HOMOTOPICAL LOCALIZATIONS OF MODULE SPECTRA

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ABSTRACT. We prove that stable homotopical localizations preserve ring spectrum structures and module spectrum structures under suitable hypotheses, and we use this fact to describe all possible localizations of the integral Eilenberg–Mac Lane spectrum $H\mathbf{Z}$. More generally, we describe the main features of localizations of $H\mathbf{Z}$-modules (i.e., stable GEMs), motivated by similar results in unstable homotopy.

1. INTRODUCTION

A stable GEM is a graded Eilenberg–Mac Lane spectrum, i.e., a wedge

$$\bigvee_{k\in\mathbf{Z}} \Sigma^k HA_k,$$

where each A_k is an abelian group and HA_k denotes a spectrum with $\pi_0(HA_k)\cong A_k$ and $\pi_i(HA_k)=0$ if $i\neq 0$. A spectrum X is a stable GEM if and only if it admits an $H\mathbf{Z}$-module structure. In other words, X is a stable GEM if and only if X is a homotopy retract of $H\mathbf{Z}\wedge X$. Note the analogy with unstable homotopy, where a space X is a GEM (i.e. a weak product of abelian Eilenberg–Mac Lane spaces) if and only if it is a homotopy retract of the infinite symmetric product $\mathrm{SP}^\infty X$, a space whose homotopy groups are the integral homology groups of X.

Farjoun and others [Bad01], [Far96] have shown that unstable homotopical localizations preserve GEMs. In this article we prove that the same result is true in stable homotopy, by developing further certain ideas used by Bousfield in [Bou96] and [Bou99].

We work in the Bousfield–Friedlander category of spectra [BF78]. Localization with respect to a map of spectra $f\colon A\longrightarrow B$ is a homotopy idempotent functor L_f on the stable homotopy category taking values in the full subcategory of spectra X such that the map of connective function spectra induced by f,

$$F^c(B,X)\longrightarrow F^c(A,X),$$

is a homotopy equivalence. All known forms of stable localizations are f-localizations for suitable choices of the map f. Among these, the classical Bousfield localizations (that is, homological localizations) commute with suspension. Not all f-localizations have this property; we give necessary and sufficient conditions for a localization to commute with suspension.

We prove that if f is any map of spectra and E is a ring spectrum (in the homotopical sense), then L_fE is a ring spectrum and the localization map $E\longrightarrow L_fE$ is a ring map, if we either assume that E is connective or that L_f commutes with

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suspension. Similarly, if M is an E-module spectrum, then the localization map $M \longrightarrow L_f M$ is an E-module map, provided that E is connective or L_f commutes with suspension.

We believe that a stronger result holds, namely that strict (not just up to homotopy) ring spectrum structures and module spectrum structures are preserved by homotopy idempotent functors, if one works in monoidal categories of spectra, such as S-modules [EKMM97] or symmetric spectra [HSS00]. We plan to address this question in subsequent work.

It follows from the aforementioned results that f-localizations send $H\mathbf{Z}$-modules to $H\mathbf{Z}$-modules; that is, the class of stable GEMs is preserved by f-localizations. We show that, in fact, for every abelian group G,

$$L_f HG \simeq HA \vee \Sigma HB$$

for certain abelian groups A and B. When $G = \mathbf{Z}$, we show that $B = 0$, and the group A admits a ring structure with unit, for which $\mathrm{Hom}(A, A) \cong A$ via evaluation at the unit. Rings A with this property were called "rigid" in [CRT00]. There is a proper class of nonisomorphic rigid rings, and for every rigid ring A there is a map f such $L_f H\mathbf{Z} \simeq HA$.

As special cases, we show that if $E = K$ (complex K-theory) or $E = E(n)$ (the Johnson–Wilson spectrum) for some n, then the E_*-localization of the sphere spectrum has the following homology groups:

$$H_0(L_E S) \cong \mathbf{Q}, \qquad H_i(L_E S) = 0 \quad \text{for } i \neq 0.$$

A more conceptual explanation of the fact that, for every abelian group G, the spectrum $L_f HG$ has at most two nonzero homotopy groups is that the homotopy category of $H\mathbf{Z}$-modules is equivalent to the homotopy category of ($\mathbf{Z}$-graded) chain complexes of abelian groups. We do not give a reference for this fact in the article, but give instead an argument to prove it. The appropriate context to discuss such equivalences of categories is again the theory of structured ring spectra, as in [EKMM97] or [HSS00]. Note, however, the distinction between homotopy HR-modules and strict HR-modules in such categories. For certain rings, including $R = \mathbf{Z}$, the corresponding homotopy categories are equivalent, but in general they are not. This aspect will also be discussed in more detail elsewhere.

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2. Localization of spectra with respect to a map

The notion of homotopical localization with respect to a map can be formulated in any category $\mathcal{C}$ with a simplicial model structure, or in even more general model categories; see [GJ99] or [Hir00] for further details about methods and terminology.

Given a map in a simplicial model category $\mathcal{C}$, pick a cofibration $f\colon A \longrightarrow B$ between cofibrant objects in its homotopy class. Then an object X is called *f-local* if X is fibrant and the induced fibration of simplicial sets

$$\mathrm{HOM}(B, X) \longrightarrow \mathrm{HOM}(A, X)$$

is a weak equivalence. We state the following standard properties for later use.

Lemma 2.1. *Any homotopy retract of an f-local object is f-local.*

Proof. If X is f-local and $Y \longrightarrow X$ has a left homotopy inverse, then the map

$$\mathrm{HOM}(B,Y) \longrightarrow \mathrm{HOM}(A,Y)$$

induced by f is a homotopy retract of the corresponding map for X, and hence it is a weak equivalence. ☐

Lemma 2.2. *Every homotopy limit of f-local objects is f-local.*

Proof. If I is a small category and $D\colon I \longrightarrow \mathcal{C}$ is a diagram in $\mathcal{C}$ taking values in f-local objects, then

$$\begin{aligned}\mathrm{HOM}(B, \mathrm{holim}_I\, D) &\cong \mathrm{holim}_I\, \mathrm{HOM}(B,D)\\ &\simeq \mathrm{holim}_I\, \mathrm{HOM}(A,D) \cong \mathrm{HOM}(A, \mathrm{holim}_I\, D).\end{aligned}$$

Details about the fact that HOM commutes with holim in any simplicial model category can be found in [Hir00, Ch. 19]. ☐

A map $g\colon X \longrightarrow Y$ is an *f-equivalence* if there is a cofibrant approximation $\tilde{g}\colon \tilde{X} \longrightarrow \tilde{Y}$ such that the induced fibration of simplicial sets

$$\mathrm{HOM}(\tilde{Y},E) \longrightarrow \mathrm{HOM}(\tilde{X},E)$$

is a weak equivalence for every f-local object E. An *f-localization* of an object X is an f-equivalence $l\colon X \longrightarrow L_f X$ where $L_f X$ is f-local. This map l is initial in the homotopy category $\mathrm{Ho}\mathcal{C}$ among maps from X to f-local objects, and it is terminal in $\mathrm{Ho}\mathcal{C}$ among f-equivalences with domain X. Each of these two universal properties ensures that, if an f-localization of X exists, then it is unique up to homotopy.

The existence of f-localization for every map f and all objects X is guaranteed if the simplicial model category $\mathcal{C}$ satisfies certain additional assumptions. Specifically, the following result is proved as indicated in [Bou77] or in [Hir00]. An object X is called λ-small, where λ is an infinite cardinal, if every morphism from X to the direct limit of a sequence of cofibrations indexed by a limit ordinal greater than or equal to λ factors through some object in the sequence.

Theorem 2.3. *Let $\mathcal{C}$ be a cofibrantly generated simplicial model category. Suppose that every object X of $\mathcal{C}$ is λ-small for some infinite cardinal λ (which may depend on X). Then f-localization exists for every map f in $\mathcal{C}$. Moreover, L_f can be constructed as a functor in $\mathcal{C}$ which is idempotent up to homotopy.* ☐

The Bousfield–Friedlander category of spectra [BF78] satisfies the assumptions stated in Theorem 2.3. In this category, $\mathrm{HOM}(X,Y)$ is the simplicial set whose n-simplices are the maps $X \wedge \Delta[n]_+ \longrightarrow Y$ of spectra, and for X cofibrant and Y fibrant, one has

$$\pi_k(\mathrm{HOM}(X,Y)) \cong \pi_k(F(X,Y)) \quad \text{for } k \geq 0,$$

where $F(X,Y)$ denotes the function spectrum from X to Y. Therefore, the homotopy groups of the simplicial set $\mathrm{HOM}(X,Y)$ are isomorphic to those of the connective cover $F^c(X,Y)$ of the function spectrum.

Hence, for a map $f\colon A \longrightarrow B$, a spectrum Y is f-local if and only if it is fibrant and the induced map of connective covers of function spectra

$$F^c(B,Y) \longrightarrow F^c(A,Y)$$

induces isomorphisms of all homotopy groups.

Proposition 2.4. *Let f be any map.*

(a) *If E is f-local, then $\Sigma^{-k}E$ is also f-local for $k \geq 0$.*
(b) *If $g\colon X \to Y$ is an f-equivalence, then $\Sigma^k g$ is also an f-equivalence for $k \geq 0$.*

Proof. If E is f-local, then f induces a homotopy equivalence

$$F^c(B, \Sigma^{-k}E) \simeq F^c(A, \Sigma^{-k}E) \quad \text{for } k \geq 0,$$

since $\pi_i(F^c(B, \Sigma^{-k}E)) \cong \pi_{i+k}(F^c(B, E))$ if $i \geq 0$ and $k \geq 0$. The proof of part (b) is similar. ☐

Since $F^c(B, E) \simeq F^c(\Sigma^k B, \Sigma^k E)$ for all B, E and $k \in \mathbf{Z}$, we may also infer that if E is f-local then $\Sigma^k E$ is $\Sigma^k f$-local for every $k \in \mathbf{Z}$, and similarly for f-equivalences. From this fact we deduce the following result.

Proposition 2.5. *For every map of spectra f and every spectrum X we have a homotopy equivalence*

$$L_f \Sigma^{-k} X \simeq \Sigma^{-k} L_{\Sigma^k f} X \quad \textit{for all } k \in \mathbf{Z}.$$

Proof. Since $l\colon X \longrightarrow L_{\Sigma^k f}X$ is a $\Sigma^k f$-equivalence, $\Sigma^{-k} l$ is an f-equivalence. Moreover, $\Sigma^{-k} L_{\Sigma^k f} X$ is f-local since $L_{\Sigma^k f}X$ is $\Sigma^k f$-local, so our claim follows. ☐

This is to be compared with the expression $L_f \Omega^k X \simeq \Omega^k L_{\Sigma^k f} X$ for spaces, which was proved in [Far96].

Localization with respect to a map of the form $f\colon A \longrightarrow *$ is called *A-nullification*, and it is denoted by P_A instead of L_f. The corresponding local spectra are called *A-null*. Thus, a spectrum X is A-null if and only if $F^c(A, X) \simeq *$.

For every map f, there is a natural transformation $P_C \longrightarrow L_f$, where C is the cofibre of f. This follows from the fact that every f-local spectrum is C-null.

As a consequence of Proposition 2.4, for every spectrum X there is a natural map

$$\Sigma L_f X \longrightarrow L_f \Sigma X.$$

We say that L_f *commutes with suspension* if this natural map is a homotopy equivalence for all X.

Theorem 2.6. *Let $f\colon A \longrightarrow B$ be a map of spectra. Then the following statements are equivalent:*

(i) *L_f commutes with suspension.*
(ii) *$\Sigma L_f X \simeq L_f \Sigma X$ for every spectrum X.*
(iii) *$\Sigma^k L_f X \simeq L_f \Sigma^k X$ for every spectrum X and every $k \in \mathbf{Z}$.*
(iv) *If E is any f-local spectrum, then $\Sigma^k E$ is also f-local for any $k \in \mathbf{Z}$.*
(v) *The map $F(B, E) \longrightarrow F(A, E)$ induced by f is a homotopy equivalence for every f-local spectrum E.*
(vi) *If E is f-local and X is any spectrum, then $F(X, E)$ is f-local.*
(vii) *If $g\colon X \longrightarrow Y$ and $h\colon M \longrightarrow N$ are arbitrary f-equivalences, then the map $g \wedge h\colon X \wedge M \longrightarrow Y \wedge N$ is also an f-equivalence.*
(viii) *If g is an f-equivalence, then $\Sigma^k g$ is also an f-equivalence for all $k \in \mathbf{Z}$.*
(ix) *$L_f X \simeq L_{\Sigma^k f} X$ for every spectrum X and every $k \in \mathbf{Z}$.*
(x) *If $X \to Y \to Z$ is any cofiber sequence of spectra, then $L_f X \to L_f Y \to L_f Z$ is also a cofiber sequence.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. Statement (v) is equivalent to the fact that f induces homotopy equivalences

$$F^c(B, \Sigma^k E) \simeq F^c(A, \Sigma^k E)$$

for all $k \in \mathbf{Z}$, and hence it follows from (iv). To prove (vi), we have to verify that

$$F^c(B, F(X, E)) \simeq F^c(A, F(X, E)),$$

which is equivalent to

$$F^c(X, F(B, E)) \simeq F^c(X, F(A, E)),$$

and this follows from (v). Statement (vii) is proved from (vi) by taking any f-local spectrum E and observing that

$$\begin{aligned} F^c(Y \wedge N, E) &\simeq F^c(Y, F(N, E)) \simeq F^c(X, F(N, E)) \\ &\simeq F^c(N, F(X, E)) \simeq F^c(M, F(X, E)) \simeq F^c(X \wedge M, E). \end{aligned}$$

Statement (viii) follows from (vii) by smashing g with the identity of $\Sigma^k S$, for $k \in \mathbf{Z}$. The claim made in (ix) is equivalent to (viii). We next deduce (x) from (ix). Given a cofibre sequence $X \longrightarrow Y \longrightarrow Z$, let C be the cofibre of $L_f \Sigma^{-1} Z \longrightarrow L_f X$. Thus, we have a cofibre sequence

$$L_f X \longrightarrow C \longrightarrow \Sigma L_f \Sigma^{-1} Z$$

and $\Sigma L_f \Sigma^{-1} Z \simeq L_{\Sigma f} Z \simeq L_f Z$, by Proposition 2.5 and by our assumption (ix). As in Theorem 2.2 of [Bou96], there is a map $Y \longrightarrow C$ which is an f-equivalence, and C is Σf-local. Hence, $C \simeq L_f Y$. Finally, to prove that (x) implies (i), pick the cofibre sequence $X \longrightarrow * \longrightarrow \Sigma X$, for any spectrum X. ☐

Statement (v) tells us that f-localization commutes with suspension if and only if f-local spaces may be defined by means of the full function spectrum F instead of its connective cover F^c. We also emphasize that, by the following observation, every f-localization which commutes with suspension is a nullification.

Corollary 2.7. *If a localization functor L_f commutes with suspension, then the natural transformation $P_C \longrightarrow L_f$, where C is the cofiber of f, is a homotopy equivalence.*

Proof. There are natural transformations $L_{\Sigma f} \longrightarrow P_C \longrightarrow L_f$ which correspond to inclusions of the respective classes of local spectra. Since the composite $L_{\Sigma f} \longrightarrow L_f$ is an equivalence by part (ix) of Theorem 2.6, the arrow $P_C \longrightarrow L_f$ is also an equivalence. ☐

3. Examples of f-localizations

We discuss three examples of localizations in the stable homotopy category, namely Postnikov sections, homological localizations and localizations at sets of primes. These examples serve to illustrate certain features of f-localizations that we wish to emphasize.

3.0.1. *Postnikov sections.* Localization of a spectrum E with respect to the map $f\colon \Sigma^{k+1}S \longrightarrow *$, where S is the sphere spectrum and $k \in \mathbf{Z}$, is homotopy equivalent to the k-th Postnikov section of E; that is, $P_{\Sigma^{k+1}S}E \simeq E_{(k)}$, where $\pi_i(E_{(k)}) = 0$ for $i > k$ and there is a map $E \longrightarrow E_{(k)}$ that induces isomorphisms of homotopy groups in dimension less than or equal to k.

Postnikov sections do not commute with suspension. Indeed, if $\pi_k(E) \neq 0$, then $P_{\Sigma^{k+1}S}\Sigma E \not\simeq \Sigma P_{\Sigma^{k+1}S}E$.

3.0.2. *Homological localizations.* Homological localizations in stable homotopy were first discussed by Bousfield in [Bou79]. Let E be any spectrum. A spectrum X is called *E_*-acyclic* if $E_k(X) = 0$ for all $k \in \mathbf{Z}$ or, equivalently, if $E \wedge X \simeq *$. A map of spectra $g\colon X \longrightarrow Y$ is an *E_*-equivalence* if it induces an isomorphism in E-homology, i.e., if the map $g_*\colon E_k(X) \longrightarrow E_k(Y)$ is an isomorphism for all $k \in \mathbf{Z}$. A spectrum Z is *E_*-local* if each E_*-equivalence $f\colon X \longrightarrow Y$ induces a homotopy equivalence $F(Y,Z) \simeq F(X,Z)$ or, equivalently, if $F(A,Z) \simeq *$ for each E_*-acyclic spectrum A.

An *E_*-localization* of a spectrum X is an E_*-equivalence $X \longrightarrow L_E X$ from X to an E_*-local spectrum. Each E_*-localization is an f-localization for a suitable map f. In fact, it is a nullification, as shown in [Bou79].

Theorem 3.1. *Let E be any spectrum. Then there exists an E_*-acyclic spectrum Z such that there is a natural equivalence $P_Z X \simeq L_E X$ for every spectrum X.* □

Homological localizations commute with suspension.

3.0.3. *Localization at sets of primes.* Let G be any abelian group and let MG denote its associated Moore spectrum. Thus, MG is a spectrum such that $(H\mathbf{Z})_0(MG) \cong \pi_0(MG) \cong G$, $\pi_i(MG) = 0$ if $i < 0$, and $(H\mathbf{Z})_i(MG) = 0$ if $i \neq 0$.

Lemma 3.2. *There is a natural exact sequence*

$$0 \longrightarrow \operatorname{Ext}(G, \pi_{k+1}(X)) \longrightarrow [\Sigma^k MG, X] \longrightarrow \operatorname{Hom}(G, \pi_k(X)) \longrightarrow 0$$

for each spectrum X and each abelian group G, where MG is the Moore spectrum associated with G.

Proof. Pick a free abelian presentation of the group G and use the associated cofibre sequence of Moore spectra; cf. [Bou79, (2.2)]. □

Let $\mathbf{Z}_P$ denote the integers localized at a set of primes P (possibly empty). For every spectrum X, the P-localization of X is the map

$$1 \wedge \eta\colon X \wedge S \longrightarrow X \wedge M\mathbf{Z}_P$$

where η is given by the unit in $\pi_0(M\mathbf{Z}_P) = \mathbf{Z}_P$. If we denote $X_P = X \wedge M\mathbf{Z}_P$, then for each $k \in \mathbf{Z}$ we have

$$\pi_k(X_P) \cong \pi_k(X) \otimes \pi_0(M\mathbf{Z}_P) \cong \pi_k(X) \otimes \mathbf{Z}_P.$$

From this fact it follows that, if E is any spectrum, then

$$E_k(X_P) \cong \pi_k(E \wedge X \wedge M\mathbf{Z}_P) \cong E_k(X) \otimes \mathbf{Z}_P.$$

Hence P-localization of spectra induces P-localization of their homotopy and homology groups. On the other hand, about cohomology, Lemma 3.2 yields

$$E^k(X_P) \cong [M\mathbf{Z}_P, F(X, \Sigma^k E)] \cong \operatorname{Hom}(\mathbf{Z}_P, E^k(X)) \oplus \operatorname{Ext}(\mathbf{Z}_P, E^{k-1}(X)).$$

It follows from the definition that P-localization is a homological localization, namely $(M\mathbf{Z}_P)_*$-localization. Hence, P-localization commutes with suspension. We also emphasize that

$$X_P \simeq X \wedge S_P,$$

for all X; that is, P-localization is *smashing*, as defined in [Rav84].

An explicit map f such that $L_f X \simeq X_P$ for all X can be displayed as follows.

Theorem 3.3. *Let P be a set of primes and let $g\colon \vee_{q\notin P} S \longrightarrow \vee_{q\notin P} S$ be a wedge of maps inducing multiplication by q in $\pi_0(S)$ for each prime q not in P, and let $f = \vee_{n<0}\Sigma^n g$. Then $L_f X \simeq X \wedge M\mathbf{Z}_P$ for all X.*

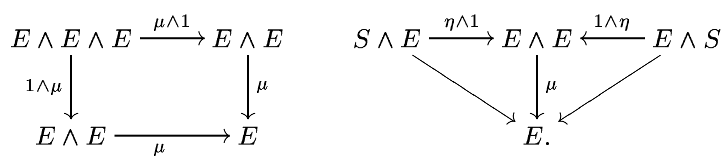
Proof. The map f has been chosen so that f-local spectra are precisely those spectra whose homotopy groups are $\mathbf{Z}_P$-modules. Thus, the map $\eta\colon S \longrightarrow M\mathbf{Z}_P$ is an f-localization, since $M\mathbf{Z}_P$ is f-local and the map η is an f-equivalence because, by Lemma 3.2, $[\Sigma^k M\mathbf{Z}_P, Y] \longrightarrow \pi_k(Y)$ is an isomorphism when $\pi_k(Y)$ is a $\mathbf{Z}_P$-module. By part (vii) of Theorem 2.6, the map

$$1 \wedge \eta\colon X \wedge S \longrightarrow X \wedge M\mathbf{Z}_P$$

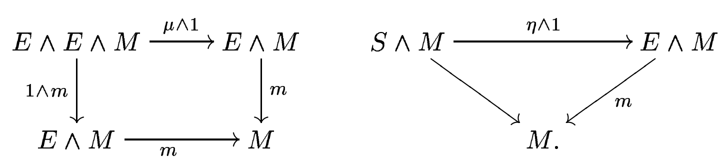
is also an f-equivalence, and our claim follows. □

4. Localizations of Ring Spectra and Module Spectra

We recall the definition of ring spectra and module spectra in the homotopical sense, as in [Ada74]. A spectrum E is called a *ring spectrum* if it is equipped with two maps $\mu\colon E \wedge E \longrightarrow E$ and $\eta\colon S \longrightarrow E$ such that the following diagrams commute up to homotopy:



It is said that E is *commutative* if $\mu \circ \tau \simeq \mu$, where $\tau\colon E \wedge E \longrightarrow E \wedge E$ is the twist map. A spectrum M is called a *module spectrum* over a ring spectrum E or an *E-module* if it is equipped with a map $m\colon E \wedge M \longrightarrow M$ such that the following diagrams commute up to homotopy:



Every ring spectrum E is an E-module spectrum with $m = \mu$. Note also that, if M is an E-module, then it is a homotopy retract of $E \wedge M$.

A *ring map* between ring spectra (E, μ, η) and (E', μ', η') is a map $f\colon E \longrightarrow E'$ such that $f \circ \mu \simeq \mu' \circ (f \wedge f)$ and $f \circ \eta \simeq \eta'$. An *$E$-module map* is defined similarly.

If R is an associative ring with unit and M is an R-module, then the Eilenberg–Mac Lane spectrum HR is a ring spectrum, and HM is a module spectrum over HR. The structure maps on HR and HM come from the product $R \otimes R \longrightarrow R$ and the unit $\mathbf{Z} \longrightarrow R$ in the ring R, and from the structure homomorphism

$R \otimes M \longrightarrow M$ of M as an R-module. For every ring R, the spectrum HR is an $H\mathbf{Z}$-module. Moreover, every HR-module is an $H\mathbf{Z}$-module via the map $H\mathbf{Z} \longrightarrow HR$ corresponding to the unit $\mathbf{Z} \longrightarrow R$.

Remark 4.1. If M is an E-module spectrum, then, for every spectrum X, the graded abelian group $[X, M]_*$ is a $\pi_*(E)$-module, as follows. For every map $\alpha \in \pi_i(E)$ and every map $f \in [X, M]_j$ we obtain another map in $[X, M]_{i+j}$ by smashing α with f and composing with the structure map of M:

$$\Sigma^{i+j} X \simeq \Sigma^i S \wedge \Sigma^j X \xrightarrow{\alpha \wedge f} E \wedge M \xrightarrow{m} M.$$

In particular, if M is an HR-module spectrum, then $\pi_n(M)$ is an R-module for every n.

As we next prove, in the case of f-localization functors that commute with suspension, the f-localizations of ring spectra or module spectra acquire a compatible ring structure or module structure. In the rest of this chapter, we assume that f is a fixed map of spectra, and we write L instead of L_f.

Theorem 4.2. *Let $f\colon A \longrightarrow B$ be any map of spectra. If the f-localization functor L commutes with suspension, then the following hold:*

(i) *If E is a ring spectrum, then the spectrum LE has a unique ring spectrum structure such that the localization map $l_E\colon E \longrightarrow LE$ is a ring map. If E is commutative, then LE is also commutative.*

(ii) *If M is an E-module, then the spectrum LM has a unique E-module structure such that the localization map $l_M\colon M \longrightarrow LM$ is an E-module map. Moreover, LM admits a unique LE-module structure extending the E-module structure.*

Proof. For the first part we need to construct a product $\overline{\mu}$ and a unit $\overline{\eta}$ on LE. Let μ and η be the product and the unit of the ring spectrum E, respectively. We have an equivalence $F(E, LE) \simeq F(LE, LE)$ because LE is f-local and the functor L commutes with suspension by assumption. Then,

$$\begin{aligned}[E \wedge E, LE] &\cong [E, F(E, LE)] \cong [E, F(LE, LE)] \cong [E \wedge LE, LE] \\ &\cong [LE, F(E, LE)] \cong [LE, F(LE, LE)] \cong [LE \wedge LE, LE].\end{aligned}$$

Hence, the product $\mu\colon E \wedge E \longrightarrow E$ extends to a unique map $\overline{\mu}\colon LE \wedge LE \longrightarrow LE$ rendering homotopy commutative the diagram

$$\begin{array}{ccc} E \wedge E & \xrightarrow{\mu} & E \\ \downarrow{\scriptstyle l_E \wedge l_E} & & \downarrow{\scriptstyle l_E} \\ LE \wedge LE & \xrightarrow[\overline{\mu}]{} & LE. \end{array}$$

We define the unit $\overline{\eta}$ as the composition $l_E \circ \eta$. The commutativity of the diagrams for $\overline{\mu}$ and $\overline{\eta}$ follows from the commutativity of the diagrams for μ and η and the universal property of L (using part (vii) of Theorem 2.6).

The commutativity of LE when E is commutative and the statements in part (ii) are proved in the same way. □

As we next show, localization functors not commuting with suspension need not preserve ring structures nor module structures in general. The following lemma is useful to prove that certain spectra fail to be ring spectra or module spectra. The idea is due to Rudyak [Rud98, Ch. II, 4.31].

Lemma 4.3. *Let E and F be ring spectra, and let M be an E-module spectrum. If $F_0(E) = 0$, then $F_k(M) = 0$ for all $k \in \mathbf{Z}$.*

Proof. The diagram

$$\begin{array}{ccc} \pi_0(E) \otimes F_k(M) & \xrightarrow{h\otimes 1} & F_0(E) \otimes F_k(M) \\ {\scriptstyle (\eta_E)_*\otimes 1}\uparrow & & \downarrow{\scriptstyle m_*} \\ \pi_0(S) \otimes F_k(M) & \xrightarrow[\cong]{} & F_k(M) \end{array}$$

is commutative, where $(\eta_E)_*$ is induced by the unit of the ring spectrum E and h is the Hurewicz homomorphism of F. The map m_* is induced by the multiplication of $M \wedge F$ as an $(E \wedge F)$-module. So, if $F_0(E) = 0$, then the bottom row isomorphism factors through zero and hence $F_k(M) = 0$ for all $k \in \mathbf{Z}$. □

Example 4.4. Given a natural number n and a fixed prime p, let $K(n)$ denote the ring spectrum corresponding to n-th Morava K-theory. If we consider its nullification $P_{\Sigma S}K(n)$, then, according to [Rud98], $(H\mathbf{Z}/p)_0(P_{\Sigma S}K(n)) = 0$ yet $(H\mathbf{Z}/p)_k(P_{\Sigma S}K(n)) \neq 0$ for some $k > 0$. This implies that $P_{\Sigma S}K(n)$ cannot be a ring spectrum, by Lemma 4.3. The same argument, now considering $K(n)$ as a $K(n)$-module and using that $(H\mathbf{Z}/p)_0(K(n)) = 0$ and Lemma 4.3, shows that $P_{\Sigma S}K(n)$ is not a $K(n)$-module.

This difficulty can be repaired by imposing suitable connectivity conditions. The following result extends an observation made by Bousfield in [Bou99].

Theorem 4.5. *Let $f\colon A \longrightarrow B$ be any map of spectra and let L be f-localization. Then the following hold:*

(i) *If E is a connective ring spectrum and LE is connective, then the spectrum LE has a unique ring structure such that the localization map $l_E\colon E \longrightarrow LE$ is a ring map. If E is commutative, then LE is also commutative.*
(ii) *If M is an E-module, where E is a connective ring spectrum, then LM has a unique E-module structure such that the localization map $l_M\colon M \longrightarrow LM$ is an E-module map. Moreover, if LE is connective, then LM also admits a unique LE-module structure extending the E-module structure.*

Proof. Using that E is a connective spectrum, we have equivalences

$$F^c(E, F^c(X,Y)) \simeq F^c(E, F(X,Y)) \simeq F^c(E \wedge X, Y)$$

that give a bijection $[E, F^c(X,Y)] \cong [E \wedge X, Y]$. Then one proceeds as in the proof of Theorem 4.2. □

5. Localization of Stable GEMs

As we next recall, the stable GEMs are precisely the $H\mathbf{Z}$-modules. Thus, we may use our results in the previous section to prove that every f-localization sends stable GEMs to stable GEMs.

Definition 5.1. Let R be a ring. A spectrum E is called a *stable R-GEM* if it is homotopy equivalent to a wedge of suspensions of Eilenberg–Mac Lane spectra $\vee_{k\in\mathbf{Z}}\Sigma^k HA_k$, where each A_k is an R-module (hence, each HA_k is an HR-module spectrum). If $R=\mathbf{Z}$, then stable $\mathbf{Z}$-GEMs are called *stable GEMs.*

For a ring spectrum E, let $\mathrm{Ho}^{\mathrm{s}}_{E\text{-mod}}$ denote the subcategory of the stable homotopy category Ho^{s} whose objects are the E-module spectra and whose morphisms are (ordinary) homotopy classes of E-module maps. If M and N are E-module spectra, let $[M,N]_{E\text{-mod}}\subset[M,N]$ denote the set of morphisms $M\longrightarrow N$ in this subcategory.

If E is a ring spectrum, then for every spectrum X the smash product $E\wedge X$ has an E-module structure given by $E\wedge E\wedge X\xrightarrow{\mu\wedge 1}E\wedge X$. We are indebted to Gustavo Granja for pointing out the following fact to us.

Lemma 5.2. *Let E be any ring spectrum. Then the functor $\mathrm{Ho}^{\mathrm{s}}\longrightarrow\mathrm{Ho}^{\mathrm{s}}_{E\text{-mod}}$ assigning to every spectrum X the spectrum $E\wedge X$ is left adjoint to the forgetful functor $\mathrm{Ho}^{\mathrm{s}}_{E\text{-mod}}\longrightarrow\mathrm{Ho}^{\mathrm{s}}$. That is, for every spectrum X and every E-module M, there is a natural isomorphism*

$$[X,M]\cong[E\wedge X,M]_{E\text{-mod}}$$

induced by the unit of E.

Proof. Let M be an E-module, X a spectrum, and $f\colon X\longrightarrow M$ any map. We are going to show that there is a homotopy unique E-module map $\tilde f\colon E\wedge X\longrightarrow M$ such that the diagram

$$\begin{array}{ccc}
S\wedge X\simeq X & \xrightarrow{\;f\;} & M \\
{\scriptstyle\eta\wedge 1}\big\downarrow & \nearrow{\scriptstyle\tilde f} & \\
E\wedge X & &
\end{array}$$

commutes up to homotopy. The following diagram

$$\begin{array}{ccccc}
S\wedge X & \xrightarrow{1\wedge f} & S\wedge M & & \\
{\scriptstyle\eta\wedge 1}\big\downarrow & & {\scriptstyle\eta\wedge 1}\big\downarrow & \searrow{\scriptstyle\simeq} & \\
E\wedge X & \xrightarrow[1\wedge f]{} & E\wedge M & \xrightarrow[m]{} & M
\end{array}$$

commutes, so if we define $\tilde f=m\circ(1\wedge f)$, then it satisfies $\tilde f\circ(\eta\wedge 1)\simeq f$.

The map $\tilde f$ is a map of E-modules, since the following diagram is commutative:

$$\begin{array}{ccccc}
E\wedge(E\wedge X) & \xrightarrow{1\wedge(1\wedge f)} & E\wedge(E\wedge M) & \xrightarrow{1\wedge m_M} & E\wedge M \\
\simeq\big\| & & \simeq\big\| & & \simeq\big\| \\
(E\wedge E)\wedge X & \xrightarrow{(1\wedge 1)\wedge f} & (E\wedge E)\wedge M & \xrightarrow{\mu_E\wedge 1} & E\wedge M \\
{\scriptstyle\mu_E\wedge 1}\big\downarrow & & {\scriptstyle\mu_E}\big\downarrow & & {\scriptstyle m_M}\big\downarrow \\
E\wedge X & \xrightarrow[1\wedge f]{} & E\wedge M & \xrightarrow[m_M]{} & M.
\end{array}$$

If there exists another E-module map g satisfying $g \circ (\eta \wedge 1) \simeq f$, then $g \simeq \tilde{f}$, because the following diagram also commutes:

$$\begin{array}{ccc} E \wedge (S \wedge X) & \xrightarrow{\simeq} & E \wedge X \\ \downarrow{\scriptstyle 1\wedge(\eta\wedge 1)} & & \downarrow{\scriptstyle 1\wedge f} \\ E \wedge (E \wedge X) & \xrightarrow{1\wedge g} & E \wedge M \\ \downarrow{\scriptstyle 1\wedge\mu} & & \downarrow{\scriptstyle m} \\ E \wedge X & \xrightarrow{g} & M. \end{array} \qquad (E \wedge X \xrightarrow{1} E \wedge X)$$

The upper square commutes by hypothesis and the lower square commutes because g is a map of E-modules, so $g \simeq m \circ (1 \wedge f) \simeq \tilde{f}$. □

Proposition 5.3. *For every $H\mathbf{Z}$-module M, there is a map of $H\mathbf{Z}$-modules*

$$H\mathbf{Z} \wedge (\vee_{k\in\mathbf{Z}} \Sigma^k MG_k) \xrightarrow{\tilde{\alpha}} M$$

which is a homotopy equivalence, where $G_k = \pi_k(M)$.

Proof. The argument was sketched in [Ada74, p. 307]. For each $k \in \mathbf{Z}$, take one map $\alpha_k \in [\Sigma^k MG_k, M]$ mapping to the identity in $\mathrm{Hom}(G_k, G_k)$. Let $\alpha = \vee_{k\in\mathbf{Z}}\alpha_k$ be the wedge of all α_k constructed in this way. This yields, by Lemma 5.2, an $H\mathbf{Z}$-module map $\tilde{\alpha}\colon H\mathbf{Z} \wedge (\vee_{k\in\mathbf{Z}}\Sigma^k MG_k) \longrightarrow M$, namely $\tilde{\alpha} = m \circ (1 \wedge \alpha)$. This map $\tilde{\alpha}$ induces an isomorphism on π_k for all $k \in \mathbf{Z}$, because $\eta \wedge 1$ and $1 \wedge \alpha$ are isomorphisms on π_k. So $\tilde{\alpha}$ is a homotopy equivalence. □

Proposition 5.3 tells us that $H\mathbf{Z}$-module spectra and stable GEMs are the same thing, because $H\mathbf{Z} \wedge MG \simeq HG$ for any abelian group G and therefore, if M is an $H\mathbf{Z}$-module, then $M \simeq \vee_{k\in\mathbf{Z}}\Sigma^k H\pi_k(M)$. Similarly, the HR-modules are precisely the stable R-GEMs because each HR-module spectrum is an $H\mathbf{Z}$-module spectrum, and the homotopy groups of HR-module spectra are R-modules (see Remark 4.1).

Corollary 5.4. *Let A and B be abelian groups. Then*

$$[HA, HB]_{H\mathbf{Z}\text{-mod}} \cong \mathrm{Hom}(A, B),$$
$$[HA, \Sigma HB]_{H\mathbf{Z}\text{-mod}} \cong \mathrm{Ext}(A, B), \textit{ and}$$
$$[HA, \Sigma^k HB]_{H\mathbf{Z}\text{-mod}} = 0 \quad \textit{if } k \neq 0, 1.$$

Proof. As a special case of Proposition 5.3, $HA \simeq H\mathbf{Z} \wedge MA$ as $H\mathbf{Z}$-modules. By Lemma 5.2, there is a natural bijection

$$[MA, \Sigma^k HB] \cong [H\mathbf{Z} \wedge MA, \Sigma^k HB]_{H\mathbf{Z}\text{-mod}}$$

so the result follows directly from Lemma 3.2. □

In the rest of this section, L denotes f-localization with respect to a fixed but arbitrary map f. Using the results on f-localizations of E-modules of Section 4, we have the following.

Theorem 5.5. *Let R be any associative ring with unit. If E is a stable R-GEM, then LE is also a stable R-GEM and the localization map $l_E\colon E \longrightarrow LE$ is an HR-module map.*

Proof. A stable R-GEM is the same as an HR-module, and HR is a connective spectrum. Hence we may apply Theorem 4.5. $\square$

In other words, if a spectrum E is homotopy equivalent to $\vee_{i\in\mathbf{Z}}\Sigma^i HA_i$ where A_i is an R-module for each $i \in \mathbf{Z}$, then $LE \simeq \vee_{i\in\mathbf{Z}}\Sigma^i HG_i$ where each G_i is an R-module as well.

Next, we are going to study the case when the spectrum E is a suspension of a single Eilenberg–Mac Lane spectrum, i.e., $E \simeq \Sigma^n HG$ where G is an R-module. By Theorem 5.5 we know that $L\Sigma^n HG \simeq \vee_{k\in\mathbf{Z}}\Sigma^k HG_k$ with each G_k an R-module. In fact, most of the R-modules G_k are zero, as we next explain. Consider the following sequence of $H\mathbf{Z}$-module maps, where β is a homotopy inverse of the map given by Proposition 5.3, and p_i is the projection onto the i-th factor:

$$\Sigma^n HG \xrightarrow{\ l\ } L\Sigma^n HG \xrightarrow{\ \beta\ } \vee_{k\in\mathbf{Z}}\Sigma^k HG_k \xrightarrow{\ p_i\ } \Sigma^i HG_i$$

By Corollary 5.4, $[\Sigma^n HG, \Sigma^i HG_i]_{H\mathbb{Z}\text{-mod}} = 0$ unless $i = n$ or $i = n+1$. The universal property of localization and the fact that $\Sigma^i HG_i$ is f-local because it is a retract of $\vee_{k\in\mathbf{Z}}\Sigma^k HG_k$ (see Lemma 2.1) tell us that $G_i = 0$ if $i \neq n$ or $i \neq n+1$.

What we get is that the localization of any suspension of an Eilenberg–Mac Lane spectrum has at most two nonzero homotopy groups.

Theorem 5.6. *Let G be any abelian group and $n \in \mathbf{Z}$. Then $L\Sigma^n HG \simeq \Sigma^n HG_1 \vee \Sigma^{n+1} HG_2$ as $H\mathbf{Z}$-modules for some abelian groups G_1, G_2. If G is an R-module for some ring R, then G_1 and G_2 are also R-modules.* $\square$

There are some special cases in which the localization of an Eilenberg–Mac Lane spectrum is a single Eilenberg–Mac Lane spectrum.

Theorem 5.7. *If G is a free abelian group and n is an integer, then $L\Sigma^n HG \simeq \Sigma^n HA$ for some abelian group A.*

Proof. From Theorem 5.6, we know that $L\Sigma^n HG$ has at most two nonzero homotopy groups, A and B. By Corollary 5.2,

$$[\Sigma^n HG, \Sigma^{n+1} HB]_{H\mathbb{Z}\text{-mod}} \cong \operatorname{Ext}(G, B).$$

If G is free, then $\operatorname{Ext}(G,B) = 0$, and this tells us that the projection $\Sigma^n HG \longrightarrow \Sigma^{n+1} HB$ is nullhomotopic. Moreover, $\Sigma^{n+1} HB$ is f-local, because it is a retract of $L\Sigma^n HG$. The universal property of localization forces that $B = 0$. $\square$

If we now project $L\Sigma^n HG$ onto the first summand

$$\begin{array}{ccc} \Sigma^n HG & \xrightarrow{\ l\ } & \Sigma^n HA \vee \Sigma^{n+1} HB \\ & \searrow^{p_1\circ l} & \downarrow{\scriptstyle p_1} \\ & & \Sigma^n HA \end{array}$$

we can obtain information about the group A. This diagram yields an isomorphism of abelian groups

$$[HA, HA] \times [\Sigma HB, HA] \cong [HG, HA],$$

and $[\Sigma HB, HA] \cong (HA)^0(\Sigma HB) \cong \operatorname{Hom}(\pi_0(\Sigma HB), A) = 0$. Hence we get

$$\operatorname{Hom}(A, A) \cong [HG, HA] \cong \operatorname{Hom}(G, A).$$

In the case when $G = \mathbf{Z}$, this says that $\operatorname{Hom}(A,A) \cong A$. Therefore, if $G = \mathbf{Z}$ and A is nonzero, then A admits a ring structure, with a multiplication coming from composition in $\mathrm{Hom}(A, A)$ and a unit coming from the identity homomorphism. Moreover, the isomorphism $\mathrm{Hom}(A, A) \cong A$ is given by evaluation at the unit.

Definition 5.8. A ring A with unit such that $\mathrm{Hom}(A, A) \cong A$ via $\varphi \mapsto \varphi(1)$ is called a *rigid ring*.

This terminology was first used in [CRT00]. However, rigid rings had previously been studied in a different context, under the name of E-rings. The most obvious examples of such rings are $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{Z}/p$, or the p-adics $\hat{\mathbf{Z}}_p$, for any p. There are many other examples. In fact, as shown in [DMV87], there are rigid rings of arbitrarily large cardinality.

Rigid rings are commutative. Solid rings in the sense of [BK72] are rigid rings, but not conversely (in fact, solid rings are countable). Proofs of these claims and further details about rigid rings can be found in [CRT00].

All rigid rings occur as homotopy groups of localizations of $H\mathbf{Z}$, since, if A is any rigid ring, then $L_f H\mathbf{Z} \simeq HA$ where f is the map $H\mathbf{Z} \longrightarrow HA$ induced by the unit homomorphism $\mathbf{Z} \longrightarrow A$.

We can summarize the results obtained for f-localizations of $H\mathbf{Z}$ in the following theorem.

Theorem 5.9. *Let f be any map of spectra. Then the f-localization of the spectrum $H\mathbf{Z}$ has at most one nonzero homotopy group, i.e., $L_f H\mathbf{Z} \simeq HA$. Moreover, the group A has a rigid ring structure if $A \neq 0$. All rigid rings appear this way.* □

We conclude with an example. As already mentioned above, a localization L is called *smashing* if $LX \simeq X \wedge LS$ for all spectra X, where S is the sphere spectrum. It follows from this definition that every smashing localization is homological (namely, $L \simeq L_E$, where $E = LS$) and hence it commutes with suspension. Moreover, LS is a commutative ring spectrum, by Theorem 4.3.

As shown in [Rav84, 1.27], a homological localization L_E is smashing if and only if it commutes with direct limits. This happens, for example, if E is the spectrum K of (complex) K-theory or the Johnson–Wilson spectrum $E(n)$ for any n.

Theorem 5.10. *If L is smashing, then $H_n(LS) = 0$ if $n \neq 0$, and it is either zero or a rigid ring if $n = 0$.*

Proof. We have that

$$H_n(LS) = \pi_n(H\mathbf{Z} \wedge LS) \cong \pi_n(LH\mathbf{Z}) \cong \pi_n(HA)$$

for some rigid ring A, by Theorem 5.9. □

The ring A happens to be $\mathbf{Q}$ if L is localization with respect to $E = K$ or $E = E(n)$ for any n. In each of these cases, the spectrum $H\mathbf{Q}$ is E_*-local, since it is a retract of $E \wedge M\mathbf{Q}$. Hence, it suffices to show that the natural map $H\mathbf{Z} \longrightarrow H\mathbf{Q}$ is an E_*-equivalence. For this, we may use the fact that $E_k(H\mathbf{Z}) = \lim_i E_{k+i}(K(\mathbf{Z}, i))$. Then, for $E = K$, our claim follows from [AH68]. For $E = E(n)$, it is a consequence of [Bou82, Example 7.5].

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