

# ON THE COHOMOLOGY OF HIGHLY CONNECTED COVERS OF FINITE COMPLEXES

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ABSTRACT. Relying on the computation of the André-Quillen homology groups for unstable Hopf algebras, we prove that the mod  $p$  cohomology of the  $n$ -connected cover of a finite  $H$ -space is always finitely generated as algebra over the Steenrod algebra.

## INTRODUCTION

Consider the  $n$ -connected cover of a finite complex. Does its (mod  $p$ ) cohomology satisfy some finiteness property? Such a question has already been raised by McGibbon and Møller in [MM97], but no satisfactory answer has been proposed. We do not ask here for an algorithm which would allow to make explicit computations. We rather look for a general structural statement which would tell us to what kind of class such cohomologies belong. The prototypical theorems we have in mind are the Evens-Venkov result, [Eve61], [Ven59], that the cohomology of a finite group is Noetherian, the analog for  $p$ -compact groups obtained by Dwyer and Wilkerson [DW94], and the fact that the mod  $p$  cohomology of an Eilenberg-Mac Lane space  $K(A, n)$ , with  $A$  abelian of finite type, is finitely generated as an algebra over the Steenrod algebra, which can easily be inferred from the work of Serre [Ser53] and Cartan [Car55].

This last observation leads us to ask first whether or not the mod  $p$  cohomology of a finite Postnikov piece is also finitely generated as an algebra over the Steenrod algebra and second, since a finite complex  $X$  and its  $n$ -connected cover  $X\langle n \rangle$  only differ in a finite number of homotopy groups, if  $H^*(X\langle n \rangle; \mathbb{F}_p)$  satisfies the same property. We offer in this paper a positive answer when  $X$  is an  $H$ -space, based on the analysis of the fibration  $P \rightarrow X\langle n \rangle \rightarrow X$ , where  $P$  is a finite Postnikov piece. In fact we prove a strong closure property for  $H$ -fibrations.

**Theorem 4.2.** *Let  $F \rightarrow E \rightarrow B$  be an  $H$ -fibration in which both  $H^*(F; \mathbb{F}_p)$  and  $H^*(B; \mathbb{F}_p)$  are finitely generated unstable algebras. Then so is  $H^*(E; \mathbb{F}_p)$ .*

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2000 *Mathematics Subject Classification.* Primary 55P45; Secondary 13D03, 55S10, 55T20.

All three authors are partially supported by MEC grant MTM2004-06686. The first and third authors were partially supported by the Mittag-Leffler Institute in Sweden. The third author is supported by the program Ramón y Cajal, MEC, Spain.

This applies in particular to highly connected covers of finite  $H$ -spaces, see Theorem 4.3. In our previous work [CCSa] we proved that the theorem holds whenever the base space is an Eilenberg-Mac Lane space. The proof relied mainly on Smith's work [Smi70] on the Eilenberg-Moore spectral sequence.

Our starting point here is the same and we need to analyze carefully certain Hopf subalgebras of  $H^*(F; \mathbb{F}_p)$ . Observe that the property for an unstable algebra  $K$  to be a finitely generated  $\mathcal{A}_p$ -algebra is equivalent to say that the module of the undecomposable elements  $QK$  is finitely generated as unstable module. It is often more handy to work with this module because it is smaller than the whole algebra and, above all, the category of unstable modules is locally Noetherian, [LZ86].

The main problem (or interest) with the functor  $Q(-)$  is the failure of left exactness. To what extent this functor is not left exact is precisely measured by André-Quillen homology  $H_*^Q(-)$ . In our setting we keep control of the size of these unstable modules.

**Proposition 2.3.** *Let  $A$  be a Hopf algebra which is a finitely generated unstable  $\mathcal{A}_p$ -algebra. Then  $H_0^Q(A) = QA$  and  $H_1^Q(A)$  are both finitely generated unstable modules.*

As the higher groups are all trivial (see Proposition 1.3), this gives a quite accurate description of André-Quillen homology in our situation. The relevance of André-Quillen homology in homotopy theory is notorious since Miller solved the Sullivan conjecture, [Mil84]. The typical result which is needed in his work, and which has been then extended by Lannes and Schwartz, [LS86], is that the module of indecomposable elements of an unstable algebra  $K$  is locally finite if and only if so are all André-Quillen homology groups of  $K$ .

Proposition 2.3 yields then our main algebraic structural result about the category of unstable Hopf algebras.

**Theorem 2.4.** *Let  $B$  be a Hopf algebra which is a finitely generated unstable  $\mathcal{A}_p$ -algebra. Then so is any unstable Hopf subalgebra.*

**Acknowledgments.** We would like to thank Jean Lannes for his guided reading of [GLM92] at the Mittag-Leffler Institute and Carles Broto for many discussions on this material.

## 1. ANDRÉ-QUILLEN HOMOLOGY OF HOPF ALGEBRAS

In this section, we compute André-Quillen homology for Hopf algebras, and introduce the action of the Steenrod algebra in the next one. A clear and short introduction to André-Quillen homology can be found in Bousfield's [Bou75, Appendix], see also Goerss' book [Goe90].

Let us briefly recall from Schwartz's book [Sch94] how one computes André-Quillen homology in our setting. The symmetric algebra comonad  $S(-)$  yields a simplicial resolution  $S^\bullet(A)$  for any commutative algebra  $A$ . The André-Quillen homology group  $H_i^Q(A)$  is the  $i$ -th homology group

of the complex obtained from  $S^\bullet(A)$  by taking the module of indecomposable elements (and the differential is the usual alternating sum). This is a graded  $\mathbb{F}_p$ -vector space. Long exact sequences arise from certain extensions, just like in the dual situation for the primitive functor, [Bou75, Theorem 3.6].

**Lemma 1.1.** *Let  $A$  be a Hopf subalgebra of a Hopf algebra  $B$  of finite type. Then there is a long exact sequence*

$$\cdots \rightarrow H_2^Q(B//A) \rightarrow H_1^Q(A) \rightarrow H_1^Q(B) \rightarrow H_1^Q(B//A) \rightarrow QA \rightarrow QB \rightarrow Q(B//A)$$

in André-Quillen homology.

*Proof.* Long exact sequences in André-Quillen homology are induced by cofibrations of simplicial algebras. However, the inclusion  $A \subset B$  of a sub-Hopf algebra is not a cofibration in general (seen as a constant simplicial object). To get around this difficulty we use Goerss' argument from [Goe90, Section 10] and we reproduce it here in our framework.

For any morphism  $f : A \rightarrow B$  of simplicial algebras, there is a spectral sequence, [Goe90, Proposition 4.7],  $\text{Tor}_p^{\pi_* A}(\mathbb{F}_p, \pi_* B)_q$  converging to the homotopy groups  $\pi_{p+q} \text{Cof}(f)$  of the homotopy cofiber. Now, because  $B$  is of finite type, it is always a free  $A$ -module by the Milnor-Moore result [MM65, Theorem 4.4]. Thus the  $E_2$ -term is isomorphic to  $\text{Tor}_0^{\mathbb{F}_p}(\mathbb{F}_p, B//A)_* \cong B//A$ . The spectral sequence collapses and hence  $\text{Cof}(f)$  is weakly equivalent to  $B//A$ . In particular, we have the desired long exact sequence. ■

Following the terminology used in [Smi70, Section 6], we introduce the following definition.

**Definition 1.2.** A sequence of (Hopf) algebras

$$\mathbb{F}_p \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \mathbb{F}_p$$

is *coexact* if the morphism  $A \rightarrow B$  is a monomorphism and its cokernel  $B//A$  is isomorphic to  $C$  as a (Hopf) algebra.

We can thus restate the previous lemma by saying that coexact sequences of Hopf algebras induce long exact sequences in André-Quillen homology.

By the Borel-Hopf decomposition theorem [MM65, Theorem 7.11], any Hopf algebra of finite type is isomorphic, as an algebra, to a tensor product of monogenic Hopf algebras, i.e. either a truncated polynomial algebra of the form  $\mathbb{F}_p[x_i]/(x_i^{p^{k_i}})$ , where  $p^{k_i}$  is the *height* of the generator  $x_i$ , or a polynomial algebra of the form  $\mathbb{F}_p[y_j]$ , or, when  $p$  is odd, an exterior algebra  $\Lambda(z_i)$ . Let us denote by  $\xi$  the Frobenius map, sending an element  $x$  to its  $p$ -th power  $x^p$ .

**Proposition 1.3.** *Let  $A$  be a Hopf algebra of finite type. Then  $H_0^Q(A) = QA$  and  $H_1^Q(A)$  is isomorphic to the  $\mathbb{F}_p$ -vector space generated by the elements  $\xi^{k_i} x_i$  of degree  $p^{k_i} \cdot |x_i|$  where  $x_i \in A$  is a generator of height  $p^{k_i}$ ,  $0 < k_i < \infty$ . Moreover  $H_n^Q(A) = 0$  if  $n \geq 2$ .*

*Proof.* Consider the symmetric algebra  $S(QA)$  and construct an algebra map  $S(QA) \twoheadrightarrow A$  by choosing representatives in  $A$  of the indecomposable elements. We have then a coexact sequence of algebras  $\mathbb{F}_p[\xi^{k_i} x_i] \hookrightarrow S(QA) \twoheadrightarrow A$  and  $A$ , as Hopf algebra, can be seen as the quotient  $S(QA)/\mathbb{F}_p[\xi^{k_i} x_i]$ . Since  $S(QA)$  is a free commutative algebra,  $H_n^Q(S(QA)) = 0$  for all  $n \geq 1$ . Likewise  $H_n^Q(\mathbb{F}_p[\xi^{k_i} x_i]) = 0$  for all  $n \geq 1$ . Now, Lemma 1.1 allows us to identify  $H_1^Q(A) \cong H_0^Q(\mathbb{F}_p[\xi^{k_i} x_i]) \cong \bigoplus_k \mathbb{F}_p \langle \xi^{k_i} x_i \rangle$ , as a graded vector space.  $\blacksquare$

The vanishing of the higher André-Quillen homology groups, or in other words the fact that the functor  $Q(-)$  has homological dimension  $\leq 1$  for Hopf algebras, has been analyzed by Bousfield in the dual situation [Bou75, Theorem 4.1]. The next lemma is now a reformulation of the preceding proposition.

**Lemma 1.4.** *Let  $A$  be a Hopf algebra of finite type and denote by  $x_i$  the truncated polynomial generators. Then  $H_1^Q(A)$  is isomorphic to the  $\mathbb{F}_p$ -vector space generated by the elements  $x_i^{\otimes p^{k_i}} \in S(A)$ , where  $p^{k_i}$  is the height of  $x_i$ .*

*Proof.* We have to compute the first homology group of the complex

$$\dots \rightarrow S^2(A) \xrightarrow{d} S(A) \xrightarrow{m} A.$$

The morphisms are given by the alternating sums of the face maps. Let us use the symbols  $\otimes$  for the tensor product in  $S(A)$  and  $\boxtimes$  for the next level in  $S(S(A))$ . If  $\eta_A : S(A) \rightarrow A$  is the counit defined by  $\eta_A(a \otimes b) = ab$ , the two face maps  $S^2(A) \rightarrow S^1(A)$  are then  $S(\eta_A)$  and  $\eta_{S(A)}$ .

Thus  $m(a) = S(\eta_A)(a) - \eta_{S(A)}(a) = a - a = 0$  and  $m(a \otimes b) = ab$ , since  $\eta_{S(A)}(a \otimes b) = a \otimes b$  is decomposable. Likewise  $d(w) = \eta_A(w)$  on elements  $w \in S(A)$  and  $d(v \boxtimes w) = v \otimes w - \eta_A(v) \otimes \eta_A(w)$  for  $v, w \in S(A)$ . The elements  $x_i^{\otimes p^{k_i}}$  clearly belong to the kernel of  $m$ . To compare them to the generators  $\{\xi^{k_i} x_i\}$  of  $H_0^Q(\mathbb{F}_p[\xi^{k_i} x_i]) \cong H_1^Q(A)$ , we apply  $S^\bullet$  to the coexact sequence of algebras  $\mathbb{F}_p[\xi^{k_i} x_i] \hookrightarrow S(QA) \twoheadrightarrow A$ . The snake Lemma yields a connecting morphism  $\text{Ker}(m) \rightarrow H_0^Q(\mathbb{F}_p[\xi^{k_i} x_i])$ , which sends precisely  $x_i^{\otimes p^{k_i}}$  to  $\xi^{k_i} x_i$ .  $\blacksquare$

**Remark 1.5.** Alternatively, one could use the identification of the first André-Quillen homology group  $H_1^Q(A)$  with the indecomposable elements of degree 2 in  $\text{Tor}_A(\mathbb{F}_p, \mathbb{F}_p)$ , [Goe90, Section 10]. As an  $\mathbb{F}_p$ -vector space it is generated by the transpotence elements  $[x_i^{p^{k_i}-1} | x_i]$ . When  $p = 2$  and  $k_i = 0$ , this is not technically speaking a transpotence element, but still it is  $[x_i | x_i]$  which appears in degree 2, see for example [Kan88, Section 29-2].

## 2. BRINGING IN THE ACTION OF THE STEENROD ALGEBRA

The results of the previous section apply to Hopf algebras which are finitely generated as algebras over the Steenrod algebra: they are of finite type. Our aim in this section is to identify the action of the Steenrod algebra on the unstable module  $H_1^Q(A)$ . Good references on André-Quillen homology

for unstable algebras are [Sch94, Chapter 7] and of course [Mil84], which showed the importance of André-Quillen homology in a topological context.

The  $\mathbb{F}_p$ -vector space  $H_1^Q(A)$  is equipped with an action of  $\mathcal{A}_p$  because the Steenrod algebra acts on the symmetric algebra via the Cartan formula. This yields the same unstable module  $H_1^Q(A)$  as the derived functor computed with a resolution in the category of unstable algebras, [LS86] and [Sch94, Proposition 7.2.2].

As expected with this type of questions, the case when  $p = 2$  is slightly simpler than the case when  $p$  is odd. To write a unified proof, we use the well-known trick [LZ86] to consider, in the odd-primary case, the subalgebra of  $\mathcal{A}_p$  concentrated in even degrees. If  $M$  is a module over  $\mathcal{A}_p$ , the module  $M'$  concentrated in even degree is defined by  $(M')^{2n} = M^{2n}$  and  $(M')^{2n+1} = 0$ . This is not an  $\mathcal{A}_p$ -submodule of  $M$ , but it is a module over  $\mathcal{A}_p$  on which the Bockstein  $\beta$  acts trivially. Hence it can be seen as a module over the algebra  $\mathcal{A}'_p$ , the subalgebra of  $\mathcal{A}_p$  generated by the operations  $\mathcal{P}^i$ . When  $p = 2$  we adopt the convention that  $\mathcal{A}'_2 = \mathcal{A}_2$ ,  $\mathcal{U}' = \mathcal{U}$ , and write  $\mathcal{P}^i$  for  $Sq^i$ . Like in [Sch94, 1.2], for a sequence  $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$  where the  $\varepsilon_k$ 's are 0 or 1, we write  $\mathcal{P}^I$  for the operation  $\beta^{\varepsilon_0} \mathcal{P}^{i_1} \beta^{\varepsilon_1} \dots \mathcal{P}^{i_n} \beta^{\varepsilon_n}$ .

In [LZ86, Appendice B], Lannes and Zarati prove that the category  $\mathcal{U}$  is locally noetherian, which they do by reducing the proof to the case of  $\mathcal{U}'$ . We use their computations in the following lemma, in fact the explicit version from [Sch94, 1.8].

**Lemma 2.1.** *Let  $M$  be an unstable module which is finitely generated over  $\mathcal{A}_p$ . Then so is the module  $M'$ , over  $\mathcal{A}'_p$ .*

*Proof.* The statement is a tautology when  $p = 2$ . Let us assume  $p$  is an odd prime. In the category  $\mathcal{U}'$  of unstable modules concentrated in even degrees,  $F'(2n)$  is the free object on one generator  $\iota_{2n}$  in degree  $2n$ . We must show that  $M'$  is a quotient of a finite direct sum of such modules. As we know that  $M$  is a quotient of a finite direct sum of  $F(n)$ 's, it is enough to prove the lemma for a free module  $F(n)$ .

A basis over  $\mathbb{F}_p$  for the module  $F(n)$  is given by the elements  $\mathcal{P}^I \iota_n$  where  $I$  is admissible with excess  $e(I) \leq n$ . Define  $F(n)'_k$  to be the span over  $\mathbb{F}_p$  of the elements  $\mathcal{P}^I \iota_n$  with  $e(I) \geq k$ . Then  $F(n)'_k / F(n)'_{k+1}$  is zero when  $k + n$  is odd and it is generated by the images of the elements  $\mathcal{P}^I \iota_n$  where the  $k$  Bocksteins appear in the first  $k + 1$  possible slots. In particular,  $F(n)'$  is generated by these elements as an  $\mathcal{A}'_p$ -module. ▀

The generators for  $H_1^Q(A)$  will be related to certain elements in  $QA$  we describe next.

**Lemma 2.2.** *Let  $A$  be a Hopf algebra which is a finitely generated unstable  $\mathcal{A}_p$ -algebra and let  $N$  be the submodule of  $QA$  generated by the truncated polynomial generators  $x_i$ . Then  $N'$  is finitely generated in  $\mathcal{U}'$ . There exists an integer  $d$  and a finite set  $\{x_{k,i} \mid 1 \leq k \leq d, 1 \leq i \leq n_k\}$  of generators*

such that any element in  $N$  of height  $p^k$  can be written  $\sum_i \theta_{k,i} x_{k,i}$  for some (admissible) operations  $\theta_{k,i} \in \mathcal{A}'_p$ .

*Proof.* Since  $U$  is locally noetherian, [Sch94, Theorem 1.8.1], the unstable module  $N$  is finitely generated, being a submodule of  $QA$ . Thus, by Lemma 2.1,  $N'$  is finitely generated over  $\mathcal{A}'_p$ . This implies in particular that the height of the truncated generators is bounded by some integer  $p^d$  (the action of the Steenrod algebra on  $x_i$  can only lower the height by the formulas [Sch94, 1.7.1]).

For  $1 \leq k \leq d$ , write  $N'(k)$  for the submodule of  $N'$  generated by the  $x_i$ 's of height  $p^k$  and choose representatives in  $A$  of generators  $x_{k,i}$  for the module  $N'(k)$ , with  $1 \leq i \leq n_k$ . Hence the finite set  $\{x_{k,i} \mid 1 \leq k \leq d, 1 \leq i \leq n_k\}$  generates  $N'$ .  $\blacksquare$

The relation  $x = \sum_i \theta_{k,i} x_{k,i}$  holds in the module of indecomposable elements (in fact in  $N'$ ). Beware that the same relation holds also in the algebra  $A$ , but only up to decomposable elements.

**Proposition 2.3.** *Let  $A$  be a Hopf algebra which is a finitely generated unstable  $\mathcal{A}_p$ -algebra. Then  $H_0^Q(A) = QA$  and  $H_1^Q(A)$  are both finitely generated unstable modules.*

*Proof.* Lemma 1.4 allows us to identify  $H_1^Q(A) \cong \bigoplus_k \mathbb{F}_p \langle x_i^{\otimes p^{k_i}} \rangle$ , as a graded vector space. We must now identify the action of the Steenrod algebra.

We claim that the finite set of elements  $x_{k,i}^{\otimes p^{k_i}}$  generates  $H_1^Q(A)$  as unstable module. More precisely we show that the relation  $x = \sum_i \theta_{k,i} x_{k,i}$  in  $QA$  yields a relation for  $x^{\otimes p^{k_i}}$  in  $H_1^Q(A)$ . To simplify the notation, let us assume that the height of  $x$  is  $p^k$  and that the relation is of the form  $x = \sum_j \theta_j x_j$  for generators  $x_j$  of the same height. The relation for  $x$  holds in  $A$  up to decomposable elements which must have lower height. But if  $a^{p^k} = 0 = b^{p^k}$ , then

$$d[a^{\otimes p^k} \boxtimes b^{\otimes p^k} - (a \otimes b)^{\boxtimes p^k}] = a^{\otimes p^k} \otimes b^{\otimes p^k} - a^{p^k} \otimes b^{p^k} - (a \otimes b)^{\otimes p^k} + (ab)^{\otimes p^k} = (ab)^{\otimes p^k}$$

and hence the decomposable elements disappear in  $H_1^Q(A)$ . Therefore  $x^{\otimes p^k} = (\sum_i \theta_j x_j)^{\otimes p^k}$  in  $H_1^Q(A)$ . The operations  $\theta_j$  live in  $\mathcal{A}'_p$ , so that we have basically to perform the following computation in the symmetric algebra:  $(\mathcal{P}^n x)^{\otimes p^k} = \mathcal{P}^{p^k n}(x^{\otimes p^k})$ . There exist thus operations  $\Theta_j \in \mathcal{A}'_p$  such that

$$x^{\otimes p^k} = \sum_j (\theta_j x_j)^{\otimes p^k} = \sum_j \Theta_j(x_j^{\otimes p^k})$$

and the claim is proven.  $\blacksquare$

**Theorem 2.4.** *Let  $B$  be a Hopf algebra which is a finitely generated unstable  $\mathcal{A}_p$ -algebra. Then so is any unstable Hopf subalgebra.*

*Proof.* Consider an unstable Hopf subalgebra  $A \subset B$  and the quotient  $B//A$ . By Lemma 1.1, we have an associated exact sequence in André-Quillen homology

$$H_1^Q(B//A) \rightarrow QA \rightarrow QB,$$

in which the unstable modules  $QB$  and  $H_1^Q(B//A)$  are finitely generated by Proposition 2.3. Thus so is  $QA$ .  $\blacksquare$

**Example 2.5.** Let us consider the Hopf algebra  $B = H^*(K(\mathbb{Z}/p, 2))$ . When  $p$  is odd it is the tensor product of a polynomial algebra  $\mathbb{F}_p[\iota_2, \beta\mathcal{P}^1\beta\iota_2, \beta\mathcal{P}^p\mathcal{P}^1\beta\iota_2, \dots]$ , concentrated in even degrees, with an exterior algebra  $\Lambda(\beta\iota_2, \mathcal{P}^1\beta\iota_2, \dots)$ .

We consider the Hopf subalgebra  $A$  given by the image of the Frobenius  $\xi$ . This is the polynomial subalgebra

$$\mathbb{F}_p[(\iota_2)^p, (\beta\mathcal{P}^1\beta\iota_2)^p, (\beta\mathcal{P}^p\mathcal{P}^1\beta\iota_2)^p, \dots]$$

The quotient  $B//A$  has an exterior part and a truncated polynomial part where all generators have height  $p$ . The module of indecomposable elements  $Q(B//A)$  is isomorphic to  $QB$ . It is a quotient of  $F(2)$ , and thus generated, as an unstable module, by a single generator  $\iota_2$  in degree 2. The submodule concentrated in even degree is a module over  $\mathcal{A}'_p$ . It is finitely generated as well, by Lemma 2.1, but one needs two generators  $\iota_2$  and  $\beta\mathcal{P}^1\beta\iota_2$ . Explicit computations of the action of the Steenrod algebra can be found in [Cre01].

Therefore  $H_1^Q(B//A)$  is an unstable module, which is generated by the elements  $\iota_2^{\otimes p}$  and  $(\beta\mathcal{P}^1\beta\iota_2)^{\otimes p}$ .

**Remark 2.6.** For plain unstable algebras, Theorem 2.4 is false, as pointed out to us by Hans-Werner Henn. Consider indeed the unstable algebra

$$H^*(\mathbf{C}P^\infty \times S^2; \mathbf{F}_p) \cong \mathbf{F}_p[x] \otimes E(y)$$

where both  $x$  and  $y$  have degree 2. Take the ideal generated by  $y$ , and add 1 to turn it into an unstable subalgebra. Since  $y^2 = 0$ , this is isomorphic, as an unstable algebra, to  $\mathbf{F}_p \oplus \Sigma^2\mathbf{F}_p \oplus \Sigma^2\tilde{H}^*(\mathbf{C}P^\infty; \mathbf{F}_p)$ , which is not finitely generated.

### 3. $H$ -FIBRATIONS OVER EILENBERG-MAC LANE SPACES

In the second part of this paper we turn now our attention to topological applications of the André-Quillen homology computation we have done previously. More precisely, we concentrate on  $H$ -fibrations.

**Definition 3.1.** An  $H$ -space  $B$  satisfies the *(strong) fg closure property* if, for any  $H$ -fibration  $F \rightarrow E \rightarrow B$ , the cohomology  $H^*(E)$  is a finitely generated unstable algebra if (and only if) so is  $H^*(F)$ .

We prove in this section that Eilenberg-Mac Lane spaces enjoy the strong fg closure property. In [CCSa] we established the fg closure property for  $K(A, n)$  with  $n \geq 2$ , which was sufficient to our purposes there.

Given  $n \geq 2$ , consider a non-trivial  $H$ -fibration  $F \xrightarrow{i} E \xrightarrow{\pi} K(A, n)$  where  $A$  is either  $\mathbb{Z}/p$  or a Prüfer group  $\mathbb{Z}_{p^\infty}$ . This situation has been extensively and carefully studied by L. Smith in [Smi70].

The  $E_2$ -term of the Eilenberg-Moore spectral sequence is given by  $\mathrm{Tor}_{H^*(K(A, n))}(H^*(E), \mathbb{F}_p)$  and converges to  $H^*(F)$ . Since we deal with an  $H$ -fibration, [Smi70, Theorem 2.4] applies and  $E_2 \cong H^*(E)/\pi^* \otimes \mathrm{Tor}_{H^*(K(A, n)) \setminus \setminus \pi^*}(\mathbb{F}_p, \mathbb{F}_p)$  as algebras, where  $H^*(K(A, n)) \setminus \setminus \pi^*$  is the Hopf subalgebra kernel of  $\pi^*$ . The first differential is  $d_{p-1}$  [Smi70, Theorem 4.7]. The same argument as in [Smi70, Section 5] (done for stable Postnikov pieces) is valid in our situation as well, and has been in fact already used in this setting, see [Smi70, Proposition 7.3\*]: on algebra generators the next differentials must be zero, so that the spectral sequence collapses at  $E_p$ . This term is generated by  $H^*(E)/\pi^* = E_p^{0,*}$ ,  $s^{-1,0} \mathrm{Coker} \beta\mathcal{P}_0 \subset E_p^{-1,*}$ , and  $s^{-1,0} QH^*(K(A, n)) \setminus \setminus \pi^{\mathrm{odd}} \subset E_p^{-1,*}$ , where  $\beta\mathcal{P}_0: QH^{\mathrm{odd}}(K(A, n)) \rightarrow QH^{\mathrm{even}}(K(A, n))$  is defined by  $\beta\mathcal{P}_0(x) = \beta\mathcal{P}^t(x)$  with  $2t+1 = |x|$ .

The algebra structure is described in [Smi70, Proposition 7.3\*] by means of coexact sequences, see Definition 1.2.

**Proposition 3.2.** [Smi70] *Let  $n \geq 2$  and consider an  $H$ -fibration  $F \xrightarrow{i} E \xrightarrow{\pi} K(A, n)$  where  $A$  is either  $\mathbb{Z}/p$  or a Prüfer group  $\mathbb{Z}_{p^\infty}$ . Then there is a coexact sequence of Hopf algebras*

$$\mathbb{F}_p \longrightarrow H^*(E)/\pi^* \xrightarrow{i^*} H^*(F) \longrightarrow R \longrightarrow \mathbb{F}_p,$$

and  $R$  is described in turn by a coexact sequence of Hopf algebras

$$\mathbb{F}_p \longrightarrow \Lambda \longrightarrow R \longrightarrow S \longrightarrow \mathbb{F}_p,$$

where  $\Lambda$  is an exterior algebra generated by  $s^{-1,0} \mathrm{Coker} \beta\mathcal{P}_0$ , and  $S \subseteq H^*(K(A, n-1))$  is a Hopf subalgebra. ▀

**Example 3.3.** Let us see how the well-known cohomology of  $S^3\langle 3 \rangle$  can be identified with these tools. Consider the fibration

$$S^3\langle 3 \rangle \xrightarrow{i} S^3 \xrightarrow{\pi} K(\mathbb{Z}, 3)$$

In this situation  $H^*(S^3)/\pi^* = 0$  and  $H^*(S^3\langle 3 \rangle) \cong \Lambda \otimes S$  by Proposition 3.2. Recall that  $H^*(K(\mathbb{Z}, 3)) = \mathbb{F}_p[\beta\mathcal{P}_k u_3 : k \geq 1] \otimes E(\mathcal{P}_k u_3 : k \geq 0)$  where  $\mathcal{P}_k = \mathcal{P}^{k-1} \cdots \mathcal{P}^1$  and  $\mathcal{P}_0 u_3 = u_3$ . Here the Hopf algebra kernel  $H^*(K(\mathbb{Z}, 3)) \setminus \setminus \pi^*$  is  $\mathbb{F}_p[\beta\mathcal{P}_k u_3 : k \geq 1] \otimes E(\mathcal{P}_k u_3 : k \geq 1)$ . One sees next that the cokernel of  $\beta\mathcal{P}_p$  is  $\{\beta\mathcal{P}^1 u_3\}$  and  $S \subset H^*(K(\mathbb{Z}, 2))$  is generated by  $\mathcal{P}^1 \iota_2 = \iota_2^p$ . This implies that  $H^*(S^3\langle 3 \rangle) \cong \mathbb{F}_p[x_{2p}] \otimes E(\beta x_{2p})$ .

Proposition 3.2 will allow us to improve [CCSa, Theorem 6.1]. We rely on the following obvious lemma, which we will use again in the next section.

**Lemma 3.4.** *Consider an  $H$ -space  $B$  and assume that there exists an  $H$ -fibration  $B' \rightarrow B \rightarrow B''$  such that both  $B'$  and  $B''$  satisfy the (strong) fg closure property. Then so does  $B$ .*

*Proof.* Consider an  $H$ -fibration  $F \rightarrow E \rightarrow B$  and construct the pull-back diagram of fibrations

$$\begin{array}{ccccc} E' & \longrightarrow & E & \longrightarrow & B'' \\ p' \downarrow & & \downarrow p & & \parallel \\ B' & \longrightarrow & B & \longrightarrow & B'' \end{array}$$

The homotopy fiber of  $p'$  is  $F$ , which allows to conclude. ▀

**Theorem 3.5.** *Let  $A$  be a finite direct sum of copies of cyclic groups  $\mathbb{Z}/p^r$  and Prüfer groups  $\mathbb{Z}_{p^\infty}$ , and  $n \geq 2$ . Consider an  $H$ -fibration  $F \xrightarrow{i} E \xrightarrow{\pi} K(A, n)$ . Then  $H^*(F)$  is a finitely generated  $\mathcal{A}_p$ -algebra if and only if so is  $H^*(E)$ .*

*Proof.* If we consider the fibration of Eilenberg-Mac Lane spaces induced by a group extension  $A' \rightarrow A \rightarrow A$ , we see from Lemma 3.4 that we can assume that  $A = \mathbb{Z}/p$  or  $\mathbb{Z}_{p^\infty}$ .

Since  $H^*(K(A, n))$  is finitely generated as algebra over  $\mathcal{A}_p$ , so is its image  $\text{Im}(\pi^*) \subseteq H^*(E)$ . Hence, to prove the theorem, it is enough to show that the module of indecomposable elements  $Q(H^*(E)/\pi^*)$  is a finitely generated  $\mathcal{A}_p$ -module if and only if so is  $QH^*(F)$ .

Let us now apply Lemma 1.1 to the coexact sequences from Proposition 3.2. The unstable Hopf algebra  $S$  is an unstable Hopf subalgebra of  $H^*(K(A, n))$ . Thus Theorem 2.4 implies that  $S$  is finitely generated over  $\mathcal{A}_p$ . Moreover, the exterior algebra  $\Lambda$  is identified with  $E(s^{-1,0} \text{Coker } \beta\mathcal{P}_0)$ , where the cohomological operation  $\beta\mathcal{P}_0$  is to be understood as an operation from the odd degree part of  $QH^*(K(A, n))$  to the even degree part. The latter module is finitely generated by Lemma 2.1. Hence the cokernel is finitely generated as well, as a module over the Steenrod algebra. The exact sequence in André-Quillen homology for the coexact sequence involving  $R$  and Proposition 2.3 show that both  $QR$  and  $H_1^Q(R)$  are finitely generated unstable modules. Finally, since  $\mathcal{U}$  is a locally noetherian category, [Sch94, Theorem 1.8.1], the exact sequence

$$H_1^Q(R) \rightarrow Q(H^*(E)/\pi^*) \rightarrow QH^*(F) \rightarrow QR$$

implies that  $QH^*F$  is a finitely generated  $\mathcal{A}_p$ -module if and only if so is  $Q(H^*(E)/\pi^*)$ . ▀

**Remark 3.6.** Another approach to Theorem 3.5 is to dualize the work of Goerss, Lannes, and Morel in [GLM92, Section 2]. They analyze the homology sequence

$$H_*(\Omega^2 K(A, n)) \rightarrow H_*(\Omega F) \rightarrow H_*(\Omega E) \rightarrow H_*(\Omega K(A, n))$$

for a fibration of spaces  $F \rightarrow E \rightarrow K(A, n)$  and measure its failure to be exact. This can be dualized and actually works for  $H$ -fibrations, not only loop fibrations. Let us quickly sketch the key ideas. Consider now an  $H$ -fibration  $F \rightarrow E \rightarrow K(A, n)$  and the complex

$$H^*(K(A, n)) \xrightarrow{\pi^*} H^*E \xrightarrow{i^*} H^*F \rightarrow H^*K(A, n-1).$$

Define  $K$  to be the Hopf cokernel of the morphism  $i^*: H^*E \rightarrow H^*F$  and  $M$  to be the kernel of the morphism  $\pi^*$  on primitive elements  $PH^*K(A, n) \rightarrow PH^*E$ . The cohomology suspension morphism  $\sigma: H^*(K(A, n)) \rightarrow \Sigma H^*(K(A, n-1))$  restricted to  $M$  defines a morphism  $M \rightarrow \Sigma K$ . The adjoint  $\Omega M \rightarrow K$  induces an isomorphism of  $\mathcal{A}_p$ -Hopf algebras  $U\Omega M \rightarrow K$  (compare with [Smi67, Proposition 5.7]), where  $U$  is Steenrod-Epstein's functor, left adjoint to the forgetful functor  $\mathcal{K} \rightarrow \mathcal{U}$ .

Denote by  $N$  the cokernel of  $M \hookrightarrow PH^*K(A, n)$ . The above complex is then exact at  $H^*E$ , [Smi67, Proposition 5.5], and its homology at  $H^*F$  is isomorphic to  $U\Omega_1 N$ , where  $\Omega_1$  is the first left derived functor of  $\Omega$ .

Theorem 3.5 now follows from the fact that  $\Omega_1 N$  is a finitely generated unstable module. This simply reflects the fact that the functor  $\Phi$  (which is the “doubling” functor when  $p = 2$ ), [Sch94, 1.7.2] takes finitely generated unstable modules to finitely generated ones.

We have thus shown that Eilenberg-MacLane spaces  $K(A, n)$  satisfy the strong fg closure property. But in fact, it can be easily generalized to  $p$ -torsion Postnikov pieces.

**Proposition 3.7.** *Consider an  $H$ -fibration  $F \xrightarrow{i} E \xrightarrow{\pi} B$ , where  $B$  is an  $p$ -torsion  $H$ -Postnikov piece whose homotopy groups are finite direct sums of cyclic groups and Prüfer groups. Then  $H^*(E)$  is a finitely generated  $\mathcal{A}_p$ -algebra, if and only if so is  $H^*(F)$ .*

*Proof.* An induction on the number of homotopy groups of  $B$  with Lemma 3.4 reduces to the case when  $B$  is an Eilenberg-Mac Lane space  $K(A, n)$ . We know from Theorem 3.5 that the statement holds in this case. ▀

Our first corollary has already been proved in [CCSa].

**Corollary 3.8.** *Let  $F$  be an  $H$ -Postnikov piece of finite type. Then  $H^*(F)$  is finitely generated as unstable algebra.*

*Proof.* The result is true for an Eilenberg-Mac Lane space  $K(A, n)$  where  $A$  is an abelian group of finite type. The proof then follows by induction on the number of homotopy groups. ▀

#### 4. CLOSURE PROPERTIES OF $H$ -FIBRATIONS

The aim of this section is to extend the results of the preceding section to arbitrary base spaces. We will prove that any  $H$ -space  $B$  such that  $H^*(B)$  is a finitely generated algebra over  $\mathcal{A}_p$  satisfies the fg closure property. We need here some input from the theory of localization. Recall (cf. [Far96]) that, given a pointed connected space  $A$ , a space  $X$  is  $A$ -local if the evaluation at the base point in  $A$  induces a weak equivalence of mapping spaces  $\text{map}(A, X) \simeq X$ . When  $X$  is an  $H$ -space, it is sufficient to require that the pointed mapping space  $\text{map}_*(A, X)$  be contractible.

Dror-Farjoun and Bousfield have constructed a localization functor  $P_A$  from spaces to spaces together with a natural transformation  $l : X \rightarrow P_A X$  which is an initial map among those having an  $A$ -local space as target (see [Far96] and [Bou77]). This functor is known as the  $A$ -nullification. It preserves  $H$ -space structures since it commutes with finite products. Moreover, when  $X$  is an  $H$ -space, the map  $l$  is an  $H$ -map and its fiber is an  $H$ -space.

Recall that, for any elementary abelian group  $V$ , tensoring with  $H^*V$  has a left adjoint, Lannes'  $T$ -functor  $T_V$ , [Lan92]. When  $V = \mathbb{Z}/p$ , the notation  $T$  is usually used instead of  $T_{\mathbb{Z}/p}$  and the reduced  $T$ -functor is left adjoint to tensoring with the reduced cohomology of  $\mathbb{Z}/p$ . This allows to characterize the Krull filtration of the category  $\mathcal{U}$  of unstable modules as follows:  $M \in \mathcal{U}_n$  if and only if  $\overline{T}^{n+1}M = 0$ , [Sch94, Theorem 6.2.4].

**Lemma 4.1.** *Let  $X$  be an  $H$ -space such that  $T_V H^*(X)$  is of finite type for any elementary abelian  $p$ -group  $V$ . Then  $H^*(P_{B\mathbb{Z}/p}X)$  is finite if and only if, for any  $n$ ,  $H^*(P_{\Sigma^n B\mathbb{Z}/p}X)$  is a finitely generated  $\mathcal{A}_p$ -algebra.*

*Proof.* By [Bou94], for any  $n$  there are fibrations  $P_{\Sigma^n B\mathbb{Z}/p}X \rightarrow P_{\Sigma^{n-1}B\mathbb{Z}/p}X \rightarrow K(A_n, n+1)$  where  $A_n$  is a  $p$ -torsion abelian group, which is a finite direct sum of copies of cyclic groups  $\mathbb{Z}/p^r$  and Prüfer groups  $\mathbb{Z}_{p^\infty}$  (the technical hypothesis on the  $T$  functor allows to apply [CCSa, Theorem 5.4]). In this situation, we can apply Theorem 3.5 to show that  $H^*(P_{\Sigma^n B\mathbb{Z}/p}X)$  is a finitely generated algebra if and only if  $H^*(P_{\Sigma^{n-1}B\mathbb{Z}/p}X)$  is so. The statement follows by induction since  $H^*(P_{B\mathbb{Z}/p}X)$  is always locally finite.  $\blacksquare$

**Theorem 4.2.** *Consider an  $H$ -fibration  $F \xrightarrow{i} E \xrightarrow{\pi} B$ . If  $H^*(F)$  and  $H^*(B)$  are finitely generated  $\mathcal{A}_p$ -algebras, then so is  $H^*(E)$ .*

*Proof.* Since both  $H^*(F)$  and  $H^*(B)$  are finitely generated  $\mathcal{A}_p$ -algebras, the modules of indecomposable elements  $QH^*(F)$  and  $QH^*(B)$  are finitely generated  $\mathcal{A}_p$ -modules. Therefore, [CCSa, Lemma 7.1], they belong to some stage  $\mathcal{U}_{n-1}$  of the Krull filtration. By [CCSa, Theorem 5.3], we know that both  $F$  and  $B$  are  $\Sigma^n B\mathbb{Z}/p$ -local spaces. Since nullification preserves fibrations whose base space is local (see [Far96, Corollary 3.D.3]), it follows that  $E$  is also  $\Sigma^n B\mathbb{Z}/p$ -local.

Let us consider the fibration  $\bar{P}_{B\mathbb{Z}/p}B \rightarrow B \rightarrow P_{B\mathbb{Z}/p}B$ . We know from Lemma 4.1 that the nullification  $P_{B\mathbb{Z}/p}B$  has finite mod  $p$  cohomology. The homotopy fiber  $\bar{P}_{B\mathbb{Z}/p}B$  is an  $H$ -Postnikov piece whose homotopy groups are finite direct sums of cyclic groups  $\mathbb{Z}/p^r$  and Prüfer groups  $\mathbb{Z}_{p^\infty}$  by [CCSa, Theorem 5.4]. Lemma 3.4 and Proposition 3.7 show then that it is enough to prove the theorem when  $H^*(B)$  is finite.

In that case,  $B$  is a  $B\mathbb{Z}/p$ -local space, and we have thus a diagram of fibrations

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \parallel \\ P_{B\mathbb{Z}/p}F & \longrightarrow & P_{B\mathbb{Z}/p}E & \longrightarrow & B \end{array}$$

The mod  $p$  cohomology  $H^*(P_{B\mathbb{Z}/p}F)$  is finite and hence so is  $H^*(P_{B\mathbb{Z}/p}E)$  by an easy Serre spectral sequence argument. Finally, we can apply Lemma 4.1 to conclude that  $H^*(E) \cong H^*(P_{\Sigma^n B\mathbb{Z}/p}E)$  is a finitely generated  $\mathcal{A}_p$ -algebra.  $\blacksquare$

**Corollary 4.3.** *Consider an  $H$ -space  $X$  with finite mod  $p$  cohomology. Then the mod  $p$  cohomology of its  $n$ -connected cover  $X\langle n \rangle$  is a finitely generated  $\mathcal{A}_p$ -algebra.*

*Proof.* Consider the  $H$ -fibration  $\Omega(X[n]) \rightarrow X\langle n \rangle \rightarrow X$ . The fiber is an  $H$ -Postnikov piece of finite type and the cohomology of the base is finite. The result follows.  $\blacksquare$

This can be seen as the mirror result of [CCSb], where we proved that any  $H$ -space with finitely generated cohomology as algebra over the Steenrod algebra is an  $n$ -connected cover of an  $H$ -space with finite mod  $p$  cohomology, up to a finite number of homotopy groups.

**Remark 4.4.** We have already seen in the proof of Theorem 4.2 that the module of indecomposable elements of a finitely generated cohomology, as algebra over the Steenrod algebra, belongs to some stage of the Krull filtration. In fact, for any  $H$ -space  $X$  with finite mod  $p$  cohomology,  $QH^*X\langle n \rangle$  belongs to  $\mathcal{U}_{n-2}$  by [CCSa, Theorem 5.3], because  $\Omega^{n-1}(X\langle n \rangle)$  is  $B\mathbb{Z}/p$ -local.

**Remark 4.5.** Theorem 4.2 cannot be improved to an “if and only if” statement. Consider for example the path-fibration for the 3-dimensional sphere  $\Omega S^3 \rightarrow PS^3 \rightarrow S^3$ . It is well-known that  $H^*(\Omega S^3)$  is a divided power algebra, which is not finitely generated over  $\mathcal{A}_p$ .

We conclude with a characterization of the  $H$ -spaces which satisfy the strong fg closure property.

**Proposition 4.6.** *Let  $X$  be an  $H$ -space  $X$  which satisfies the strong fg closure property. Then  $X$  is, up to  $p$ -completion, a  $p$ -torsion Postnikov piece.*

*Proof.* If  $X$  satisfies the strong fg closure property, then  $H^*(\Omega X)$  is a finitely generated  $\mathcal{A}_p$ -algebra. But in this case, by [CCSa, Corollary 7.4],  $\Omega X$  is, up to  $p$ -completion, a  $p$ -torsion Postnikov piece.  $\blacksquare$

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