Three themes in the work of Charles Ehresmann: Local-to-global; Groupoids; Higher dimensions. *

Ronald Brown†
University of Wales, Bangor.

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Abstract

This paper illustrates the themes of the title in terms of: van Kampen type theorems for the fundamental groupoid; holonomy and monodromy groupoids; and higher homotopy groupoids. Interaction with work of the writer is explored. [1]

1 Introduction

It is a pleasure to honour Charles Ehresmann by giving a personal account of some of the major themes in his work which interact with mine. I hope it will be useful to suggest how these themes are related, how the pursuit of them gave a distinctive character to his aims and his work, and how they influenced my own work, through his writings and through other people.

Ehresmmann's work is so extensive that a full review would be a great task, which to a considerable extent is covered by Andrée Ehresmann in her commentaries in the collected works [24]. His wide vision is shown by his description of his overriding aim as: 'To find the structure of everything'. 'To find structure' is related to the Bourbaki experience and aim, in which he was a partner. A description of a new structure is in some sense a development of part of a new language: the aim of doing this contrasts with that of many, who feel that the development of mathematics is mainly guided by the solution of famous problems.

The notion of structure is also related to the notion of *analogy*. It one of the triumphs of category theory in the 20th century to make progress in unifying mathematics through the finding of analogies between the behaviours of structures across different areas of mathematics.

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This theme is elaborated in the article [15]. That article argues that many analogies in mathematics, and in many other areas, are not between objects themselves but between the relations between objects. Here we mention only the notion of *pushout*, which we use later in discussing the van Kampen Theorem. A pushout has the same definition in different categories even though the construction of pushouts in these categories may be wildly different. Thus a concentration on the constructions rather than on the universal properties may lead to a failure to see the analogies.

Ehresmann developed new concepts and new language which have been very influential in mathematics: I mention only fibre bundles, foliations, holonomy groupoid, germs, jets, Lie groupoids. There are other concepts whose time perhaps is just coming or has yet to come: included here might be ordered groupoids, multiple categories.

In this direction of developing language, we can usefully quote G.-C. Rota [37, p.48]:

> "What can you prove with exterior algebra that you cannot prove without it?" Whenever you hear this question raised about some new piece of mathematics, be assured that you are likely to be in the presence of something important. In my time, I have heard it repeated for random variables, Laurent Schwartz' theory of distributions, ideles and Grothendieck's schemes, to mention only a few. A proper retort might be: "You are right. There is nothing in yesterday's mathematics that could not also be proved without it. Exterior algebra is not meant to prove old facts, it is meant to disclose a new world. Disclosing new worlds is as worthwhile a mathematical enterprise as proving old conjectures.

2 Local-to-global questions

Ehresmann developed many new themes in category theory. One example is structured categories, with principal examples those of differentiable categories and groupoids, and of multiple categories. His work on these is quite disparate from the general development of category theory in the 20th century, and it is interesting to search for reasons for this. One must be the fact that he used his own language and notation. Another is surely that his early training and motivation came from analysis, rather than from algebra, in contrast to the origins of category theory in the work of Eilenberg, Mac Lane and of course Steenrod, centred on homology and algebraic topology. Part of the developing language of category theory became essential in those areas, but other parts, such as that of algebraic theories, groupoids, multiple categories, were not used till fairly recently.

It seems likely that Ehresmann's experience in analysis led him to the major theme of *local-to-global* questions. I first learned of this term from Dick Swan in Oxford in 1957-58, when as a research student I was writing up notes of his Lectures on the Theory of Sheaves [40]. Dick explained to me that two important methods for local-to-global problems were sheaves and spectral sequences—he was thinking of Poincaré duality, which is discussed in the lecture notes, and the more complicated Dolbeaut's theorem for complex manifolds. But in truth such problems are central in mathematics, science and technology. They are fundamental to differential equations and dynamical systems, for example. Even deducing consequences of a set of rules is a local-to-global problem: the rules are applied locally, but we are interested in the global consequences.

My own work on local-to-global problems arose from writing an account of the Seifert-van Kampen theorem on the fundamental group. This theorem can be given as follows, as first shown by R.H. Crowell:

Theorem 2.1 [19] *Let the space X be the union of open sets U, V with intersection W, and suppose W, U, V are path connected. Let $x_0 \in W$. Then the diagram of fundamental group morphisms induced by inclusions*

$$\begin{tikzcd} \pi_1(W, x_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd} \tag{1}$$

is a pushout of groups.

Here the 'local parts' are of course U, V put together with intersection W and the result describes completely, under the open set and connectivity conditions, the nonabelian fundamental group of the global space X. This theorem is usually seen as a necessary part of basic algebraic topology, but one without higher dimensional analogues.

In writing the first 1968 edition of the book [7], I noted that to compute the fundamental group of the circle one had to develop something of covering space theory. Although that is an excellent subject in its own right, I became irritated by this detour. After some time, I found work of Higgins on groupoids, [25], which defined free products with amalgamation of groupoids, and this led to a more general formulation of theorem 2.1 as follows:

Theorem 2.2 [3] *Let the space X be the union of open sets U, V with intersection W, and suppose W, U, V are path connected. Let X_0 be a subset of W meeting each path component of W. Then the diagram of fundamental group morphisms induced by inclusions*

$$\begin{tikzcd} \pi_1(W, X_0) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(U, X_0) \arrow[d] \\ \pi_1(V, X_0) \arrow[r] & \pi_1(X, X_0) \end{tikzcd} \tag{2}$$

is a pushout of groupoids.

Here $\pi_1(X, X_0)$ is the fundamental groupoid of X on a set X_0 of base points: so it consists of homotopy classes rel end points of paths in X joining points of $X_0 \cap X$.

In the case X is the circle S^1, one chooses U, V to be slightly extended semicircles including $X_0 = \{+1, -1\}$. The point is that in this case $W = U \cap V$ is not path connected and so it is not clear where to choose a single base point. The day is saved by hedging one's bets, and using two base points. The proof of theorem 2.1 uses the same tricks as to prove theorem 2.2, but in a broader context. In order to compute fundamental groups from this theorem, one can set up some general combinatorial groupoid theory, see [7, 26]. A key feature of this theory is the groupoid $\mathcal{I}$, the indiscrete groupoid on two objects $0, 1$, which acts as a unit interval object in the category of groupoids. It also plays a rôle analogous to that of the infinite cyclic group C in the category of groups. One then compares the pushout diagrams, the first in spaces, the second in groupoids:

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ {[0,1]} \arrow[r] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \mathcal{I} \arrow[r, "u"'] & \mathsf{C} \end{tikzcd} \tag{3}$$

to see how this version of the van Kampen Theorem gives an analogy between the geometry, and the algebra provided by the notion of groupoid. This kind of result is seen as 'change of base' in [5].

The fundamental group is a kind of anomaly in algebraic topology because of its nonabelian nature. Topologists in the early part of the 20th century were aware that:

- the non commutativity of the fundamental group was useful in applications;
- for path connected X there was an isomorphism

$$H_1(X) \cong \pi_1(X, x)^{\mathrm{ab}};$$

- the abelian homology groups existed in all dimensions.

Consequently there was a desire to generalise the nonabelian fundamental group to all dimensions.

In 1932 Čech submitted a paper on higher homotopy groups $\pi_n(X, x)$ to the ICM at Zurich, but it was quickly proved that these groups were abelian for $n \geqslant 2$, and on these grounds Čech was persuaded to withdraw his paper, so that only a small paragraph appeared in the Proceedings [17]. We now see the reason for the commutativity as the result (Eckmann-Hilton) that a group internal to the category of groups is just an abelian group. Thus the vision of a non commutative higher dimensional version of the fundamental group has since 1932 been generally considered to be a mirage.

Theorem 2.2 is also anomalous: it is a colimit type theorem, and so yields *complete information* on the fundamental groups which are contained in it, even in the non connected case, whereas the usual method in algebraic topology is to relate different dimensions by exact sequences or even spectral sequences, which usually yield information only up to extension. Thus exact sequences by themselves cannot show that a group is given as an HNN-extension: however such a description may be obtained from a pushout of groupoids, generalising the pushout of groupoids in diagram 3.

It was then found that the theory of covering spaces could be given a nice exposition using the notion of covering morphism of groupoids. Even later, it was found by Higgins and Taylor [27] that there was a nice theory of orbit groupoids which gave models of orbit spaces.

The objects of a groupoid add a 'spatial component' to group theory, which is essential in many applications. This is evident in many parts of Ehresmann's work. Another view of this anomalous success of groupoids is that they have structure in two dimensions, 0 with the objects and 1 with the arrows. We have a colimit type theorem for this larger structure, and so a good model of the geometry. Useful information on fundamental groups is carried by the fundamental groupoid.

It is therefore natural to seek for higher homotopy theory algebraic models which:

- have structure in a range of dimensions;
- contain useful information on classical invariants, and
- satisfy van Kampen type theorems.

That is, we seek *nonabelian methods for higher dimensional local-to-global problems in homotopy theory.* We return to this theme in section 4, which gives an indication of some of the motivation for the writer for *higher dimensional algebra.*

3 Holonomy and monodromy

Once I had been led to groupoids by Philip Higgins, and having been told by G.W. Mackey in 1967 of his work on ergodic groupoids, [31], it was natural to consider topological groupoids and differentiable groupoids, and to seek their properties and applications, [10, 11].

So I came across papers of Ehresmann, [21], and of Jean Pradines, [35], and in 1981 I visited Jean in Toulouse under British Council support, to try and understand something of his papers. I saw the grand vision of the whole scheme of generalising the relation between Lie groups and Lie algebras to a relation between Lie groupoids and Lie algebroids, which has now become a large theory, see for example [30]. We concentrated on his first paper [35], which states theorems but gives no indication of proofs. What I found remarkable was first of all the beautiful constructions Jean explained, which seemed to me clear in principle, and then the fact that he gave for holonomy and monodromy universal properties: these are fairly rare in differential topology.

Jean was interested in the *monodromy principle*, which involves the following situation:

$$\begin{tikzcd} & M(G,W) \arrow[d, "p"] \arrow[dr, dashed, "f'"] & \\ W \arrow[ur, dashed, "i'"] \arrow[r, "i"] \arrow[rr, bend right, "f"'] & G & H \end{tikzcd}$$

Here G is a topological groupoid, W is an open subset of G which contains the set of identities G_0 of G, and i is the inclusion. The aim is to find a topological groupoid $M(G)$, called the *monodromy groupoid of* G, with morphism of topological groupoids $p : M(G,W) \to G$ and a 'local morphism' $i' : W \to M(G,W)$ such that for any local morphism $f : W \to H$ to a topological groupoid H, f

'locally extends' to $f' : M(G,W) \to H$. The existence such $M(G,W), p, i'$ with this universal property is called the *monodromy principle.*

This idea was expressed in Chevalley's famous book on 'Lie groups', [18], for the case the groupoid G is the trivial groupoid $X \times X$ on a manifold X, when $M(G,W)$ becomes, for sufficiently small W, the fundamental groupoid of X and $p : M(G,W) \to G$ is the 'anchor map' as defined in Mackenzie's book [30]. Pradines is the first, I believe, to see how to extend it this notion to the case of a general groupoid, as announced in [35].

We have to explain the term 'local morphism'. Note that if $u, v \in W$ and uv is defined in G, this does not necessarily mean that $uv \in W$. So the algebraic structure that W inherits from G is what is called a 'local groupoid structure'. A local morphism is therefore one that preserves the local groupoid structure.

It is easy to construct algebraically the groupoid $M(G,W)$ and $p : M(G,W) \to G$ so that it has the required universal property, algebraically, as follows. The source, target and identity maps for G induce on W the structure of a reflexive graph. So one forms $F(W)$, the free groupoid on this reflexive graph, the reflexive condition ensuring that identities in W become identities in $F(W)$. There is a function $j : W \to F(X)$ which sends $w \in W$ to the corresponding generator $[w] \in F(W)$ and then one factors the groupoid $F(W)$ by the relations $[u][v] = [uv]$ for all $u, v \in W$ such that uv is defined in W. This defines $M(G,W)$, and the function j induces the local morphism i', which is injective since it is determined by the inclusion i.

The problem is how to topologise $M(G,W)$ so that it becomes a topological groupoid for which the monodromy principle is satisfied not only algebraically but also with regard to continuity, or differentiability.

Pradines solved this by a beautiful holonomy construction, which he explained to me in 1981 during my visit, and which develops Ehresmann's ideas.

He took the view that the pair $(M(G,W), W')$, where $W' = i'(W)$, should be regarded as a 'locally Lie groupoid', and that this raised the general problem of when a locally topological groupoid is *extendible*, i.e. is obtained from an open subset containing the identities of a topological groupoid. (The term used in [35] is *morceau d'un groupoïde différentiable.*)

We therefore start again and consider a groupoid G and a subset W of G such that W contains the identities of G and W has the structure of topological space and even of a manifold. We ask: what conditions should be put on W which ensure that G can be given the structure of topological or Lie groupoid for which W is an open set? In short, we ask is the pair (G,W) extendible?

This is a classical question in the case G is a group, and the answer is given in books on topological groups. A topology on G is obtained by taking as subbase the sets gU for all U open in W and $g \in G$. The conditions for this to give a topological group structure on G are fairly mild, and are loosely that the algebraic operations on G should be as continuous on W as can be expected given that these operations are defined only partially on W.

One of the reasons this works is that in a topological group, the left multiplication operator $L_g : G \to G$, given for $g \in G$ by $u \mapsto gu$, is a homeomorphism and so maps open sets to open sets.

This property of L_g no longer holds when G is a topological groupoid, because L_g is not defined on all of G. This reflects the considerable change in moving from groups to groupoids, that is, from algebraic operations always defined to those only partially defined. This is not a loss of information: the wider concept has greater powers of expression.

There is also a wide range of examples where (G, W) is not extendible, many coming from the theory of foliations. For example, if $\mathcal{F}$ is the foliation of the Möbius Band M given by circles going once or twice round the band, and R is the equivalence relation determined by the leaves, then R is a subset of $M \times M$ and as such is a topological groupoid. But R is not a submanifold of $M \times M$, since it has self intersections. However the foliation structure does determine a locally Lie groupoid (R, W). The argument for both these facts is spelled out in [14].

Given a groupoid $\alpha, \beta : G \rightrightarrows G_0$, then an *admissible section* $s : G_0 \to G$ *of* α satisfies $\alpha \circ s = 1$ and $\beta \circ s : G_0 \to G_0$ is a bijection. We follow Mackenzie in [30] in calling s a *bisection*. We can also regard s as a homotopy $1 \simeq a$ where $a : G \to G$ is an automorphism of G. This interpretation has intuitive value, and is suggestive for analogues in higher dimensions, as applied in [12].

Now suppose G is as above but G_0 is also a topological space. Then a *local bisection* of G is a function $s : U \to G$ with U open in G_0 such that $\alpha \circ s = 1_U$ and $\beta \circ s$ maps U homeomorphically to its image which is also open in G_0. There is an 'Ehresmannian composition' $s * t$ of local bisections. We first make clear that if $g : x \to y$ and $h : y \to z$ in G then their composition in G is $hg : x \to z$. So the composition of local bisections is

$$(s * t)(x) = (s\beta t(x))(tx)$$

for $x \in G_0$. This means that in general the domain of $s * t$ is smaller than that of t, and may even be empty. This composition makes the set of local bisections into an *inverse semigroup*. Recall that this is a semigroup in which for each element s there is a unique element s', called a *relative inverse for* s, such that $s'ss' = s', ss's = s$. Pseudogroups, a concept first defined by Ehresmann in [20], give examples of such structures. We write this inverse semigroup as $\Gamma(G)$: of course it depends on the topology of G_0.

A left partial 'adjoint' operation L_s of the local bisection s on G is defined by $L_s(g) = (s\beta g)g, g \in G$. It is easy to prove that if G is a topological groupoid and s is a continuous local bisection, then L_s maps open sets of G to open sets of G. Thus the important observation is that for adjoint operations on topological or Lie groupoids it is not enough to rely on the elements of G: we need the local continuous or smooth bisections, which are kind of 'tubes' rather than 'elements', to transport the local structure of G.

Now suppose given a pair (G, W) such that G is a groupoid, $G_0 \subseteq W \subseteq G$, W is a manifold, and the groupoid operations are 'as smooth as possible' on W. By a *smooth local bisection of* (G, W) we mean a local bisection s of G such that s takes values in W and is smooth. The set of smooth local bisections forms a subset $\Gamma^{(r)}(W)$ where r denotes the class of differentiability of the manifold $M = G_0$.

Pradines' key definition is to form the sub-inverse semigroup $\Gamma^{(r)}(G, W)$ of $\Gamma(G)$ *generated by* $\Gamma^{(r)}(W)$. My interpretation is that an element of $\Gamma^{(r)}(W)$ can be thought of as a *local procedure* and an element of $\Gamma^{(r)}(G,W)$ can be thought of as an *iteration of local procedures*. Thus an iteration of local procedures need not be local, and this is one of the basic intuitions of non trivial holonomy.

We say that (G,W) is *sectionable* if for all $w \in W$ there is a smooth local bisection s in W whose domain includes αw and with $s\alpha w = w$.

The next step is to form from $\Gamma^{(r)}(G,W)$ the associated sheaf of germs $J^{(r)}(G,W)$: the elements of this are written $[s]_x$ where $s \in \Gamma^{(r)}(G,W)$ and $x \in \operatorname{dom} s$. The inverse semigroup structure on $\Gamma^{(r)}(G,W)$ induces a groupoid structure on $J^{(r)}(G,W)$. This contains a subgroupoid J^0 whose elements are germs $[s]_x$ such that $\beta s x = x$ and there is a neighbourhood U of x such that $s|U \in \Gamma^{(r)}(W)$; in words, s is an iteration of local procedures about x which is still a local procedure. It is a proposition that $J^0(G,W)$ is a normal subgroupoid of $J^{(r)}(G,W)$. The *holonomy groupoid* $\mathrm{Hol}(G,W)$ is defined to be the quotient groupoid $J^{(r)}(G,W)/J^0$. The class of $[s]_x$ in the holonomy groupoid is written $\langle s\rangle_x$. There is a projection $p : \mathrm{Hol}(G,W) \to G$ given by $\langle s\rangle_x \mapsto s(x)$.

The intuition is that first of all W embeds in $\mathrm{Hol}(G,W)$, by $w \mapsto \langle f\rangle_{\alpha w}$, where f is a local smooth bisection such that $f\alpha w = w$, and second that $\mathrm{Hol}(G,W)$ has enough local sections for it to obtain a topology by translation of the topology of W.

Let $s \in \Gamma^c(G,W)$. We define a partial function $\sigma_s : W \to \mathrm{Hol}(G,W)$. The domain of σ_s is the set of $w \in W$ such that $\beta w \in \mathrm{dom}(s)$. The value $\sigma_s w$ is obtained as follows. Choose a smooth bisection f through w. Then we set

$$\sigma_s w = \langle s\rangle_{\beta w}\langle f\rangle_{\alpha w} = \langle sf\rangle_{\alpha w}.$$

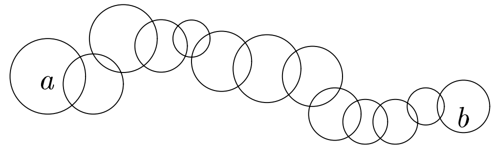
Then $\sigma_s w$ is independent of the choice of the local section f. It is proved in detail in [2] that these σ_s form a set of charts for $\mathrm{Hol}(G,W)$ making it into a Lie groupoid with a universal property.

The books [30, 32] argue that the most efficient treatment of the holonomy groupoid of a foliation is via the monodromy groupoid, which is itself defined using the fundamental groupoid of the leaves. However there are counter arguments.

One fact is that their arguments do not so far obtain a monodromy principle, which is obtained by the opposite route in [35, 13]. Thus there is loss of a universal principle, with its potentiality for enabling analogies.

The second problem is that the route through the fundamental groupoid is based on paths, and so on the standard notion of a topological space, and its exemplification as a manifold.

The Pradines' approach gives a clear realisation of the intuitive idea of *iteration of local procedures*, without requiring the notion of path to 'carry' these procedures, as happens for example in the usual process of analytic continuation. It is possible that this idea would lead to wider applications of non abelian groupoid like methods for local-to-global problems. For example, the following picture illustrates a chain of local procedures from a to b:



We would like to be able to define such a chain, and equivalences of such chains, without resource to the notion of 'path'. The claim is that a candidate for this lies in the constructions of Ehresmann and Pradines for the holonomy groupoid.

Here are some final questions in this area.

Can one use these ideas in other situations to obtain monodromy (i.e. analogues of 'universal covers') in situations where paths do not exist but 'iterations of local procedures' do? It is even possible that holonomy may exist in wider situations, but not monodromy.

How widely useful is the notion of 'locally Lie groupoid' as a context for describing local situations? Is there a locally Lie groupoid (G, W) where G is an action groupoid?

A further point is that Pradines was very keen on having a theory for germs of locally Lie groupoids. It is in these terms that the theorems in [35] are stated. Such work is not considered in [2, 13].

One could consider other structures on W and ask for bisections which preserve these structures. Simple examples for degrees of differentiability, due to Pradines [36], are given in [2]. One might consider other geometric structures, such as Riemannian, or Poisson. İçen and I, in looking for notions of double holonomy, have considered in [12] a double groupoid G and linear bisections.

4 Higher dimensions

After writing out the proofs of theorem 2.2 a number of times to make the exposition clear, it became apparent to me in 1966 that the method of proof ought to extend to higher dimensions, if one had the right gadgets. So this was an idea of a proof in search of a theorem. The first search was for a higher dimensional version of the fundamental groupoid on a set of base points.

It was at this point that I found Ehresmann's book, [22]. It was difficult to understand, but it did contain a definition of double category, and a key example, the double category $\square C$ of commuting squares in a category C. What caused problems for me was to find any construction of a fundamental double groupoid of a space which really contained 2-dimensional homotopy information. In fact a solution to this problem was not published till 2002, [9]. The construction of higher homotopy groupoids went in 1975 in a different direction, using pairs of spaces, and then filtered spaces, in work with Philip Higgins, which is surveyed in [6].

As an intermediate step, I decided in 1970 to investigate the notion of double groupoid purely algebraically, to see whether the putative homotopy double groupoid functor would take values in a category of some interest.

Work with Chris Spencer, [16], found a relationship between double groupoids and the crossed modules introduced by J.H.C. Whitehead to discuss the second relative homotopy group and its boundary $\partial : \pi_2(X, A.a) \to \pi_1(A, a)$. We found a functor from crossed modules to a certain kind of double groupoid, which was later called *edge symmetric*, in that the groupoids of vertical and of horizontal edges are the same. The next question was what kind of double groupoids arose in this way?

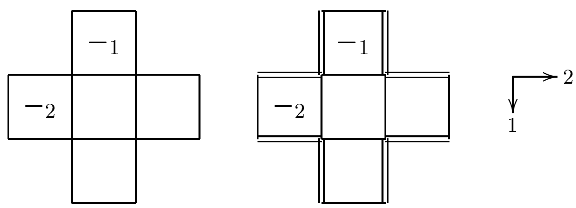
At the same time we were looking at conjectured proofs of a vaguely formulated 2-dimensional van

Kampen type theorem. It was clear that we needed to generalise commutative squares to commutative cubes in the context of double groupoids, in such a way that any composite of commutative cubes is commutative.

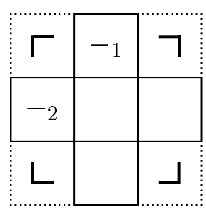
A square in a groupoid

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "c"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "d"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "2"] \arrow[d, "1"'] & {} \\ {} & \end{tikzcd}$$

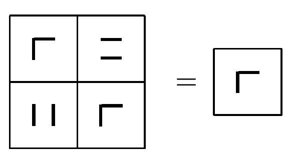
is commutative if and only if $ab = cd$, or, equivalently, $a = cdb^{-1}$. We wanted a similar expression to the last in the case of a cube. If we fold flat five of the faces of a cube we get a net such as the first of the following:



which is not a composable set of squares. However in the second diagram we have noted by double lines adjacent pairs of edges which are the same. Therefore we must assume we have canonical elements, called connections, to fill the corner holes as in the diagram:



The rules for connections include the transport law:



borrowed from a law for path connections found in Virsik's paper [41].

Virsik was a student of Ehresmann. He left Czechoslovakia at the time of the Russian invasion, and went to Australia. There he met Kirill Mackenzie, and suggested to him Lie groupoids and Lie algebroids as a PhD topic. Hearing of Kirill's work, I managed to get British Council support for him to visit Bangor in 1986, and this led to his continuing to work in the UK.

In the paper [41], Virsik introduces the notion of path connection on a principal bundle $p : E \to B$ as follows. Let G be the Ehresmann groupoid $EE^{-1} \rightrightarrows B$ of the principal bundle. Thus $G(b, b')$ can be identified with the bundle maps $E_b \to E_{b'}$ of the fibres over b, b' respectively. Let ΛB be the category of Moore paths on B, and let ΛG be the category of Moore paths on G. This may be combined easily to give the structure of a double category.

Virsik defines a *path connection* for the bundle E to be a function $\Gamma : \Lambda B \to \Lambda G$ satisfying two conditions. One is invariance under reparametrisation, and the other is known as the transport law. Chris Spencer and I were amazed when this law was found to be exactly right as a law on the connections we needed for the boundary of a cube, and these have been central in work on cubical higher groupoids ever since. So we found an equivalence between crossed modules and what we then called *special double groupoids with special connections*, [16].

Ehresmann had earlier shown in [23] that a 2-category gave rise to a double category of quintettes. Spencer showed in [38] that this gave an equivalence between 2-categories and edge symmetric double categories with connections. This has been generalised to all dimensions in [1], and there has been recent further work on commutative cubes by Higgins in [28] and Steiner [39].

What has yet to be accomplished is to find relations between the higher order connections in differential geometry and the geometry of cubes.

One of the advantages of cubes for local-to-global questions is that they have an easy to define notion of multiple composition. Analogous ideas in the globular case present conceptual and technical difficulties. Multiple compositions allow easily an *algebraic inverse to subdivision*, and this is a key to certain local-to-global results, as outlined for example in [6]. However the algebraic relations between the cubical, globular, and (in the groupoid case) crossed complexes, are an essential part of the picture.

It is possible to dream of a unification of all these themes in a way which I believe Ehresmann would have favoured, but there still seems quite a way to go to realise this. Work is in progress with Jim Glazebrook and Tim Porter on aspects of this, [8], and many others are working on stacks, gerbes and higher order groupoids.

References

[1] Al-Agl, F.A., Brown, R. and Steiner, R., *Multiple categories: the equivalence of a globular and cubical approach,* Adv. in Math. 170 (2002) 711-118.

[2] Aof, M.E.-S.A.-F. and Brown, R., *The holonomy groupoid of a locally topological groupoid*, Top. Appl. 47, 1992, 97-113.

[3] Brown R., *Groupoids and Van Kampen's theorem,* Proc. London Math. Soc. (3) 17 (1967) 385-401.

[4] Brown R., *From groups to groupoids: a brief survey*, Bull. London Math. Soc. 19 (1987) 113-134.

[5] Brown, R., *Homotopy theory, and change of base for groupoids and multiple groupoids,* Applied categorical structures, 4 (1996) 175-193.

[6] Brown, R., *Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems*, Proceedings of the Fields Institute Workshop on Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories, September 23-28, Fields Institute Communications 43 (2004) 101-130.

[7] Brown, R., *Topology and groupoids*, BookSurge LLC (2006).

[8] Brown, R., Glazebrook, J.F. and Porter, T., *Smooth crossed complexes I : nonabelian cohomology of a cover*, (in preparation).

[9] Brown, R., Hardie, K., Kamps, H. and Porter, T., *The homotopy double groupoid of a Hausdorff space*, Theory and Applications of Categories, 10 (2002) 71-93.

[10] Brown, R., and Hardy, J.P.L., *Topological groupoids I: universal constructions*, Math. Nachr. 71 (1976) 273-286.

[11] Brown, R., Danesh-Naruie, G., and Hardy, J.P.L., *Topological groupoids II: covering morphisms and G-spaces*, Math. Nachr. 74 (1976) 143-156.

[12] Brown, R. and İçen, İ., *Towards a 2-dimensional notion of holonomy*, Advances in Math, 178 (2003) 141-175.

[13] Brown, R. and Mucuk, O., *The monodromy groupoid of a Lie groupoid*, Cah. Top. Géom. Diff. Cat. 36 (1995) 345-369.

[14] Brown, R. and Mucuk, O., *Foliations, locally Lie groupoids, and holonomy*, Cah. Top. Géom. Diff. Cat. 37 (1996) 61-71.

[15] Brown, R. and Porter. T., *Category Theory: an abstract setting for analogy and comparison*, Advanced Studies in Mathematics and Logic, volume on 'What is category theory?', edited G. Sica (to appear, 2006).

[16] Brown, R. and Spencer, C.B., *Double groupoids and crossed modules*, Cah. Top. Géom. Diff. 17 (1976) 343–362.

[17] Čech, E., *Höherdimensionale homotopiegruppen*, Verhandlungen des Internationalen Mathematiker-Kongresses Zurich, Band 2, (1932) 203.

[18] Chevalley, C.,*Theory of Lie groups*, Princeton University Press, 1946.

[19] Crowell, R. H., *On the van Kampen theorem*, Pacific J. Math., 9 (1959) 43–50.

[20] Ehresmann, C., *Structures locales et structures infinitésimales*, C.R.A.S. Paris 274 (1952) 587-589.

[21] Ehresmann, C., *Catégories topologiques et catégories différentiables*, Coll. Géom. Diff. Glob. Bruxelles, 1959, 137-150.

[22] Ehresmann, C., *Catégories et structures*, Dunod, Paris, (1965).

[23] Ehresmann, C., *Catégories doubles des quintettes: applications covariantes*, C.R.A.S. Paris 256 (1963) 1891–1894.

[24] Ehresmann, C., *Oeuvres complètes et commentées: Amiens, 1980-84*, edited and commented by Andrée Ehresmann.

[25] Higgins, P. J., *Presentations of groupoids, with applications to groups*, Proc. Cambridge Philos. Soc., 60 (1964) 7–20.

[26] Higgins, P. J., *Categories and groupoids*, Van Nostrand Mathematical Studies 32, (1971), Reprinted in Theory and Application of Categories Reprints (2005).

[27] Higgins, P.J. and Taylor, J., *The fundamental groupoid and homotopy crossed complex of an orbit space*, *Category Theory Proceedings, Gummersbach, 1981*, Lecture Notes in Math. 962, edited K.H.Kamps *et al*, Springer, Berlin, pp. 115-122, 1982.

[28] Higgins, P. J., *Thin elements and commutative shells in cubical ω-categories with connections*, Theory and Applications of Categories, 14 (2005) 60-74.

[29] Mackenzie, K.C.H., *Double Lie algebroids and second order geometry, II*, Adv. Math. 154 (2000) 46-75.

[30] Mackenzie, K.C.H., *General theory of Lie groupoids and Lie algebroids*, London Math. Soc. Lecture Note Series 213, Cambridge University Press, 2005.

[31] Mackey, G.W., *Ergodic theory and virtual groups*, Math. Ann. 166 (1966) 187-207.

[32] Moerdijk, I. and Mrčun, J., *Introduction to foliations and Lie groupoids,* Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, 2003.

[33] Mucuk, O., *Covering groups of non-connected topological groups and the monodromy groupoid of a topological groupoid*, PhD Thesis, University of Wales, 1993.

[34] Phillips, J., *The holonomic imperative and the homotopy groupoid of a foliated manifold*, Rocky Mountain J. Math., 17 (1987) 151-165.

[35] Pradines, J., *Théorie de Lie pour les groupoides différentiables, relation entre propriétés locales et globales*, Comptes Rendus Acad. Sci. Paris, Sér A, 263 (1966), 907-910.

[36] Pradines, J., *Private communication*, 1982.

[37] Rota, G.-C., *Indiscrete thoughts*, Birkhäuser.

[38] Spencer, C. B., *An abstract setting for homotopy pushouts and pullbacks*, Cahiers Topologie Géom. Différentielle, 18 (1977) 409–429.

[39] Steiner, R., *Thin fillers in the cubical nerves of omega-categories*, arXiv:math.CT/0601386.

[40] Swan, R.G., *The theory of sheaves*, Notes by Ronald Brown, Oxford Mathematical Inst., 1958; Chicago University Press 1964.

[41] Virsik, J., *On the holonomity of higher order connections*, Cah. Top. Géom. Diff. 12 (1971) 197-212.