ON SPACES OF SELF HOMOTOPY EQUIVALENCES OF p-COMPLETED CLASSIFYING SPACES OF FINITE GROUPS AND HOMOTOPY GROUP EXTENSIONS

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1. INTRODUCTION

Let G and π be groups and let p be a prime number. A mod-p homotopy group extension of π by G is a fibration with base space $B\pi^\wedge_p$ and fibre $BG^\wedge_p$, where $(-)^\wedge_p$ denotes the Bousfield-Kan p-completion functor.

In this paper we study homotopy extensions of finite groups. Thus if X is the total space in a mod-p homotopy group extension of π by G, where both π and G are finite, we shall call X a finite homotopy group extension. As one would expect from a classification problem of a class of fibrations, the project involves studying spaces of homotopy equivalences of the fibres under consideration. This problem forms one of the main cores of the paper and is of independent interest. Indeed, spaces of self homotopy equivalences for compact connected simple Lie groups are well understood as a byproduct of [8]. Our results are partially the finite group analogue.

Let p be a prime and let G be a finite group. Then $O_{p'}G$ is defined to be the maximal normal subgroup of G of order prime to p. Define the mod-p' reduction of G to be the quotient $G/O_{p'}G$. If $O_{p'}G$ is trivial then G is said to be p'-reduced. Notice that the natural projection from G to $G/O_{p'}G$ induces a mod-p homology isomorphism and hence a p-local homotopy equivalence on classifying spaces.

For an arbitrary space X, let $\mathrm{Aut}(X)$ denote topological monoid of all self homotopy equivalences of X. Let $\mathrm{SAut}(X)$ denote the identity component of $\mathrm{Aut}(X)$ and let $\mathrm{Out}(X)$ denote the group of components $\pi_0(\mathrm{Aut}(X))$. By general theory, fibre homotopy equivalence classes of Mod-p homotopy group extensions of π by G are in 1-1 correspondence with the set of homotopy classes of maps from $B\pi^\wedge_p$ to $B\,\mathrm{Aut}(BG^\wedge_p)$. The next theorem describes the homotopy type of the identity component $\mathrm{SAut}(BG^\wedge_p)$.

Theorem 1.1. *Let G be a finite group. Then* $\mathrm{SAut}(BG^\wedge_p) \simeq BZ(G/O_{p'}G)$.

Recall that a group G is said to be p-perfect if its first mod-p homology group is trivial and p-superperfect if in addition its second mod-p homology vanishes. For every discrete group G there exists a maximal normal p-perfect subgroup $O^pG < G$ and $BO^pG^\wedge_p$ is the 1-connected cover of $BG^\wedge_p$. Furthermore, if G is itself p-perfect then

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there is a central extension U^pG of G by $H_2(BG;\mathbb{Z}_{(p)})$ and $BU^pG^\wedge_p$ is the 2-connected cover of $BG^\wedge_p$. For simplicity of notation we shall denote $U^p(O^pG)$ by U^pG, even if G itself is not p-prefect. As corollaries of Theorem 1.1 we obtain

Corollary 1.2. *Let X be a mod-p homotopy extension of π by G. Then there is a homotopy equivalence*

$$X\langle 2\rangle \simeq BU^p\pi^\wedge_p \times BU^pG^\wedge_p,$$

where $X\langle 2\rangle$ is the 2-connected cover of X.

Corollary 1.3. *Assume π is p-perfect and let X be a mod-p homotopy extension of π by G. Then, there is a homotopy equivalence*

$$X \simeq BH^\wedge_p$$

where H is an extension of π by G with trivial action $\pi \longrightarrow \mathrm{Out}(G)$.

Corollary 1.4. *Assume π is p-superperfect and let G be any finite group. Then every mod-p homotopy extension of π by G is trivial.*

The general case might be easily reduced to the case in which the base is the classifying space of a finite p-group π. In fact, if π is not a p group, then there is a fibration $BO^p(\pi)^\wedge_p \longrightarrow B\pi^\wedge_p \longrightarrow B\pi/O^p(\pi)$, where $\pi/O^p(\pi) = \pi_1(B\pi^\wedge_p)$ is a p-group. Then, if X is a mod-p homotopy extension of π by G, one has a diagram of fibrations

$$\begin{array}{ccccc}
BG^\wedge_p & = & BG^\wedge_p & & \\
\downarrow & & \downarrow & & \\
Y & \longrightarrow & X & \longrightarrow & B\pi/O^p(\pi) \\
\downarrow & & \downarrow & & \| \\
BO^p(\pi)^\wedge_p & \longrightarrow & B\pi^\wedge_p & \longrightarrow & B\pi/O^p(\pi)
\end{array}$$

Now, Corollary 1.3 applies to the fibration in the left column and $Y \simeq BH^\wedge_p$, hence X is expressed in the middle row fibration as a mod-p homotopy groups extension of a p-group, $\pi/O^p(\pi)$, by a group H.

We now turn to the group of components $\mathrm{Out}(BG^\wedge_p)$. There is an obvious group homomorphism

$$\gamma_G : \mathrm{Out}(G) \longrightarrow \mathrm{Out}(BG^\wedge_p),$$

which, as we shall see below, is not generally an isomorphism. A different approximation of the group $\mathrm{Out}(BG^\wedge_p)$ is given as follows. Let $\mathcal{C}_p$ denote the category whose objects are finite p-groups and whose morphisms are conjugacy classes of monomorphisms $\pi \longrightarrow \pi'$. For a finite group G, let $I_G : \mathcal{C}_p^{op} \longrightarrow \mathcal{S}ets$ denote the functor which takes a finite p-group π to the set $\mathrm{Inj}(\pi, G)$ of conjugacy classes of monomorphism $\pi \longrightarrow G$ and which takes a morphism $\pi' \xrightarrow{\rho} \pi$ in $\mathcal{C}_p$ to the induced map

$\text{Inj}(\pi, G) \longrightarrow \text{Inj}(\pi', G)$. The set of all natural equivalences of I_G forms a group under composition, which we denote by $\text{Aut}(I_G)$. Let $\mathcal{O}_p^c(G)$ denote the orbit category of p-centric subgroups of G. Let $\mathcal{Z} : \mathcal{O}_p^c(G) \longrightarrow \mathcal{A}b$ denote the contravariant functor which associate with an orbit G/P the center $Z(P)$.

Theorem 1.5. *Let G be a finite group. Then there is an exact sequence*

$$0 \longrightarrow \lim_{\mathcal{O}_p^c(G)^{op}}{}^{1} \mathcal{Z} \longrightarrow \text{Out}(BG_p^\wedge) \xrightarrow{\psi_G} \text{Aut}(I_G).$$

Furthermore, the obstruction for ψ_G being onto is an element of $\lim_{\mathcal{O}_p^c(G)^{op}}{}^{2} \mathcal{Z}$. In particular if both higher limits vanish then ψ_G is an isomorphism.

The general behavior of the maps γ_G and ψ_G is not well understood. One class of examples where γ_G is always an isomorphism is given by groups G, whose Sylow p-subgroup is normal. However, different groups with the same mod-p homotopy type may well have different outer automorphism groups (e.g. $(BM_{11})_2^\wedge \simeq BSL_3(3)_2^\wedge$ but $\mathbb{Z}/2 \cong \text{Out}(SL_2(3)) \neq \text{Out}(M_{11}) = \{1\}$) and so γ_G cannot be an isomorphism in general. The map ψ_G is known not to be a monomorphism in general as the next theorem demonstrates, but there is no known example where ψ_G fails to be an epimorphism.

Theorem 1.6. *Let $q = 3^{2^s}$, $s \geq 1$. Let $G = SL_2(q)$ and let $H = PSL_2(q)$. Then*

1. *There is a commutative diagram of isomorphisms*

$$\begin{array}{ccc} \text{Out}(G) & \xrightarrow{\cong} & \text{Out}(H) \\ \cong\downarrow & & \cong\downarrow \\ \text{Out}(BG_2^\wedge) & \xrightarrow{\cong} & \text{Out}(BH_2^\wedge) \end{array}$$

   and $\text{Out}(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z}$.
2. $\lim_{\mathcal{O}_p^c(G)}{}^{i} \mathcal{Z} = 0$ *for all $i > 0$. Thus* $\text{Out}(BH_2^\wedge) \cong \text{Aut}(I_H)$.
3. $\lim_{\mathcal{O}_p^c(H)}{}^{i} \mathcal{Z} \cong \mathbb{Z}/2\mathbb{Z}$ *if $i = 1$, but vanishes otherwise. Thus there is a non-split short exact sequence*

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Out}(BH_2^\wedge) \longrightarrow Aut(I_H) \longrightarrow 0.$$

A complete identification of $\text{Out}(BG_p^\wedge)$ for an arbitrary finite group G will appear in a future paper joint with R. Oliver.

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2. The Space $\mathrm{Map}(B\pi, BG^\wedge_p)$

In this section we record the following

Proposition 2.1. *Let π be a finite p-group and let G be a finite group. Then $\mathrm{Map}(B\pi, BG^\wedge_p)$ is p-complete and the natural map*

$$\mathrm{Map}(B\pi, BG) \longrightarrow \mathrm{Map}(B\pi, BG^\wedge_p)\,, \tag{1}$$

induces a mod-p homology isomorphism.

The proposition is known to the experts, but as we are not aware of a suitable reference, a proof is included here.

Proof. The case where π is an elementary abelian p-group is due to Lannes [9]. By induction we may thus assume the statement of the proposition for every p-group κ of order less than p^n.

Let π a group of order p^n. There is an extension

$$\kappa \longrightarrow \pi \longrightarrow V$$

where V is a non-trivial elementary abelian p-group. The group V acts on $\widetilde{B\kappa} = E\pi/\kappa$ and one has a homotopy equivalence

$$\mathrm{Map}(B\pi, BG^\wedge_p) \simeq \mathrm{Map}(\widetilde{B\kappa}, BG^\wedge_p)^{hV}\,.$$

Recall that for a general V-space X, the homotopy fixed points space is defined as the space of V-equivariant maps $\mathrm{Map}_V(EV, X)$ and this can be reinterpreted as the space of sections of the projection $X_{hV} \longrightarrow BV$. In fact, using the action of BV on $\mathrm{Map}(BV, X_{hV})$ one obtains in general a homotopy equivalence over BV

$$BV \times X^{hV} \xrightarrow{\simeq} \mathrm{Map}(BV, X_{hV})_{\bar{1}}$$

where $\mathrm{Map}(BV, X_{hV})_{\bar{1}}$ denotes the set of components of $\mathrm{Map}(BV, X_{hV})$ over the component of the identity $\mathrm{Map}(BV, BV)_1 \simeq BV$.

In our case we take X to be either $\mathrm{Map}(\widetilde{B\kappa}, BG^\wedge_p)$ or $\mathrm{Map}(\widetilde{B\kappa}, BG)$. Both spaces are related by the induction hypothesis, and after taking homotopy quotients, we have

$$\mathrm{Map}(\widetilde{B\kappa}, BG^\wedge_p)_{hV} \simeq \big(\mathrm{Map}(\widetilde{B\kappa}, BG)^\wedge_p\big)_{hV} \simeq \big(\mathrm{Map}(\widetilde{B\kappa}, BG)_{hV}\big)^\wedge_p\,. \tag{2}$$

Now, notice that $\mathrm{Map}(\widetilde{B\kappa}, BG)_{hV}$ has the homotopy type of a union of a finite number of classifying spaces of finite groups. Hence we have homotopy equivalences over BV

$$\begin{aligned} BV \times \mathrm{Map}(\widetilde{B\kappa}, BG^\wedge_p)^{hV} &\simeq \mathrm{Map}\big(BV, \mathrm{Map}(\widetilde{B\kappa}, BG^\wedge_p)_{hV}\big)_{\bar{1}} \simeq \\ &\big[\mathrm{Map}\big(BV, \mathrm{Map}(\widetilde{B\kappa}, BG)_{hV}\big)_{\bar{1}}\big]^\wedge_p \simeq \big[BV \times \mathrm{Map}(\widetilde{B\kappa}, BG)^{hV}\big]^\wedge_p\,, \end{aligned} \tag{3}$$

from which we obtain $\mathrm{Map}(B\pi, BG^\wedge_p) \simeq \mathrm{Map}(B\pi, BG)^\wedge_p$.

Notice that Equation (2) implies that $\mathrm{Map}(\widetilde{B\kappa}, BG^\wedge_p)_{hV} \simeq (\mathrm{Map}(\widetilde{B\kappa}, BG^\wedge_p)_{hV})^\wedge_p$ and the same arguments as above yield

$$\mathrm{Map}(B\pi, BG^\wedge_p) \simeq \mathrm{Map}(B\pi, BG^\wedge_p)^\wedge_p.$$

Hence $\mathrm{Map}(B\pi, BG^\wedge_p)$ is p-complete. This finishes the proof of the proposition. ☐

3. The Topological Monoid $\mathrm{SAut}(BG^\wedge_p)$

In this section we prove the following

Theorem 3.1. *Let G be a finite group. Then*

$$\mathrm{SAut}(BG^\wedge_p) \simeq BZ(G/O_{p'}G),$$

where $O_{p'}G$ is the maximal normal subgroup of G of order prime to p.

Proof. Notice that since $G/O_{p'}G$ is p'-reduced, its center is automatically an abelian p-group. Also $BG^\wedge_p \simeq B(G/O_{p'}G)^\wedge_p$ and so $\mathrm{SAut}(BG^\wedge_p) \simeq \mathrm{SAut}(B(G/O_{p'}G)^\wedge_p)$. Thus to prove the theorem it suffices to show that if G is p'-reduced then $\mathrm{SAut}(BG^\wedge_p) \simeq BZ(G)$.

This is done in three steps. In Proposition 3.6 below it is shown that $\mathrm{SAut}(BG^\wedge_p)$ has at most one non-trivial homotopy group

$$\pi_1(\mathrm{SAut}(BG^\wedge_p), id) = L\,,$$

where L is a finite p-group and furthermore, that there is a monomorphism $i : L \longrightarrow G$ such that $Z(G) \leq i(L) \leq Z(\pi)$, where π is any Sylow p-subgroup of G containing L. Next, identify L with its image $i(L) < G$. In Proposition 3.7 it is proved that the inclusion of the centralizer of L in G, $C_G(L) \leq G$, induces an isomorphism in mod-p cohomology; that is to say L is p-cohomologically central in G. Finally one uses a theorem of Mislin, stated below, which shows that L is central in G. Since $L \geq Z(G)$, the two subgroups have to coincide, and therefore $\mathrm{SAut}(BG^\wedge_p) \simeq BZ(G)$.

Theorem 3.2 (Mislin [12])**.** *Let G be a finite p'-reduced group. Then, every p-cohomologically central subgroup of G is central.*

Remark 3.3. This statement is not explicit in [12] but the necessary arguments are contained in there. We sketch here the proof for convenience of the reader. By definition, if P is p-cohomologically central, then the inclusion of its centralizer $C_G(P)$ in G induces a mod-p cohomology isomorphism and therefore an equivalence of the corresponding Frobenius categories of p-subgroups. In particular $C_G(P)$ controls p-fusion in G and according to the Z^*-theorem this implies $C_G(P) = G$, because G is p'-reduced. It might be worth pointing out that the Z^*-theorem for finite groups depends on the classification of finite simple groups [3] ☐

For a finite group G and a collection $\mathcal{C}$ of subgroups of G, which is closed under conjugacy, the $\mathcal{C}$-orbit category is the category whose objects are the G-sets given by left cosets G/H with $H \in \mathcal{C}$ and whose morphisms are given by all G-maps. Notice that the action of G on a coset is given by $g'(gH) := g'gH$. Thus any morphism φ from G/H to G/K in the orbit category is determined by $\varphi(H) = gK$ for some $g \in G$, which conjugates H into K. Moreover this element g is unique up to a right shift by an element of K. Thus each map $\varphi : G/H \longrightarrow G/K$ gives rise to a unique faithful representation of H in K (a conjugacy class of monomorphisms). If $g \in G$ conjugates H to K then g^{-1} conjugates $C_G(K)$ to $C_G(H)$. Furthermore, if g_1 and

g_2 represent the same morphism $G/H \longrightarrow G/K$ then conjugation by g_1^{-1} and g_2^{-1} coincides on $C_G(K)$. Recall the following

Definition 3.4. A p-subgroup π of a finite group G is said to be p-centric if

$$C_G(\pi) \cong Z(\pi) \times W,$$

where W is a group of order prime to p. Let $\mathcal{O}_p^c(G)$ denote the full subcategory of the orbit category whose objects are orbits of p-centric subgroups of G.

Our main tool is a homology decomposition theorem which expresses the mod-p homotopy type of BG by means of classifying spaces of p-centric subgroups. The technique is originally due to Jackowski, McClure and Oliver [7, 8]. A very coherent account of these methods by Dwyer, with some significant improvements and unification of methods appears in [4, 5]. The following is a particular case, which is all we need here. Let $\phi_G : \mathcal{O}_p^c(G) \longrightarrow G - Sets$ be the functor associating with an orbit G/H the G-set given by G/H itself. For a G-space X let X_{hG} denote the homotopy orbit space $X \times_G EG$, where EG is a free contractible G space, thus, $\phi_G(G/H)_{hG} = G/H \times_G EG \simeq BH$

Theorem 3.5. *Let G be a finite group. Then the natural map*

$$\Phi_G \colon \operatorname*{hocolim}_{\mathcal{O}_p^c(G)} (\phi_G)_{hG} \longrightarrow BG$$

induces a mod-p homology isomorphism.

Thus one has for any finite group G a homotopy equivalence

$$\operatorname{Map}(BG_p^\wedge, BG_p^\wedge) \simeq \operatorname*{holim}_{\mathcal{O}_p^c(G)} \operatorname{Map}\left((\phi_G)_{hG}, BG_p^\wedge\right). \tag{4}$$

Proposition 3.6. *Let G be a finite p'-reduced group and π a p-Sylow subgroup of G. Then*

$$\pi_i(\operatorname{SAut}(BG_p^\wedge)) = \begin{cases} L & i=1 \\ 0 & otherwise \end{cases}$$

where L is an abelian p-group which can be identified with a subgroup $L' < G$ such that $Z(G) \leq L' \leq Z(\pi)$.

Proof. We write $\operatorname{SAut}(BG_p^\wedge) = \operatorname{Map}(BG_p^\wedge, BG_p^\wedge)_{id}$ and then use the Bousfield-Kan spectral sequence for computing the homotopy groups of a homotopy limit associated to equation (4)

$$E_2^{s,t} = \operatorname*{lim^s}_{\mathcal{O}_p^c(G)} \pi_t(\operatorname{Map}((G/H)_{hG}, BG_p^\wedge)) \Rightarrow \pi_{t-s}(\operatorname{Map}(BG_p^\wedge, BG_p^\wedge)).$$

The differentials in this spectral sequence have the form $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$.

Take the identity map of $BG_p^\wedge$ as a base point. It restricts to the standard inclusion

$$inc\colon (G/H)_{hG} = BH \longrightarrow BG_p^\wedge$$

for any centric p-subgroup $H < G$. One has

$$\mathrm{Map}((G/H)_{hG}, BG_p^\wedge)_{inc} \simeq \mathrm{Map}(BH, BG_p^\wedge)_{inc} \simeq \\ (\mathrm{Map}(BH, BG)_{inc})_p^\wedge \simeq BC_G(H)_p^\wedge \simeq BZ(H).$$

Let $\mathcal{Z} : \mathcal{O}_p^c(G) \longrightarrow \mathcal{A}b$ denote the contravariant functor which associates to each orbit G/H the center $Z(H)$. Then, for our choice of base point, the E_2 term of the spectral sequence has the form

$$E_2^{s,t} \cong \lim_{\mathcal{O}_p^c(G)}{}^s \pi_t(\mathrm{Map}((G/H)_{hG}, BG_p^\wedge)_{inc}) = \begin{cases} \lim_{\mathcal{O}_p^c(G)}{}^s \mathcal{Z}\,, & t = 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

Hence the only relevant value is obtained in the spectral sequence for $s = 0$. Consequently $\pi_i(\mathrm{SAut}(BG_p^\wedge)) = 0$ if $i > 1$ and

$$\pi_1(\mathrm{SAut}(BG_p^\wedge)) = L := \lim_{\mathcal{O}_p^c(G)}{}^0 \mathcal{Z}.$$

We proceed by proving the properties of L claimed in the proposition. By definition the inverse limit is the kernel of the map

$$\phi : \prod_{G/H \in Obj(\mathcal{O}_p^c(G))} Z(H) \longrightarrow \prod_{\varphi \in Mor(\mathcal{O}_p^c(G))} Z(\mathrm{range}(\varphi))$$

defined by its projection to the φ component as $x_P - BC_G(\varphi)(x_Q)$, where $\varphi : P \longrightarrow Q$, $x_P \in Z(P)$ and $x_Q \in Z(Q)$. Notice that for every $G/P \in \mathcal{O}_p^c(G)$, $Z(P) > Z(G)$, since $Z(G)$ is a p-group. Since all maps in $\mathcal{O}_p^c(G)$ are induced by translation in G, the induced maps on centralizers are given by sub conjugation and it follows at once that $Z(G)$ is a subgroup of L. An obvious embeding is given by including $Z(G)$ diagonally in the domain of ϕ.

Finally, notice that if $\bar{x} \in L$ is represented by a sequence of elements of G indexed by objects of $\mathcal{O}_p^c(G)$ and x_P is the projection of $\bar{x}$ to the component of P, then x_P is invariant under the action of the normalizer $N_G(P)$ and hence is central in P. Let π be a Sylow p-subgroup of G containing P. Then $x_P = x_\pi$ and so x_π is central in π. Notice that projection from L to the component of each one of the groups involved is a monomorphism. Thus L can be embedded in the center of each Sylow p-subgroup. In particular L is an abelian p-group and the proof is complete. ☐

The inclusion of L in G obtained in the above Proposition factors through a p-Sylow subgroup π of G. This is represented up to conjugation by the composition

$$BL \longrightarrow BZ(\pi) \simeq \mathrm{Map}((G/\pi)_{hG}, BG_p^\wedge) \xrightarrow{ev} BG_p^\wedge$$

and therefore by

$$BL \simeq \mathrm{Map}(BG_p^\wedge, BG_p^\wedge)_{id} \xrightarrow{ev} BG_p^\wedge$$

This last map is canonically defined. However a representing homomorphism is only well defined up to conjugation. The next Proposition however shows that L is p-cohomologically central in G and hence central in G, by Mislin theorem 3.2. Thus

L coincides with all its conjugates. Hence we conclude that the identification of $L = \pi_1 \operatorname{Map}(BG^\wedge_p, BG^\wedge_p)_{id}$ with a subgroup of G is canonically induced by evaluation.

In the next Proposition identify the group L with its image L' under some appropriate embedding.

Proposition 3.7. *Fix an embeding of L in G and identify L with its image. Then the mod-p cohomology restriction*

$$\operatorname{Res} : H^*(BG) \longrightarrow H^*(BC_G(L))$$

is an isomorphism.

Proof. Let π be a Sylow p-subgroup of G containing L. Then $L < Z(\pi)$ and so $\pi < C_G(Z(\pi)) < C_G(L)$. Thus the restriction map is a monomorphism.

Next notice that there is a homotopy equivalence

$$\phi : BL \longrightarrow \operatorname{SAut}(BG^\wedge_p),$$

and a homotopy commutative diagram

$$\begin{array}{ccc} BL & \xrightarrow{\phi} & \operatorname{SAut}(BG^\wedge_p) = \operatorname{Map}(BG, BG^\wedge_p)_{id} \\ & \searrow & \downarrow{\scriptstyle Bi^*} \\ & & BZ(\pi) \simeq \operatorname{Map}(B\pi, BG)_{Bi} \end{array}$$

where $i : \pi \longrightarrow G$ is the inclusion. Taking adjoints we get a commutative diagram

$$\begin{array}{ccc} BL \times BG & \longrightarrow & BG^\wedge_p \\ {\scriptstyle 1\times Bi}\uparrow & \nearrow & \\ BL \times B\pi & & \end{array}$$

Taking adjoints the other way and writing $BC_G(L)^\wedge_p$ for $\operatorname{Map}(BL, BG^\wedge_p)$ gives a commutative diagram

$$\begin{array}{ccc} BG & \longrightarrow & BC_G(L)^\wedge_p \\ {\scriptstyle Bi}\uparrow & \nearrow & \\ B\pi & & \end{array}$$

Let $\psi : BG \longrightarrow BC_G(L)^\wedge_p$ denote the resulting map. By commutativity this map induces a monomorphism on mod-p cohomology. Composed with the restriction $BC_G(L) \longrightarrow BG$ we get a self map of $BC_G(L)$ which induces a cohomology monomorphism. Since $H^*(BC_G(L))$ is a graded group of finite type any monomorphism is an isomorphism. It follows then that the restriction map is an epimorphism and thus an isomorphism and the proof is complete. □

Corollary 3.8. *For every finite group G the space $B\operatorname{Aut}(BG^\wedge_p)$ is a 2-stage Postnikov tower with homotopy groups in dimensions 1 and 2. In particular*

$$\pi_2(B\operatorname{Aut}(BG^\wedge_p)) = Z(G/O_{p'}G).$$

Let a mod-p homotopy group extension of π by G be given. Then one has the following commutative diagram of fibrations, where the bottom square is a homotopy pull-back.

$$\begin{array}{ccc} BG_p^\wedge & \longrightarrow & BG_p^\wedge \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ BU^p\pi_p^\wedge & \longrightarrow & B\pi_p^\wedge \end{array}$$

But the left column is also a mod-p homotopy group extension, where the base space is 2-connected. Since $B\operatorname{Aut}(BG_p^\wedge)$ is a 2-stage Postnikov tower with homotopy in dimension 1 and 2, the classifying map for this extension is trivial. Hence the extension is split, namely

$$Y \simeq BU^p\pi_p^\wedge \times BG_p^\wedge.$$

But Y is the fibre of the composition

$$X \longrightarrow B\pi_p^\wedge \longrightarrow B\pi_p^\wedge[2],$$

where $B\pi_p^\wedge[2]$ denotes the second Postnikov section of $B\pi_p^\wedge$. Hence Y and X have the same two connected covers and Corollary 1.2 follows.

Let a finite mod-p homotopy extension of π by G be given, with π a p-perfect group and assume without loss of generality that G is p'-reduced. Then $B\pi_p^\wedge$ is simply-connected and hence the classifying map $B\pi_p^\wedge \longrightarrow B\operatorname{Aut}(BG_p^\wedge)$ is trivial on fundamental groups. It follows that the action of $\pi_1(B\pi_p^\wedge)$ on $BG_p^\wedge$ is trivial and so there exists at least one mod-p homotopy extension of π by G with this action, namely the trivial extension. Hence fibre homotopy equivalence classes of mod-p homotopy extensions of π by G with a trivial action of $\pi_1(B\pi_p^\wedge)$ are in 1-1 correspondence with $H^2(B\pi_p^\wedge, Z(G))$. Now, ordinary group extensions of π by G with trivial outer action of π on G are classified by $H^2(\pi, Z(G)) \cong H^2(B\pi_p^\wedge, Z(G))$, since π is finite and $Z(G)$ is a p-group. But given any such ordinary group extension, p-completion of the fibration resulting from applying the classifying space construction to the extension remains a fibration. Corollary 1.3 follows.

Finally assume that π is p-superperfect. Then the classifying map

$$B\pi_p^\wedge \longrightarrow B\operatorname{Aut}(BG_p^\wedge)$$

is trivial. Corollary 1.4 follows.

4. The Group of Components $\operatorname{Out}(BG_p^\wedge)$

Let G be a finite group. As before we shall assume that G is p'-reduced. In this section we give an approximation for $\operatorname{Out}(BG_p^\wedge)$. We shall essentially reduce the calculation to the computation of two higher limits. In the case where these limits vanish we shall obtain an identification of $\mathrm{Out}(BG_p^\wedge)$ with the group of natural equivalences of the functor I_G defined in the introduction.

Recall that the category $\mathcal{C}_p$ has all finite p-groups as objects and conjugacy classes of monomorphisms $\pi \longrightarrow \pi'$ as morphisms. The functor $I_G : \mathcal{C}_p^{op} \longrightarrow \mathcal{S}ets$ takes a finite p-group π to the set $\mathrm{Inj}(\pi, G)$ of conjugacy classes of monomorphism $\pi \longrightarrow G$ and a morphism $\pi \xrightarrow{\rho} \pi'$ in $\mathcal{C}_p^{op}$ to the induced map $\mathrm{Inj}(\pi, G) \longrightarrow \mathrm{Inj}(\pi', G)$.

Lemma 4.1. *For any finite group G, there are homomorphisms of groups*

$$\delta_G : \mathrm{Out}(G) \longrightarrow \mathrm{Aut}(I_G)$$

and

$$\gamma_G : \mathrm{Out}(G) \longrightarrow \mathrm{Out}(BG_p^\wedge).$$

Proof. For $[\alpha] \in \mathrm{Out}(G)$ and a finite p-group π define

$$\delta_G([\alpha])_\pi : I_G(\pi) \longrightarrow I_G(\pi)$$

to be the map taking $[f] \in I_G(\pi)$ to $[\alpha \circ f]$, where α is any automorphism of G representing its class. That δ_G is a homomorphism is clear from the definition and naturality follows from functoriality of $Inj(\pi, -)$.

The definition of γ_G is as the map induced by a representative for an outer automorphism by applying it to the second variable in the respective mapping space. □

Let G be a finite group and let π be a finite p-group. By Proposition 2.1 there are isomorphisms of sets

$$\pi_0(\mathrm{Map}(B\pi, BG_p^\wedge)) \cong \pi_0(\mathrm{Map}(B\pi, BG)) \cong \mathrm{Rep}(\pi, G)\,.$$

The following lemma is well known.

Lemma 4.2. *Let $f : \pi \longrightarrow G$ be a homomorphism. Then the following are equivalent:*

1. *f is a monomorphism*
2. *$H^*B\pi$ is a finitely generated module over $Bf^*(H^*(BG))$.*

Lemma 4.3. *There is a group homomorphism*

$$\psi_G : \mathrm{Out}(BG_p^\wedge) \longrightarrow \mathrm{Aut}(I_G)$$

Proof. Let $[f] \in \mathrm{Out}(BG_p^\wedge)$ be the homotopy class of some self equivalence

$$f : BG_p^\wedge \longrightarrow BG_p^\wedge\,.$$

We must present a natural equivalence $\psi_G(f)$ of I_G. Thus for each $\pi \in \mathcal{C}_p$ let

$$\psi_G(f)_\pi : I_G(\pi) \longrightarrow I_G(\pi)$$

be the function which takes the conjugacy class of a monomorphism $\varphi : \pi \longrightarrow G$ to the conjugacy class of a homomorphism inducing the composition

$$B\pi \xrightarrow{B\varphi} BG_p^\wedge \xrightarrow{f} BG_p^\wedge$$

up to homotopy. By the Proposition 2.1 such a class exists. Furthermore, by the Lemma 4.2 it follows that $\psi_G(f)_\pi[\varphi]$ is the conjugacy class of a monomorphism. The definition does not depend on the choice of f or φ, but rather only on their homotopy classes and so $\psi_G(f)_\pi$ is well defined.

Naturality of $\psi_G(f)$ follows at once by similar considerations. Also, to see that ψ_G is a group homomorphism observe that multiplication in both domain and range is given by composition. □

Definition 4.4. Let G be a finite group and π a finite p-group. Let $B\pi$ denote any model for the classifying space for π. Define $\mathrm{Inj}(B\pi, BG^\wedge_p)$ to be the subset of the mapping space $\mathrm{Map}(B\pi, BG^\wedge_p)$ consisting of all maps which are homotopic to Bf for some monomorphism $f : \pi \longrightarrow G$.

Let $\mathcal{C}_p^G$ denote the full subcategory of $\mathcal{C}_p$ whose objects are all p-subgroups of G. Consider the functor

$$K_G : \mathcal{C}_p^G \longrightarrow \mathcal{S}ets$$

taking a p-subgroup π of G to the set $\pi_0(\mathrm{Inj}((G/\pi)_{hG}, BG^\wedge_p))$ and a morphism in $\mathcal{C}_p^G$ to the induced map. Then there is a natural transformation of functors

$$\beta : I_G \longrightarrow K_G,$$

which by Proposition 2.1 is a natural equivalence. Here I_G is considered as a functor defined on $\mathcal{C}_p^G$.

Consider the homology decomposition of Theorem 3.5

$$\Phi_G : \underset{\mathcal{O}_p^c(G)}{\mathrm{hocolim}}(\phi_G)_{hG} \longrightarrow BG.$$

Thus Φ_G induces a mod-p homology equivalence and so its p-completion is a homotopy equivalence. Thus one has an isomorphism of monoids

$$\left[(\underset{\overrightarrow{\mathcal{O}_p^c(G)}}{\mathrm{hocolim}}(\phi_G)_{hG})^\wedge_p, (\underset{\overrightarrow{\mathcal{O}_p^c(G)}}{\mathrm{hocolim}}(\phi_G)_{hG})^\wedge_p\right] \cong \left[BG^\wedge_p, BG^\wedge_p\right].$$

The proof of Theorem 1.5 thus amounts to showing that given any natural equivalence $\alpha \in \mathrm{Aut}(I_G)$, the obstruction to existence of a lift of α to a self map $\tilde{\alpha}$ of $BG^\wedge_p$ such that $\psi_G(\tilde{\alpha}) = \alpha$ is an element of $\lim^2_{\mathcal{O}_p^c(G)} \mathcal{Z}$ and if a lift exists then any two differ by an element of $\lim^1_{\mathcal{O}_p^c(G)} \mathcal{Z}$.

Lemma 4.5. *Let G be a finite p'-reduced group. For any p-subgroup π of G, the following are equivalent*

1. *π is a p-centric subgroup of G.*
2. *π is a p-centric subgroup of every p-Sylow subgroup of G that contains π.*

Proof. Assume first that π is p-centric in G. That is, $C_G(\pi) \cong Z(\pi) \times W$, where W is a p'-group. For any subgroup H of G containing π, $Z(\pi) < C_H(\pi) < C_G(\pi)$, hence

$C_H(\pi) \cong Z(\pi) \times W'$, where $W' = C_H(\pi) \cap W$ is a p'-groups and so π is p-centric in H.

Conversely, assume that π is p-centric in every Sylow p subgroup of G containing it. Let R be a Sylow p subgroup of $C_G(\pi)$. We must show that $R = Z(\pi)$. Notice that $\pi \cdot R < G$ is a p-subgroup. Let S be a Sylow p-subgroup of G containing $\pi \cdot R$. Since π is p-centric in S, one has $C_S(\pi) = Z(\pi)$. Then $R < C_S(\pi)$, since it is contained in S and centralizes π. On the other hand $C_S(\pi) = Z(\pi)$ is contained in every p-Sylow of $C_G(\pi)$, in particular, it is contained in R and so $R = Z(\pi)$ as claimed. □

Lemma 4.6. *Let π be a finite p group and assume that there exists a monomorphism $f : \pi \longrightarrow G$, including π in G as a p-centric subgroup. Let $[f]$ denote the conjugacy class of f. Then for any natural equivalence $\varphi \in Aut(I_G)$ any representative of the class $\varphi_\pi[f]$ is a monomorphism $t : \pi \longrightarrow G$ such that $t(\pi)$ is p-centric in G.*

Proof. First note that if $t : \pi \longrightarrow G$ is a monomorphism such that $t(\pi) < G$ is p-centric, then any conjugate of t also includes π in G as a p-centric subgroup. Let $t : \pi \longrightarrow G$ be any representative for $\varphi_\pi[f]$. It suffices to show that $t(\pi) < G$ is p-centric.

Let $S < G$ be a Sylow p-subgroup containing $t(\pi)$. Let $\iota_S : S \longrightarrow G$ and $\bar{t} : \pi \longrightarrow S$ denote the respective inclusions, such that $t = \iota_S \circ \bar{t}$.

Since φ is a natural equivalence, there is a commutative diagram

$$\begin{array}{ccc} Inj(S,G) & \xrightarrow{\varphi_S} & Inj(S,G) \\ \big\downarrow{\scriptstyle \bar{t}^*} & & \big\downarrow{\scriptstyle \bar{t}^*} \\ Inj(\pi,G) & \xrightarrow{\varphi_\pi} & Inj(\pi,G) \end{array}$$

Thus $\bar{t}^*[\iota_S] = [t]$. By commutativity and the fact that φ is an equivalence one has $\bar{t}^*\varphi_S^{-1}[\iota_S] = [f]$. Since S is a Sylow p-subgroup in G, there exists an automorphism α of S, such that $\varphi_S^{-1}[\iota_S] = [\iota_S \circ \alpha]$ and it follows that $\iota_S \circ \alpha \circ \bar{t}$ is conjugate to f in G.

Since f includes π in G as a p-centric subgroup, it follows by Lemma 4.5 that $\alpha \circ \bar{t}$ includes π as a p-centric subgroup of S. Since α is an automorphism, $\bar{t}$ includes π in S as a p-centric subgroup. Since this applies to any Sylow subgroup S containing $t(\pi)$, it follows, again by Lemma 4.5, that $t(\pi)$ is p-centric in G.

□

Lemma 4.7. *For each $\varphi \in \mathrm{Aut}(I_G)$ and each p-centric subgroup $\pi < G$ there is a map*

$$\chi_\pi^\varphi : (G/\pi)_{hG} \longrightarrow BG_p^\wedge$$

such that if $f : G/\pi \longrightarrow G/\pi'$ *is a morphism in* $\mathcal{O}_p^c(G)$ *then the diagram*

$$
(5) \qquad \begin{array}{ccc}
(G/\pi)_{hG} & & \\
\downarrow f_{hG} & \searrow^{\chi^{\varphi}_{\pi}} & \\
 & & BG^{\wedge}_p \\
 & \nearrow_{\chi^{\varphi}_{\pi'}} & \\
(G/\pi')_{hG} & &
\end{array}
$$

commutes up to homotopy.

Proof. By the remarks above, we have a natural equivalence of functors

$$\beta : I_G \longrightarrow K_G,$$

where I_G is restricted to $\mathcal{C}_p^G$. Thus each natural equivalence φ of I_G induces a natural equivalence of K_G, which by abuse of notation, we also denote by φ.

For two p-subgroups $\pi, \pi' < G$, the set of morphisms $\mathrm{Mor}_{\mathcal{O}_p^c(G)}(G/\pi, G/\pi')$ consists of all elements of G conjugating π into π' modulo the right action of π'. On the other hand the group of inner automorphisms of π' operates from the right on the set of all injective homomorphisms from from π to π' and the orbits of this action form the morphism set $\mathrm{Mor}_{\mathcal{C}_p^G}(\pi, \pi')$. In other words this is the set of those faithful representations of π in π' induced by conjugation in G. Hence there is a functor $\alpha : \mathcal{O}_p^c(G) \longrightarrow \mathcal{C}_p^G$, which sends an orbit of a p-centric G/π to the group π and a morphism in $\mathcal{O}_p^c(G)$ to the induced representation. Let $\tilde{K}_G$ denote the composition $K_G \circ \alpha$. Then every natural self equivalence φ of K_G induces a self equivalence of $\tilde{K}_G$, which again we denote by φ.

For each p-centric $\pi < G$ let $[\iota_\pi] \in \tilde{K}_G(G/\pi)$ denote the class of the map induced by inclusion. Let

$$\chi^{\varphi}_{\pi} : (G/\pi)_{hG} \longrightarrow BG^{\wedge}_p$$

be any representative for the homotopy class of $\varphi_\pi[\iota_\pi]$.

Let $f : G/\pi \longrightarrow G/\pi'$ be a morphism in $\mathcal{O}_p^c(G)$. Then, since φ is a natural equivalence we have $\tilde{K}_G(f) \circ \varphi_{\pi'} = \varphi_\pi \circ \tilde{K}_G(f)$. Homotopy commutativity of Diagram (5) is equivalent to the following equation holding:

$$\tilde{K}_G(f)[\chi^{\varphi}_{\pi'}] = [\chi^{\varphi}_{\pi}].$$

But since Diagram (5) commutes up to homotopy if χ^{φ}_{π} and $\chi^{\varphi}_{\pi'}$ are replaced by ι_π and $\iota_{\pi'}$ respectively, we have $\tilde{K}_G(f)[\iota_{\pi'}] = [\iota_\pi]$. Hence

$$\tilde{K}_G(f)[\chi^{\varphi}_{\pi'}] = \tilde{K}_G(f)\varphi_{\pi'}[\iota_{\pi'}] = \varphi_\pi \tilde{K}_G(f)[\iota_{\pi'}] = \varphi[\iota_\pi] = [\chi^{\varphi}_{\pi}],$$

as required. This completes the proof. □

Lemma 4.7 implies that each natural equivalence $\varphi \in \mathrm{Aut}(I_G)$ gives rise to a map

$$E^1_\varphi : \operatorname*{hocolim}_{\mathcal{O}_p^c(G)} {}^{(1)}(\phi_G)_{hG} \longrightarrow BG^{\wedge}_p,$$

where the superscript on the left hand side means the 1-skeleton of the homotopy colimit.

Let G be a finite group and let π be a p-centric subgroup. For a natural equivalence $\varphi \in \mathrm{Aut}(I_G)$ let $t^{\varphi}_{\pi} : \pi \longrightarrow G$ denote any choice of a monomorphism such that

$$B(t^{\varphi}_{\pi})^{\wedge}_p : B\pi \longrightarrow BG^{\wedge}_p$$

is homotopic to χ^{φ}_{π}, under the usual identification of $(G/\pi)_{hG}$ with $B\pi$. By Lemma 4.5, t^{φ}_{π} includes π in G as a p-centric subgroup.

For every $j \geq 1$, define a functor

$$\Pi^{\varphi}_j : \mathcal{O}^c_p(G) \longrightarrow \mathcal{A}b$$

by

$$\Pi^{\varphi}_j(G/\pi) = \pi_i(Map((G/\pi)_{hG}, BG^{\wedge}_p)_{\chi^{\varphi}_{\pi}}).$$

Notice that

$$Map((G/\pi)_{hG}, BG^{\wedge}_p)_{\chi^{\varphi}_{\pi}} \simeq Map((G/\pi)_{hG}, BG^{\wedge}_p)_{B(t^{\varphi}_{\pi})^{\wedge}_p} \simeq BC_G(t^{\varphi}_{\pi}(\pi))^{\wedge}_p \simeq BZ(\pi).$$

Thus Π^{φ}_j is the zero functor for all j except possibly for $j = 1$ and $\Pi^{\varphi}_1(G/\pi) \cong Z(\pi)$.

General obstruction theory for maps out of a homotopy colimit [13] gives that if

$$E^n_{\varphi} : \underset{\mathcal{O}^c_p(G)}{\operatorname{hocolim}_{\to}}\, {}^{(n)}(\phi_G)_{hG} \longrightarrow BG^{\wedge}_p$$

has been constructed, the obstructions to extending it to a map E^{n+1}_{φ} out of the $(n+1)$-st skeleton of the homotopy colimit are in the group $\lim^{n+1}_{\mathcal{O}^c_p(G)} \Pi^{\varphi}_n$ and if any extension exists then any two differ by an element in $\lim^{n}_{\mathcal{O}^c_p(G)} \Pi^{\varphi}_n$. For a self equivalence φ of the functor I_G, let

$$\Theta^i_j(\varphi) = \lim_{\mathcal{O}^c_p(G)}{}^{i}\, \Pi^{\varphi}_j.$$

We proceed by analyzing the obstruction groups. For a finite (p-reduced) group G, let

$$\mathcal{Z} \colon \mathcal{O}^c_p(G)^{op} \longrightarrow \mathcal{A}b$$

denote the functor, which associates to a p-centric subgroup π of G its center $Z(\pi)$.

Definition 4.8. For any finite group G and $i \geq 0$, define

$$J^i(G) = \lim_{\mathcal{O}^c_p(G)^{op}}{}^{i}\, \mathcal{Z}$$

Notice in particular that the discussion in section 3 implies that $J^0(G) = Z(G)$ if G is p'-reduced.

Proposition 4.9. *For every natural equivalence $\varphi \in \mathrm{Aut}(I_G)$, we have $\Theta^i_j(\varphi) = 0$ for $j \geq 2$ and*

$$J^i(G) \cong \Theta^i_1(\varphi)$$

for all $i \geq 0$.

Proof. Let $t_\pi^\varphi : \pi \longrightarrow G$ be the homomorphisms chosen above. Let μ_π^φ denote the composition

$$\begin{array}{ccc} Z(\pi)\times\pi & \xrightarrow{\mu_\pi^\varphi} & G \\ & \searrow^{\mu} \quad \nearrow_{t_\pi^\varphi} & \\ & \pi & \end{array}$$

where μ denotes the multiplication map. Then the maps μ_π^φ induce homotopy equivalences

$$\widetilde{B\mu_\pi^\varphi}\colon BZ(\pi) \xrightarrow{\simeq} \mathrm{Map}(B\pi, BG_p^\wedge)_{(Bt_\pi^\varphi)_p^\wedge}.$$

Furthermore, by Lemma 4.7, if $f\colon G/\pi \longrightarrow G/\pi'$ is a morphism in $\mathcal{O}_p^c(G)$, which is determined by an inner automorphism c of G sub conjugating π to π', then there is a homotopy commutative diagram

$$\begin{array}{ccccc} & BZ(\pi)\times B\pi & \xrightarrow{B\mu} & B\pi & \\ {}^{Bc^{-1}\times 1}\nearrow & & & \Big\downarrow{\scriptstyle Bc} & \searrow^{B(t_\pi^\varphi)_p^\wedge} \\ BZ(\pi')\times B\pi & & & & BG_p^\wedge \\ {}_{1\times Bc}\searrow & & & & \nearrow_{B(t_{\pi'}^\varphi)_p^\wedge} \\ & BZ(\pi')\times B\pi' & \xrightarrow[B\mu]{} & B\pi' & \end{array}$$

Taking respective adjoints one gets a homotopy commutative diagram

$$\begin{array}{ccc} BZ(\pi) & \xrightarrow{\simeq} & \mathrm{Map}(B\pi, BG_p^\wedge)_{(Bt_\pi^\varphi)_p^\wedge} \\ {\scriptstyle Bc^{-1}}\Big\uparrow & & \Big\uparrow{\scriptstyle Bc^*} \\ BZ(\pi') & \xrightarrow[\simeq]{} & \mathrm{Map}(B\pi', BG_p^\wedge)_{(Bt_{\pi'}^\varphi)_p^\wedge} \end{array}$$

which in turn induces a commutative diagram of homotopy groups for every $j \geq 1$

$$\begin{array}{ccc} \pi_j(BZ(\pi)) & \xrightarrow{\cong} & \pi_j(\mathrm{Map}(B\pi, BG_p^\wedge)_{(Bt_\pi^\varphi)_p^\wedge}) = \Pi_j^\varphi(G/\pi) \\ {\scriptstyle (Bc^{-1})_\#}\Big\uparrow & & \Big\uparrow{\scriptstyle (Bc^*)_\#} \\ \pi_j(BZ(\pi')) & \xrightarrow[\cong]{} & \pi_j(\mathrm{Map}(B\pi', BG_p^\wedge)_{(Bt_{\pi'}^\varphi)_p^\wedge}) = \Pi_j^\varphi(G/\pi') \end{array}$$

But $\Pi_j^\varphi(G/\pi) = 0$ for all π in $\mathcal{O}_p^c(G)$ and every $j \geq 2$. Thus for $j \geq 2$ one gets $\Theta_j^i(\varphi) = 0$. On the other hand, for $j = 1$, one obtains a natural equivalence of functors on $\mathcal{O}_p^c(G)$

$$\mathcal{Z} \xrightarrow{\cong} \Pi_1^\varphi$$

which induces the required isomorphism upon taking higher limits. $\square$

Let $\psi_G : \mathrm{Out}(BG^\wedge_p) \longrightarrow \mathrm{Aut}(I_G)$ denote the map constructed in Lemma 4.3. The discussion above now implies the following theorem, which in particular gives Theorem 1.5.

Theorem 4.10. *Let G be a finite group and let $\varphi \in \mathrm{Aut}(I_G)$ be any natural equivalence. Let*

$$E^1_\varphi : \underset{\mathcal{O}^c_p(G)}{\overrightarrow{\mathrm{hocolim}}}\ {}^{(1)}(-)_{hG} \longrightarrow BG^\wedge_p$$

be a map constructed by the procedure described above. Then

1. *The obstruction to lifting E^1_φ to a map*
$$E_\varphi : \underset{\mathcal{O}^c_p(G)}{\mathrm{hocolim}}(\phi_G)_{hG} \longrightarrow BG^\wedge_p$$
such that $\psi_G([(E_\varphi)^\wedge_p]) = \varphi$ is an element in $J^2(G)$.
2. *If some lifting E_φ of φ exists then all homotopy classes of lifts are in 1-1 correspondence with $J^1(G)$.*

In other words the sequence

$$0 \longrightarrow J^1(G) \longrightarrow \mathrm{Out}(BG^\wedge_p) \xrightarrow{\psi_G} \mathrm{Aut}(I_G)$$

is exact and if $J^2(G) = 0$ then ψ_G is an epimorphism.

Remark 4.11. The group of automorphisms of the functor I_G is strongly related to self equivalences of the Frobenius category of G, namely, the category whose objects are all p-subgroups of G and whose morphisms are all homomorphisms induced by inclusions and conjugations. These equivalences are frequently quite easy to describe. Thus one may wonder about the case when the map ψ_G is in fact an isomorphism. By the theorem this is the case when the groups $J^i(G)$ vanish for $i = 1, 2$. Below we compute an example where $J^1(G) \neq 0$. This may suggest that $J^i(G)$ is non-vanishing in general. However, it would be interesting to examine conditions on the group G which insure vanishing of the obstruction groups.

5. Sample Calculations

5.1. **Normal Sylow p-subgroup.** Let G be a finite p'-reduced group with a Sylow p-subgroup π and assume that π is normal in G.

Lemma 5.1. *Let G be a p'-reduced finite group with a Sylow p-subgroup $\pi \lhd G$ then $\mathrm{Out}(BG^\wedge_p) \cong \mathrm{Aut}(I_G)$.*

Proof. Since $\pi \lhd G$, it is the only p-subgroup which is both p-centric and p-stubborn. Thus we may compute higher limits over the category containing π alone as an object and $W = G/\pi$ as morphisms. But in this case all higher limits vanish and so Theorem 1.5 applies to give the result. □

Proposition 5.2. *Let G be a finite p'-reduced group with a normal Sylow p-subgroup π. Then*

$$\mathrm{Out}(BG^\wedge_p) \simeq \mathrm{Out}(G).$$

Proof. Let $[BG, BG^\wedge_p]^*$ denote the classes of maps which correspond to equivalences of $BG^\wedge_p$. Let $W = G/\pi$. Then W operates on EG/π and one has

$$(6)\quad [BG, BG^\wedge_p] = \pi_0(\mathrm{Map}(BG, BG^\wedge_p)) \cong \pi_0(\mathrm{Map}((EG/\pi)_{hW}, BG^\wedge_p)) \cong \pi_0((\mathrm{Map}(EG\pi, BG^\wedge_p))^{hW}).$$

To compute $[BG, BG^\wedge_p]^*$ we only need to consider components of inclusions of π in G. Let $\bar{i}$ denote the collection of all faithful representations $\pi \longrightarrow G$. Then for every $i \in \bar{i}$ $\mathrm{Map}(EG/\pi, BG)_{Bi} \simeq BC_G(i(\pi)) \simeq BZ(\pi)$ is p-complete. Thus

$$\mathrm{Map}(EG/\pi, BG^\wedge_p)_{\bar{i}} \simeq (\mathrm{Map}(EG/\pi, BG)_{\bar{i}})^\wedge_p \simeq \mathrm{Map}(EG/\pi, BG)_{\bar{i}}$$

so

$$(7)\quad \pi_0((\mathrm{Map}(EG/\pi, BG^\wedge_p)_{\bar{i}})^{hW}) \simeq \pi_0((\mathrm{Map}(EG/\pi, BG)_{\bar{i}})^{hW}) \simeq \pi_0(\mathrm{Map}(BG, BG)_{equiv}) = \mathrm{Out}(BG) = \mathrm{Out}(G).$$

This completes the proof. □

5.2. **The Groups $SL_2(q)$ and $PSL_2(q)$ at 2.** Let $SL_2(q)$ denote the special linear group over $\mathbb{F}_q$, where $q = p^n$ is an odd prime power. Then $SL_2(q) < GL_2(q)$, the general linear group over $\mathbb{F}_q$ and the quotients of both by their respective centers give the projective groups $PSL_2(q)$ and $PGL_2(q)$ respectively. Diagrammatically one has the following diagram, where rows are group extensions and columns are central extensions.

$$(8)\quad \begin{array}{ccccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{F}_q^* & \longrightarrow & \mathbb{Z}/\frac{q-1}{2}\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ SL_2(q) & \longrightarrow & GL_2(q) & \xrightarrow{\det} & \mathbb{F}_q^* \\ \downarrow & & \downarrow & & \downarrow \\ PSL_2(q) & \longrightarrow & PGL_2(q) & \longrightarrow & \mathbb{Z}/2 \end{array}$$

Observe that the composition $\mathbb{F}_q^* \longrightarrow GL_2(q) \xrightarrow{\det} \mathbb{F}_q^*$ is the squaring map.

The outer automorphism group of $PSL_2(q)$ is given by

$$\mathrm{Out}(PSL_2(q)) \cong \mathbb{Z}/2 \times \mathbb{Z}/n$$

where $\mathbb{Z}/2$ is generated by α given as the outer action defined by the extension in the bottom row of the above diagram, and $\mathbb{Z}/n \cong \mathrm{Gal}(\mathbb{F}_q|\mathbb{F}_p)$ if $q = p^n$, generated by the Frobenius automorphism ϕ, acts in the obvious way.

The Sylow 2-subgroup of $PSL_2(q)$ is a dihedral 2-group

$$D_{2^s} = \langle x, y \,|\, x^{2^{s-1}} = y^2 = (xy)^2 = 1 \rangle$$

of order 2^s depending on q. For $s = 2$, D_4 is elementary abelian of rank two, hence $\mathrm{Aut}(D_4) = \mathrm{Out}(D_4) \cong \Sigma_3$. The automorphism group of D_{2^s} is also easily described as the semidirect product

$$\mathrm{Aut}(D_{2^s}) = \mathbb{Z}/2^{s-1} \rtimes (\mathbb{Z}/2^{s-1})^*,$$

where an element $l \in \mathbb{Z}/2^{s-1}$ corresponds to the automorphism g_l with $g_l(x) = x$ and $g_l(y) = x^l y$, and an element $a \in (\mathbb{Z}/2^{s-1})^*$ corresponds to the automorphism f_a such that $f_a(x) = x^a$ and $f_a(y) = y$. One verifies that g_l with l even and f_{-1} are the inner automorphisms and since $(\mathbb{Z}/4)^* \cong (\mathbb{Z}/2)$ and $(\mathbb{Z}/2^{s-1})^* \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{s-3}$, $s \geq 4$, generated by -1 and 3 modulo 2^{s-1} one obtains

$$\begin{aligned}
&\mathrm{Out}(D_4) = \Sigma_3 \\
&\mathrm{Out}(D_8) \cong \mathbb{Z}/2 \\
&\mathrm{Out}(D_{2^s}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{s-3}, \quad s \geq 4
\end{aligned}$$

generated by the classes $\bar{g}_1$ (order 2) and $\bar{f}_3$ (order 2^{s-3}) of g_1 and f_3 respectively.

Proposition 5.3. *Assume that $q \equiv 1 \pmod 8$, $q = p^n$, p an odd prime number, and let s be the largest integer such that $2^s \mid q-1$. There is a homomorphism defined by restriction*

$$\mathrm{Out}(PSL_2(q)) \longrightarrow \mathrm{Out}(D_{2^s})$$

that sends α to $\bar{g}_1$ and ϕ to $\bar{f}_p$.

In particular, if $q = 3^{2^{s-2}}$, $s \geq 3$, then there is an extension

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Out}(PSL_2(q)) \longrightarrow \mathrm{Out}(D_{2^s}) \longrightarrow 1$$

where $\mathbb{Z}/2$ in the kernel is represented by $\phi^{2^{s-3}}$.

Proof. Let ϵ be a 2^s root of unity in $\mathbb{F}_q$. The 2-Sylow subgroup of $PSL_2(q)$ is D_{2^s}, generated by the classes of the matrices $X = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$:

$$D_{2^s} \cong \langle \bar{X}, \bar{Y} \mid \bar{X}^{2^{s-1}} = \bar{Y}^2 = (\bar{X}\bar{Y})^2 = 1 \rangle \subset PSL_2(\mathbb{F}_q)\,.$$

Since ϵ is not a square in $\mathbb{F}_q$, the class matrix $A = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$ provides a set theoretic section of the bottom extension in diagram (8), and therefore the outer automorphism α of $PSL_2(\mathbb{F}_q)$ is described as conjugation by $\bar{A}$ in $PGL_2(\mathbb{F}_q)$. One can now check that this conjugation leaves the given 2-Sylow subgroup stable and induces the outer automorphism $\bar{g}_1$ of D_{2^s}. On the other hand the Frobenius ϕ is defined as p-th power on $\mathbb{F}_q$ and then it induces $\bar{f}_p$ on D_{2^s}.

In case $q = 3^{2^{s-2}}$, $s \geq 3$, $\bar{f}_3$ is a generator of $\mathbb{Z}/2^{s-3} \subset \mathrm{Out}(D_{2^s})$, so the restriction is an epimorphism with kernel clearly generated by $\phi^{2^{s-3}}$ that has order 2. □

Lemma 5.4. *There are isomorphism*

$$\mathrm{Aut}(SL_2(q)) \xrightarrow{\cong} \mathrm{Aut}(PSL_2(q))$$

and

$$\mathrm{Out}(SL_2(q)) \xrightarrow{\cong} \mathrm{Out}(PSL_2(q))\,.$$

Proof. Since $PSL_2(q)$ is the quotient of $SL_2(q)$ by its center, any automorphism of $SL_2(q)$ induces an automorphism of $PSL_2(q)$. Conversely, any automorphism of $PSL_2(q)$ preserves the extension class for $SL_2(q)$ and thus induces an automorphism of $SL_2(q)$. This proves the first statement. For the second statement, notice that

$$\mathrm{Inn}(SL_2(q)) \cong SL_2(q)/Z(SL_2(q)) \cong PSL_2(q) \cong \mathrm{Inn}(PSL_2(q)). \quad \square$$

Next, compute the automorphisms of the generalized quaternion 2-groups,

$$Q_{2^{s+1}} = \langle x, y \,|\, x^{2^{s-1}} = y^2, x^{2^s} = 1, yxy^{-1} = x^{-1} \rangle \,,$$

which appear as the Sylow 2-subgroups of $SL_2(q)$, q odd. Thus for $Q_{2^{s+1}}$ one has the automorphisms g_l defined by $g_l(x) = x$ and $g_l(y) = x^l y$ for $l \in Z/2^s$ and f_a such that $f_a(x) = x^a$ and $f_a(y) = y$ for $a \in (\mathbb{Z}/2^s)^*$. This gives $\mathrm{Aut}(Q_{2^{s+1}}) \cong \mathbb{Z}/2^s \rtimes (\mathbb{Z}/2^s)^*$ if $s \geq 3$.

The center of $Q_{2^{s+1}}$ is cyclic of order 2 generated by $x^{2^{s-1}} = y^2$ and the quotient of $Q_{2^{s+1}}$ by its center is isomorphic to D_{2^s}. There is an induced homomorphism $\mathrm{Aut}(Q_{2^{s+1}}) \longrightarrow \mathrm{Aut}(D_{2^s})$, which is an epimorphism with kernel $\mathbb{Z}/2 \times \mathbb{Z}/2$ generated by $f_{2^{s-1}+1}$:

$$f_{2^{s-1}+1}(x) = x^{2^{s-1}+1}, \qquad f_{2^{s-1}+1}(y) = y$$

and $g_{2^{s-1}}$:

$$g_{2^{s-1}}(x) = x, \qquad g_{2^{s-1}}(y) = x^{2^{s-1}} y\,.$$

Notice that $g_{2^{s-1}}$ is inner while $f_{2^{s-1}+1}$ it not inner unless $s = 2$, and therefore

$$\mathrm{Out}(Q_8) \cong \mathrm{Out}(D_4) \cong \Sigma_3$$

and for $s \geq 3$, we obtain an extension

$$1 \longrightarrow \langle \bar{f}_{2^{s-1}+1} \rangle \mathbb{Z}/2 \longrightarrow \mathrm{Out}(Q_{2^{s+1}}) \longrightarrow \mathrm{Out}(D_{2^s}) \longrightarrow 1.$$

The following Proposition describes the relationships between the outer automorphism groups computed above for $q = 3^{2^s}$, $s \geq 1$.

Proposition 5.5. *Assume that $q = 3^{2^{s-2}}$, $s \geq 3$. There is an isomorphism*

$$\pi : \mathrm{Out}(SL_2(\mathbb{F}_q)) \longrightarrow \mathrm{Out}(Q_{2^{s+1}})$$

and a commutative diagram

$$\begin{array}{ccc} & & \mathbb{Z}/2 \\ & & \downarrow \\ \mathrm{Out}(SL_2(q)) & \xrightarrow{\cong} & \mathrm{Out}(Q_{2^{s+1}}) \\ \cong \downarrow & & \downarrow \\ \mathbb{Z}/2 \rightarrowtail \mathrm{Out}(PSL_2(q)) & \twoheadrightarrow & \mathrm{Out}(D_{2^s}) \end{array}$$

where the bottom row and right column are exact. $\square$

Homology decomposition of $SL_2(q)$ and $PSL_2(q)$. A homology decomposition of $PSL_2(q)$ is described in detail in [1]. We refer to [4] for the general theory. Restrict attention to the case where $q = 3^{2^{s-2}}$, $s \geq 3$. Choose a Sylow subgroup $S \cong D_{2^s}$. Then there are subgroups $Z < S$ of order 2 given by the center of S and two non-conjugate elementary abelian 2-subgroups $V, W < S$ of rank 2, which give an ample collection $\mathcal{E}_2 = \{Z, V, W\}$ of elementary abelian 2-subgroups of $PSL_2(\mathbb{F}_q)$. The associated conjugacy category $\mathbf{A}_{\mathcal{E}_2}$ can be described by means of the following diagram

$$\Sigma_3 \circlearrowleft W \underset{\Sigma_3/\Sigma_2}{\overleftarrow{\overleftarrow{\longleftarrow}}} Z \underset{\Sigma_3/\Sigma_2}{\overrightarrow{\overrightarrow{\longrightarrow}}} V \circlearrowleft \Sigma_3 .$$

The centralizer diagram $\alpha_{\mathcal{E}_2} : \mathbf{A}_{\mathcal{E}_2}^{\mathrm{op}} \longrightarrow \mathbf{Spaces}$ is up to homotopy and 2-completion described by

$$\Sigma_3 \circlearrowleft BW \overrightarrow{\overrightarrow{\longrightarrow}} BD_{2^s} \overleftarrow{\overleftarrow{\longleftarrow}} BV \circlearrowleft \Sigma_3.$$

and the natural map

$$a_{\mathcal{E}_2} : \operatorname{hocolim} \alpha_{\mathcal{E}_2} \longrightarrow BPSL_2(\mathbb{F}_q) \tag{9}$$

induces a mod-2 homology isomorphism.

For $SL_2(q)$ one obtains a homology decomposition by pulling back $\alpha_{\mathcal{E}_2}$ along the projection $SL_2(q) \longrightarrow PSL_2(q)$ ($q = 3^{2^{s-2}}$, $s \geq 3$ as above). We obtain a new strictly commutative diagram $\beta_{\mathcal{E}_2}$ of the form

$$\Sigma_3 \circlearrowleft BQ_8 \overrightarrow{\overrightarrow{\longrightarrow}} BQ_{2^{s+1}} \overleftarrow{\overleftarrow{\longleftarrow}} BQ_8 \circlearrowleft \Sigma_3.$$

One can view $\beta_{\mathcal{E}_2}$ as a functor from $\mathbf{A}_{\mathcal{E}_2}$ to **Spaces** and there is a map $b_{\mathcal{E}_2}$ out of the diagram to $BSL_2(q)$ given by the pull-back process described above.

Lemma 5.6. *The map*

$$b_{\mathcal{E}_2} : \operatorname*{hocolim}_{\mathbf{A}_{\mathcal{E}_2}} \beta_{\mathcal{E}_2} \longrightarrow BSL_2(q)$$

induces a mod-2 equivalence.

Proof. This follows from the homology decomposition in equation (9). □

The group $\operatorname{Out}(BG_2^\wedge)$ **for** $G = SL_2(q)$ **and** $PSL_2(q)$. The left column of diagram (8) induces a principal fibration

$$B\mathbb{Z}/2 \xrightarrow{Bi} BSL_2(q)_2^\wedge \xrightarrow{Bp} BPSL_2(q)_2^\wedge. \tag{10}$$

Lemma 5.7. *There is an isomorphism*

$$\theta\colon \operatorname{Out}(BSL_2(q)_2^\wedge) \longrightarrow \operatorname{Out}(BPSL_2(q)_2^\wedge) .$$

Proof. There is a homotopy equivalence $\operatorname{Map}(B\mathbb{Z}/2, BPSL_2(q)_2^\wedge)_c \simeq BPSL_2(q)_2^\wedge$, where c denotes the constant map (cf. [1]). Then the Zabrodsky lemma [14, 10] applies to the principal fibration (10) to give a homotopy equivalence

$$\operatorname{Map}(BSL_2(q)_2^\wedge, BPSL_2(q)_2^\wedge)_{\{f|f\circ Bi\simeq *\}} \simeq \operatorname{Map}(BPSL_2(q)_2^\wedge, BPSL_2(q)_2^\wedge) .$$

Now, any homotopy equivalence $g \in \operatorname{Out}(BSL_2(q)_2^\wedge)$ induces the identity in mod-2 cohomology. Hence $g \circ Bi \simeq Bi$ and so $Bp \circ g \circ Bi \simeq *$. It follows that there exists $\bar{g} \in \operatorname{Out}(BPSL_2(q)_2^\wedge)$ satisfying $Bp \circ g \simeq \bar{g} \circ Bp$. Define $\theta(g) = \bar{g}$.

Finally, we observe that $[BSL_2(q)^\wedge_2), B\mathbb{Z}/2] \cong [BSL_2(q)^\wedge_2), B^2\mathbb{Z}/2] \cong 0$, hence θ turns out to be an isomorphism. □

Lemma 5.8. *Let G be a finite group and let S be its Sylow p-subgroup. Assume that S is self normalizing in G. Then there is a homomorphism $res : \mathrm{Out}(G) \xrightarrow{res} \mathrm{Out}(S)$, which factors through $\mathrm{Out}(BG^\wedge_p)$.*

Proof. Let $\bar{\phi}$ be an outer automorphism of G represented by some automorphism ϕ. Then ϕ carries S into another Sylow p-subgroup S'. But there is an inner automorphism c_g of G which carries S' back to S. Define $res(\bar{\phi})$ to be the class of the composition $c_g \circ \phi$ in $\mathrm{Out}(S)$. If $g' \in G$ is another element conjugating S' to S then $g'g^{-1} \in N_G(S) = S$. Hence $c_g \circ \phi$ and $c_{g'} \circ \phi$ differ only by an inner automorphism of S. Also if ϕ' is another representative for $\bar{\phi}$ then ϕ and ϕ' differ by an inner automorphism of G and the procedure carries the difference again into an inner automorphism of S. Thus the restriction map is well defined and obviously a group homomorphism.

Similarly, if $\psi \in \mathrm{Out}(BG^\wedge_p)$ then the composition

$$BS \xrightarrow{B\iota} BG^\wedge_p \xrightarrow{\phi} BG^\wedge_p,$$

where ι denotes the inclusion, is homotopic to a map induced by a homomorphism (see Proposition 2.1). Thus the same argument implies that there is an automorphism α of S such that $B\iota\alpha \simeq \psi B\iota$.

Finally if $\psi \in \mathrm{Out}(BG^\wedge_p)$ is induced by an automorphism ϕ of G, then the procedure described above gives that α is conjugate to $res(\phi)$ and the lemma follows. □

Remark 5.9. Lemma 5.8 applies to $G = SL_2(q)$ and $G = PSL_2(q)$, $q \equiv \pm 1 \pmod 8$. In fact, the centralizer in $PSL_2(q)$ of the center $Z \cong \mathbb{Z}/2$ of its Sylow 2-subgroup is isomorphic to D_{q-1}, the dihedral group of order $2q-2$, generated by the classes of the matrices $\left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, where ζ is a generator of $\mathbb{F}_q^*$ (Proposition 4.2 of [1]). Then $N_{PSL_2(q)}(D_{2^s}) \subset D_{q-1}$ and a quick calculation shows that $N_{PSL_2(q)}(D_{2^s}) = D_{2^s}$. From this equality it follows that $N_{SL_2(q)}(Q_{2^{s+1}}) = Q_{2^{s+1}}$ as well.

In particular, for $q = 3^{2^{s-2}}$, $s \geq 3$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Out}(SL_2(q)) & \xrightarrow{\cong} & \mathrm{Out}(Q_{2^{s+1}}) \\ \downarrow & \nearrow_{res} & \\ \mathrm{Out}(BSL_2(q)^\wedge_2) & & \end{array} \tag{11}$$

Proposition 5.10. *For $q = 3^{2^{s-2}}$, $s \geq 3$, the natural map*

$$B\colon \mathrm{Out}(SL_2(q)) \longrightarrow \mathrm{Out}(BSL_2(q)^\wedge_2)$$

is an isomorphism.

Proof. It suffices to show that

$$res\colon \mathrm{Out}(BSL_2(q)^\wedge_2) \longrightarrow \mathrm{Out}(BQ_{2^{s+1}})$$

is an isomorphism. It is clear from Diagram (11) that res is an epimorphism. We use the homology decomposition for $BSL_2(q)^\wedge_2$ of Lemma 5.6 in order to prove injectivity.

The class of a self equivalence f of $BSL_2(q)^\wedge_2$ is in the kernel the restriction map if and only if the diagram

$$\begin{array}{ccc} BQ_{2^{s+1}} & = & BQ_{2^{s+1}} \\ \downarrow{\scriptstyle B\iota} & & \downarrow{\scriptstyle B\iota} \\ BSL_2(q)^\wedge_2 & \xrightarrow{f} & BSL_2(q)^\wedge_2 \end{array}$$

is homotopy commutative, where ι denotes the inclusion map. Then, for every object in the diagram $\beta_{\mathcal{E}_2}$ we have

$$f|BQ_{2^{s+1}} \simeq Id|BQ_{2^{s+1}}$$
$$f|BQ_8 \simeq Id|BQ_8 \,.$$

The obstructions for f to be homotopic to the identity lie in the groups

$$\varprojlim_{\mathbf{A}_{\mathcal{E}_2}}{}^i \pi_j\big(\operatorname{Map}(\beta_{\mathcal{E}_2}, BSL_2(q)^\wedge_2)_{B\iota.}\big)\,, \qquad i \geq 1$$

and according to Proposition 5.12 these groups are trivial. Hence $f \simeq Id$, that is res is injective. □

Proposition 5.11. *For $q = 3^{2^{s-2}}$, $s \geq 3$,*

1. $\operatorname{Map}(BQ_8, BSL_2(q)^\wedge_2)_{B\iota_\epsilon} \simeq B\mathbb{Z}/2$, *where $\epsilon = 1, 2$ denotes the two different inclusions.*
2. $\operatorname{Map}(BQ_{2^{s+1}}, BSL_2(q)^\wedge_2)_{B\iota} \simeq B\mathbb{Z}/2$.

Proof. By Proposition 2.1 these spaces are equivalent to the 2-completion of the respective centralizer. The claim now follows by direct calculation using Remark 5.9 □

Proposition 5.12. *For $q = 3^{2^{s-2}}$, $s \geq 3$,*

$$\varprojlim_{\mathbf{A}_{\mathcal{E}_2}}{}^i \pi_j \operatorname{Map}(\beta_{\mathcal{E}_2}, BSL_2(q)^\wedge_2)_{B\iota.} \cong \begin{cases} \mathbb{Z}/2 & i = 0, j = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows at once from Proposition 5.11 and [1, §10]. □

The discussion above implies Theorem 1.6. Specifically, part 1 follows by combining Proposition 5.5, Lemma 5.7 and Proposition 5.10. Part 2 follows from the calculation of Proposition 5.12 and part 3 follows from [1, Lemma 6.4].

6. Mod-p Homotopy Group Extensions

In this final section we return to the motivation for our study and discuss mod-p homotopy group extensions. The following table compares the classification of group extensions to the classification of homotopy group extensions. The last is of course just a special case of the general classification problem for fibrations.

	Group Extensions π by G	Mod-p Homotopy Group Extensions π by G
1	A homomorphism $\pi \xrightarrow{\alpha} \mathrm{Out}(G)$	A map $B\pi_p^\wedge \xrightarrow{\alpha} B\,\mathrm{Out}(BG_p^\wedge)$
2	Obstructions to the existence of any extension of π by G in $H^3(\pi; Z(G))$, where $Z(G)$ becomes a π-module via α	Obstructions to the existence of any mod-p homotopy extension of π by G in $H^3(B\pi_p^\wedge; Z(G/O_{p'}G))$, where $Z(G/O_{p'}G)$ becomes a $\pi_1(B\pi_p^\wedge)$-module via α
3	If an extension exists then all extensions are classified by $H^2(\pi; Z(G))$	If an extension exists then all the others are classified by $H^2(B\pi_p^\wedge; Z(G/O_{p'}G))$

Corollary 6.1. *Let G be a finite group such that the natural map*

$$\gamma_G : \mathrm{Out}(G/O_{p'}G) \longrightarrow \mathrm{Out}(BG_p^\wedge)$$

is an isomorphism. Let π be a finite p-group. Then there is a 1-1 correspondence between fibre homotopy classes of mod-p homotopy extensions of π by G and equivalence classes of ordinary group extensions of π by $G/O_{p'}G$.

Proof. Under our hypotheses, there is a homotopy equivalence

$$B\,\mathrm{Aut}(B(G/O_{p'}G)) \xrightarrow{\simeq} B\,\mathrm{Aut}(BG_p^\wedge).$$

The result follows. □

Recall that a p-group P is called a Swan group if for any finite group G containing P as a Sylow p-subgroup, the inclusion of the normalizer $N_G(P) \longrightarrow G$ induces a mod-p homology isomorphism.

Corollary 6.2. *Let G be a finite group with a Sylow p-subgroup P and assume either*

1. *P is normal in G or*
2. *the inclusion $N_G(P) \longrightarrow G$ induces a mod-p homology equivalence or*
3. *P is a Swan group.*

In either case let H denote the mod-p' reduction of $N_G(P)$ and let π be any finite p-group. Then fibre homotopy classes of mod-p homotopy extensions of π by G are in 1-1 correspondence with ordinary group extensions of π by H.

Proof. Under either one of 1, 2 or 3, we may replace $BG_p^\wedge$ by $BH_p^\wedge$ up to homotopy. The result follows at once from Corollary 6.1. □

Our calculation for the special linear groups implies a similar result. Specifically we have

Proposition 6.3. *Assume $q = 3^{2^{s-2}}$, $s \geq 3$, then there is a homotopy equivalence*

$$B\,\mathrm{Aut}(BSL_2(q)) \xrightarrow{\simeq} B\,\mathrm{Aut}(BSL_2(q)_2^\wedge)\,.$$

Proof. Since p-completion is a continuous functor, there is a diagram of fibrations

$$\begin{array}{ccccc} B\,\mathrm{SAut}(BSL_2(q)) & \longrightarrow & B\,\mathrm{Aut}(BSL_2(q)) & \longrightarrow & B\,\mathrm{Out}(SL_2(q)) \\ \downarrow & & \downarrow & & \downarrow \\ B\,\mathrm{SAut}(BSL_2(q)^\wedge_2) & \longrightarrow & B\,\mathrm{Aut}(BSL_2(q)^\wedge_2) & \longrightarrow & B\,\mathrm{Out}(BSL_2(\mathbb{F}_q)^\wedge_2) \end{array}$$

where the left vertical arrow is a homotopy equivalence by Theorem 1.1 and the right vertical arrow is a homotopy equivalence by Proposition 5.10, hence the result follows. □

Remark 6.4. Notice that we have actually computed the spaces $B\,\mathrm{Aut}(BSL_2(q)^\wedge_2)$ and $B\,\mathrm{Aut}(BPSL_2(q)^\wedge_2)$ for every odd prime power $q = p^k$. In fact, according to [1] the homotopy types of $BSL_2(q)^\wedge_2$ and $BPSL_2(q)^\wedge_2$ depends only on the order of the Sylow 2-subgroup rather than on the concrete odd prime power q. Observe that any order of a Sylow 2-subgroup in $SL_2(q)$ can be obtained by letting $q = 3^{2^s}$ for some $s \geq 1$ if $q \equiv \pm 1 \mod 8$ and that if $q \equiv \pm 3 \mod 8$ then $BSL_2(q)^\wedge_2 \simeq BSL_2(3)^\wedge_2$, in which case the Sylow 2-subgroup Q_8, and respectively $BPSL_2(3)^\wedge_2 \simeq (BA_4)^\wedge_2$ with Sylow 2-subgroup elementary abelian of rank 2. In these two cases the Sylow 2-subgroups are Swan groups.

Corollary 6.5. *Let π be a finite 2-group and let q be any odd prime power. Then there is a 1-1 correspondence between fibre homotopy classes mod-2 homotopy extensions of π by $SL_2(q)$ and equivalence classes of ordinary group extensions of π by $SL_2(t)$ where $t = 3^{2^s}$ for some s such that $SL_2(q)$ and $SL_2(t)$ have Sylow 2-subgroups of the same order if $q \equiv \pm 1 \mod 8$ and $t = 3$ otherwise. The corresponding result applies to homotopy extensions of a finite 2-group by $PSL_2(q)$.*

Our interest in this project was motivated by a rather simple minded question, namely homotopy uniqueness of the space BQ_{2^n} [2]. Our results here enable us to give an easy solution of this problem.

Corollary 6.6. *Let Q_{2^r} be a generalized quaternion group of order 2^r. Let X be a 2-complete space with $H^*(X) \cong H^*(BQ_{2^r})$ as an algebra over the Steenrod algebra and assume further that there is an isomorphism between the Bockstein spectral sequences of $H^*(X)$ and $H^*(BQ_{2^r})$ in the sense of* [2]*.Then there is a homotopy equivalence $X \simeq BQ_{2^r}$*

Proof. It is shown in [2] that under our hypotheses if X is not equivalent to BQ_{2^r} then $\pi_1(X) \cong Q_{2^s}$ for some $s < r-2$ and its universal cover $\tilde{X}$ has the cohomology of $BSL_2(q)$ for some appropriate q, again as an algebra over the Steenrod algebra and with the same Bockstein spectral sequence. By [1] it follows that $\tilde{X} \simeq BSL_2(q)^\wedge_2$. The previous corollary thus implies that X is the 2-completion of the classifying space of an extension of Q_{2^s} by $SL_2(t)$ for an appropriate t. But one now easily checks that no such extension has the cohomology assumed for X. The result follows. □

To conclude this paper we comment that homotopy group extensions were defined to be fibrations where both base and fibre are p-completed classifying spaces. In some contexts, in particular if whether or not the total space is p-complete has no significance, it makes sense to consider fibrations with base $B\pi$ (rather than $B\pi^\wedge_p$), and fibre $BG^\wedge_p$.

Corollary 6.7. *Let G be a finite group and assume that the natural map*

$$\gamma_G : \mathrm{Out}(G/O_{p'}G) \longrightarrow \mathrm{Out}(BG^\wedge_p)$$

is an isomorphism. Then for any discrete group π there is a 1-1 correspondence between fibre homotopy equivalence classes of fibrations with base $B\pi$ and fibre $BG^\wedge_p$ and equivalence classes of ordinary group extensions of π by $G/O_{p'}G$.

Finally, notice that in the case of the foregoing corollary, the correspondence is given via the Bousfield-Kan fibrewise p-completion functor.

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