DEPTH AND THE COHOMOLOGY OF WREATH PRODUCTS

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1. Introduction.

Suppose that G is a finite group and that k is a field of characteristic $p > 0$. The cohomology ring, $H^*(G,k)$, is a finitely generated, graded-commutative k-algebra and hence is subject to the usual ring-theoretic scrutinies of commutative algebra. Our knowledge of the ideal spectra and associated varieties is reasonably advanced thanks to the work of Quillen and others (see [1] or [7] for general reference). Questions about depth and associated primes seem more mysterious. The depth of $H^*(G,k)$ is defined to be the length of the longest regular sequence in the ring. The most general result on depth is probably the theorem of Jeanne Duflot ([5], see also [2]) which says that the depth of $H^*(G,k)$ is at least equal to the p-rank of the center of a Sylow p-subgroup of G. This is truely a satisfying result in that it relates a cohomological structure directly to structure of the underlying group. However it does not tell us what the depth is, and there are many examples where the depth exceeds Duflot's lower bound. The study of the associated primes for $H^*(G,k)$ is in even greater disarray. It is known that they must be invariant under the action of the subalgebra generated by the reduced power operations of the Steenrod algebra [9], which implies that they must be the radicals of the restrictions to elementary abelian p-subgroups. But little else is known beyond the only partially verified assertions of the relation to depth in [3].

In this paper we consider the behavior of the depth and of the associated primes of the cohomology ring $H^*(G,k)$ under the wreath product operation: $G \to G \wr \mathbb{Z}/p$. In particular, we show that the depth increases by one and also that the minimal dimension of the associated primes increases by at most one. This allows us to determine completely the depths of the mod-p cohomology rings of the symmetric groups. The example of an n-fold wreath product $(\cdots(\mathbb{Z}/p \wr \mathbb{Z}/p)\cdots) \wr \mathbb{Z}/p$ shows that the depth can exceed the p-rank of the center of a p-group by an arbitrarily large amount. The results give some validity

[1]Partially supported by a grant from NSF.
[2]Supported by a Heisenberg grant from DFG.

to the question in [3] on the relation of depth and the dimensions of associated primes. It seems also to be connected to question of detectability of cohomology (see [4], [8]).

The first author would like to thank the Mathematisches Institut of the Universität Heidelberg for its kind hospitality and support during the period when the results of this paper were discovered.

2. The cohomology of wreath products.

We begin this section by outlining the structure of the ring $H^*(G \wr \mathbb{Z}/p, k)$. We rely on the fundamental result of Nakaoka, which expresses the ring in terms of the related spectral sequence. Specifically, there is the extension

$$1 \to G^p \to G \wr \mathbb{Z}/p \to \mathbb{Z}/p \to 1.$$

The associated Lyndon-Hochschild-Serre spectral sequence has E_2 term

$$E_2^{*,*} = H^*(\mathbb{Z}/p, H^*(G^p, k)).$$

Nakaoka's Theorem [11] says that the spectral sequence collapses at the E_2 term and that $E_2^{*,*}$, which is a ring and a k-algebra, is isomorphic to $H^*(G \wr \mathbb{Z}/p, k)$ as a k-algebra.

Now consider first the terms along the fiber

$$E_2^{0,*} = H^0(\mathbb{Z}/p, H^*(G^p, k)) \cong H^*(G^p, k)^{\mathbb{Z}/p}.$$

This subring is the ring of invariants of $H^*(G^p, k) \cong (H^*(G, k))^{\otimes p}$ under the permutation σ of order p which permutes the factors cyclically. The invariants ($\mathbb{Z}/p$-fixed points) come in two types: norms and traces. The norms are the elements of the form $\phi(x) = x \otimes \cdots \otimes x$ for $x \in H^*(G, k)$. In fact the map $\phi : H^*(G, k) \to (H^{*p}(G^p, k))^{\mathbb{Z}/p}$ is a ring homomorphism except for a possible sign change in odd degrees. If $x_1, \cdots, x_p \in H^*(G, k)$ then the trace of $x_1 \otimes \cdots \otimes x_p$ is

$$\tau(x_1 \otimes \cdots \otimes x_p) = \sum_{i=0}^{p-1} \sigma^i(x_1 \otimes \cdots \otimes x_p),$$

which is invariant under the permutation σ. The set of all traces in $E_2^{0,*}$ is an ideal of $H^*(G \wr \mathbb{Z}/p, k)$ which we label $Tr(H^*(G^p, k))$. It is the annihilator in $H^*(G \wr \mathbb{Z}/p, k)$ of the canonical element μ in $E_2^{2,0} = H^2(\mathbb{Z}/p, k)$ which is the Bockstein of the degree-one element η represented by the homomorphism $G \wr \mathbb{Z}/p \to \mathbb{Z}/p$.

Summing up, we have that

$$E_2^{0,*} = Tr(H^*G^p, k)) \oplus \phi(H^*(G, k)).$$

Factoring out the ideal of traces we get the exact sequence

$$0 \to Tr(H^*(G^p, k)) \to H^*(G \wr \mathbb{Z}/p, k) \to H^*(\mathbb{Z}/p, k) \otimes \phi H^*(G, k) \to 0 \qquad (2.1)$$

Here the base of the spectral sequence $E_2^{*,0} = H^*(\mathbb{Z}/p, k)$ is generated as an algebra by η and μ. With this preparation we are ready to prove our main theorem which reads as follows.

Theorem 2.1. *Suppose that the depth of $H^*(G,k)$ is d. Then the depth of $H^*(G\wr\mathbb{Z}/p,k)$ is $d+1$.*

Proof. The proof requires a series of steps. First notice that $H^*(\mathbb{Z}/p,k)$ modulo its radical is a polynomial ring and hence has depth 1, while $\phi(H^*(G,k))$ has depth d, the same as that of $H^*(G,k)$. So the tensor product has depth exactly equal to $d+1$. A regular sequence is given by μ and $\phi(x_1),\cdots,\phi(x_d)$ where $x_1,\ldots,x_d$ is any regular sequence for $H^*(G,k)$. So we have that

Lemma 2.2. *The depth of $H^*(\mathbb{Z}/p,k)\otimes\phi(H^*(G,k))$ is $d+1$.*

Theorem 16.7 of [10] gives a homological criterion for depth. In the case that M is a finitely generated graded $A=H^*(G,k)$ module, it says that the depth of M is the least integer n such that $\mathrm{Ext}^n_A(k,M)\neq 0$. From sequence (2.1) and the long exact sequence on Ext, the theorem is a consequence of the following lemma.

Lemma 2.3. *The depth of $Tr(H^*(G^p,k))$ as an $H^*(G\wr\mathbb{Z}/p,k)$-module is $d+1$.*

Proof. We first show that $d+1$ is a lower bound. Consider a regular sequence $x_1,\ldots,x_d$ of homogeneous elements in $H^*(G,k)$ and write $H^*(G,k)\cong P\otimes V$ as a P-module where P is the polynomial algebra with polynomial generators $x_1,\ldots,x_d$. This statement is roughly a consequence of the discussion of polynomial extensions and graded modules on page 125 of [10]. Here V is considered as graded vector space which is isomorphic to $H^*(G,k)/(x_1,\ldots,x_d)$ and hence contains a canonical element 1. As $(P^{\otimes p})^{\mathbb{Z}/p}$ - modules we have an isomorphism

$$Tr(H^*(G^p,k)\cong(Tr(P^{\otimes p})\otimes\phi V)\oplus(P^{\otimes p}\otimes Tr(V^{\otimes p}))\ .$$

That is, $Tr(H^*(G^p,k))/(Tr(P^{\otimes p})\otimes\phi V)$ is isomorphic to $P^{\otimes p}\otimes Tr(V^{\otimes p})$ via the map which sends $(y_1\otimes\cdots\otimes y_p)\otimes\tau(v_1\otimes\cdots\otimes v_p)$ to the element $\tau((y_1\otimes v_1)\otimes\cdots\otimes(y_p\otimes v_p))$ for $y_1,\cdots y_p$ in P and $v_1,\cdots,v_p\in V$. To get the lower bound on depth it suffices to show that, as $(P^{\otimes p})^{\mathbb{Z}/p}$ - modules, we have

a) $\mathrm{depth} Tr(P^{\otimes p})\geq d+1$, and
b) $\mathrm{depth} P^{\otimes p}\geq d+1$.

We claim that the elements $\phi(x_1),\ldots,\phi(x_d)$, $\tau(x_1\otimes 1\otimes\cdots\otimes 1)$ form a regular sequence for both modules.

We consider case (a) first. We define a lexicographical order on the monomials in P induced by declaring $x_1>x_2>\cdots>x_d$. We get an induced lexicographical order on $P^{\otimes p}$. With this order, a basis of $Tr(P^{\otimes p})$ is given by the elements $\tau(m_1\otimes m_2\otimes\cdots\otimes m_p)$ where each m_i is a monomial in P, not all the m_i are equal and $m_1\otimes m_2\otimes\cdots\otimes m_p$ is larger than any of its cyclic permutations in $P^{\otimes p}$. That the first d elements in the sequence above are regular is easy to see. The action of $\phi(x_i)$ on $\tau(m_1\otimes m_2\otimes\cdots\otimes m_p)$ is given by

$$\phi(x_i)\tau(m_1\otimes m_2\otimes\cdots\otimes m_p)=\tau(x_im_1\otimes x_im_2\otimes\cdots\otimes x_im_p)$$

and hence after dividing out $\phi(x_1),\ldots,\phi(x_k)$ the quotient has a vector space basis consisting of elements $\tau(m_1\otimes m_2\otimes\cdots\otimes m_p)$ as above with the additional condition that the gcd of $m_1, \dots m_d$ is not divisible by x_i for all $i \le k$. In particular $\phi(x_{k+1})$ is still regular on this quotient (for $k+1 \le d$). Now the action of $\tau(x_1 \otimes 1 \otimes \cdots \otimes 1)$ on a basis element is given by

$$\tau(x_1 \otimes 1 \otimes \cdots \otimes 1)\tau(m_1 \otimes m_2 \otimes \cdots \otimes m_p) = \Sigma_i \tau(m_1 \otimes \cdots \otimes x_1 m_i \otimes \cdots \otimes m_p) \ .$$

In particular we get $\tau(x_1 \otimes 1 \otimes \cdots \otimes 1)\tau(m_1 \otimes m_2 \otimes \cdots \otimes m_p) = \tau(x_1 m_1 \otimes m_2 \otimes \cdots \otimes m_p)$ modulo terms of smaller lexicograghical order. Note that if x_1 does not divide m_1 then x_1 does not divide any of the m_i because $m_1 \ge m_i$ for all i. In particular $\tau(x_1 m_1 \otimes m_2 \otimes \cdots \otimes m_p)$ is a nontrivial basis element in $\tau(P^{\otimes p})/(\phi(x_1), ..., \phi(x_d))$. From this description it is obvious that $\tau(x_1 \otimes 1 \otimes \cdots \otimes 1)$ is still regular on $\tau(P^{\otimes p})/(\phi(x_1), \dots, \phi(x_d))$.

To prove (b) we should note that for each i, the elementary symmetric polynomials in x_i in the polynomial ring $(k[x_i])^{\otimes p} = k[x_i] \otimes \cdots \otimes k[x_i]$ form a set of homogeneous parameters. Two of these symmetric polynomials are $\phi(x_i)$ and $\tau(x_1 \otimes 1 \otimes \cdots \otimes 1)$ and hence the two form a regular sequence. Likewise the symmetric polynomials in the individual variables $x_1, \dots, x_d$ form a set of homogeneous paramenters for the polynomial ring

$$P^{\otimes p} = (k[x_1])^{\otimes p} \otimes \cdots \otimes (k[x_d])^{\otimes p}.$$

It is well known that for a Cohen-Macaulay ring such as this any set of homogeneous parameters, taken in any order, is a regular sequence. Consequently the elements $\phi(x_1), \dots, \phi(x_p)$, $\tau(x_1 \otimes \cdots \otimes 1)$, being members of a homogeneous set of parameters, form a regular sequence. This proves the lower bound for the depth.

To get the upper bound we note that the element $\zeta = \tau(x_1 \otimes 1 \otimes \cdots \otimes 1)$ is not in the submodule

$$(\phi(x_1), \cdots, \phi(x_d), \zeta)Tr(H^*(G^p, k))$$

but any multiple of ζ by an element of positive degree in $(H^*(G^p, k))^{\mathbb{Z}/p}$ is in this submodule. So there is no regular sequence of length $d+2$.

3. Dimensions, Associated Primes and Symmetric Groups.

It is well known that in any reasonable ring the depth is at most equal to the minimum of the dimensions of the associated primes in the ring. By the dimension of a prime $\mathfrak{p}$ in a cohomology ring $H^*(G, k)$, we mean the Krull dimension of the quotient ring $H^*(G, k)/\mathfrak{p}$, or the dimension of its maximal ideal spectrum $V_G(\mathfrak{p})$. Now the minimal primes in $H^*(G, k)$ are always among the associated primes (see [10], (6.5)), and they are in one-to-one correspondence with the components of the maximal ideal spectrum $V_G(k)$. By Quillen [12] , the components of the variety are in one-to-one correspondence with the conjugacy classes of maximal elementary abelian p-subgroups of G. That is, if E is a maximal elementary abelian p-subgroup of G, then the radical of the restriction to E is a minimal prime ideal in $H^*(G, k)$ and its dimension is the rank of E. It follows that the depth of $H^*(G, k)$ never exceeds the minimum of the ranks of the maximal elementary abelian p-subgroups of G.

With regards to associated primes, it was speculated in [3] that the depth of any mod-p cohomology ring $H^*(G, k)$ might coincide with the minimum of the dimensions of the associated primes. The following analysis and examples support the speculation, even though the evidence is not strong.

Proposition 3.1. *(a) If a is the minimum of the dimensions of the associated primes of $H^*(G,k)$, then the minimum of the dimensions of the associated primes of $H^*(G \wr \mathbb{Z}/p, k)$ is at most $a+1$.*

(b) Suppose that m is the minimum of the dimensions of the components of the variety of $H^*(G,k)$. Then $m+1$ is the minimum of the dimension of the components of the maximal ideal spectrum of $H^*(G \wr \mathbb{Z}/p, k)$.

Proof. Suppose that $\zeta \in H^*(G,k)$ is an element whose annihilator is a prime $\mathfrak{p}$ with $H^*(G,k)/\mathfrak{p}$ of minimal dimension a. Then we claim that (in the notation of the last section) the annihilator $\mathfrak{p}'$ of $\phi(\zeta)\mu$ has dimension $a+1$. This is because $Tr(H^*(G^p,k))$ annihilates μ and hence

$$H^*(G \wr \mathbb{Z}/p, k)/\mathfrak{p}' \cong \phi(H^*(G,k)/\mathfrak{p}) \otimes H^*(\mathbb{Z}/p, k).$$

This proves (a).

For (b) we use Quillen's correspondence of the components of $V_G(k)$ with the maximal elementary abelian p-subgroups of G. Let $y \in G \wr \mathbb{Z}/p$ be an element of order p in $G \wr \mathbb{Z}/p$ that cyclically permutes the factors of $G^p = G \times \cdots \times G$. Then the centralizer of y is the direct product $\Delta G \times < y >$ where $\Delta G = \{(g, \cdots, g) | g \in G\} \cong G$ is the diagonal subgroup. Let E be a maximal elementary abelian p-subgroup of rank m in G. Then $\Delta E \times < y >$ is contained in no larger elementary abelian p-subgroup of $G \wr \mathbb{Z}/p$.

Finally we consider the case of the symmetric groups S_n. For notation let $W_1 = \mathbb{Z}/p$ and inductively define the n-fold wreath product $W_n = W_{n-1} \wr \mathbb{Z}/p$.

Corollary 3.2. *Suppose that n has p-adic expansion $n = b_0 + b_1 p + \cdots + b_t p^t$. Then*

$$\operatorname{depth}(H^*(S_n, k)) = b_1 + 2b_2 + \cdots + tb_t.$$

This number is equal to the minimum of the dimensions of the components of $H^*(S_n, k)$ and also the minimum of the dimensions of the associated primes.

Proof. The point is that a Sylow p-subgroup P of S_n is isomorphic to

$$W_1^{b_1} \times W_2^{b_2} \times \cdots \times W_t^{b_t}$$

which has the prescribed depth by Theorem 2.1. But now the depth of $H^*(G,k)$ is at least equal to the depth of $H^*(P,k)$ by known arguments (see [7], proof of 10.3.1). The reverse inequality comes from Proposition 3.1 and the fact that G has a maximal elementary abelian p-subgroup of rank $b_1 + \cdots + tb_t$. The statement about the dimensions of the components is a consequence of Quillen's Theorem and the fact that S_n has a maximal elementary abelian p-subgroup of the prescribed rank.

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