x_{q-1}$.

Now, we look at (5). In case x_i is nilpotent, it must restrict to zero along the induced map $B\mathbb{Z}/2 \to F$ and therefore its transgression is a decomposable primitive, or just trivial. Since $x_i^2 = Sq^{\deg(x_i)} x_i$, x_i is transgressive, too. If the transgression of x_i is zero, then so is that of x_i^2. Suppose that x_i transgresses to $(Sq^{\Delta_m})^{2^s}$, with $s \geq 1$. In this case the degree of x_i had to be odd, and then x_i^2 transgresses to $Sq^{\deg(x_i)}\big((Sq^{\Delta_m})^{2^s}\big) = 0$, by the Cartan formula. Hence, in any case x_i^2 transgresses to zero; that is, it is a permanent cycle in the spectral sequence, so the same is true for $x_i^{2^k}$, $k \geq 1$. □

Now we show the induction step. Let x_q the next generator (that might be the first one!). The generators $x_1, \dots, x_{q-1}, x_q$ may fail to satisfy the same conditions for two different reasons:

- x_q is not transgressive, or
- there is a monomial y on the previous generators such that both y and x_q have non trivial transgression.

Suppose that x_q is not transgressive, that is, there is n with $n \leq \deg(x_q)$ and $d_n(x_q) \neq 0$.

Since d_n is non trivial, by lemma 4.1, there is a transgressive element y with $d_n(y) \neq 0$. y is, by degree reasons, a polynomial on the generators $x_1, \dots, x_{q-1}$. According to lemma 4.2, we can modify x_q to $x_q' = x_q + \sum b_k y^{2s_k+1}$ with $d_n(x_q') = 0$ and b_k polynomials on the generators $x_1, \dots, x_{q-1}$, so the whole $\sum b_k y^{2s_k+1}$ is a polynomial on the generators $x_1, \dots, x_{q-1}$. Now, we check the next differential and modify x_q' again if it is necessary and so on until we have obtained $\tilde{x}_q = x_q + p_q(x_1, \dots, x_{q-1})$ which is transgressive.

Suppose, next, that x_q have non trivial transgression but there is another monomial, y, on the previous generators, $x_1, \dots, x_{q-1}$. According to the claim, y is either one of the generators or a power of a polynomial generator.

If y is just one of the previous generators its height would be smaller than or equal to the height of x_q by our assumptions about the arrangement of the generators. Then, Lemma 4.2 applies again. Actually, we just choose $\tilde{x}_q = x_q + y$ as the new generator that substitutes x_q, having the same height and trivial transgression.

In case y is a power of a polynomial generator: $y = x_i^{2^k}$, the transgression of both, y and x_q, being non trivial, should be an odd dimensional primitive; that is, an indecomposable primitive. But this means that x_q restricts non trivially along the induced map $B\mathbb{Z}/2 \to F$ and therefore that x_q has infinite height. So again we just choose

$\tilde{x}_q = x_q + y$ as the new generator instead of x_q and having infinite height as well but trivial transgression.

We have finally obtained, in any case, a new generator $\tilde{x}_q = x_q + p_q(x_1, \dots, x_{q-1})$ with the same height as x_q and such that $x_1, \dots, x_{q-1}, \tilde{x}_q$ are transgressive and satisfy (1), (2) and (3), and also (4) and (5) again by the claim. We have therefore finished with the induction step and proved the Proposition. □

The above Proposition suggests the following definition

Definition 5.4. *Let $F \to E \to B^2\mathbb{Z}/2$ be an H-fibration with $H^*(F;\mathbb{F}_2)$ of finite type. A system of algebra generators for $H^*(F;\mathbb{F}_2)$, $x_1, \dots x_r, \dots$, is a* good *system of transgressive generators if*

1. $H^*(F;\mathbb{F}_2) \cong \dfrac{P[x_1, \dots, x_r, \dots]}{(x_1^{2^{\alpha_1}}, \dots, x_r^{2^{\alpha_r}}, \dots)}$.
2. *Each x_i is transgressive.*
3. *In each degree, there is at most one monomial on the generators x_i with non trivial transgression.*

Proposition 5.2, thus, proves the existence of good systems of trangressive generators and also, according to the claim, that any good system of transgressive generators satisfies as well conditions (4) and (5) of Proposition 5.2.

Propositions 5.1 and 5.2 determine the behavior of the Serre spectral sequence of an H-fibration $F \xrightarrow{j} E \xrightarrow{p} B^2\mathbb{Z}/2$ where $H^*(F;\mathbb{F}_2)$ is of finite type. Each stage of such a spectral sequence is a bigraded differential Hopf algebra $E_n \cong A_n \otimes B_n$ of the sort considered in section 4 hence the differentials are always determined by transgression.

It should finally converge to

$$E_\infty \cong A_\infty \otimes B_\infty$$

where

$$A_\infty \cong \varinjlim_n A_n \cong \frac{H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)}{(\theta_1, \theta_2, \dots, \theta_r, \dots)}$$

with $\theta_1, \theta_2, \dots, \theta_r, \dots$ a regular sequence of primitives and

$$B_\infty = \bigcap_n B_n \subset H^*(F;\mathbb{F}_2)$$

is a sub Hopf algebra of $B_2 = H^*(F;\mathbb{F}_2)$ that might be described using a good system of transgressive generators for $H^*(F;\mathbb{F}_2)$: $x_1, \dots, x_r, \dots$.

What remains to do is describing the link between those generators of $H^*(F;\mathbb{F}_2)$ and the regular sequence $\theta_1, \dots, \theta_r, \dots$ of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$. This link should clearly be the transgression homomorphism and here is where Steenrod operations come into the picture. In fact it is well known that the transgression homomorphism commutes with primary operations.

In terms of our good system of transgressive generators any transgressive element with non trivial transgression is either x_i for some i or $x_i^{2^s} = Sq^{2^{s-1}|x_i|} \dots Sq^{2|x_i|} Sq^{|x_i|} x_i$, for some $s \geq 1$. Hence we can choose as source of the transgression homomorphim, without loose of information, the sub $\mathcal{A}$-module of $H^*(F;\mathbb{F}_2)$ generated by $x_1, x_2, \dots, x_r, \dots$ as $\mathcal{A}$-module: $\langle x_1, x_2, \dots, x_r, \dots \rangle_{\mathcal{A}}$

Notice that A_∞ should coincide with the image of p^* and the kernel is $\ker p^* = (\theta_1, \theta_2, \dots, \theta_r, \dots)$, a PGBA-ideal of $H^*(B^2\mathbb{Z}/2; \mathbb{F}_2)$. So this ideal can be chosen as target of the transgression. But there is an indeterminacy given by possible multiples of elements hit by previous differentials. That indeterminacy is therefore contained in $\ker p^* \cdot \tilde{H}^*(B^2\mathbb{Z}/2; \mathbb{F}_2)$ and then the trangression is determined by a well defined morphism of unstable $\mathcal{A}$-modules

$$\tau\colon \Sigma\langle \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r\rangle_{\mathcal{A}} \twoheadrightarrow \ker p^* / \ker p^* \cdot \tilde{H}^*(B^2\mathbb{Z}/2; \mathbb{F}_2) \tag{15}$$

where we have finally written the suspension of $\langle \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r\rangle_{\mathcal{A}}$ as source in order to make τ a degree zero homomorphism.

Moreover, τ is an epimorphism because $\ker p^* / \ker p^* \cdot \tilde{H}^*(B^2\mathbb{Z}/2; \mathbb{F}_2)$ is a vector space generated by the classes of $\theta_1, \theta_2, \dots, \theta_r, \dots$, which are obtained precisely as targets of the transgression homomorphims according to Proposition 5.1.

6. Transgression in the Serre Spectral sequence for H-fibrations over $B^2\mathbb{Z}/2$

In this section we further study the transgression map in the Serre spectral sequence of an H-fibration

$$F \xrightarrow{j} E \xrightarrow{p} B^2\mathbb{Z}/2$$

in the case in which F satisfies the finiteness conditions F1, F2 and F3.

Assume that $x_1, \dots, x_r, y_1, \dots, y_s, \dots$ is a good system of transgressive generators for $H^*(F; \mathbb{F}_2)$, where we distinguish between polynomial generators x_i and nilpotent generators y_i, thus we write:

$$H^*(F; \mathbb{F}_2) \cong P[x_1, \dots, x_r] \otimes \frac{P[y_1, \dots, y_s]}{(y_1^{2^{\alpha_1}}, \dots, y_s^{2^{\alpha_s}})}$$

with $0 < \alpha_i < \infty$ for each $i = 1, \dots, s$.

The suspension of the nil-localization of $H^*(F; \mathbb{F}_2)$ (see Theorem 2.2) provides a sequence:

$$0 \to \Sigma \ker\mu \to \Sigma\langle x_1, \dots, x_r, y_1, \dots, y_s\rangle_{\mathcal{A}} \xrightarrow{\Sigma\mu} \Sigma\langle x_1, \dots, x_r\rangle_{\mathcal{A}} \to 0$$

where $\langle x_1, \dots, x_r\rangle_{\mathcal{A}} \subset P[x_1, \dots, x_r] \rightarrowtail H^*(BV; \mathbb{F}_2)$ is reduced and $\ker\mu \subset (y_1, \dots, y_r)$ is nilpotent, hence $\Sigma\ker\mu$ is nilpotent of class two.

On the other hand, the PGBA-ideal $I = \ker p^*$ can also be decomposed, according to the sequence (9), as a maximal sub$\mathcal{A}$-module, K, which is nilpotent of class 2 and the quotient ΣL, the suspension of a reduced $\mathcal{A}$-module,

Then we obtain a corresponding decomposition of the transgression map τ (15):

$$\begin{array}{ccccc} \Sigma\ker\mu & \rightarrowtail & \Sigma\langle x_1, \dots, x_r, y_1, \dots, y_s\rangle_{\mathcal{A}} & \twoheadrightarrow & \Sigma\langle x_1, \dots, x_r\rangle_{\mathcal{A}} \\ \downarrow{\scriptstyle \tau|_{\Sigma\ker\mu}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau'} \\ K & \rightarrowtail & I/I\cdot H^*(B^2\mathbb{Z}/2; \mathbb{F}_2) & \twoheadrightarrow & \Sigma L \end{array} \tag{16}$$

where τ' should also be seen as the restriction of τ to the polynomial generators, since those appear in degrees a power of two (see Theorem 2.2) and then by degree reasons can only map by τ to elements of ΣL. This proves in turn that $\tau|_{\Sigma \ker \mu}$ in the diagram is an epimorphism.

Lemma 6.1. *Let $F \xrightarrow{j} E \xrightarrow{p} B^2\mathbb{Z}/2$ be an H-fibration where $H^*(F;\mathbb{F}_2)$ satisfies conditions F1, F2 and F3. Then either the H-fibration is trivial or the PGBA ideal $I = \ker p^*$ is of type* I*0 or* II*0.*

Proof. According to Proposition 3.4 the PGBA ideals of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ are either 0 or of type Is or IIs. Assume that $I = \ker p^*$ is a PGBA ideal with $s > 0$ or it is just 0. In that cases $\Sigma L = 0$ in the diagram (16) and all of the polynomial generators of $H^*(F;\mathbb{F}_2)$ transgress trivially and in turn they map trivially along $B\mathbb{Z}/2 \xrightarrow{f} F$. Hence this map is trivial in cohomology and therefore null-homotopic [15].

We will now apply Zabrodsky's lemma (see Lemma 2.3) to the principal fibration $B\mathbb{Z}/2 \xrightarrow{f\simeq *} F \xrightarrow{g} E$ in order to extend the identity of F to a section of the fibration $s\colon E \to F$. This turns out to be an H-map. In fact, look now at the fibration $B\mathbb{Z}/2 \times B\mathbb{Z}/2 \to F \times F \to E \times E$. The multiplication of F, $m\colon F \times F \to F$ extends to $E \times E$ in two different ways, namely, $E \times E \xrightarrow{s\times s} F \times F \xrightarrow{m} F$ and, since $g\colon F \to E$ is an H-map, also as $E \times E \xrightarrow{m} E \xrightarrow{s} F$. Applying again Zabrodsky's lemma, these two factorizations should be homotopic, hence the section $s\colon E \to F$ is an H-map.

We have obtained a diagram of H-fibrations and H-maps

$$\begin{array}{ccccc}
F & \xrightarrow{\quad g \quad} & E & \xrightarrow{\quad h \quad} & B^2\mathbb{Z}/2 \\
\Vert & & \big\downarrow {\scriptstyle (s,h)} & & \Vert \\
F & \longrightarrow & F \times B^2\mathbb{Z}/2 & \longrightarrow & B^2\mathbb{Z}/2
\end{array}$$

that commutes up to homotopy, thus $E \simeq F \times B^2\mathbb{Z}/2$ and our original fibration is trivial. $\square$

It will be useful to distinguish the non trivial H-fibrations $F \xrightarrow{j} E \xrightarrow{p} B^2\mathbb{Z}/2$ according to the type of the ideal $\ker p^*$.

Definition 6.2. *A non trivial H-fibration $F \xrightarrow{j} E \xrightarrow{p} B^2\mathbb{Z}/2$, where $H^*(F;\mathbb{F}_2)$ satisfies conditions F1, F2 and F3 is of type* I *if the PGBA-ideal $\ker p^*$ is of type* I*0 and it is of type* II *if $\ker p^*$ is of type* II*0.*

Proposition 6.3. *Let $F \xrightarrow{j} E \xrightarrow{p} B^2\mathbb{Z}/2$ be a non trivial H-fibration with $H^*(F;\mathbb{F}_2)$ satisfying the conditions F1, F2 and F3. The transgression is determined by the induced epimorphism of unstable $\mathcal{A}$-modules*

$$\tilde{\tau}\colon \Sigma QH^*(F;\mathbb{F}_2) \twoheadrightarrow M_n^\bullet$$

for some $n \geq -1$ where $M_n^\bullet = M_n^{\mathrm{I}}$ if the fibration is of type I *and $M_n^\bullet = M_n^{\mathrm{II}}$ or M_{-1} if the fibration is of type* II*. It satisfies $\tilde{\tau}(x_i) = Sq^{\Delta_n}\iota$, for x_i a least dimensional* *polynomial generator of* $H^*(F;\mathbb{F}_2)$ *that maps non trivially along* $B\mathbb{Z}/2 \to F$ *for* $n \geq 0$ *or* $\tilde{\tau}(x_i) = \iota \in M_{-1}$ *if* $n = -1$.

See 3.6 for the definition of the unstable $\mathcal{A}$-modules $M_n^\bullet$. Notice also, that by abuse of language we denote equally by x_i the element of $\Sigma QH^*(F;\mathbb{F}_2)$ represented by the generator $x_i \in H^*(F;\mathbb{F}_2)$.

Proof. Look at the decomposition of the transgression map τ (15), given in diagram (16). There should be a polynomial generator that transgresses to the least dimensional element in ΣL. And it has to be one in least possible dimension that restricts non trivially to $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$ along $B\mathbb{Z}/2 \to F$. Thus, it appears in dimension one and $\tilde{\tau}(x_i) = \iota$ or $\deg x_i = 2^{n+1}$, $n \geq 0$, and $\tilde{\tau}(x_i) = Sq^{\Delta_n}\iota$. It follows that the decomposables contained in $\langle x_1, \ldots, x_r\rangle_{\mathcal{A}}$ maps into $\Sigma\hat{L} \subset I/I \cdot H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$.

Now, look at the elements in $\ker \Sigma\mu$. According to Proposition 5.2, (4), (5), among those elements, only the generators $y_1, \ldots, y_s$ can have non trivial transgressions. Hence, all the decomposables contained in $\ker \Sigma\mu$ map to zero in $I/I \cdot H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$.

We have obtained that in the composition

$$\Sigma\langle x_1, \ldots, x_r, y_1, \ldots, y_s\rangle_{\mathcal{A}} \twoheadrightarrow I/I \cdot H^*(B^2\mathbb{Z}/2;\mathbb{F}_2) \twoheadrightarrow M_n^\bullet$$

all the decomposable elements remain in the kernel, hence this factors as

$$\Sigma QH^*(F;\mathbb{F}_2) \xrightarrow{\tilde{\tau}} M_n^\bullet$$

and this finishes the proof. □

7. Outcome of the Serre Spectral sequence for H-fibrations over $B^2\mathbb{Z}/2$

So far, we have obtained the structure of the E_∞ term of the Serre spectral sequence of an H-fibration $F \xrightarrow{g} E \xrightarrow{h} B^2\mathbb{Z}/2$:

$$E_\infty \cong A_\infty \otimes B_\infty$$

where

$$A_\infty = \frac{H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)}{I},$$

with I a PGBA-ideal of $H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$ and B_∞ is a sub-$\mathcal{A}$-Hopf algebra of $H^*(F;\mathbb{F}_2)$. It follows, that g^* and h^* factor as edge homomorphisms inducing $\operatorname{Im} h^* \cong A_\infty$ and $\operatorname{Im} g^* \cong B_\infty$.

We thus have an associated graded ring to $H^*(E;\mathbb{F}_2)$. In this section que study the extension problems in order to get information about $H^*(E;\mathbb{F}_2)$ itself.

Using the filtration degree of the elements of $H^*(E;\mathbb{F}_2)$ it is defined an additive isomorphism

$$\pi\colon H^*(E;\mathbb{F}_2) \to E_\infty$$

that fits in the diagram

$$\begin{array}{ccccc} \operatorname{Im} h^* & \rightarrowtail & H^*(E;\mathbb{F}_2) & \twoheadrightarrow & \operatorname{Im} g^* \\ \downarrow\cong & & \cong\downarrow\pi & & \downarrow\cong \\ A_\infty & \rightarrowtail & E_\infty & \twoheadrightarrow & B_\infty \end{array}$$

The map π is not in general an algebra map. It is true however that given elements $x, y \in H^*(E;\mathbb{F}_2)$ such that $\pi(x)\pi(y) \neq 0$ in E_∞, we have $\pi(xy) = \pi(x)\pi(y)$.

Lemma 7.1. *If $a_1, \dots, a_m, \dots$ is a system of algebra generators for $\operatorname{Im} h^* \cong A_\infty$ and $b_1, \dots, b_n, \dots$ for $\operatorname{Im} g^* \cong B_\infty$, then $a_1, \dots, a_m, \dots, b'_1, \dots, b'_n, \dots$ is a system of algebra generators for $H^*(E;\mathbb{F}_2)$, where b'_i is any element for which $g^*(b'_i) = b_i$.*

Moreover, the sequence

$$1 \to \operatorname{Im} h^* \to H^*(E;\mathbb{F}_2) \to \operatorname{Im} g^* \to 1$$

is an exact sequence of Hopf algebras in the sense that $\operatorname{Im} g^* \cong H^*(E;\mathbb{F}_2)//\operatorname{Im} h^*$.

Proof. Since we have $E_\infty \cong A_\infty \otimes B_\infty$ for any two elements $x \in \operatorname{Im} h^*$ and $y \in H^*(E;\mathbb{F}_2)$ with $g^*(y) \neq 0 \in \operatorname{Im} g^*$ we have $\pi(x) = x \otimes 1$ and $\pi(y) = 1 \otimes g^*(y) +$ decomposables, so that $0 \neq \pi(x)\pi(y) = \pi(xy)$.

Consequently, we can obtain a system of algebra generators for $H^*(E;\mathbb{F}_2)$ from a system of algebra generators for $\operatorname{Im} h^*$ and one for $\operatorname{Im} g^*$, as indicated in the lemma.

Also π maps the ideal $(\operatorname{Im} h^{*+})$ of $H^*(E;\mathbb{F}_2)$ onto the ideal $({A_\infty}^+)$ of E_∞, and then $H^*(E;\mathbb{F}_2)//\operatorname{Im} h^* \cong \operatorname{Im} g^*$. □

Lemma 7.2. *Let A, B and C be connected graded algebras and let $A \to B \to C \cong B//A$ be an exact sequence of algebras. Let $\bar{A}$, $\bar{B}$ and $\bar{C}$ denote the respective quotients by the ideals of the nilpotent elements. Then $\bar{A} \rightarrowtail \bar{B}$ is an injection while $\bar{B}//\bar{A} \twoheadrightarrow \bar{C}$ is an epimorphism with nilpotent kernel.*

Proof. Straightforward. □

Lemma 7.3. *Assume that $A = P_A \otimes N_A$, $B = P_B \otimes N_B$, $C = P_C \otimes N_C$ are connected algebras and we have a diagram*

$$\begin{array}{ccccc} P_A & \xrightarrow{g|_{P_A}} & P_B & \xrightarrow{h|_{P_B}} & P_C \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{g} & B & \xrightarrow{h} & C \cong B//A \end{array}$$

Then, there is a diagram

$$N_A \xrightarrow{\bar{g}} N_B \xrightarrow{\bar{h}} N_C$$

where

- $N_B//\bar{g}(N_A) \cong N_C$ *and*
- $\ker \bar{g}$ *consists of the elements of N_A represented in the ideal (P_B^+) of B.*

Proof. Straightforward. □

7.1. **Fibrations of type I.** Let us now specialize to H-fibrations

$$F \xrightarrow{g} E \xrightarrow{h} B^2\, Z/2$$

where F satisfies conditions F1, F2 and F3, which are of type I.

So there is a good system of transgressive generators with

$$H^*(F;\mathbb{F}_2) \cong P[x_1,\dots,x_r] \otimes \frac{P[y_1,\dots,y_s,\dots]}{(y_1^{2^{\alpha_1}},\dots,y_s^{2^{\alpha_s}},\dots)}$$

and transgression

$$\tau\colon \Sigma QH^*(F;\mathbb{F}_2) \twoheadrightarrow M_n^{\mathrm{I}} \cong \langle Sq^{\Delta_n}\iota, (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, (Sq^{\Delta_{n-2}}\iota)^{2^{m_2}},\dots,(Sq^1\iota)^{2^{m_n}}\rangle_{\mathbb{F}_2}$$

We may suppose that

$$\tilde\tau(x_1) = Sq^{\Delta_n}\iota,\ \tilde\tau(y_1) = (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}},\dots,\tilde\tau(y_n) = (Sq^1\iota)^{2^{m_n}}$$

and $\tilde\tau$ trivial elsewhere. These formulae determine the transgression hence the spectral sequence. That is, using inductively the computation of Lemma 4.2 (3) we get that the E_∞ term of the Serre spectral sequence for the fibration is $E_\infty \cong A_\infty \otimes B_\infty$ with

$$A_\infty \cong P[\iota]\otimes N', \qquad N' \cong \frac{P[Sq^1\iota,\dots,Sq^{\Delta_{n-1}}\iota]}{\big((Sq^1\iota)^{2^{m_n}},\dots,(Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}\big)}$$

and

$$B_\infty \cong P[x_2,\dots,x_r]\otimes N'', \qquad N'' \cong \frac{P[{y_1}^2,\dots,{y_n}^2,y_{n+1},\dots,y_s,\dots]}{(y_1^{2^{\alpha_1}},\dots,y_s^{2^{\alpha_s}},\dots)}$$

With this notation we obtain

Proposition 7.4. *For a fibration of type* I *as above, and provided F is 1-connected*

1. $H^*(E;\mathbb{F}_2) \cong P[x_1',x_2',\dots,x_r']\otimes N$ *where* $x_1' = h^*(\iota)$ *and* $g^*(x_i') = x_i$ *if* $i \geq 2$.
2. *N is a nilpotent Hopf algebra of finite type that fits in an exact sequence of Hopf algebras*

$$1 \to N' \xrightarrow{h^*} N \xrightarrow{g^*} N'' \to 1\,.$$

Proof. By Lemma 7.1 $H^*(E;\mathbb{F}_2)$ has in each dimension at most a finite number of generators so it is of finite type and then there is an isomophism of algebras

$$H^*(E;\mathbb{F}_2)\cong P\otimes N$$

where P is a polynomial algebra and N is a nilpotent Hopf algebra of finite type.

It also follows from lemma 7.1 the existence of an exact sequence of Hopf algebras

$$1 \to \operatorname{Im} h^* \to H^*(E;\mathbb{F}_2) \to \operatorname{Im} g^* \to 1\,.$$

By lemma 7.2 there is an injection

$$P[\iota] \hookrightarrow P\,.$$

As E is 1-connected we can choose generators for P, $z_1, z_2, z_3,\dots$ in such a way that $\iota \mapsto z$. Hence $P//P[\iota] \cong P[z_2,z_3,\dots]$ contains no nilpotent elements and therefore Lemma 7.2 implies

$$P[z_2,z_3,\dots] \cong P//P[\iota] \cong P[x_2,\dots,x_r]\,.$$

This means that we can choose $x_1' = h^*(\iota)$, $x_2', \dots, x_r'$ with $g^*(x_i') = x_i$ for $i \geq 2$, and an isomorphism of algebras

$$H^*(E;\mathbb{F}_2) \cong P[x_1', x_2', \dots, x_r'] \otimes N\,.$$

In order to prove (2) we first observe that $N' \cap (x_1', \dots, x_r') = 0$. In fact, assume that $n \in N' \hookrightarrow H^*(E;\mathbb{F}_2)$ can be written as $n = \sum_i k_i x_i'$ in $H^*(E;\mathbb{F}_2)$. Observe that we could have chosen $x_1', \dots, x_r'$ in such a way that they are represented in E_∞ by the regular sequence $\iota \otimes 1, 1 \otimes x_2, \dots, 1 \otimes x_r$ so each $\pi(k_i)\pi(x_i') \neq 0$ as soon as $k_i \neq 0$, and then $\pi(n) = \sum_i \pi(k_i)\pi(x_i') \in (\iota \otimes 1, 1 \otimes x_2, \dots, 1 \otimes x_r)$. But this ideal of E_∞ does not contain $\pi(n) = n \otimes 1$ unless $n = 0$. Hence by Lemma 7.3 N' injects in N and $N//N' \cong N''$. □

Remark 7.5. *A counterexemple to this proposition in case F is not 1-connected is the H-fibration*

$$B\mathbb{Z}/4 \to B\mathbb{Z}/2 \to B^2\mathbb{Z}/2\,.$$

Remark 7.6. *Notice that in the case of fibrations of type* I *with $n = 0$; that is,*

$$F \xrightarrow{g} E \xrightarrow{h} B^2\mathbb{Z}/2$$

with

$$H^*(F;\mathbb{F}_2) \cong P[x_1, \dots, x_r] \otimes \frac{P[y_1, \dots, y_s, \dots]}{(y_1^{2^{\alpha_1}}, \dots, y_s^{2^{\alpha_s}}, \dots)}$$

where $x_1, \dots, x_r, y_1, \dots, y_s, \dots$ is a good system of transgressive generators and

$$\tilde{\tau}\colon \Sigma QH^*(F;\mathbb{F}_2) \to M_0 = \langle Sq^1\iota\rangle_{\mathbb{F}_2}\,,$$

the result of Proposition 7.4 has no extension problems and if we assume that just x_1 transgresses to $Sq^1\iota$, we have

$$H^*(E;\mathbb{F}_2) \cong P[x_1', \dots, x_r'] \otimes \frac{P[y_1', \dots, y_s', \dots]}{(y_1'^{2^{\alpha_1}}, \dots, y_s'^{2^{\alpha_s}}, \dots)}$$

with $h^*(\iota) = x_1'$, $g^*(x_i') = x_i$ for $i = 2, \dots, r$ and $g^*(y_i') = y_i$, for $i = 1, \dots, s, \dots$

7.2. **Fibrations of type** II**.** We consider now fibrations of type II with 1-connected fibre; that is, H-fibrations

$$F \xrightarrow{g} E \xrightarrow{h} B^2\, Z/2$$

with

1. $H^*(F;\mathbb{F}_2) \cong P[x_1, \dots, x_r] \otimes \dfrac{P[y_1, \dots, y_s, \dots]}{(y_1^{2^{\alpha_1}}, \dots, y_s^{2^{\alpha_s}}, \dots)}$, where $x_1, \dots, x_r, y_1, \dots, y_s, \dots$ is a good system of transgressive generators and
2. the transgression is determined by $\tilde{\tau}\colon \Sigma QH^*(F;\mathbb{F}_2) \twoheadrightarrow M_n^{\mathrm{II}}$, $n \geq 0$.

Now $M_n^{\mathrm{II}} \cong \langle Sq^{\Delta_n}\iota, (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, (Sq^{\Delta_{n-2}}\iota)^{2^{m_2}}, \dots, (Sq^1\iota)^{2^{m_n}}, \iota^{2^m}\rangle_{\mathbb{F}_2}$ and we may assume that

$$\tilde{\tau}(x_1) = Sq^{\Delta_n}\iota,\ \tilde{\tau}(y_1) = (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}, \dots, \tilde{\tau}(y_n) = (Sq^1\iota)^{2^{m_n}},\ \tilde{\tau}(y_{n+1}) = \iota^{2^m}\,,$$

and $\tilde{\tau}$ trivial elsewhere. So, with the same arguments as above we obtain $E_\infty \cong A_\infty \otimes B_\infty$ with

$$A_\infty \cong N', \qquad N' \cong \frac{P[\iota, Sq^1\iota, \dots, Sq^{\Delta_{n-1}}\iota]}{\left(\iota^{2^m}, (Sq^1\iota)^{2^{m_n}}, \dots, (Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}\right)}\,,$$

$$B_\infty \cong P[x_2, \dots, x_r] \otimes N'', \qquad N'' \cong \frac{P[{y_1}^2, \dots, {y_{n+1}}^2, y_{n+2}, \dots, y_s, \dots]}{(y_1^{2^{\alpha_1}}, \dots, y_s^{2^{\alpha_s}}, \dots)}$$

and

Proposition 7.7. *For a fibration of type* II *with 1-connected fibre as above*

1. $H^*(E;\mathbb{F}_2) \cong P[x_2', \dots, x_r'] \otimes N$ *where* $g^*(x_i') = x_i$ *if* $i \geq 2$.
2. N *is a nilpotent Hopf algebra of finite type that fits in an exact sequence of Hopf algebras*

$$1 \to N' \xrightarrow{h^*} N \xrightarrow{g^*} N'' \to 1\,.$$

Proof. Like that of Proposition 7.4. □

8. The iteration

Let X be a 1-connected mod 2 H-space that satisfies conditions F1, F2 and F3.

It follows from section 2 that we can choose a polynomial generator of $H^*(X;\mathbb{F}_2)$, detect it by an H-map $B\mathbb{Z}/2 \to X$ and form the sequence of H-fibrations

$$B\mathbb{Z}/2 \to X \to E \to B^2\mathbb{Z}/2\,.$$

Sections 5, 6 and 7 are concerned with the computation of $H^*(E;\mathbb{F}_2)$. Now, we will iterate this construction with E and the subsequent quotients and will obtain the proofs of Theorems 1.1 and 1.6.

Given a polynomial generator x of $H^*(X;\mathbb{F}_2)$, where X is a mod 2 H-space satisfying conditions F1, F2 and F3 and according to Theorem 2.6 we can construct an H-map $f\colon B\mathbb{Z}/2 \to X$ such that x restricts non trivially to $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$ and by construction we can complete x to a system of generators where any other generator restricts trivially to $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$. Moreover, this map fits in a sequence of H-fibrations

$$B\mathbb{Z}/2 \xrightarrow{f} X \xrightarrow{g} E \xrightarrow{h} B^2\mathbb{Z}/2$$

According to Proposition 5.2 our system of generators can be modified to a good system of transgressive generators. Actually, we can keep x itself in the new system. For this, we should have a look at the proof of Proposition 5.2. Since x is already known to be transgressive and any other polynomial generators of degree less than the degree of x transgresses trivially, the only reason for changing x would be to have a nilpotent generator y in its same degree which transgresses non trivially as well, but this is impossible by diagram (16).

The results of section 7 show that, essentially, we substitute the old polynomial generator, x, by a new generator in dimension 2, x', which is either polynomial in case the fibration $X \to E \to B^2\mathbb{Z}/2$ was of type I or nilpotent if it was of type II.

Notice that E is again a 1-connected mod 2 H-space that satisfies conditions F1 and F2 and it also satisfies condition F3, by Lemma 2.4 and [10, Theorem 3.2].

So, therefore , we can repeat the operation with $E_1 = E$ and the subsequent quotients E_k using each time the new polynomial generator $x^{(k)}$ of degree two and, of course, we stop if, eventually, our polynomial generator degenerates to a 2-dimensional nilpotent generator.

Thus we obtain a sequence (either finite or infinite)

$$X = E_0 \to E_1 \to \ldots \to E_k \to E_{k+1} \to \ldots \tag{17}$$

of principal fibrations

$$B\mathbb{Z}/2 \to E_k \to E_{k+1} \to B^2\mathbb{Z}/2$$

where $B\mathbb{Z}/2 \to E_k$ detects $x^(k)$ while $E_{k+1} \to B^2\mathbb{Z}/2$ classifies $x^{(k+1)}$.

Proposition 8.1. (i) *The compositions in (17) are principal H-fibrations*

$$B\mathbb{Z}/2^k \xrightarrow{f_k} X \xrightarrow{g_k} E_k \xrightarrow{h_k} B^2\mathbb{Z}/2^k$$

(ii) *The evaluation map* $\mathrm{map}(B\mathbb{Z}/2^k, X)_{f_k} \simeq X$ *is a homotopy equivalence and* E_k *coincides with the Borel construction*

$$E_k \simeq \mathrm{map}(B\mathbb{Z}/2^k, X)_{f_k} \times_{B\mathbb{Z}/2^k} EB\mathbb{Z}/2^k$$

Proof. (i) For $k = 1$ this sequence is just the construction of the first step of the sequence 17. For $k > 1$, assume by induction that we have H-fibrations

$$B\mathbb{Z}/2^j \to X \to E_j \to B^2\mathbb{Z}/2^j$$

for $j \leq k$, where the fundamental class of $H^*(B^2\mathbb{Z}/2; \mathbb{F}_2)$ restricts to $x^{(j)} \in H^*(E_j; \mathbb{F}_2)$ which class is therefore the mod 2 reduction of the class in $H^*(E_j; \mathbb{Z}/2^j)$ classified by $E_j \to B^2\mathbb{Z}/2^j$.

In case the fibration $E_{k-1} \to E_k \to B^2\mathbb{Z}/2$ was of type II we would have finished the iteration and therefore the proof of (i). Thus, we assume that this is still a fibration of type I and then there is a next step $B\mathbb{Z}/2 \to E_k \to E_{k+1} \to B^2\mathbb{Z}/2$ and we can form the pull-back diagram of H-spaces

$$\begin{array}{ccccccc}
B\mathbb{Z}/2^k & \longrightarrow & F & \longrightarrow & B\mathbb{Z}/2 & \longrightarrow & B^2\mathbb{Z}/2^k \\
\| & & \downarrow & & \downarrow & & \| \\
B\mathbb{Z}/2^k & \longrightarrow & X & \longrightarrow & E_k & \longrightarrow & B^2\mathbb{Z}/2^k \\
 & & \downarrow & & \downarrow & & \\
 & & E_{k+1} & = & E_{k+1} & &
\end{array} \tag{18}$$

Then $F \simeq K(A, 1)$ where A is a group that fits in an extension classified by the composition $B\mathbb{Z}/2 \to E_k \to B^2\mathbb{Z}/2^k$. We need to check the effect of this composition in cohomology. Since the fundamental class of $H^*(B^2\mathbb{Z}/2^k; \mathbb{F}_2)$ restricts to $x^{(k)} \in H^*(E_k; \mathbb{F}_2)$ which is in turn detected by $B\mathbb{Z}/2 \to E_k$ the composition $B\mathbb{Z}/2 \to E_k \to B^2\mathbb{Z}/2^k$ is non trivial and therefore $A \cong \mathbb{Z}/2^{k+1}$.

Both, X and E_{k+1} are 1-connected, so we have an exact sequence

$$0 \to \pi_2(X) \to \pi_2(E_{k+1}) \to \mathbb{Z}/2^{k+1} \to 0\,.$$

And by the Hurewicz theorem the second homomorphism represents a cohomology class classified by a map $E_{k+1} \to B^2\mathbb{Z}/2^{k+1}$ which is an H-map and fits in the sequence of H-fibrations

$$B\mathbb{Z}/2^{k+1} \to X \to E_{k+1} \to B^2\mathbb{Z}/2^{k+1}\,.$$

This finishes the induction and therefore the proof of (i). (ii) Since X is a connected H-space, all the components in $\mathrm{map}(B\mathbb{Z}/2^k, X)$ are homotopy equivalent. So, in order to prove that the evaluation map $\mathrm{map}(B\mathbb{Z}/2^k, X)_{f_k} \to X$ is a homotopy equivalence it suffices to show the same statement for the component of the constant map. And this follows by induction. We know the case $k = 1$ from Lemma 2.3. And then we apply Zabrodsky's lemma to the principal fibration $B\mathbb{Z}/2 \to B\mathbb{Z}/2^{k+1} \to B\mathbb{Z}/2^k$. Since $\mathrm{map}(B\mathbb{Z}/2, X)_c \simeq X$, it follows that $\mathrm{map}(B\mathbb{Z}/2^k, X)_c \simeq \mathrm{map}(B\mathbb{Z}/2^{k+1}, X)_c$ and by the induction hypothesis $\mathrm{map}(B\mathbb{Z}/2^{k+1}, X)_c \simeq X$.

Notice that the same is true for E_k. So, in the diagram

$$\begin{array}{ccc}
B\mathbb{Z}/2 & = & B\mathbb{Z}/2 \\
\downarrow & & \downarrow \\
\mathrm{map}(B\mathbb{Z}/2^k, X)_{f_k} & \xrightarrow[\simeq]{ev} & X \\
\downarrow & & \downarrow \\
\mathrm{map}(B\mathbb{Z}/2^k, X)_{f_k} \times_{B\mathbb{Z}/2^k} EB\mathbb{Z}/2^k & \dashrightarrow & E_k \\
\downarrow & & \downarrow \\
B^2\mathbb{Z}/2^k & & B^2\mathbb{Z}/2^k
\end{array}$$

the dashed arrow might be obtained applying again Zabrodsky's lemma. And this is the required homotopy equivalence $\mathrm{map}(B\mathbb{Z}/2^k, X)_{f_k} \times_{B\mathbb{Z}/2^k} EB\mathbb{Z}/2^k \simeq E_k$. □

Assume now that the sequence (17) is infinite; that is, all the fibrations $E_k \to E_{k+1} \to B^2\mathbb{Z}/2$ are of type I. In this case we define

$$E_\infty = \mathrm{hocolim}_k\, E_k$$

Let us first study the cohomology of E_∞. Suppose that

$$H^*(X;\mathbb{F}_2) \cong P[x_1,\dots,x_r] \otimes \frac{P[y_1,\dots,y_s,\dots]}{(y_1^{2^{\alpha_1}},\dots,y_s^{2^{\alpha_s}},\dots)}$$

for a good system of trangressive generators and x_1 is the class detected by $B\mathbb{Z}/2 \to X$. According to Proposition 7.4

$$H^*(E_1;\mathbb{F}_2) \cong P[x_1', x_2', \dots, x_r'] \otimes N$$

where

$$\frac{P[Sq^1\iota,\dots,Sq^{\Delta_{n-1}}\iota]}{\big((Sq^1\iota)^{2^{m_n}},\dots,(Sq^{\Delta_{n-1}}\iota)^{2^{m_1}}\big)} \rightarrowtail N \twoheadrightarrow \frac{P(y_1^2,\dots,y_n^2,y_{n+1},\dots,y_s,\dots]}{(y_1^{2^{\alpha_1}},\dots,y_s^{2^{\alpha_s}},\dots)}$$

In the following steps we just detect the two dimensional class x_1' and produce a new x_1'' and so on, hence according to the remark 7.6

$$H^*(E_k;\mathbb{F}_2) \cong P[x_1^{(k)}, x_2^{(k)}, \dots, x_r^{(k)}] \otimes N$$

each map $g\colon E_k \to E_{k+1}$ maps $x_1^{(k)}$ to zero and the other generators to the same indexed ones up to a polynomial in the previous generators, thus inducing an isomorphism

$$H^*(E_k;\mathbb{F}_2)/(x_1^{(k)}) \xleftarrow{\cong} H^*(E_{k+1};\mathbb{F}_2)/(x_1^{k+1)})$$

So, using the Milnor exact sequence to compute the cohomology of a telescope we obtain

$$H^*(E_\infty;\mathbb{F}_2) \cong \varprojlim{}_k H^*(E_k;\mathbb{F}_2) \cong P[\tilde{x}_2, \dots, \tilde{x}_r] \otimes N$$

Observe that the polynomial class x_1, finally disappeared and instead we keep, in particular, a three dimensional class in N that appeared after the first step, restricted from $Sq^1\iota \in H^*(B^2\mathbb{Z}/2;\mathbb{F}_2)$. This is the one that we could classify in order the recover the original X. Let us make this statement precise.

The sequences of Proposition 8.1 (i) combine in a direct system

$$\begin{array}{ccccccc}
\dots & \longrightarrow & B\mathbb{Z}/2^k & \longrightarrow & B\mathbb{Z}/2^{k+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
\dots & = & X & = & X & = & \dots \\
 & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & E_k & \longrightarrow & E_{k+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & B^2\mathbb{Z}/2^k & \longrightarrow & B^2\mathbb{Z}/2^{k+1} & \longrightarrow & \dots
\end{array}$$

Hence we obtain fibrations:

$$B\mathbb{Z}/2^\infty \xrightarrow{f_\infty} X \xrightarrow{g_\infty} E_\infty \xrightarrow{h_\infty} B^2\mathbb{Z}/2^\infty$$

and the mod 2 completion,

$$B\hat{S}^1_2 \xrightarrow{\hat{f}} X \xrightarrow{\hat{g}} \hat{E}_\infty \xrightarrow{\hat{h}} B^2\hat{S}^1_2\,.$$

It remains to show that E_∞, $\hat{E}_\infty$ are H-spaces.

Lemma 8.2. *For $A = \mathbb{Z}/2^\infty$ or $\hat{S}^1_2$,*

(i) *the evaluation map induces*

$$\mathrm{map}(BA, X)_f \simeq \mathrm{map}(BA, X)_c \simeq X$$

for any $f\colon BA \to X$.

(ii) *Also induced by evaluation:*

$$\begin{aligned}\mathrm{map}(B\mathbb{Z}/2, E_\infty)_c &\simeq E_\infty\\ \mathrm{map}(B\mathbb{Z}/2^\infty, E_\infty)_c &\simeq E_\infty\\ \mathrm{map}(B\hat{S}^1_2, \hat{E}_\infty)_c &\simeq \hat{E}_\infty\end{aligned}$$

Proof. We know form Proposition 8.1 (ii) and because X is an H-space, that

$$\mathrm{map}(B\mathbb{Z}/2^k, X)_{f_k} \simeq \mathrm{map}(B\mathbb{Z}/2^k, X)_c \simeq X$$

Now $\mathrm{map}(B\mathbb{Z}/2^\infty, X) \simeq \mathrm{holim}_k\, \mathrm{map}(B\mathbb{Z}/2^k, X)$ and since $\lim^1_k \pi_1 \mathrm{map}(B\mathbb{Z}/2^k, X) \cong \varprojlim{}^1_k \pi_1 X = 0$ from [5, XI,7.4] it follows that

$$\pi_0(\mathrm{holim}_k\, \mathrm{map}(B\mathbb{Z}/2^k, X)) \cong \varprojlim{}^0_k \pi_0(\mathrm{map}(B\mathbb{Z}/2^k, X))$$

and then

$$\begin{aligned}\mathrm{map}(B\mathbb{Z}/2^\infty, X)_c &\simeq \mathrm{holim}_k\, \mathrm{map}(B\mathbb{Z}/2^k, X)_c\\ &\simeq \mathrm{holim}_k\, X\\ &\simeq X\end{aligned}$$

Finally since X is 2-complete we have as well

$$\mathrm{map}(B\hat{S}^1_2, X)_c \simeq \mathrm{map}(B\mathbb{Z}/2^\infty, X)_c \simeq X$$

(ii) Is a consequence of the diagrams of principal fibrations like

$$\begin{array}{ccccc}\mathrm{map}(B\mathbb{Z}/2^\infty, B\mathbb{Z}/2^\infty)_c & \longrightarrow & \mathrm{map}(B\mathbb{Z}/2^\infty, X)_c & \longrightarrow & \mathrm{map}(B\mathbb{Z}/2^\infty, E_\infty)_c\\ \simeq\Big\downarrow ev & & \simeq\Big\downarrow ev & & \Big\downarrow ev\\ B\mathbb{Z}/2^\infty & \longrightarrow & X & \longrightarrow & E_\infty\end{array}$$

where the fibre of $\mathrm{map}(B\mathbb{Z}/2^\infty, X)_c \to \mathrm{map}(B\mathbb{Z}/2^\infty, E_\infty)_c$ consists of those components of $\mathrm{map}(B\mathbb{Z}/2^\infty, B\mathbb{Z}/2^\infty)$ containing maps $\varphi\colon B\mathbb{Z}/2^\infty \to B\mathbb{Z}/2^\infty$ such that $f_\infty \circ \varphi$ is homotopy to a constant map. But this is detectable by cohomology with 2-adic coefficients and the only possibility is $\varphi \simeq$ constant.

It then follows that $ev\colon \mathrm{map}(B\mathbb{Z}/2^\infty, E_\infty)_c \to E_\infty$ is a homotopy equivalence. The other statements are proved in the same way. □

Proposition 8.3. *The spaces E_∞, $\hat{E}_\infty$ are H-spaces and the fibrations*

$$B\mathbb{Z}/2^\infty \xrightarrow{f_\infty} X \xrightarrow{g_\infty} E_\infty \xrightarrow{h_\infty} B^2\mathbb{Z}/2^\infty$$

and

$$B\hat{S}'_2 \xrightarrow{\hat{f}} X \xrightarrow{\hat{g}} \hat{E}_\infty \xrightarrow{\hat{h}} B^2\hat{S}'_2$$

are H-fibrations.

Proof. We explain two different proofs.

First, observe that for a direct system indexed by $\mathbb{N}$ we have that

$$\mathrm{hocolim}_k\, E_k \times \mathrm{hocolim}_k\, E_k \longleftarrow \mathrm{hocolim}_k\, E_k \times E_k$$

is a homotopy equivalence and then the multiplications $\mu_k \colon E_k \times E_k \to E_k$ induce a multiplication

$$\mu_\infty \colon E_\infty \times E_\infty \longrightarrow E_\infty$$

It is not clear however that μ_∞ has a homotopy neutral element. We can clearly guess what the neutral element should be but then we need to show that the composition

$$E_\infty \xrightarrow{j_1} E_\infty \times E_\infty \xrightarrow{\mu_\infty} E_\infty$$

is homotopic to the identity.

We know that the restriction to each $i_k \colon E_k \hookrightarrow E_\infty$ is homotopic to the identity. So the obstructions for $\mu_\infty \circ j_i$ to be homotopic to the identity lie in

$$\varprojlim{}^i_k \pi_i \operatorname{map}(E_k, E_\infty)_{i_k}, \quad i \geq 1.$$

The Zabrodsky's lemma applied to the principal fibration $B\mathbb{Z}/2 \to E_k \to E_{k+1}$ together with the fact that $\operatorname{map}(B\mathbb{Z}/2, E_\infty)_c \simeq E_\infty$ (see Lemma 8.2 ii) implies that $\operatorname{map}(E_{k+1}, E_\infty)_{i_{k+1}} \simeq \operatorname{map}(E_k, E_\infty)_{i_k}$ so that $\pi_i \operatorname{map}(E_k, E_\infty)_{i_k}$ are constant functors and the higher limit functors vanish.

We have therefore proved that $\mu_\infty \colon E_\infty \times E_\infty \to E_\infty$ has a homotopy neutral element and then E_∞ becomes and H-space. We can easily see that $f_\infty, g_\infty, h_\infty$ are H-maps.

A different point of view consists in adapting the argument of Proposition 2.5 using be results of Lemma 8.2. □

Thus, our method produce a new H-space X_1 out of $X = X_0$ with one less polynomial generator and X_1 still satisfies conditions F1, F2 and F3. Either, the iteration stops at a finite place and the new H-space inherits a nilpotent two dimensional generator or the iteration does not stop and the new inherited classes start at dimension three.

Observe that in any case the two dimensional classes of X are still in the new H-space X_1, unless x_1 if it had dimension two (see Propositions 7.4 and 7.7).

And now we can repeat our construction with the new H-space X_1 and produce an H-space X_2 with one less polynomial generator. And so continue up to a final step X_n, where X_n has no polynomial generator.

We have obtained X_n, a 1-connected mod 2 H-space with $H^*(X_n; \mathbb{F}_2)$ nilpotent, so $\operatorname{map}(B\mathbb{Z}/2, X_n) \simeq \operatorname{map}(B\mathbb{Z}/2, X_n)_c$, and $QH^*(X_n; \mathbb{F}_2)$ locally finite as $\mathcal{A}$-module, hence $\operatorname{map}(B\mathbb{Z}/2, X_n) \simeq \operatorname{map}(B\mathbb{Z}/2, X_n)_c \simeq X_n$. That is X_n satisfies the Sullivan conjecture, or in other words X_n is $L_{B\mathbb{Z}/2}$-local or $F(X_n) \simeq X_n$, where F is, as defined in the Introduction, the composition of the $B\mathbb{Z}/2$-nullification functor, $L_{B\mathbb{Z}/2}$, and Bousfield-Kan 2-completion.

Theorem 8.4. *Let X be a 1-connected mod 2 H-space that satisfies conditions F1, F2 and F3. Then*

1. *There is a sequence of mod 2 H-spaces*

$$X = X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n = F(X)$$

where all X_i satisfy as well conditions F1, F2 and F3, the depth of X_i is the depth of X_{i-1} minus one and $X_n = F(X)$ is $L_{B\mathbb{Z}/2}$-local.

2. *The maps*
$$X_i \longrightarrow X_{i+1}$$
are principal H-fibrations with fibre either $(\mathbb{C}P^\infty)\hat{}_2$ or $B\mathbb{Z}/2^k$ for some $k \geq 1$.
3. *The composition $X \longrightarrow F(X)$ is as well a principal fibration with fibre the product of the fibres of the maps $X_i \longrightarrow X_{i+1}$.*

Assume furthermore that $H^*(X;\mathbb{F}_2)$ is actually noetherian, then

3. *As algebras, $H^*(X;\mathbb{F}_2) \cong P \otimes N$, where P is a polynomial algebra and N is a 2-connected finite Hopf algebra.*
4. *In the above sequence of mod 2 H-spaces*
$$X = X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n = F(X)$$
all X_i have noetherian mod 2 cohomology.
5. *$X_n = F(X)$ is a mod 2 finite H-space.*
6. *The fibrations (17) involved in the construction of*
$$X_i \longrightarrow X_{i+1}$$
are are of type I *and $X_i \longrightarrow X_{i+1}$ is a principal H-fibrations with fibre $(\mathbb{C}P^\infty)\hat{}_2$.*
7. *The composition $X \longrightarrow F(X)$ is as well a principal fibration:*
$$\left((\mathbb{C}P^\infty)\hat{}_2\right)^n \longrightarrow X \longrightarrow F(X)\,.$$

Proof. 1 and 2 follow from the previous constructions. 3 and argument similar to that of Proposition 8.1 (i).

Assume now that $H^*(X;\mathbb{F}_2)$ is noetherian. Using the propositions 7.4 and 7.7, we see that at each step of our construction the obtained H-space has as well noetherian mod 2 cohomology $H^*(X_i;\mathbb{F}_2) \cong P_i \otimes N_i$ where P_i is a polynomial algebra and N_i is a finite algebra. In particular the mod 2 cohomology of $X_n = F(X)$ is finite: $H^*((;\mathbb{F}_2)X_n) \cong N_n$. So $X_n = F(X)$ is a mod 2 finite H-space. It is known ([8]) that a 1-connected mod 2-finite H-space is actually 2-connected so its mod 2 cohomology, N_n cannot contain two dimensional classes. But, according to propositions 7.4 and 7.7 a two dimensional class in any N_i would be inherited by N_n. Hence, all N_i should be 2-connected.

Observe as well that if one of the fibrations (17) involved in the construction was of type II then it would produce a nilpotent two dimensional class in the cohomology of the constructed H-space (Proposition 7.7). Again this class would survive to N_n, which is a contradiction, hence all the fibrations involved are of type I, the sequence (17) is infinite
$$X_i = E_0 \to E_1 \to \ldots \to E_k \to \ldots \to E_\infty = \mathrm{hocolim}_k\, E_k$$
and the fibre of the principal fibration $X_i \to X_{i+1} = (E_\infty)\hat{}_2$ is $(\mathbb{C}P^\infty)\hat{}_2$. We have proved 3, 4, 5, 6. 7 follows as before. □

Proof of Theorem 1.1. It follows from Theorem 8.4 for the case in which $H^*(X;\mathbb{F}_2)$ is noetherian. □

Proof of Theorem 1.6. If X is a mod 2 H-space that satisfies the conditions F1, F2 and F3 and $x \in H^*(X;\mathbb{F}_2)$ is a polynomial generator, we can construct a fibration
$$B\mathbb{Z}/2 \to X \to E$$

and with the arguments of the beginning of this section, we can complete x to a good system of transgressive generators such that x is the only one in this system that transgresses non trivially. The theorem then follows from Proposition 6.3 and Theorem 8.4(6). ☐

Example 8.5. Observe that our method allows us to guess what the cohomology of $F(X)$ should be. Let us have a look at some examples with just one polynomial generator.

1. One four dimensional polynomial generator. We have already seen in example 1.7 that a four dimensional polynomial generator always appears together with its Sq^1. So the minimal possible cohomology of an H-space , X, with a four dimensional polynomial generator is $P[x_4] \otimes E(Sq^1 x_4)$.

   After the first step we have $H^*(E_1; \mathbb{F}_2) \cong P[x_2] \otimes E(x_3)$ and then $H^*(X_1; \mathbb{F}_2) \cong E(x_3)$, the cohomology of S^3.
2. One eight dimensional polynomial generator. Also from example 1.7 we know that there are two minimal possibilities for the cohomology of an H-space, X, with an eight dimensional polynomial generator are. Namely
   - $H^*(X; \mathbb{F}_2) = P[x_8] \otimes E(x_9, x_{11})$ which would be the three connected cover of X_1 with $H^*(X_1; \mathbb{F}_2) = P[x_3]/(x_3^4) \otimes E(x_5)$.
   - $H^*(X; \mathbb{F}_2) = P[x_8] \otimes E(x_5, x_9)$ which in this case would be the three connected cover of X_1 with $H^*(X_1; \mathbb{F}_2) = E(x_3, x_5)$.

   In both cases with $Sq^2 x_3 = x_5$. Those cohomology algebras correspond to G_2, $SU(3)$ and its three connective coverings.
3. One 16-dimensional generator. In the same way one obtains what would be the minimal possibilities for the cohomology of an H-space with a 16-dimensional polynomial generator. The possible M_3^{I} are

$$\circ \xrightarrow{Sq^1} \bullet \xleftarrow{Sq^8} \bullet \xleftarrow{Sq^4} \bullet\ ,$$

$$\circ \xrightarrow{Sq^1} \bullet \xleftarrow{Sq^8} \bullet \xrightarrow{Sq^2} \bullet\ ,$$

$$\circ \xrightarrow{Sq^1} \bullet \xrightarrow{Sq^2} \bullet \xleftarrow{Sq^8} \bullet$$

   and

$$\circ \xrightarrow{Sq^1} \bullet \xrightarrow{Sq^2} \bullet \xrightarrow{Sq^4} \bullet\ .$$

   Such an H-space should be the 3-connected cover of a mod 2 H-space with cohomology either
   - $E(x_3, x_5, x_9)$,
   - $\dfrac{P(x_3)}{(x_3^4)} \otimes E(x_5, x_9)$,
   - $\dfrac{P(x_3, x_5)}{(x_3^4, x_5^4)} \otimes E(x_9)$ or
   - $\dfrac{P(x_3, x_5)}{(x_3^8, x_5^4)} \otimes E(x_9)$,

   respectively, with $Sq^2 x_3 = x_5$, $Sq^4 x_5 = x_9$. Observe that those Hopf algebras are primitively generated and by classical results (cf. [14]) they cannot appear as the cohomology of an H-space. Hence these minimal cohomology Hopf algebras are not realizable. Notice that the first of the possibilities embeds in the cohomology of $SU(5)$, so $SU(5)\langle 3\rangle$ realizes one on the minimal examples but with an additional class in dimension seven.

4. One 32-dimensional polynomial generator. As before, one 32-dimensional generator together with just the minimal amount of extra generators implied by Theorem 1.6 cannot exist. However some of those minimal possibilities embed in the cohomologies of the three connected covers of the Lie groups $SU(9)$, E_6, E_7 and E_8, where the corresponding modules M_4^{I} are:

$$\circ \xrightarrow{Sq^1} \bullet \xleftarrow{Sq^{16}} \bullet \xleftarrow{Sq^8} \bullet \xleftarrow{Sq^4} \bullet\ ,$$

$$\circ \xrightarrow{Sq^1} \bullet \xleftarrow{Sq^{16}} \bullet \xleftarrow{Sq^8} \bullet \xrightarrow{Sq^2} \bullet\ ,$$

$$\circ \xrightarrow{Sq^1} \bullet \xrightarrow{Sq^2} \bullet \xleftarrow{Sq^{16}} \bullet \xleftarrow{Sq^8} \bullet$$

and

$$\circ \xrightarrow{Sq^1} \bullet \xrightarrow{Sq^2} \bullet \xrightarrow{Sq^4} \bullet \xrightarrow{Sq^8} \bullet\ ,$$

respectively (see [13]).

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Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Spain.

E-mail address: broto@mat.uab.es

E-mail address: chiqui@mat.uab.es