NONIMMERSIONS OF REAL PROJECTIVE SPACES IMPLIED BY eo_2

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Abstract. Recently Hopkins and Mahowald constructed a new 2-primary ring spectrum eo_2, satisfying $H^*(eo_2) \approx A//A_2$. We use eo_2 to obtain new results regarding nonimmersions of real projective spaces in Euclidean space. The method is to say enough about the eo_2-cohomology of a product of real projective spaces to obtain nonexistence of certain axial maps.

1. Main theorem and an example

In [14], a new 2-primary ring spectrum eo_2 was constructed, satisfying $H^*(eo_2) \approx A//A_2$. All cohomology groups have coefficients in $\mathbf{Z}_2 = \mathbf{Z}/2$, and A_2 denotes the subalgebra of the mod 2 Steenrod algebra A generated by Sq^1, Sq^2, and Sq^4. We will say more about this spectrum eo_2, which is related to elliptic cohomology, in Section 5. In this paper, we will use eo_2 to derive the following new result regarding nonimmersions of real projective space P^n in Euclidean space.

Let $\nu(n)$ denote the exponent of 2 in n, $\alpha(n)$ the number of 1's in the binary expansion of n, and $p(n)$ the smallest 2-power greater than or equal to n.

Theorem 1.1. *If $\alpha(M) = 4h+\epsilon$, then $P^{8M+8h+\delta}$ cannot be immersed ($\not\subseteq$) in $\mathbf{R}^{16M-8h+d}$ in the following cases:*

1. *$M \equiv 0$ mod $\frac{1}{2}p(h+1)$ and $(\epsilon, \delta, d) = (0, -4, -1)$ or $(-1, -6, -1)$.*
2. *$M \equiv 0$ mod $p(h)$ and $(\epsilon, \delta, d) = (-1, 2, 10)$ or $(-2, 0, 12)$, or $(-1, 0, 6)$ with h even.*
3. *$M \equiv 2^{e_0} + 2^{e_1}$ mod 2^{e_1+1} with $e_0 < e_1$ and $h \le 2^{e_1} - 2^{e_0}$, or M a 2-power, and $(\epsilon, \delta, d) = (4, 4, -12)$, $(3, 2, -10)$, or $(0, 0, 2)$.*

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These results are new provided $\alpha(M) \geq 6$ except in isolated cases for P^n with $\alpha(n)$ very small. They improve on previously known nonimmersions by arbitrarily many dimensions. The first new result is $P^n \not\subseteq \mathbf{R}^{2n-36}$ if $n = 1536$, the case $\alpha(M) = 6$, $M = 2^7 + 2^6 - 2$ in part 2 of the theorem. The results should be compared with the result $P^{2(m+\alpha(m)-1)} \not\subseteq \mathbf{R}^{4m-2\alpha(m)}$ of [3]. That result was proved by applying the cohomology theory $BP\langle 2\rangle^*(-)$ to detect axial maps, while ours here uses $eo_2^*(-)$ in a similar way. Our results here improve on those results in a manner analogous to the way in which ko results improve on ku results, since, for $E = \langle eo_2, BP\langle 2\rangle, ko, ku\rangle$, we have $H^*(E) = A//B$ with $B = \langle A_2, E_2, A_1, E_1\rangle$, where A_n is the subalgebra generated by the first $n+1$ Sq^{2^i}'s and E_n is the subalgebra generated by the first $n+1$ Milnor primitives. For this and other reasons, eo_2 is sometimes called "higher real K-theory." We will say more about the strength of the results in Section 4.

The method of proof is to prove nonexistence of axial maps. An axial map $P^m \times P^n \to P^k$ is one whose restrictions to $P^m \times *$ and $* \times P^n$ are both homotopic to the inclusion maps. The following result is well known.

Proposition 1.2. ([21],[15]) *If P^n can be immersed in $\mathbf{R}^{n+k}$, then there exist axial maps $P^n \times P^n \to P^{n+k}$ and $P^n \times P^{2^L-n-k-2} \to P^{2^L-n-2}$, where $L > n/2$.*

Let $e = eo_2$. In Section 2, we will discuss the computation of $e^*(P^n)$ and $e^*(P^m \times P^n)$. We will show that there is an element $X \in e^8(P^n)$ for which the corresponding map $P^n \to \Sigma^8 e$ is nontrivial in $H^8(-)$. If f is an axial map, then $f^*(X^i) = (X_1+X_2)^i$, where X_i corresponds to X in the ith factor. Theorem 1.1.1 is proved by showing that if the hypothesized immersion existed, then the first type of axial map in Proposition 1.2 would satisfy $X^{2M-h} = 0$ but $(X_1+X_2)^{2M-h} \neq 0$, an obvious contradiction. Parts 2 and 3 of Theorem 1.1 are deduced similarly using the second type of axial map in 1.2.

The hard part is showing that $(X_1 + X_2)^i \neq 0 \in e^*(P^m \times P^n)$. We will use the Adams spectral sequence (ASS) to give an almost complete determination of a quotient algebra of $e^*(P^m \wedge P^n)$ (with m and n even) generated by X_1 and X_2. We illustrate with a typical group, and its use in obtaining a nonimmersion result.

Example 1.3. *Let $G = e^{8a+8b-72}(P^{8a-4} \wedge P^{8b-4})$, and for $1 \leq i \leq 8$, let $y_i = X_1^{a-i}X_2^{b-9+i} \in G$. Then G has a subgroup S (viz. the elements of filtration ≥ 12*

in its ASS determination) such that G/S has presentation with generators $y_1, \dots, y_8$ and relations

$$2^4 y_1,\ 2^8 y_2,\ 2^8 y_7,\ 2^4 y_8, \tag{1.4}$$

$$2^{12} y_i,\ 3 \le i \le 6, \tag{1.5}$$

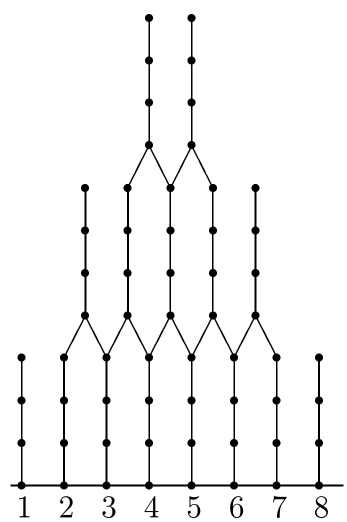
$$2^4(y_2 + y_3 + y_4 + y_5 + y_6 + y_7) - 2^5 g, \tag{1.6}$$

$$2^8(y_3 + y_5) - 2^9 g', \tag{1.7}$$

for some $g, g' \in G/S$.

Diagram 1.8 illustrates the group of Example 1.3. Vertical and diagonal lines both indicate multiplication by 2, although sometimes this is only up to elements of higher filtration. Readers familiar with ASS charts should be aware that the entire group pictured below is (a portion of) a single homotopy (or generalized homology) group.

Diagram 1.8.



The chart indicates that $2^4(y_2 + \cdots + y_7)$ is 0 in filtration 4, since each element in filtration 4 appears as a summand of $2^4 y_i$ for 2 values of i, but, as (1.6) asserts, it can equal some element of larger filtration. The relations (1.4) are actually 0, not just of higher filtration, by Proposition 2.1. The chart implies (1.7) by showing that both $2^8 y_3$ and $2^8 y_5$ equal the first element in filtration 8, up to higher filtration.

We use this example to establish the first nonimmersion of Theorem 1.1.1 when $h = 3$. We will prove that if m is odd and $\alpha(m-3) = 12$, then there does not exist an axial map $P^{8m-4} \times P^{8m-4} \to P^{16m-73}$. Letting $M = m-3$ yields the desired nonimmersion, thanks to 1.2. Since $X^{2m-9} = 0 \in e^*(P^{16m-73})$ for dimensional reasons, it suffices to show

$$\sum_{i=1}^{8} \tbinom{2m-9}{m-i} y_i \neq 0 \in G/S, \tag{1.9}$$

where G/S is as in Example 1.3 with $a = b = m$.

If $\nu\tbinom{2m-9}{m-1} < 4$ or $\nu\tbinom{2m-9}{m-2} < 8$, then (1.9) is nonzero due to its terms with $i = 1$ or 2, and we are done. We now assume to the contrary, and so the terms of (1.9) with $i = 1, 2, 7,$ and 8 are 0 by (1.4). We have

$$\begin{aligned} \nu\tbinom{2m-9}{m-3} &= \alpha(m-3) + \alpha(m-6) - \alpha(2m-9) \\ &= \alpha(m-3) + (\alpha(m-5) - 1 + \nu(m-5)) - (\alpha(2(m-5)) + 1) \\ &= \alpha(m-3) - 2 + \nu(m-5) \end{aligned}$$

and similarly

$$\nu\tbinom{2m-9}{m-4} = \alpha(m-3) - 2 + \nu(m-3).$$

If $\alpha(m-3) = 12$ and m is odd, then one of these ν-values is 11 and the other is greater than 11, and so (1.9) is either $2^{11}(y_3 + y_6)$ or $2^{11}(y_4 + y_5)$. The only relations in filtration 11 are $2^{11}(y_3 + y_4 + y_5 + y_6)$ and $2^{11}(y_3 + y_5)$, from (1.6) and (1.7), and these do not imply that $2^{11}(y_3 + y_6)$ or $2^{11}(y_4 + y_5)$ is 0, establishing the nonimmersion.

There are three possible ways in which this project might be improved in the future. The most glaring is to eliminate the indeterminacy in relations such as (1.6) and (1.7), which should allow the removal of the congruence hypotheses in Theorem 1.1. The hope for doing this is to obtain an analogue of the [2]-series, which would give explicit relations in $e^*(P^n)$. The biggest obstruction to doing this is that v_2^4 is not an element of e_*. A second possible expansion of this work is to construct an orientation $MO[8] \to eo_2$, which would allow eo_2 to be incorporated more directly into obstruction theory. In particular, it should enable us to obtain new positive immersion results. The third extension is much more straightforward. It is just to consider $e^*(P^m \wedge P^n)$ with m and/or n odd. This will probably enable the deduction of some more nonimmersion results, at the expense of some complication of exposition.

We close this introductory section by observing that speculations about this approach to nonimmersions were already published in 1982 in [9, §4], although the details are not exactly as envisioned there. The main new ingredient is the existence of eo_2, along with the computer programs of the first author. We would like to thank the Johns Hopkins University JAMI program, where all three authors spent at least one month in spring 2000, at which time this work was carried out.

2. The eo_2-cohomology of $P^{2m} \wedge P^{2n}$

As before, $e = eo_2$. In this section, we determine, up to certain indeterminacy, a quotient algebra of $e^*(P^m \wedge P^n)$ (with m and n even) generated by X_1 and X_2. The result, Theorem 2.7, is similar to Example 1.3.

We begin with a result which generalizes (1.4).

Proposition 2.1. *There is a (nonunique) element $X \in e^8(P^{2n})$, compatible as n varies, for which the corresponding map $P^{2n} \to \Sigma^8 e$ is nonzero in $H^*(-)$. Let $c_1 = c_2 = 0$, $c_3 = 1$, and $c_4 = 3$. For $1 \le d \le 4$ and $1 \le i < m$,*

$$2^{4i-c_d} X^{m-i} = 0 \in e^*(P^{8m-2d}). \tag{2.2}$$

Proof. We use S-duality. It is convenient, although not necessary, to use stunted projective spaces with negative indices. See, for example, [18]. For $i > 0$, $e^i(P^n) \approx e_{-i-1}(P_{-n-1})$. Note P_m always means P_m^∞. Since $H^*(e) \approx A//A_2$, by a well-known change-of-rings theorem the E_2-term of the ASS converging to $e_*(Y)$ is given by $\operatorname{Ext}_{A_2}(H^*Y)$ for any space Y. Here and elsewhere, we omit $\mathbf{Z}/2$ in the second component of Ext groups.

In [18], it was proved that

$$\varprojlim \operatorname{Ext}_{A_2}(H^*P_{-n}) \approx \bigoplus_{i \in \mathbf{Z}} \operatorname{Ext}_{A_1}(\Sigma^{8i-1}\mathbf{Z}_2). \tag{2.3}$$

Thus, for any integer i, $\varprojlim \operatorname{Ext}_{A_2}^{*,*+8i-1}(H^*P_{-n})$ is a free h_0-module on generators of filtration $4j$ for all $j \ge 0$, and

$$\varprojlim \operatorname{Ext}_{A_2}^{*,*+8i-2}(H^*P_{-n}) = 0. \tag{2.4}$$

Let g_i denote the nonzero element of $\varprojlim \operatorname{Ext}_{A_2}^{0,8i-1}(H^*P_{-n})$.

The groups $\mathrm{Ext}_{A_2}(H^*P_N)$ were completely computed in [9, 2.3,2.4]. We have that for $1 \le d \le 4$ and $i > a$,

$$\mathrm{Ext}_{A_2}^{0,8i-1}(H^*P_{8a+2d-1}) \approx \mathbf{Z}/2$$

with generator g_i satisfying $h_0^{4i-4a-c_d-1} g_i \ne 0$, and

$$\mathrm{Ext}_{A_2}^{s,s+8i-1}(H^*P_{8a+2d-1}) = 0 \text{ if } s \ge 4i - 4a - c_d. \tag{2.5}$$

Nothing in this tower can be killed by an Adams differential because it maps onto a tower of the same height in $ko_*(P_{8a+2d-1})$ which is not killed by a differential. Moreover, g_i cannot support a differential by (2.4).

It follows from (2.3) that

$$\lim_{\leftarrow} e_*(P_{-n}) \approx \prod_{i\in\mathbf{Z}} ko_*(S^{8i-1})^\wedge.$$

Let G_i be an element of $\lim\limits_{\leftarrow} e_{8i-1}(P_{-n})$ of Adams filtration 0. Then G_i is necessarily detected by g_i, but is not unique, even up to odd multiples, because it could be changed by elements in towers beginning in filtration ≥ 4. Restriction yields compatible elements $G_i \in e_{8i-1}(P_b)$ for $b \le 8i-1$. With $i = -1$, duality produces the compatible elements $X \in e^8(P^n)$ for $n \ge 8$. Since the dual of X^{m-i} lies in $e_{-8m+8i-1}(P_{-8m+2d-1})$, (2.2) follows from (2.5). ■

In the preceding proposition, we used that as A_2-modules $H^*(P_{b+8}^{t+8}) \approx H^*(\Sigma^8 P_b^t)$, which implies an isomorphism of E_2-terms of the ASS after applying $\wedge eo_2$ to the spectra. There is, in fact, an associated equivalence of spectra, which will be useful later in the paper.

Proposition 2.6. *For any integers b and t, with t possibly infinite, there is an equivalence of spectra*

$$P_{b+8}^{t+8} \wedge eo_2 \simeq \Sigma^8 P_b^t \wedge eo_2.$$

Proof. The elements X of Proposition 2.1 combine to give an element $X \in e^8(P_1^\infty)$. The exact $e^*(-)$ sequence induced by the cofibration $P^7 \to P_1^\infty \to P_8^\infty$ shows that X pulls back to an element $X \in e^8(P_8)$, with the property that the associated map $P_8 \to \Sigma^8 e$ is nontrivial in $H^8(-)$.

Now we observe that covering the diagonal map $P \to P \times P$ is a map of Thom spectra $T((8+n)\xi) \to T(8\xi) \wedge T(n\xi)$. We use the representation of these Thom spectra as stunted projective spaces, apply $\wedge e$ to the map, and follow by $X : P_8 \to \Sigma^8 e$, obtaining

$$P_{8+n} \wedge e \to P_8 \wedge P_n \wedge e \to \Sigma^8 e \wedge P_n \wedge e \to \Sigma^8 P_n \wedge e,$$

where we have used that e is a ring spectrum at the last step. One readily verifies that this composite $P_{8+n} \wedge e \to \Sigma^8 P_n \wedge e$ induces an isomorphism in cohomology ($\sigma^8 g_{8i-1} \mapsto g_8 \otimes g_{8i-1} \mapsto g_{8i+7}$), and hence is an equivalence. Since the maps are compatible with respect to changing n, there are also equivalences of the finite projective spaces. ■

Now we state the main result of this section, generalizing Example 1.3, which is the case $s_1 = s_2 = 2$ and $d = 9$.

Theorem 2.7. *Let s_1 and s_2 equal 1, 2, 3, or 4, and $d \geq 2$. Let c_s be as in Proposition 2.1. Let $G = e^{8a+8b-8d}(P^{8a-2s_1} \wedge P^{8b-2s_2})$, and for $1 \leq i < d$, let $y_i = X_1^{a-i} X_2^{b-d+i} \in G$. Then G has a subgroup S such that G/S has presentation with generators $y_1, \ldots, y_{d-1}$ and relations*

$$2^{4i-c_{s_1}} y_i, \ \ i \geq 1, \tag{2.8}$$

$$2^{4i-c_{s_2}} y_{d-i}, \ \ i \geq 1, \tag{2.9}$$

$$2^{4i+4} \sum_j \binom{j}{i} y_j - 2^{4i+5} x_i, \ \ i \geq 0, \ \ \textit{some } x_i \in G/S, \tag{2.10}$$

$$2^{8k+\epsilon} y_{2k+\delta} \ \textit{if } d = 6k+\beta, \ \ s_1 \geq t_1, \ \ \textit{and } s_2 \geq t_2, \tag{2.11}$$

where $(\epsilon, \delta, \beta, t_1, t_2)$ is one of the following: $(2,1,2,1,1)$, $(-1,0,-1,1,2)$, $(-1,0,-1,2,1)$, $(-2,0,-1,1,3)$, $(-2,0,-1,3,1)$, $(3,1,3,1,4)$, $(3,2,3,4,1)$, $(-1,0,0,2,4)$, $(-1,1,0,4,2)$, $(-5,0,-2,4,4)$, *and*

$$2^e y_i \ \textit{for all } i, \tag{2.12}$$

where e is the $(d-1)$st largest element in the set with repetitions which consists of all exponents which occur in (2.8), (2.9), (2.10), and (2.11).

In (2.10), we initiate the convention, followed throughout the paper, that, unless indicated to the contrary, sums involving binomial coefficients are to be taken over all values of the summation variable for which the summand is nonzero.

The reader might find it worthwhile to compare Theorem 2.7 with the special case given in Example 1.3. In that case, the integer e of (2.12) is the 8th largest entry in $\{4, 8, 12, \ldots, 4, 8, 12, \ldots, 4, 8, 12, \ldots\}$, which is 12.

We extract from Theorem 2.7 the information that will be used in proving nonimmersions. This result will be proved at the end of the section.

Corollary 2.13. *The element $\sum k_i y_i$ is nonzero in the group G of 2.7 in the following cases:*

1. *$d = 3h$, and, for $t = 1$ or 2, $s_1 = s_2 = t + 1$, $\nu(k_i) \geq 4h - t$ for $h \leq i \leq 2h$, and $\nu(\sum \binom{h-1}{j} k_{h+j}) = 4h - t$;*
2. *$(d, t, s_1, s_2) = (3h, 1, 3, 2)$ or $(3h, 2, 4, 3)$, or $(3h + 1, 1, 4, 4)$ with h even, $\nu(k_i) \geq 4h - t$ for $h + 1 \leq i \leq 2h$, and*
$$\nu\left(\sum \binom{h-1}{j} k_{h+1+j}\right) = 4h - t. \tag{2.14}$$
3. *Either $d = 3h + 3$, $(t, s_1, s_2) = (3, 2, 3)$ or $(2, 3, 4)$, $\nu(k_i) \geq 4h + t$ for $h + 1 \leq i \leq 2h + 1$, and $\nu(\sum \binom{h}{j} k_{h+1+j}) = 4h + t$, or $d = 3h + 1$, $s_1 = 4$, $s_2 = 2$, $\nu(k_i) \geq 4h - 1$ for $h + 1 \leq i \leq 2h + 1$, and $\nu(\sum \binom{h-1}{j} k_{h+2+j}) = 4h - 1$.*

The derivation of Theorem 2.7 is roughly to run a computer program that computes $\mathrm{Ext}_{A_2}(H^*(P_{2s_1-1} \wedge P_{2s_2-1}))$ through a range by minimal resolution. Of course, this is not a proof that an observed pattern persists throughout all dimensions. In order to do that, we first obtain a complete computation of the relevant part of $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge R))$, where R is the fiber of the so-called Kahn-Priddy map $\lambda : P_1 \to S^0$. This is done by embedding $H^*(P_3 \wedge R)$ in a short exact sequence between modules whose Ext groups are completely known. Next we use cofibration sequences to deduce from this the relevant part of each $\mathrm{Ext}_{A_2}(H^*(P_{2s_1-1} \wedge P_{2s_2-1}))$. Finally, we show that the elements corresponding under duality to multiples of $X_1^i X_2^j$ are not killed by differentials in the ASS converging to $e_*(P_{2s_1-1} \wedge P_{2s_2-1})$.

We begin with the following key result.

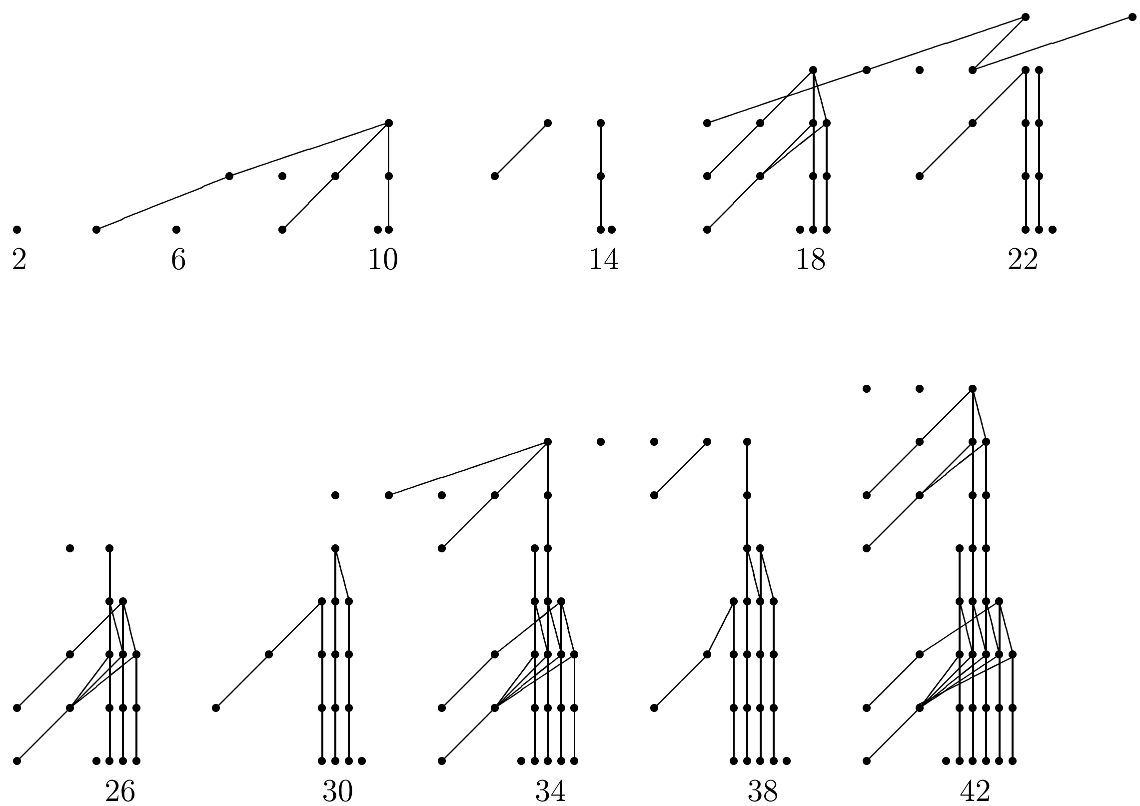
Theorem 2.15. *In filtration* $\le [(4d-2)/3]$,

$$\mathrm{Ext}_{A_2}^{*,*+8d-1}(H^*(P_3 \wedge R)) = 0$$

and $\mathrm{Ext}_{A_2}^{*,*+8d-2}(H^*(P_3 \wedge R))$ *as a module over* $\mathbf{Z}_2[h_0]$ *has generators* $g_1, \dots, g_d$ *of filtration* 0 *and relations* $h_0^{4i} g_i$ $(i \ge 1)$, $h_0^{4i+1} g_{d-i}$ $(i \ge 0)$, $h_0^{4i+4} \sum \binom{j}{i} g_j$ $(i \ge 0)$, *and, if* $d = 3k+2$, *then* $h_0^{4k+3} g_{k+1}$.

There are other elements in $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge R))$ in bigradings not specified in Theorem 2.15. The entire Ext chart makes a very pleasing picture, with the towers described in the theorem having their tops along a line of slope 1/6, with h_1^2 extending into the towers at regular spacing, and a few additional elements at the top of the chart moving along a line of slope 1/5. We illustrate with the complete chart in $t - s \le 42$. The action of h_0, h_1, and h_2 is indicated.

Diagram 2.16.



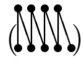
The heights $H_d(i)$ of the towers in $\mathrm{Ext}_{A_2}^{*,*+8d-2}(H^*(P_3 \wedge R))$ for $d \le 8$ are listed for illustration in Table 2.17. They satisfy $H_{d+6}(i+2) = H_d(i)+8$. From this information and the pattern of extensions from filtration $4i-1$ to $4i$, the reader can extend the relevant portion of Diagram 2.16. The relation $h_0^{4i+4} \sum \binom{j}{i} g_j$ in Theorem 2.15 can be considered to be a consequence of similar extension patterns () from filtration 3 to 4, 7 to 8, ... , $4i+3$ to $4i+4$.

Table 2.17.

	$H_d(i)$	d = 1	2	3	4	5	6	7	8
	1	1	3	4	4	4	4	4	4
	2		1	4	5	7	8	8	8
i	3			1	4	5	8	9	11
	4				1	4	5	8	9
	5					1	4	5	8
	6						1	4	5
	7							1	4
	8								1

Note that $H_d(i)$ is not, in general, the order of the element g_i because of the extensions into the tower just to the left. For example, the case $d = 5$ in Table 2.17 corresponds to the group in dimension 38 in Diagram 2.16. First note that the heights of the towers above the elements g_i in dimension 38 in 2.16 are 4, 7, 5, 4, and 1, resp., which is the information in 2.17. The relations in Theorem 2.15 when $d = 5$ are $h_0^4 g_1$, $h_0^7 g_2$, $h_0^5 g_4$, $h_0 g_5$, and $h_0^4(g_2 + g_3 + g_4)$. One can easily verify that this is consistent with 2.16 in dimension 38.

Proof of Theorem 2.15. There is a short exact sequence of A_2-modules

$$0 \to F \to H^*R \xrightarrow{\theta} Q \to 0, \tag{2.18}$$

where

$$F_i = \begin{cases} \mathbf{Z}_2 & \text{if } i = 1, 2, 4, \text{ or } 8 \\ 0 & \text{otherwise} \end{cases}$$

and $\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 \neq 0$ in F, and Q is the quotient of H^*P_{-1} by the A_2-submodule generated by classes of degree less than -1. This is the 8-fold suspension of the module called Q in [9, §1,7]. We apply $H^*P_3 \otimes$ to (2.18) and consider the long exact sequence in $\mathrm{Ext}_{A_2}(-)$. Since Q is built from subquotients $\Sigma^{8i-1}A_2//A_1$ for $i \geq 0$,

$$\mathrm{Ext}_{A_2}(H^*P_3 \otimes Q) \approx \bigoplus_{i\geq 0} \mathrm{Ext}_{A_1}(H^*(\Sigma^{8i-1}P_3)). \tag{2.19}$$

We will need the following lemma, whose proof is postponed.

Lemma 2.20. *Let* $\{g_1, \ldots, g_d\}$ *be the basis of* $\mathrm{Ext}_{A_2}^{0,8d-2}(H^*P_3 \otimes Q)$ *such that* g_i *is dual to* $x_1^{8i-1}x_2^{8d-8i-1}$. *Then*

$$v_1^4 g_i = h_0^4(g_{i+1} + \sum_{j=1}^{i-1} g_j). \tag{2.21}$$

The g_i on the LHS of (2.21) has d-value 1 less than that of the g_j on the RHS.

We have

$$\mathrm{Ext}_{A_2}^{s,t}(H^*P_3 \otimes F) \approx \bigoplus_i \Big(\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{8i+7}\mathbf{Z}_2) \oplus \mathrm{Ext}_{A_1}^{s+1,t+1}(\Sigma^{8i-1}\mathbf{Z}_2)\Big) \tag{2.22}$$

if $t-s \geq 6s+6$. One way to see this is to build it from $\mathrm{Ext}_{A_2}(H^*(\Sigma^i P_3))$ for $i = 1, 2, 4$, and 8. We use the computation of $\mathrm{Ext}_{A_2}^{s,t}(H^*P_3)$ in [9]; it is isomorphic to $\bigoplus \mathrm{Ext}_{A_1}^{s,t}(\Sigma^{8i-1}\mathbf{Z}_2)$ if $t-s \geq 6s+4$. The $\mathrm{Ext}_{A_1}(\Sigma^{8i+7}\mathbf{Z}_2)$ in (2.22) comes from the 8-cell of F, while

$$\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{8i-1}F^{(4)}) \approx \mathrm{Ext}_{A_1}^{s+1,t+1}(\Sigma^{8i-1}\mathbf{Z}_2),$$

where $F^{(4)}$ denotes the 4-skeleton of F.

Another lemma with postponed proof follows.

Lemma 2.23. *In the exact sequence*

$$\to \mathrm{Ext}_{A_2}^{s-1,t}(H^*P_3 \otimes F) \xrightarrow{\partial} \mathrm{Ext}_{A_2}^{s,t}(H^*P_3 \otimes Q) \xrightarrow{\theta^*} \mathrm{Ext}_{A_2}^{s,t}(H^*(P_3 \wedge R)) \to, \tag{2.24}$$

$\mathrm{im}(\partial)$ *in* $(s,t) = (1, 8d-1)$ *contains* $h_0 g_d$, *and* $\mathrm{im}(\partial)$ *in* $(s,t) = (4, 8d+2)$ *contains* $h_0^4(g_1 + \cdots + g_d)$.

We have given almost enough information to determine the requisite information about $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge R))$ from (2.24). The relation $h_0^{4i} g_i$ is already present in $\mathrm{Ext}_{A_2}(H^*P_3 \otimes Q)$. The relations $h_0^{4i+1} g_{d-i}$ and $h_0^{4i+4} \sum \binom{j}{i} g_j$ follow inductively from the two lemmas. Indeed, the case $i = 0$ is immediate from $\mathrm{im}(\partial)$ in Lemma

2.23. Applying v_1^4 to the $i=0$ relations in $(s,t)=(1,8d-9)$ and $(4,8d-6)$ yields, by Lemma 2.20, the relations

$$h_0^5(g_d+\sum_{j=1}^{d-2} g_j) \text{ and } h_0^8(\sum_{j=1}^{d-1}(d-j)g_j+g_d-g_1).$$

The first of these becomes $h_0^5 g_{d-1}$ after reduction using the relation $h_0^4(g_1+\cdots+g_d)$, while the second is reduced to $h_0^8\sum jg_j$ similarly, yielding the case $i=1$. A similar argument works for larger values of i.

The final relation of Theorem 2.15 is not so easily seen. Here the minimal resolution computer program ([1],[2]) of the first author is very useful. It allows us to see all these Ext groups through a range of dimensions ($t-s\le 78$ has been used often). This allows us to see any unusual boundary morphisms. We can deduce that they are promulgated by v_2^8-periodicity.

The relations already considered imply that ∂ in (2.24) is surjective if

$$t-s=8d-2 \text{ and } s\ge\begin{cases}4k & \text{if } d=3k-1,3k\\ 4k+1 & \text{if } d=3k+1.\end{cases} \tag{2.25}$$

The only elements in $\operatorname{Ext}_{A_2}^{s,s+8d-1}(H^*P_3\otimes F)$ not accounted for by (2.22) and with s less than the largest value for which (2.25) is not satisfied is a $\mathbf{Z}_2$ in

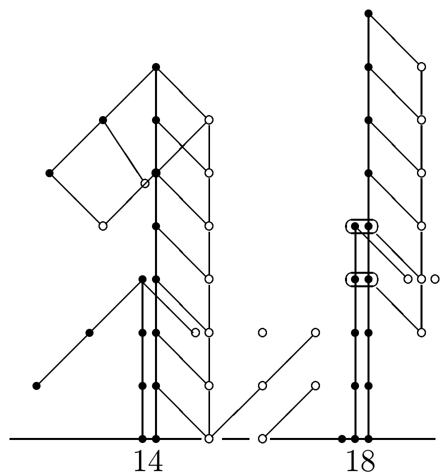
$$(t-s,s)=(15+24k,2+4k). \tag{2.26}$$

This can be seen by the method described after (2.22). The computer program mentioned above establishes the claim in $t-s\le 78$. All homomorphisms in the exact sequences that build $\operatorname{Ext}_{A_2}(H^*P_3\otimes F)$ from $\operatorname{Ext}_{A_2}(H^*\Sigma^iP_3)$, $i=1,2,4,8$, commute with v_2^8, and, since by [9] $\operatorname{Ext}_{A_2}(H^*P_3)$ has a faithful action of v_2^8, this implies that the behavior in $t-s\le 78$ continues indefinitely. These elements in (2.26) are mapped nontrivially by ∂. Again the safest way to see this is to use the computer program in low dimensions, and then v_2^8-periodicity. This accounts for the final relation in Theorem 2.15.

That $\operatorname{Ext}_{A_2}^{s,s+8d-1}(H^*(P_3\wedge R))=0$ for $s\le[(4d-2)/3]$ follows also from the exact sequence (2.24). Here we have $\operatorname{Ext}_{A_2}^{*,*+8d-1}(H^*P_3\otimes Q)=0$ by (2.19), and ∂ is injective on $\operatorname{Ext}_{A_2}^{s,s+8d-1}(H^*P_3\otimes F)$ for $s\le[(4d-2)/3]$. ■

We illustrate the above proof with a diagram of (2.24) for $12 \le t-s \le 19$. Elements of $\mathrm{Ext}_{A_2}(H^*P_3 \otimes Q)$ are indicated by $\bullet$'s, elements of $\mathrm{Ext}_{A_2}(H^*P_3 \otimes F)$ by $\circ$'s, and diagonal lines of negative slope are ∂.

Diagram 2.27.



Proof of Theorem 2.7. We begin by observing that the group G of Theorem 2.7 is isomorphic to $e_{8d-1}(P_{2s_1-1} \wedge P_{2s_2-1})$, using S-duality as in the proof of 2.1. Under this isomorphism and natural morphisms which we will consider, the classes y_i of 2.7 correspond to classes g_i of Theorem 2.15.

We begin by determining the relevant portion of $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge P_{-1}))$. This is the E_2-term of the ASS converging to $e_*(P_3 \wedge P_{-1})$. After shifting by 8 dimensions, using 2.6, this will correspond to the case $s_1 = 2$, $s_2 = 4$ of Theorem 2.7.

There is a cofibration $R \xrightarrow{i} P_{-1} \to S^0$ which induces a short exact sequence in $H^*(-)$. We apply $P_3\wedge$ and obtain a long exact sequence

$$\begin{aligned} \mathrm{Ext}_{A_2}^{s-1,t}(H^*(P_3 \wedge P_{-1})) &\xrightarrow{\lambda_*} \mathrm{Ext}_{A_2}^{s-1,t}(H^*P_3) \\ &\xrightarrow{\partial} \mathrm{Ext}_{A_2}^{s,t}(H^*(P_3 \wedge R)) \xrightarrow{i_*} \mathrm{Ext}_{A_2}^{s,t}(H^*(P_3 \wedge P_{-1})). \end{aligned} \tag{2.28}$$

The upper edge of $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge R))$ has slope $1/5$, while that of $\mathrm{Ext}_{A_2}(H^*P_3)$ has slope $1/2$, and so between slope $1/5$ and $1/2$, $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge P_{-1}))$ is isomorphic to $\mathrm{Ext}_{A_2}(H^*P_3)$, which is given in [9]. However, the region in which we are interested lies below slope $1/5$.

We study (2.28) with $t-s=8d-2$ and $s\le[(4d-2)/3]$. In this range, $\operatorname{Ext}_{A_2}^{*,*+8d-1}(H^*P_3)\approx\bigoplus\operatorname{Ext}_{A_1}^{*,*+8d-1}(\Sigma^{8i-1}\mathbf{Z}_2)$. We will show that these elements pull back under λ_* to elements of $e_{8d-1}(P_3\wedge P_{-1})$. This implies that in this range (a) $\partial=0$ and hence i_* is injective, and (b) elements in $\operatorname{im}(i_*)$ are not hit by differentials in the ASS converging to $e_*(P_3\wedge P_{-1})$. This yields $e_{8d-2}(P_3\wedge P_{-1})\approx e_{8d-2}(P_3\wedge R)$ in filtration $\le[(4d-2)/3]$, and hence the result claimed for $e^{8a+8b-8d}(P^{8a-4}\wedge P^{8b-8})$ follows by duality. Note that the last relation of Theorem 2.15 becomes, in $P_3\wedge P_7$, $h_0^{4k+3}g_{k+1}$ if $d=3k+3$, which corresponds to the relations parametrized by $(3,1,3,1,4)$ and $(-1,0,0,2,4)$ in 2.7.

To see this pulling back, note that, using 2.6,

$$e_{8d-1}(P_3\wedge P_{-1})\approx[P^0_{-\infty},\Sigma^{-8d}P_3\wedge e]\approx[P^0_{-\infty},P_{3-8d}\wedge e].$$

There is a filtration-0 map $P^0_{-\infty}\to P_{3-8d}\wedge e$ given by inclusion and collapse. Since this map is nonzero in $H^{-1}(-)$, the corresponding element of $e_{8d-1}(P_3\wedge P_{-1})$ projects to the nonzero element of $\operatorname{Ext}_{A_2}^{0,8d-1}(H^*P_3)$ in (2.28). The v_1^4-action on $e_*(-)$ implies that towers in $\operatorname{Ext}_{A_2}^{*,*+8d-1}(H^*P_3)$ which begin in filtration $4i$ also pull back to elements of $\operatorname{Ext}_{A_2}(H^*(P_3\wedge P_{-1}))$.

Other values of (s_1,s_2) in Theorem 2.7 are handled by cofibration sequences involving a change of 1 in one s_i. We illustrate with one case. The Ext charts in all cases can be viewed at [2].

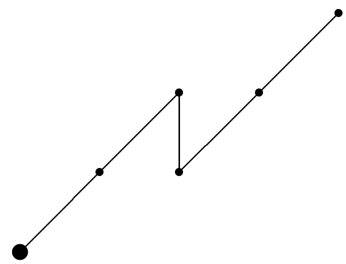
The case $(s_1,s_2)=(2,1)$ is obtained using the exact sequence

$$\operatorname{Ext}_{A_2}^{s,t}(H^*(P_3\wedge P^0_{-1}))\xrightarrow{i_*}\operatorname{Ext}_{A_2}^{s,t}(H^*(P_3\wedge P_{-1}))\to\operatorname{Ext}_{A_2}^{s,t}(H^*(P_3\wedge P_1))\tag{2.29}$$

and the information just obtained about $\operatorname{Ext}_{A_2}(H^*(P_3\wedge P_{-1}))$. The chart for $P_3\wedge P^0_{-1}$ is obtained from charts of $\Sigma^{-1}P_3$ and P_3 by inserting boundary morphisms (d_1-differentials) corresponding to the action of h_0. Due to the period-8 shift between values $s_1=0$ and 4, the relation in $P_3\wedge P_{-1}$ corresponding to (2.9) is $2^{4i+1}g_{d-i}$, which also appears in 2.15. The corresponding relation in $P_3\wedge P_1$ is supposed to be $2^{4i}g_{d-i}$. The elements of $\operatorname{Ext}_{A_2}(H^*(P_3\wedge P^0_{-1}))$ which hit the elements $h_0^{4i+1}g_{d-i}$ in (2.29) are the beginnings of the "lightning flashes" obtained from the $\operatorname{Ext}_{A_1}(\mathbf{Z}_2)$-like parts of $\operatorname{Ext}_{A_2}(H^*(\Sigma^jP_3))$. Indeed, $\operatorname{Ext}_{A_2}(H^*(P_3\wedge P^0_{-1}))$ has summands like Diagram 2.30

with initial element in $(t-s,s)=(8d-2,4i)$ throughout the relevant range, and these initial elements are mapped nontrivially by i_* in (2.29).

Diagram 2.30.



The extra relation in 2.7 parametrized by $(2,1,2,1,1)$ says that if $d=6k+2$, then $2^{8k+2}g_{2k+1}$ is in the image of i_* in (2.29). This is hit under i_* by powers of v_2^8 acting on an element of $\mathrm{Ext}_{A_2}^{2,16}(H^*(P_3\wedge P_{-1}^0))$ corresponding to an element of $\mathrm{Ext}_{A_2}^{2,17}(H^*P_3)$ which in [9, p.302] is the circled element in D_3' in $t-s=23$.

We emphasize that we do not perform a direct verification that this element of $\mathrm{Ext}_{A_2}^{2,16}(H^*(P_3\wedge P_{-1}^0))$[1] maps nontrivially under i_*. Instead, we rely on the computer calculations of various $\mathrm{Ext}_{A_2}(H^*(P_{2s_1-1}\wedge P_{2s_2-1}))$, and when we see new relations arising as s_i increases we can always interpret it in the relevant exact sequence. The action of v_2^8 guarantees promulgation of nonzero boundary morphisms. Also, our knowledge of $\mathrm{Ext}_{A_2}(H^*(P_{2s_1-1}\wedge P_{2s_2-1}^{2s_2}))$ (from [9]) guarantees that in the relevant range (below a line of slope 1/6), $\mathrm{Ext}_{A_2}(H^*(P_{2s_1-1}\wedge P_{2s_2-1}^{2s_2}))$ has very few classes other than its regular family of lightning flashes, and these classes are all obtained from v_2^8 acting on elements in the first 48 dimensions. This method accounts for all relations in (2.11).

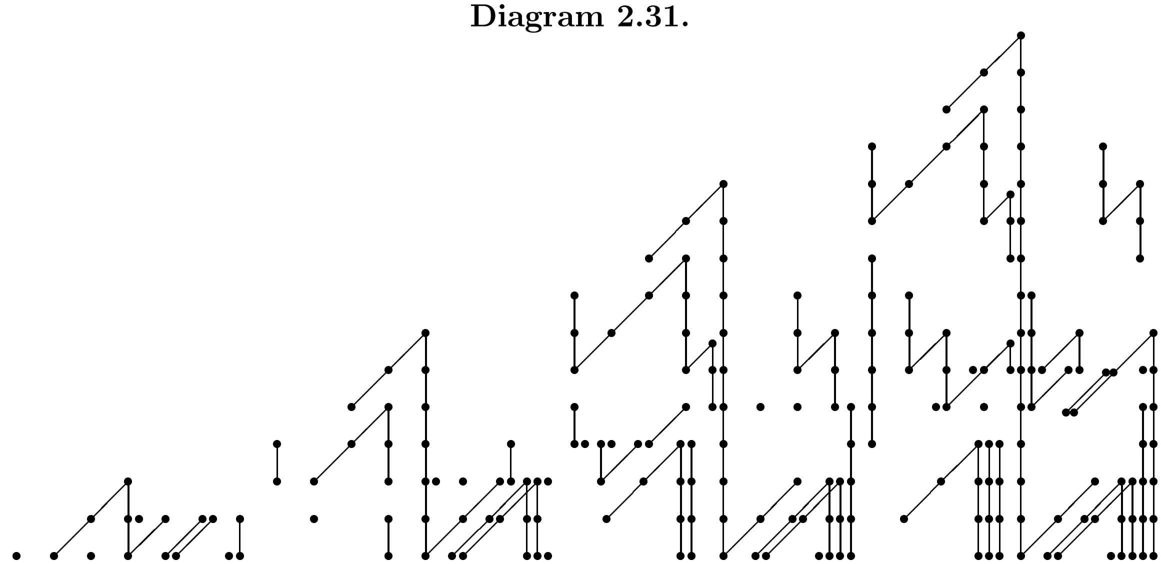
The relations (2.8) and (2.9) of 2.7 are guaranteed by 2.1. It is important that these relations are strictly valid, not just up to higher filtration, as is the case with (2.10) (the $2^{4i+5}x_i$ being the higher filtration). The key relations (2.10) are implied by naturality from the corresponding relations in Theorem 2.15.

The final relation (2.12) specifies the filtration in which all h_0-towers from filtration 0 cease to exist. It could be derived from the other relations (2.8) to (2.11), although the indeterminacy in (2.10) makes such a derivation delicate. This relation (2.12) is listed primarily as a convenience for our later arguments. ■

[1]or similar elements in other cases

We illustrate in Diagram 2.31 with the chart of $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge P_1)$ in $t-s \leq 34$. Here we use a different basis in filtration 0 than was used for $P_3 \wedge R$. The one used here is designed to simplify the appearance of the chart. Note that our analysis only involves the h_0-towers emanating from filtration 0 in dimensions $8d-2$, a very small portion of the chart.

Diagram 2.31.



We close this section by providing the three omitted proofs of lemmas and a corollary.

Proof of Lemma 2.20. Let M denote the A_2-module $H^*P_3 \otimes Q$. One can verify (by hand, even) the following relation in M_{8d-1} for $1 < j \leq d$. Here $\mathrm{Sq}(R)$ is Milnor basis notation.

$$
\begin{aligned}
&\mathrm{Sq}(1)(x_1^{8j-1}x_2^{8d-8j-1}) + \mathrm{Sq}(5)\sum_{i=0}^{1} x_1^{8j-8i-1}x_2^{8d-8j+8i-5} \\
&+ \mathrm{Sq}(3,2)(x_1^{8j-1}x_2^{8d-8j-9}) + \mathrm{Sq}(6,1)\sum_{\substack{i=j-1\\ i\neq j}}^{d-1} x_1^{8i-1}x_2^{8d-8i-9} \\
&+ \mathrm{Sq}(0,2,1)(x_1^{8j-9}x_2^{8d-8j-5}) + \mathrm{Sq}(4,2,1)\sum_{i=j}^{d-2} x_1^{8i-1}x_2^{8d-8i-17}
\end{aligned}
\tag{2.32}
$$

The relation must be modified when $j = 1$ because then $x_1^{8j-9} \notin H^*P_3$. However, it turns out to be irrelevant because $h_0^4 g_1 = 0$.

Let g_i, as in the statement of the lemma, be dual to $x_1^{8i-1}x_2^{8d-8i-1}$, and let g_i' be dual to $x_1^{8i-1}x_2^{8d-8i-9}$. We claim

$$v_1^4 g_i' = h_0^4 \sum_{j \in S_i} g_j, \tag{2.33}$$

where $j \in S_i$ iff $\mathrm{Sq}(6,1)(x_1^{8i-1}x_2^{8d-8i-9})$ appears in the relation (2.32). By (2.32), this will yield the claim of the lemma.

The portion of the minimal A_2-resolution of $\mathbf{Z}_2$ needed to obtain v_1^4 is (using $t-s$ as subscripts):

$$\begin{aligned}
H_0 &: \mathrm{Sq}(1)G;\\
H_1 &: \mathrm{Sq}(2)G;\\
H_3 &: \mathrm{Sq}(4)G;\\
J_0 &: \mathrm{Sq}(1)H_0;\\
J_2 &: \mathrm{Sq}(2)H_1 + \mathrm{Sq}(3)H_0;\\
J_3 &: \mathrm{Sq}(1)H_3 + \mathrm{Sq}(0,1)H_1 + \mathrm{Sq}(4)H_0;\\
K_0 &: \mathrm{Sq}(1)J_0;\\
K_3 &: \mathrm{Sq}(1)J_3 + \mathrm{Sq}(2)J_2 + \mathrm{Sq}(4)J_0;\\
L_8 &: \mathrm{Sq}(3,1)K_3 + \mathrm{Sq}(6,1)K_0.
\end{aligned}$$

Let $\mathbf{D}$ and $\mathbf{C}$ be minimal resolutions of M and $\mathbf{Z}_2$, respectively, and let $\phi : \mathbf{D} \to \mathbf{C}$ be a morphism such that ϕ_0 is dual to $y_i' := x_1^{8i-1}x_2^{8d-8i-9}$.

$$\begin{tikzcd}
D_0 \arrow[d, "\phi_0"'] & D_1 \arrow[l, "\partial_1"'] \arrow[d, "\phi_1"'] & D_2 \arrow[l, "\partial_2"'] \arrow[d, "\phi_2"'] & D_3 \arrow[l, "\partial_3"'] \arrow[d, "\phi_3"'] & D_4 \arrow[l, "\partial_4"'] \arrow[d, "\phi_4"'] \\
C_0 & C_1 \arrow[l, "\delta_1"'] & C_2 \arrow[l, "\delta_2"'] & C_3 \arrow[l, "\delta_3"'] & C_4 \arrow[l, "\delta_4"']
\end{tikzcd}$$

In this proof, OT means "other terms." If h is a generator of D_1, then $\phi_1(h) = \mathrm{Sq}(2,1)H_3 + OT$ iff $\mathrm{Sq}(6,1)y_i'$ is part of $\partial_1(h)$, which corresponds to one of the relations (2.32). This is seen by noting that the only ways in which $\mathrm{Sq}(6,1)G$ occurs in $\mathrm{im}(\delta_1)$ are as $\delta_1(\mathrm{Sq}(2,1)H_3) = \mathrm{Sq}(6,1)G$ and $\delta_1(\mathrm{Sq}(4,1)H_1) = (\mathrm{Sq}(6,1) + \mathrm{Sq}(0,3))G$, but since $\mathrm{Sq}(0,3)$ does not appear in the relations (2.32), it cannot appear in $\mathrm{im}(\phi_0\partial_1)$.

For each generator h of D_1, there is a corresponding generator j of D_2 which maps to $\mathrm{Sq}(1)h + OT$. Then $\phi_1(h) = \mathrm{Sq}(2,1)H_3 + OT$ iff $\phi_2(j) = \mathrm{Sq}(2,1)J_3 + OT$. Some care is required to see this. The sketch of argument is $\phi_1(h) = \mathrm{Sq}(2,1)H_3 + OT$ iff $\phi_1\partial_2(j) = \mathrm{Sq}(3,1)H_3 + OT$ iff $\delta_2\phi_2(j) = \mathrm{Sq}(3,1)H_3 + OT$ iff $\phi_2(j) = \mathrm{Sq}(2,1)J_3 + OT$. A main worry is that $\delta_2(\mathrm{Sq}(5)J_3)$ contains the term $\mathrm{Sq}(3,1)H_3$, and so $\mathrm{Sq}(5)J_3$ could conceivably appear in $\phi_2(j)$ in place of $\mathrm{Sq}(2,1)J_3$. However, $\delta_2(\mathrm{Sq}(5)J_3)$ also contains $(\mathrm{Sq}(5,1)+\mathrm{Sq}(1,0,1))H_1$, which could be in $\mathrm{im}(\phi_1\partial_2)$ only if $\phi_1(h)$ contained $(\mathrm{Sq}(4,1)+\mathrm{Sq}(0,0,1))H_1$, and this would require that the relation for h contain $\mathrm{Sq}(0,3)$, which we have already observed does not happen.

A similar argument shows that $\phi_2(j) = \mathrm{Sq}(2,1)J_3+OT$ iff $\phi_3(k) = \mathrm{Sq}(2,1)K_3+OT$, where $\partial_3(k) = \mathrm{Sq}(1)(j) + OT$. Here again, the most delicate part is ruling out the possibility that $\phi_3(k)$ might contain $\mathrm{Sq}(5)K_3$ rather than $\mathrm{Sq}(2,1)K_3$. This time we observe that $\delta_3(\mathrm{Sq}(5)K_3)$ contains $(\mathrm{Sq}(7) + \mathrm{Sq}(1,2))J_2$, which cannot appear as a term in $\mathrm{im}(\phi_2\partial_3)$.

The fourth step is the easiest: $\phi_3(k) = \mathrm{Sq}(2,1)K_3+OT$ iff $\phi_4(\ell) = L_8$, where $\partial(\ell) = \mathrm{Sq}(1)(k) + OT$. Now, if $\epsilon : C_4 \to \mathbf{Z}_2$ is dual to L_8, then $\epsilon\phi_4 : D_4 \to \mathbf{Z}_2$ represents $v_1^4 g_i'$. By the analysis just completed of the map of resolutions, $\epsilon\phi_4$ sends nontrivially generators which correspond to h_0^4 times the duals of basis elements $x_1^{8j-1}x_2^{8d-8j-1}$ of M for which the relation $\mathrm{Sq}(1)x_1^{8j-1}x_2^{8d-8j-1}+OT$ contains the term $\mathrm{Sq}(6,1)x_1^{8i-1}x_2^{8d-8i-9}$, establishing (2.33). ■

Proof of Lemma 2.23. We prove that $h_0\theta^*(g_d) = 0$ and $h_0^4\theta^*(g_1 + \cdots + g_{d-1}) = 0$.

For the first, we first observe that $h_0 g_d \neq 0$ in $\mathrm{Ext}_{A_2}(H^*P_3 \otimes Q)$ because

$$\mathrm{Sq}^1(x_1^{8d-1}x_2^{-1}) = \mathrm{Sq}^2(x_1^{8d-2}x_2^{-1}) \text{ in } H^*P_3 \otimes Q,$$

and so there is a relation involving $\mathrm{Sq}^1(x_1^{8d-1}x_2^{-1})$. However, in $H^*(P_3\wedge R)$, $\mathrm{Sq}^2(x_1^{8d-2}x_2^{-1})$ contains another term, $x_1^{8d-2}x_2^{1}$. There is no relation in $H^*(P_3\wedge R)$ involving $\mathrm{Sq}^1(x_1^{8d-1}x_2^{-1})$ because for all monomials m in degree $8d-1-t$, $t = 1$, 2, or 4, $\mathrm{Sq}^t(m)$ is the sum of an even number of monomials, with the single exception of $\mathrm{Sq}^1(x_1^{8d-1}x_2^{-1})$. Thus $h_0 g_d = 0$ in $\mathrm{Ext}_{A_2}(H^*(P_3 \wedge R))$.

For the second, we observe that there is an A_2-module homomorphism

$$H^*(P_3 \wedge R) \to H^*(\Sigma^{8d-9}P_1)$$

through dimension $8d-1$ sending

$$\begin{array}{rcl} x_1^3x_2^{8d-11} & \mapsto & \sigma^{8d-9}x^1, \\ x_1^{8i+3}x_2^{8d-8i-9} & \mapsto & \sigma^{8d-9}x^3,\ 0 \le i < d, \\ x_1^{8i-1}x_2^{8d-8i-1} & \mapsto & \sigma^{8d-9}x^7,\ 1 \le i < d. \end{array}$$

To do this, we need merely note that, under this morphism, the relations which give rise to the elements of $\mathrm{Ext}_{A_2}^{1,t}(H^*(P_3 \wedge R))$ in $t = 8d-6$, $8d-5$, $8d-3$, and $8d-1$ map to relations in $H^*(\Sigma^{8d-9}P_1)$. Ignoring terms acting on classes of degree less than $8d-8$ (which certainly map to 0), the relations in $H^*(P_3 \wedge R)$ are

$$\begin{array}{l} \mathrm{Sq}^2(x_1^3x_2^{8d-11}), \\ \mathrm{Sq}^1(x_1^{8i+3}x_2^{8d-8i-9}) + \mathrm{Sq}(0,1)(x_1^3x_2^{8d-11}),\ 1 \le i < d, \\ (\mathrm{Sq}^3 + \mathrm{Sq}(0,1))(x_1^3x_2^{8d-9}), \\ \mathrm{Sq}^1(x_1^7x_2^{8d-9}) + \mathrm{Sq}^5(x_1^{11}x_2^{8d-17}) + \mathrm{Sq}(2,1)(x_1^3x_2^{8d-9}), \\ \mathrm{Sq}^1(x_1^{8i-1}x_2^{8d-8i-1}) + \mathrm{Sq}^5(x_1^{8i-5}x_2^{8d-8i-1} + x_1^{8i+3}x_2^{8d-8i-9}) \\ \qquad +(\mathrm{Sq}^7 + \mathrm{Sq}(4,1) + \mathrm{Sq}(0,0,1))(x_1^3x_2^{8d-11}),\ 2 \le i < d. \end{array}$$

For example, these relations correspond, when $d = 5$, to the classes in filtration 1 in Diagram 2.16 with $33 \le t-s \le 38$.

One easily checks that these relations map to 0 in $H^*(\Sigma^{8d-9}P_1)$. For example, the image of the fourth relation is

$$\sigma^{8d-9}(\mathrm{Sq}^1 x^7 + \mathrm{Sq}^5 x^3 + \mathrm{Sq}(2,1)x^3) = \sigma^{8d-9}(x^8 + 0 + x^8) = 0.$$

Thus there is a morphism

$$\mathrm{Ext}_{A_2}^{s,t}(H^*\Sigma^{8d-9}P_1) \to \mathrm{Ext}_{A_2}^{s,t}(H^*(P_3 \wedge R))$$

for $t-s \le 8d-2$ sending the generator G in $(s,t) = (0, 8d-2)$ to $g_1 + \cdots + g_{d-1}$. Since $h_0^4G = 0$, we must have $h_0^4(g_1 + \cdots + g_{d-1}) = 0$. ■

Proof of Corollary 2.13. This proof is quite similar to the discussion of Example 1.3 and to the proof of [3, 1.3] given on [3, p.524]. We begin with case 1.

We first observe that if, for some $i < h$,

$$\nu(k_i) < 4i - c_{s_1} \text{ or } \nu(k_{d-i}) < 4i - c_{s_2},$$

then our element $\sum k_i y_i$ must be nonzero, as it cannot be obtained as a consequence of relations (2.8)–(2.12). Proving this requires a bit of argument, which occupies the next two paragraphs.

The relations (2.8)–(2.12) can be manipulated into a form similar to that of [3, 1.4], except that where [3, 1.4] has a coefficient 2^e, we have 2^{4e-c_s}, and the binomial coefficients occurring as coefficients are only valid mod 2. For example, if $h = 4$ and $t = 1$, the relations can be arranged, after some manipulation, as the rows of the matrix

$$\begin{pmatrix}
2^4 &&&&&&&&&& \\
& 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & \\
&& 2^8 & 2^8 & 2^8 & 2^8 & 2^8 & 2^8 & 2^8 && \\
&&& 2^8 & 2\cdot 2^8 & 2^8 & 2\cdot 2^8 & 2^8 & 2\cdot 2^8 && \\
&&&& 2^{12} & 2\cdot 2^{12} & 2^{12} & 2\cdot 2^{12} &&& \\
&&&&& 2^{12} & 2^{12} & 2\cdot 2^{12} &&& \\
&&&&&& 2^{16} &&&& \\
&&&&&&& 2^{16} &&& \\
&&&&&&&& 2^{12} && \\
&&&&&&&&& 2^8 & \\
&&&&&&&&&& 2^4
\end{pmatrix} \tag{2.34}$$

with the understanding that each 2^{4i} entry is really an unspecified odd integer times 2^{4i}, while each $2 \cdot 2^{4i}$ entry is really an unspecified even integer times 2^{4i}.

If, say, $\nu(k_1) \geq 4$, $\nu(k_2) \geq 8$, but $\nu(k_3) = 11$, then use (2.8) to eliminate $k_1y_1 + k_2y_2$. Now the third row of the above matrix of relations can be used to eliminate k_3y_3 at the expense of changing our element to $\sum_{i\geq 4} u_i 2^{11} y_i$ with u_i odd. Use the fourth row of (2.34) to change this to

$$w_5 2^{11} y_5 + A_6 2^{11} y_6 + w_7 2^{11} y_7 + A_8 2^{11} y_8 + \cdots$$

with w_i odd and A_i even. No subsequent row can eliminate the lead term of this, and so the original class was nonzero. This procedure works in general, and if it is $\nu(k_{d-i})$ which is too small, the matrix of relations can be considered from right to left, and the same argument invoked.

Now we can assume that $\nu(k_i) \geq 4i - c_{s_1}$ and $\nu(k_{d-i}) \geq 4i - c_{s_2}$ for $i < h$, and so the class in question reduces (using (2.8) and (2.9)) to $\sum_{i=h}^{2h} k_i y_i$ with $\nu(k_i) \geq 4h - t$ for all i. We will rewrite k_{h+j} as $2^{4h-t} k'_j$. By (2.12), $2^{4h-t+1} y_i = 0$ for all i, and so the analysis becomes effectively a mod 2 analysis, and the indeterminacy in (2.10) becomes inconsequential.

For example, in the case, $h = 4$, $t = 1$, considered above, restricting attention to columns 4 through 7,[2] we wish to know whether a given vector

$$\mathbf{v} = (k_0' 2^{15}, k_1' 2^{15}, k_2' 2^{15}, k_3' 2^{15})$$

is in the span of the matrix

$$M = \begin{pmatrix} 2^8 & A_1 2^8 & u_1 2^8 & A_2 2^8 \\ 0 & 2^{12} & A_3 2^{12} & u_2 2^{12} \\ 0 & 0 & 2^{12} & u_3 2^{12} \end{pmatrix} \tag{2.35}$$

considered mod 2^{16}, with u_i odd and A_i even. The only way to obtain the first component, $k_0' 2^{15}$, of our vector is by $2^7 k_0'$ times the first row of M. Since we are working mod 2^{16}, the precise values of the coefficients A_1, u_1, and A_2 do not matter, only their parity. Working similarly with the other entries and rows, we find that $\mathbf{v}$ is in the span of M iff the mod 2 vector (k_0', k_1', k_2', k_3') is in the span of

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

which is obtained from M by just considering parity of coefficients. Now case 1 of the corollary follows from the case $(r = 3, h = 4)$ of Lemma 2.38.

The proof of case 2 of the corollary is almost identical to that of case 1. In each of the three subcases, the analysis of the relations reduces, similarly to case 1, to determining whether a vector

$$\mathbf{v} := (k_0' 2^{4h-t}, k_1' 2^{4h-t}, \dots, k_{h-1}' 2^{4h-t})$$

is in the span of a mod 2^{4h-t+1} matrix whose rows are multiples by various 2-powers of the corresponding rows of the matrix M_h of Lemma 2.38 with $r = 2$. The condition (2.14) is what Lemma 2.38 says is necessary for our vector to not be in the span of the relations.

The reason that case 1 of 2.13 uses $r = 3$ in Lemma 2.38 while case 2 uses $r = 2$ is that the columns in the matrix (such as (2.35)) of case 1 are h through $2h - 1$, while those in case 2 are $h + 1$ through $2h$, and this causes a corresponding change in exactly which binomial coefficients occur in the various rows. Indeed the relevant

[2] column 8 is not required

matrix M_h is obtained by first manipulating the relations into a form illustrated by the prototype matrix (2.34), and then considering the $(h-1)\times h$ submatrix whose upper left hand element is in position (h,h) or $(h+1,h+1)$ (depending on which case we are in) of the larger matrix. Since, for relevant values of i, the (i,j)th entry of the big matrix is $2^{\mathrm{pwr}}\binom{[(i-2)/2]+j-i}{j-i}$, the effect on the entries of M_h is obtained.

But why the difference in which submatrix we consider? The difference is that in case 1, we use (2.8) for $i\le h-1$, while in case 2, we use it for $i\le h$. This difference is due to the fact that, with e as in (2.12), we have $4h-c_{s_1}=e$ in case 1, but $4h-c_{s_1}<e$ in case 2.

We require h to be even in the third subcase of case 2, for if h is odd, the last subcase of (2.11) implies $2^{4h-1}y_{h+1}=0$ (when $d=3h+1$), and then (2.12) implies that 2^{4h-1} annihilates all y_i in this dimension, and so an element $\sum k_iy_i$ satisfying the conditions of the third subcase of case 2 of 2.13, except that h is not even, would be 0, contrary to our conclusion.

The first two subcases of Case 3 are almost identical to Case 1. One apparent difference is the use of $d=3h+3$ here rather than $d=3h$ as in Case 1. The reason for using $3h+3$ here is to make the congruence hypothesis in Theorem 1.1.3 look nicer and be the same in all subcases.

If $h=3$ in the first subcase of Case 3, the matrix of relations would be just like (2.34) except that the last three entries would become 2^{11}, 2^7, and 2^3. As already discussed, these tail entries do not play a role in the results that we derive. If $h=3$ in the second subcase of Case 3, the exponents of 2 which are divisors of the successive rows of the matrix of relations when it is put in a form similar to (2.34) are 3,4,7,8,11,12,15,13,9,5,1. The fact that the largest of these is 15, rather than 16 as it was in subcase 1, is what drives our hypothesis about wanting various ν-values to be 14. The way in which we have selected which values of (s_1,s_2) to use is to use the largest values which have a given maximum order in the group corresponding to a fixed value of d.

The third subcase of Case 3 contains one slightly different feature. We illustrate when $h=4$. The reader can easily adapt the argument to the general case.

The relations are 2^1y_1, 2^5y_2, 2^9y_3, $2^{13}y_4$, $2^{12}y_{10}$, 2^8y_{11}, 2^4y_{12}, $2^{16}y_i$, $5\le i\le 9$, $2^4\sum y_j \mod 2^5$, $2^8\sum jy_j \mod 2^9$, and $2^{12}\sum\binom{j}{2}y_j \mod 2^{13}$. These can be manipulated in matrix form to

$$\begin{pmatrix}
2^1 &&&&&&&&&&& \\
& 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & 2^4 & \\
&& 2^5 & 2^5 & 2^5 & 2^5 & 2^5 & 2^5 & 2^5 & 2^5 & 2^5 & \\
&&& 2^8 & 2\cdot 2^8 & 2^8 & 2\cdot 2^8 & 2^8 & 2\cdot 2^8 & 2^8 && \\
&&&& 2^9 & 2\cdot 2^9 & 2^9 & 2\cdot 2^9 & 2^9 & 2\cdot 2^9 && \\
&&&&& 2^{12} & 2^{12} & 2\cdot 2^{12} & 2\cdot 2^{12} &&& \\
&&&&&& 2^{13} & 2^{13} & 2\cdot 2^{13} &&& \\
&&&&&&& 2^{16} &&&& \\
&&&&&&&& 2^{16} &&& \\
&&&&&&&&& 2^{12} && \\
&&&&&&&&&& 2^8 & \\
&&&&&&&&&&& 2^4
\end{pmatrix} \tag{2.36}$$

where, as with (2.34), each 2^i (resp. $2\cdot 2^i$) means an odd (resp. even) integer times 2^i.

We are given a row $(k_1, \ldots, k_{12})$ with $\nu(k_i) \geq 15$ for $5 \leq i \leq 9$. Similarly to Case 1, one can show that if $\nu(k_i) < 4i-3$ for some $i \leq 4$ or $\nu(k_i) < 4(13-i)$ for some $i \geq 10$, then the row is not in the span of (2.36).

Now assume $\nu(k_i) \geq 4i-3$ for $i \leq 4$ and $\nu(k_i) \geq 4(13-i)$ for $i \geq 10$. The relations $2^1 y_1, \ldots, 2^4 y_{12}$ may be used to eliminate $k_1, k_2, k_3, k_4, k_{10}, k_{11}$, and k_{12}. We use rows 5, 6, and 7 of (2.36) to try to eliminate the other k_i.

For $0 \leq j \leq 4$, let $k_j' = k_{5+j}/2^{15} \mod 2$. We find that $\sum k_i y_i = 0$ iff $(k_0', \ldots, k_4')$ is in the span of the mod 2 matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 \\ && 1 & 1 & 0 \end{pmatrix}. \tag{2.37}$$

The only combination of rows which could possibly equal $(k_0', \ldots, k_4')$ is

$$k_0' R_1 + k_1' R_2 + (k_0' + k_1' + k_2') R_3,$$

where R_1, R_2, R_3 are the rows of (2.37). Consideration of columns 4 and 5 shows that our vector is 0 iff $k_0' + k_1' + k_2' + k_3' \equiv 0 \mod 2$ and $k_1' + k_2' + k_3' + k_4' \equiv 0 \mod 2$.

The first of these necessary conditions, if applied to an axial map by the method employed in Case 3 of Section 3, would yield nonimmersions already proved in [3], and so we do not pursue it. The second of the necessary conditions is the one quoted in the third subcase of 2.13.3.

The novelty in this final subcase was our obtaining two necessary conditions, and just choosing the second. ■

We close this section with a lemma used in the above proof.

Lemma 2.38. *Let $r = 2$ or 3, and let M_h denote the $(h-1)\times h$ matrix over $\mathbf{Z}/2$ with (i,j)th entry*
$$\binom{[(h+i-r)/2]+j-i}{j-i}$$
for $1 \le i \le h-1$ and $1 \le j \le h$. A vector $\mathbf{v} = (k_0, k_1, \dots, k_{h-1})$ is in the span of M_h iff
$$\sum_{j=0}^{h-1} \tbinom{h-1}{j} k_j = 0 \in \mathbf{Z}/2.$$

Proof. For $1 \le i \le h-1$, let R_i denote the ith row of M_h. Using combinatorial identities, one can show that the combination of rows which agrees with $\mathbf{v}$ in the first $h-1$ entries is
$$\sum_{i=1}^{h-1} R_i \sum_{j=0}^{i-1} \binom{[(i+h-r+1)/2]}{i-j-1} k_j.$$
Its entry in the last column is $\sum_{j=0}^{h-2} \binom{h-1}{j} k_j$, which must equal k_{h-1}, yielding the lemma. ■

3. Deduction of nonimmersions

In this section we deduce Theorem 1.1 from Corollary 2.13.

Proof of Theorem 1.1.1. We use the first type of axial map in 1.2. Thus it will suffice to prove that, if $t = 1$ or 2,
$$(X_1 + X_2)^{2M-h} \ne 0 \in e^*(P^{8M+8h-2t-2} \wedge P^{8M+8h-2t-2}) \tag{3.1}$$
if $M \equiv 0 \mod \frac{1}{2}p(h+1)$ and $\alpha(M) = 4h+1-t$. By Corollary 2.13.1 with $a = b = M+h$, (3.1) will hold provided $\nu\binom{2M-h}{M+h-i} \ge 4h-t$ for $h \le i \le 2h$ and
$$\nu\left(\sum_{j=0}^{h-1} \tbinom{h-1}{j}\tbinom{2M-h}{M-j}\right) = 4h-t. \tag{3.2}$$

The sum in (3.2) equals $\binom{2M-1}{M}$, and since, $\nu\binom{2M-1}{M} = \alpha(M)-1$, our result follows from the following lemma. ■

Lemma 3.3. *If* $2^e \le h < 2^{e+1}$ *and* $M \equiv 0 \mod 2^e$, *then* $\nu\binom{2M-h}{M-j} \ge \alpha(M)-1$ *for* $0 \le j \le h$.

Proof. We must show that if $\delta_1 \ge \delta_2 \ge 0$ and $\delta_1+\delta_2 = h$, then

$$\alpha(M-\delta_1)+\alpha(M-\delta_2) \ge \alpha(M)-1+\alpha(2M-h). \qquad (3.4)$$

Case 1: $\delta_1 < 2^e$.

Let $M = 2^e m$. Then (3.4) reduces to

$$\alpha(m-1)+\alpha(2^e-\delta_1)+\alpha(m-1)+\alpha(2^e-\delta_2) \ge \alpha(m)-1+\alpha(m-1)+\alpha(2^{e+1}-h),$$

which is true since $\alpha(m-1) \ge \alpha(m)-1$ and

$$\alpha(2^e-\delta_1)+\alpha(2^e-\delta_2)-\alpha(2^{e+1}-h) = \nu\binom{2^{e+1}-h}{2^e-\delta_1} \ge 0.$$

Case 2: $\delta_1 \ge 2^e$, $M = 2^{e+1}m$.

Same argument as Case 1, with e replaced by $e+1$.

Case 3: $\delta_1 = 2^e+\Delta \ge 2^e$, $M = 2^e+2^{e+1}n$.

Here (3.4) reduces to

$$\alpha(n-1)+\alpha(2^{e+1}-\Delta)+\alpha(n)+\alpha(2^e-\delta_2) \ge \alpha(n)+\alpha(n)+\alpha(2^e-\Delta-\delta_2),$$

which is true since

$$\alpha(2^{e+1}-\Delta)+\alpha(2^e-\delta_2)-\alpha(2^e-\Delta-\delta_2) = \nu\binom{2^{e+1}+2^e-\Delta-\delta_2}{2^{e+1}-\Delta}+1.$$

■

Proof of Theorem 1.1.2. Using the second type of axial map in 1.2, we have for $1 \le s_i \le 4$ that if

$$(X_1+X_2)^{2^L-m+1} \ne 0 \in e^*(P^{8m-2s_1} \wedge P^{8(2^{L-3}-2m+1+d)-2s_2}),$$

then

$$P^{8m-2s_1} \not\subseteq \mathbf{R}^{16m-8d-10+2s_2}. \qquad (3.5)$$

With $m = M + h + 1$, the three subcases in 2.13.2 turn (3.5) into the three nonimmersions of 1.1.2. We will see that the hypotheses of 1.1.2 (namely $\alpha(M) = 4h - t$ and $M \equiv 0 \mod p(h)$) imply that the conditions of 2.13.2 are satisfied when

$$k_i = \binom{2^L-m+1}{m-i} = (-1)^{m-i}\binom{2m-i-2}{m-2}, \tag{3.6}$$

which will imply 1.1.2. The sum condition of (2.14) becomes

$$\nu\left(\sum_{j=0}^{h-1}\binom{h-1}{j}\binom{2^L-M-h}{M-j}\right) = 4h - t, \tag{3.7}$$

which is satisfied since the LHS equals $\nu\binom{2^L-M-1}{M} = \alpha(M)$. The other condition of 2.13.2 is the content of the following lemma. ■

Lemma 3.8. *If $2^e < h \le 2^{e+1}$ and $M \equiv 0 \mod 2^{e+1}$, then*

$$\nu\binom{2M+h-j-1}{M-j} \ge \alpha(M) \text{ for } 0 \le j \le h-1. \tag{3.9}$$

Proof. Letting $H = h - 1$ and $M = 2^{e+1}n$, the desired inequality is equivalent to

$$\alpha(n-1) + \alpha(2^{e+1} - j) + \alpha(n) + \alpha(H) \ge \alpha(n) + \alpha(H-j) + \alpha(n)$$

for $0 \le j \le H < 2^{e+1}$, and this reduces to

$$\nu(n) + e + 1 \ge \nu\binom{H}{j} + \nu(j), \tag{3.10}$$

which is implied by the following result. ■

Proposition 3.11. *If $0 \le j \le H < 2^{e+1}$, then $\nu\binom{H}{j} + \nu(j) \le e$.*

Proof. The proof is by induction on e. We write $H = 2H' + \epsilon_0$ and $j = 2j' + \epsilon_1$ with $\epsilon_i = 0$ or 1. The hardest case is when $\epsilon_0 = 0$ and $\epsilon_1 = 1$. In this case we have

$$\begin{aligned}
\nu\binom{2H'}{2j'+1} &= \alpha(2j'+1) + \alpha(2H' - 2j - 1) - \alpha(2H') \\
&= 1 + \alpha(j') + \alpha(H' - j') + \nu(H' - j') - \alpha(H') \\
&= 1 + \nu\binom{H'}{H'-j'} + \nu(H' - j') \le e.
\end{aligned}$$

■

Proof of Theorem 1.1.3. With $m = M+h+1$, the three subcases in 2.13.3 turn (3.5) into the three nonimmersions of 1.1.3. With k_i again given by (3.6), the sum condition of 2.13.3 is satisfied because

$$\nu\left(\sum \tbinom{h}{j}\tbinom{2^L-M-h}{M-j}\right) = \nu\tbinom{2^L-M}{M} = \alpha(M)-1 = 4h+t$$

in the first two subcases, and

$$\nu\left(\sum \tbinom{h-1}{j}\tbinom{2^L-M-h}{M-j-1}\right) = \nu\tbinom{2^L-M-1}{M-1} = \alpha(M)-1 = 4h-1$$

in the third. The other condition of 2.13.3 is the content of the following proposition.
■

Proposition 3.12. *If* $M \equiv 2^{e_0}+2^{e_1} \mod 2^{e_1+1}$ *with* $e_0 < e_1$ *and* $1 \le h \le 2^{e_1}-2^{e_0}$*, or* M *is a 2-power, then*

$$\nu\tbinom{2M+h-j-1}{M+h-1} \ge \alpha(M)-1$$

for $0 \le j \le h$*.*

Proof. The proposition is vacuously true if M is a 2-power. If $M = 2^{e_1+1}a+2^{e_1}+2^{e_0}$ and $1 \le h \le 2^{e_1}-2^{e_0}$, then

$$\nu\binom{2M+h-j-1}{M+h-1} = \nu\binom{2^{e_1+2}a+D_1}{2^{e_1+1}a+D_0}$$

with $2^{e_1+1} < D_1 < 2^{e_1+2}$ and $D_1 - 2^{e_1+1} < D_0 < 2^{e_1+1}$. Thus $\binom{D_1}{D_0}$ is even, and

$$\nu\tbinom{2M+h-j-1}{M+h-1} = \alpha(a) + \nu\tbinom{D_1}{D_0} \ge \alpha(M)-1.$$

■

4. Comparison with known results

In [3], it was proved that

$$P^{2(m+\alpha(m)-1)} \not\subseteq \mathbf{R}^{4m-2\alpha(m)}. \tag{4.1}$$

Until now, (4.1) implied all known nonimmersions of P^n for $n = 64A+d$ with $0 \le d \le 62$ and $\alpha(A) \ge 5$.[3] Our Theorem 1.1 improves upon (4.1) for many such values of n. We illustrate in Table 4.2, which also illustrates that we are far from

[3]For other values of n, (4.1) was often, but not always, the best known result.

being able to prove that our nonimmersions are best possible. We tabulate here for $n = 128A + d$ with d even, $\alpha(A) = 6$, and $16 \le d \le 76$, the number e such that $P^n \not\subseteq \mathbf{R}^{256A+e}$ was proved in (4.1), denoted Previous, and then the best known value of e now, taking Theorem 1.1 into account. Improvement is obtained in nearly half of the cases, by as much as 17 dimensions. The value of e for which P^n is known to immerse in P^{256A+e} is also listed. The smallest gap between known immersion and nonimmersion dimensions in this range is 21 dimensions. All the immersions in this range are from [10]. We feel that the final answer will be close to the current nonimmersion dimension.

Table 4.2.

d	*Previous*	*Now*	*Immerse*
16	-6	-4	21
18	-6	-2	21
20	2	2	28
22	2	2	28
24	4	4	36
26	8	15	36
28	18	18	44
30	18	18	44
32	20	22	53
34	24	26	53
36	24	28	60
38	32	32	60
40	32	32	68
42	34	47	68
44	50	50	76
46	50	50	76
48	52	54	85
50	56	58	85
52	56	60	92
54	64	64	92
56	64	66	100
58	66	66	100
60	66	79	108
62	80	80	108
64	80	82	117
66	82	82	117
68	86	86	124
70	86	86	124
72	94	94	132
74	94	111	132
76	114	114	140

Note that it is always true in this table that either Previous(n) $\geq$ Now(n) $-$ 2 or Previous($n+2$) $\geq$ Now(n). It was stated in [3] that, in this sense, (4.1) was within 2 dimensions of all known nonimmersions, namely the 2 dimensions can come from the Euclidean space or the projective space, or 1 from each. Our result, 1.1, does not change this: if $P^n \not\subseteq \mathbf{R}^m$ is proved in 1.1, then either $P^n \not\subseteq \mathbf{R}^{m-2}$ or $P^{n+2} \not\subseteq \mathbf{R}^m$ is implied by (4.1). However, in the interim, it was proved in [11] that if $i > j > 3$, then

$P^{2^i+2^j+10} \not\subseteq \mathbf{R}^{2^{i+1}+2^{j+1}+11}$, but all that is implied by (4.1) is $P^{2^i+2^j+8} \not\subseteq \mathbf{R}^{2^{i+1}+2^{j+1}+2}$ and $P^{2^i+2^j+12} \not\subseteq \mathbf{R}^{2^{i+1}+2^{j+1}+10}$. So all that can be claimed is that (4.1) "comes within 3" of all known nonimmersions.

It is not true that every case of Theorem 1.1 is new. If $\alpha(M) < 6$, then the results of 1.1 were proved in [5] or [6]. In those papers (and [10]), obstruction theory using $MO[8]$ was employed to obtain nonimmersions. Since there is an A-module splitting $H^*(MO[8]) \approx A//A_2 \oplus W$ for some A-module W, obstruction theory using $MO[8]$ and using eo_2 will be similar, and identical in a range of dimensions. The advantage of eo_2 is our complete knowledge of possible obstructions to our obstructions. Because of its plethora of additional classes, we could only use $MO[8]$ through a rather restricted range of dimensions.

The other cases in which 1.1 is not new is when the projective space which it shows not to immerse has dimension 2^i, 2^i+2, 2^i+4, 2^i+2^j+2, 2^i+2^j+4, or $2^i+2^j+2^k+4$. Strong (usually best possible) results are known for these spaces by (4.1) and earlier results. For example, 1.1.2b with $h = 2^e$, $M = (2^i - 1)2^e$ with $i = 2^{e+2} - 2$ says that, with $j = i + e + 3$, $P^{2^j} \not\subseteq \mathbf{R}^{2^{j+1}-24\cdot 2^e+12}$, whereas it is classical that $P^{2^j} \not\subseteq \mathbf{R}^{2^{j+1}-2}$.

Aside from this, our results are new, as delineated in the following result. Here when we say that one nonimmersion result improves on another by d dimensions, we mean that it obtains a nonimmersion in a Euclidean space d dimensions larger than the other.

Proposition 4.3. *With the exceptions described in the preceding paragraphs, Theorems 1.1.2 and 1.1.3c improve on previously known results by 2 dimensions. If* $\nu(M) = 0, \dots, 10$*, Theorems 1.1.3a and 1.1.3b improve on known results by*

$$d = 4, 6, 12, 16, 18, 28, 30, 36, 40, 42, 54 \tag{4.4}$$

dimensions, respectively, while Theorems 1.1.1a and 1.1.1b improve on previous results by $d + 3$ *and* $d + 1$ *dimensions, respectively, where* d *is as in (4.4).*

For $\nu(M) > 10$, the improvement will continue to increase in a pattern suggested by (4.4).

Proof. In parts 2a, 2b, 2c, and 3c of Theorem 1.1, the best previously known results were from (4.1) with $m = 4M + 2,\ 4M + 2,\ 4M + 1$, and $4M$, respectively. Note that we choose m as large as possible so that $2(m+\alpha(m)-1) \leq$ the dimension of the projective space being studied. In part 3a, the best known results were from (4.1) with $m = 4M - \delta$, where $\delta = 2, 2, 3, 4, 4, 6, 6, 7, 8, 8, 10$ as $\nu(M)$ goes from 0 to 10. The relevance of $\nu(M)$ is from the equation

$$\alpha(M-1) = \alpha(M) - 1 + \nu(M),$$

which we have been using frequently without comment. Parts 1a and 1b use these same values of $m = 4M - \delta$. ■

5. The spectrum eo_2

In 1991, Hopkins and Miller defined a new collection of spectra, one of which is a 2-primary spectrum denoted EO_2. A thorough account of one approach to these spectra is given in [20]. See also [13] and [12], which emphasize the relationship of EO_2 with elliptic curves and elliptic cohomology, and [19], which applies the Hopkins-Miller spectra to obtain new results in homotopy theory.

One approach starts with a formal group of height 2 over $\mathbf{F}_4$, obtained from an elliptic curve. Using work of Lubin-Tate and the Landweber Exact Functor Theorem, one can associate to this a spectrum E_2. Work of Goerss, Hopkins, and Miller shows that E_2 has the structure of E_∞-ring spectrum with the Morava stabilizer group S_2 acting as a group of automorphisms. The profinite group S_2 has a maximal finite subgroup G_2 of order 24. The homotopy fixed point spectrum of E_2 under the action of G_2 is defined to be the Hopkins-Miller spectrum EO_2.

One key property of EO_2 is

$$EO_2 \wedge DA_1 \simeq E_2. \tag{5.1}$$

Here DA_1, called "the double of A_1," is a finite spectrum with

$$H_*(DA_1) = \mathbf{Z}_2[\xi_1^2, \xi_2^2]/(\xi_1^8, \xi_2^4), \quad (|\xi_i| = 2^i - 1).$$

In 1992-3, Hopkins and Mahowald defined a connective version of EO_2, denoted eo_2. It sits above the connected cover $EO_2[0, \ldots, \infty]$ in a certain pullback square. From (5.1), one deduces

$$eo_2 \wedge DA_1 \simeq BP\langle 2\rangle,$$

and from this the key property $H^*(eo_2) = A//A_2$. The construction and properties of eo_2 are described in [14].

The existence of such a spectrum caused some consternation, for Davis and Mahowald had published in [8] that such a spectrum could not exist. In that paper, they presented a valid deduction that if there is a certain self-map

$$v_2^8 : \Sigma^{48} M(2, v_1^4) \to M(2, v_1^4), \tag{5.2}$$

then $A//A_2$ cannot be realized as the cohomology of a spectrum. In [7], they had presented an argument for the existence of the map (5.2).

This argument in [7] for the existence of (5.2) involved detailed computations using the stable homotopy groups of spheres. In particular, they needed to know that the class $\{\overline{h_0^3 h_3 h_5}\} \in \pi_{39}(M)$ satisfies $v_1^4\{\overline{h_0^3 h_3 h_5}\} = 0$. The mistake in their argument was their claim ([7, pp 653-4]) that $v_1^4\{\overline{h_0^3 h_3 h_5}\}$ could not equal $\{e_0 r\}$, a class of Adams filtration 10 in the 47-stem.

The argument presented noted that $\eta\{e_0 r\} \neq 0$ and claimed to show $\eta v_1^4\{\overline{h_0^3 h_3 h_5}\} = 0$. The latter involved a delicate argument involving 5-fold Toda brackets. The precise mistake involved evaluation of a certain bracket $\langle 2\alpha, 2\iota, \eta\rangle$.

The authors of [7] had another, unpublished, argument for $v_1^4\{\overline{h_0^3 h_3 h_5}\} \neq \{e_0 r\}$. This was that, if ϵ denotes the element in the 8-stem which is not in the image of J, then $\epsilon v_1^4\{\overline{h_0^3 h_3 h_5}\} = 0$, while $\epsilon\{e_0 r\} \neq 0$. The latter equals $\{Pe_0 r\}$, which was thought to survive the ASS. Kochman's work, ([16]), not ASS-based, also seemed to show $\{Pe_0 r\}$ nonzero in homotopy in the 55-stem. The existence of eo_2 caused Kochman and Mahowald to take another look, resulting in their paper [17], which produced a previously unknown differential in the ASS, showing $\{Pe_0 r\} = 0$.

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