A GENERALIZATION OF THE TRIAD THEOREM OF BLAKERS-MASSEY

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1. Introduction

The purpose of this paper is to find a geometric reason for the triad theorem of Blakers–Massey (see [1] and [7]). We are asking the following question:

How far a homotopy push–out square is from being homotopy pull–back?

In particular, for a cofibration map $A \to X$, we want to investigate the difference between A and the homotopy fiber of the cofiber map $X \to X/A$.

The initial data for our investigation can be organized by having a commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[drr, "q"] & E \arrow[d] \arrow[drr, "id"] & & \\
 & & Y \arrow[r] \arrow[d] & E \arrow[d] \\
X \arrow[r] \arrow[drr, "id"'] & B \arrow[drr, "id"] & X \arrow[r] & B \\
 & & X \arrow[r] & B
\end{tikzcd}$$

where the following are respectively a homotopy push–out square and a homotopy pull–back square:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & E \arrow[d] \\
X \arrow[r] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r] \arrow[d] & E \arrow[d] \\
X \arrow[r] & B
\end{tikzcd}$$

The map $q : A \to Y$ measures how far the above homotopy push–out diagram is from being homotopy pull–back. Our aim is to understand this map. In this paper we will be able to give some "approximation" for the homotopy fiber and the homotopy cofiber of $\Sigma q : \Sigma A \to \Sigma Y$. We will approximate using cellular inequalities – a notion that was introduced by E. Dror Farjoun (see [6]).

Let A be a connected space. By $C(A)$ we denote the smallest class of connected spaces that contains A and is closed under weak equivalences, taking homotopy push–outs, arbitrary wedges and telescopes. One can think about the class $C(A)$ as generated by A using certain simple operations: taking homotopy push–outs, arbitrary wedges and telescopes. We say that a space X is built by A if X belongs to $C(A)$. If X is built by A, we write $X >> A$. We refer to the relation ">>" as a strong cellular inequality.

The following theorem is the main result of this paper. It gives a cellular estimation for the homotopy fiber and the homotopy cofiber of $\Sigma q : \Sigma A \to \Sigma Y$ (see corollary 7.4 and theorem 7.5):

Theorem. *Let A, E and X be connected. If $Fib(A \to X)$ and $Fib(A \to E)$ are connected, then:*

- $Fib(\Sigma q : \Sigma A \to \Sigma Y) >> Fib(A \to X) \wedge Fib(A \to E)$
- $Cof(\Sigma q : \Sigma A \to \Sigma Y) >> Fib(A \to X) \star Fib(A \to E)$

where $W \wedge V$ and $W \star V$ are respectively the wedge and the join of W and V.

Let $A \to X$ be a cofibration and F be the homotopy fiber of the quotient map $X \to X/A$. Under an additional assumption, out of the above theorem, we can get a cellular estimation for the homotopy cofiber $Cof(\Sigma q : \Sigma A \to \Sigma F)$ entirely in terms of A and X/A (see corollary 6.9):

Theorem. *Let A be connected. If $A \to X$ is a cofibration, for which X/A is weakly equivalent to the suspension of a connected space, then:*

$$Cof(\Sigma q : \Sigma A \to \Sigma F) >> A \wedge (X/A)$$

The above cellular approximations are derived from an inequality that generalizes Serre's theorem [9, theorem 6.1] (see theorem 5.2):

Theorem. *Let the following be a homotopy pull–back square:*

$$\begin{array}{ccc} Y & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

If B is connected, then:

$$Fib(E/Y \to B/X) >> Fib(E \to B) \star Fib(X \to B)$$

where E/Y and B/X are the homotopy cofibers, respectively of $Y \to E$ and $X \to B$.

As an application of the proven inequalities one can get the triad theorem of Blakers-Massey (see corollary 7.7). In addition one can also get statements as follows (see corollary 7.6):

Theorem. *Let B be connected. Let the following be a homotopy push–out square:*

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

Let $Y = holim(X \to B \leftarrow E)$ and $q : A \to Y$ be the natural map. Let p and s be distinct prime numbers. If the reduced integral homology of $Fib(A \to E)$ and $Fib(A \to X)$ are respectively p and s torsion, then $q : A \to Y$ is a homology isomorphism.

2. Notation

This paper is written simplicialy. The only place though where we use in a crucial way the nature of spaces, we are working with, is the proof of the main theorem (see 5.1). All the statements remain valid in the category of topological spaces having the homotopy type of a CW–complex.

The homotopy cofiber of a map $A \to X$ is denoted either by X/A, if it is clear which map we are considering, or by $Cof(A \to X)$, if we want to point out the map.

Lets choose a basepoint in X. The homotopy fiber of a map $A \to X$ at the chosen basepoint is denoted by $Fib(A \to X)$. If X is connected, then the homotopy type of $Fib(A \to X)$ does not depend on the choice of a basepoint in X.

For a space X, by ΣX we denote the unreduced suspension of X. If we choose a basepoint in X, by ΩX we denote the homotopy fiber of the basepoint map $\star \to X$.

Let X and Y be pointed spaces. The subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ of $X \times Y$ is denoted by $X \vee Y$ and is called the wedge of X and Y. The quotient space $X \times Y/(X \vee Y)$ is denoted by $X \wedge Y$ and is called the smash of X and Y. For any choice of basepoints in X and S^1, ΣX is weakly equivalent to $X \wedge S^1$ (in the case of topological spaces, we have to assume that the chosen basepoint in A is a cofibration).

Let K be a simplicial set. By the same symbol K we denote the category of simplices of K (see [3, definition 3.1]). Objects of this category are simplices of K and morphisms are generated by arrows $\sigma \to d_i\sigma$ and $\sigma \to s_i\sigma$, where $d_i\sigma$ is the i–th face of a simplex σ and $s_i\sigma$ is the i–th degeneracy of σ. Functors over the category associated with a simplicial set K are called diagrams over this simplicial set. If for every simplex $\sigma \in K$, $F(\sigma \to s_i\sigma) = id$ and $F(\sigma \to d_i\sigma)$ is a weak equivalence, then F is called a good diagram (see [3, definition 3.8]). Having such a diagram $F : K \to Spaces$, we can form its homotopy colimit $\oint_K F$. The homotopy colimit $\oint_K F$ is a simplicial set, whose set of n–simplices is given by:

$$\Big(\oint_K F\Big)_n = \coprod_{\substack{\sigma \in K_n \\ n \geq 0}} F(\sigma)_n$$

The references regarding the homotopy colimit construction of diagrams over simplicial sets are [3] and [4]. In this paper the only place, where we are going to use this notion, is the proof of the main theorem (see theorem 5.1).

3. Closed classes

In this section we state the definition and give some examples and basic properties of closed classes. We will also discus the notion of strong cellular inequalities. The references are [2], [3] and [6]. In this section we state a language in which a generalization of the triad theorem is going to be expressed. In the rest of the paper we are going to use, in an essential way, cellular techniques that are sketched in this section. These techniques were originally introduced by E. Dror Farjoun (see [6]).

Definition 3.1. *A non empty class C of connected spaces, that does not contain the empty space, is a closed class if the following conditions are satisfied:*

- *Let X and Y be weakly equivalent. If $X \in C$, then $Y \in C$.*
- *Let $(X_i)_{i\in I}$ be a family of spaces. If $X_i \in C$, then for any choice of basepoints in X_i, $\bigvee_{i\in I} X_i \in C$.*
- *Let $(X_1 \leftarrow X_2 \rightarrow X_3)$ be a push–out diagram. If $X_i \in C$, then:*

$$hocolim(X_1 \leftarrow X_2 \rightarrow X_3) \in C$$

- *Let $(\, X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots)$ be a diagram. If $X_i \in C$, then:*

$$hocolim(X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots) \in C$$

Example 3.2. Let A be a connected space. By $C(A)$ we denote the smallest closed class that contains A. The elements of this class are called A–cellular spaces (see [2] and [5] for detailed discussion of this class).

A good way of thinking about the class of A–cellular spaces is that it is generated by A using certain simple operations: taking arbitrary wedges, homotopy push–outs and telescopes.

Example 3.3. Let $n \geq 0$. Let the class C_n consists of those connected spaces X, such that for any choice of a basepoint in X, $\widetilde{H}_i(X) = \star$, for $i \leq n$, where $\widetilde{H}_i(X)$ is the reduced integral homology of X.

Definition 3.4. *(E.Dror Farjoun [6]). Let A be a connected space and X be a space. We write $X >> A$, if $X \in C(A)$. We call the relation ">>" a strong cellular inequality.*

Remark. An expression $X >> A$ is defined only for a connected space A. Whenever an expressions $X >> A$ appears in this paper, it is understood that A is connected.

The following theorem is one of the main tools to study closed classes. It is a generalization of E. Dror Farjoun's theorem (see [3, theorem 9.1] and [6]).

Theorem 3.5. *Let K be a connected simplicial set. Let $\Psi : E \rightarrow B$ be a natural transformation between diagrams $E : K \rightarrow Spaces$ and $B : K \rightarrow Spaces$. If for every simplex $\sigma \in K$, $Fib\Big(\Psi_\sigma : E(\sigma) \rightarrow B(\sigma)\Big)$ belongs to a closed class C, then so does $Fib\Big(\Psi : \oint_K E \rightarrow \oint_K B\Big)$.*

A very useful particular cases of this theorem are listed in the following proposition:

Proposition 3.6.

- *Let the following be a map between homotopy push–out diagrams of connected spaces:*

$$\begin{array}{ccccccc} E & \simeq & hocolim & \Big(\; E_1 & \longleftarrow & E_2 & \longrightarrow & E_3 \; \Big) \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow \\ B & \simeq & hocolim & \Big(\; B_1 & \longleftarrow & B_2 & \longrightarrow & B_3 \; \Big) \end{array}$$

  If for $i = 1, 2, 3$, $Fib(E_i \to B_i) >> A$, then $Fib(E \to B) >> A$.
- *Let the following be a homotopy push–out diagram of connected spaces:*

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & B \end{array}$$

  If $Fib(f : A \to E)$ is connected, then $Fib(g : X \to B) >> Fib(f : A \to E)$
- *Let $A \to X$ be a cofibration map between connected spaces. Looking at the following homotopy push–out square:*

$$\begin{array}{ccc} A & \longrightarrow & \star \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X/A \end{array}$$

  we get: $Fib(X \to X/A) >> A$.

The following proposition lists some basic cellular inequalities. These inequalities were originally discovered by E. Dror Farjoun.

Proposition 3.7.

- *$\star >> A$ (see* [5, section 2.3]*).*
- *Let $A \to X$ be a map over a connected space X. If this map has a right homotopy inverse (A is a retract of X), then $A >> X$ (see* [5, lemma 6.1]*).*
- *Let $A \to X$ be a map. If X is connected, then $X/A >> \Sigma Fib(A \to X)$ (see* [6]*).*
- *Let A be simply–connected. If $X >> A$, then $\Omega X >> \Omega A$ (see* [2, corollary 10.4]*).*
- *Let A be connected. $X >> \Sigma A$ if and only if X is simply connected and $\Omega X >> A$ (see* [2, theorem 10.8]*).*
- *$\Omega\Sigma A >> A$.*
- *If A is simply–connected, then $A >> \Sigma\Omega A$.*

As a corollary we get:

Corollary 3.8. *Let $A \to X$ be a map over a connected space X. If $Fib(A \to X) \in C_n$, then $X/A \in C_{n+1}$ (see example 3.3).*

Proposition 3.9. *Let D be connected and B be a pointed space. The following class is closed:*

$$C = \{Y \mid Y \textit{ is connected and for any choice of a basepoint in } Y,\ Y \wedge B >> D\}$$

Proof. It is obvious that C is closed under weak equivalence. We have to prove that C is closed under taking arbitrary wedges, homotopy push–outs and telescopes.

Let $(Y_1 \leftarrow Y_2 \to Y_3)$ be a push–out diagram of spaces that belong to C. Let $Y = hocolim(Y_1 \leftarrow Y_2 \to Y_3)$. Since $Y \wedge B$ is weakly equivalent to the homotopy push–out $hocolim(Y_1 \wedge B \leftarrow Y_2 \wedge B \to Y_3 \wedge B)$, we get: $Y \wedge B >> D$ and $Y \in C$.

The proofs, for the class C being closed under taking arbitrary wedges and telescopes, go in the same way. $\square$

Corollary 3.10. *Let B be a pointed spaces. If $X >> A$, then $X \wedge B >> A \wedge B$.*

Proof. According to proposition 3.9, the following class is closed:

$$C = \{Y \mid Y \text{ is connected and for any choice of a basepoint in } Y,\ Y \wedge B >> A \wedge B\}$$

Obviously $A \in C$, therefore the smallest closed class $C(A)$, that contains A, is included in C. $\square$

Since taking the suspension of a space is weakly equivalent with smashing this space with S^1, we get:

Corollary 3.11. *If $X >> A$, then $\Sigma X >> \Sigma A$.*

4. Join

In this section we present definition and basic properties of the join construction.

Definition 4.1. *The join $X \star Y$ of two spaces X and Y is defined as follows:*

$$X \star Y = hocolim(X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y)$$

Proposition 4.2. *For any choice of basepoints in X and Y, $X \star Y \simeq \Sigma(X \wedge Y)$.*

Proof. Let us consider the following commutative diagram:

$$
\begin{tikzcd}
\star & X \arrow[l] \arrow[r, "id"] & X & \star \arrow[l] \arrow[r] & \star & \star \\
\star \arrow[u] \arrow[d] & X \arrow[l] \arrow[r, "i_1"] \arrow[u, "id"'] \arrow[d] & X \times Y \arrow[u, "pr_1"'] \arrow[d, "pr_2"'] & Y \arrow[l, "i_2"'] \arrow[r] \arrow[u] \arrow[d, "id"'] & \star \arrow[u] \arrow[d] & X \wedge Y \arrow[u] \arrow[d] \\
\star & \star \arrow[l] \arrow[r] & Y & Y \arrow[l, "id"'] \arrow[r] & \star & \star \\
\star & \star \arrow[l] \arrow[r] & X \star Y & \star \arrow[l] \arrow[r] & \star &
\end{tikzcd}
$$

Applying the homotopy colimit functor first to the columns of this diagram and then to the obtained row, we get: $X \star Y$. Reversing this procedure and applying the homotopy colimit functor first to the rows of the diagram and then to the obtained column, we get: $\Sigma(X \wedge Y)$. This proves: $X \star Y \simeq \Sigma(X \wedge Y)$. □

Corollary 4.3. *If $\widetilde{H}_i(X) = \star$ for $i \leq n$ and $\widetilde{H}_i(Y) = \star$ for $i \leq m$, then $\widetilde{H}_i(X \star Y) = \star$ for $i \leq n + m + 2$.*

Proposition 4.4. *If X is connected, then closed classes $C(X \star Y)$ and $C\Big(\Sigma\Omega(X \star Y)\Big)$ are the same.*

Proof. The space X is connected, therefore $X \star Y$ is simply connected (it is the suspension of a connected space) and proposition 3.7 implies: $X \star Y >> \Sigma\Omega(X \star Y)$.

Since $X \star Y \simeq \Sigma(X \wedge Y)$ and $X \wedge Y$ is connected, using again proposition 3.7, we obtain $\Omega(X \star Y) >> X \wedge Y$. Suspending this inequality gives:

$$\Sigma\Omega(X \star Y) >> \Sigma(X \wedge Y) \simeq X \star Y$$

□

Proposition 4.5. *If $X >> A$, then $X \star Y >> A \star Y$.*

Proof. Let us choose a basepoint in Y. Corollary 3.10 implies: $X \wedge Y >> A \wedge Y$. By suspending this inequality, we get: $X \star Y \simeq \Sigma(X \wedge Y) >> \Sigma(A \wedge Y) \simeq A \star Y$. □

As a straightforward consequence of the proposition, we get:

Corollary 4.6. *If $X >> A$ and $Y >> B$, then $X \star Y >> A \star B$.*

5. Main theorem

In this section we are going to prove a strong cellular inequality which will be our main technical tool to prove a generalization of the triad theorem. This inequality can be thought of as a generalization of Serre's theorem (see [9, theorem 6.1]).

Theorem 5.1. *Let B be a connected space. Let us consider the following commutative diagram:*

$$\begin{tikzcd}
H \arrow[rr] \arrow[dr] \arrow[dd] & & F \arrow[d] & Z \arrow[d] \\
& Y \arrow[r, "l"] \arrow[d, "f"] & E \arrow[r] \arrow[d, "p"] & E/Y \arrow[d] \\
G \arrow[r] & X \arrow[r, "g"] & B \arrow[r] & B/X
\end{tikzcd}$$

where:

- *$F \to E$ is the homotopy fiber of $p : E \to B$,*
- *$G \to X$ is the homotopy fiber of $g : X \to B$,*
- *$H \to Y$ is the homotopy fiber of the composition $g \circ f : Y \to B$,*
- *$E \to E/Y$ is the homotopy cofiber of $l : Y \to E$,*
- *$B \to B/X$ is the homotopy cofiber of $g : X \to B$,*
- *maps $H \to F$, $H \to G$ and $E/Y \to B/X$ are induced by l, f, p and g,*
- *$Z \to E/Y$ is the homotopy fiber of $E/Y \to B/X$.*

If hocolim$(G \leftarrow H \to F)$ is connected, then:

$$Z >> hocolim(G \leftarrow H \to F)$$

Remark. Notice that since B is connected, the homotopy fibers $G \to X$, $F \to E$, $H \to Y$ and $Z \to E/Y$ are unique up to homotopy.

Proof. By changing the diagram in a homotopy meaningful way we can arrange so that the maps $g : X \to B$, $p : E \to B$ and $g \circ f : Y \to B$ are fibrations.

According to [3, example 3.12], we can construct good diagrams: $G : K \to Spaces$, $F : K \to Spaces$, $H : K \to Spaces$ together with natural transformations $\Phi : H \to G$ and $\Psi : H \to F$, such that:

- for every simplex $\sigma \in K$, $G(\sigma)$, $H(\sigma)$ and $F(\sigma)$ are weakly equivalent respectively to the homotopy fibers G, H and F,

- the following commutative squares are weakly equivalent:

$$\begin{array}{ccc} Y & \xrightarrow{l} & E \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B \end{array} \qquad \begin{array}{ccc} \oint_K H & \xrightarrow{\Psi} & \oint_K F \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow \\ \oint_K G & \longrightarrow & K \end{array}$$

It follows that the map $E/Y \to B/X$ can be represented, up to homotopy, as a homotopy push–out:

$$\begin{array}{ccccccccc} E/Y & \simeq & hocolim & \Big(& \star & \longleftarrow & \oint_K H & \xrightarrow{\Psi} & \oint_K F & \Big) \\ \Big\downarrow & & & & \Big\downarrow & & \downarrow\Phi & & \Big\downarrow & \\ B/X & \simeq & hocolim & \Big(& \star & \longleftarrow & \oint_K G & \longrightarrow & K & \Big) \end{array}$$

Without changing its homotopy colimit, we can modify the above diagram as follows:

$$\begin{array}{ccccccccccccc} E/Y & \simeq & hocolim & \Big(& \star & \longleftarrow & \oint_K G & \xrightarrow{id} & \oint_K G & \xleftarrow{\Phi} & \oint_K H & \xrightarrow{\Psi} & \oint_K F & \Big) \\ \Big\downarrow & & & & \Big\downarrow & & \downarrow id & & \downarrow id & & \downarrow \Phi & & \downarrow & \\ B/X & \simeq & hocolim & \Big(& \star & \longleftarrow & \oint_K G & \xrightarrow{id} & \oint_K G & \xleftarrow{id} & \oint_K G & \longrightarrow & \oint_K \star & \Big) \end{array}$$

Once again, without changing its homotopy colimit, we can modify the last diagram further, obtaining the following weak equivalence of maps:

$$\begin{array}{ccccccccc} E/Y & \simeq & hocolim & \Big(& \star & \longleftarrow & \oint_K G & \longrightarrow & \oint_K hocolim(G \leftarrow H \to F) & \Big) \\ \Big\downarrow & & & & \Big\downarrow & & \downarrow id & & \downarrow & \\ B/X & \simeq & hocolim & \Big(& \star & \leftarrow & \oint_K G & \to & \oint_K hocolim(G \overset{id}{\leftarrow} G \to \star) & \Big) \end{array}$$

Since for every simplex $\sigma \in K$, $hocolim\Big(G(\sigma) \overset{id}{\leftarrow} G(\sigma) \to \star\Big)$ is contractible, the map:

$$\oint_K hocolim(G \leftarrow H \to F) \to \oint_K hocolim(G \overset{id}{\leftarrow} G \to \star)$$

is weakly equivalent to $\oint_K hocolim(G \leftarrow H \to F) \to K$. According to theorem 3.5, we get that the homotopy fiber $Fib\Big(\oint_K hocolim(G \leftarrow H \to F) \to K\Big)$ is built by $hocolim(G \leftarrow H \to F)$, where G, H and F are the homotopy fibers respectively of g, $g \circ f$ and p. By applying proposition 3.6, we get:

$$Fib(E/Y \to B/X) >> hocolim(G \leftarrow H \to F)$$

□

The following theorem is a very useful particular case of theorem 5.1.

Theorem 5.2. *Let the following be a homotopy pull–back square:*

$$\begin{array}{ccc} Y & \xrightarrow{l} & E \\ \Big\downarrow f & & \Big\downarrow p \\ X & \xrightarrow{g} & B \end{array}$$

If B is connected, then:

$$Fib(E/Y \to B/X) >> Fib(g : X \to B) \star Fib(p : E \to B)$$

Proof. By changing the homotopy pull–back diagram of the theorem in a homotopy meaningful way, we can arrange so that $g : X \to B$, $p : E \to B$ and $g \circ f : Y \to B$ are fibrations and the diagram becomes a pull–back square.

Let $G = Fib(g : X \to B)$, $H = Fib(g \circ f : Y \to B)$ and $F = Fib(p : E \to B)$. It is obvious that $H = G \times F$ and the induced maps $H \to G$ and $H \to F$ are the projections. By definition: $G \star F = hocolim(G \xleftarrow{p_1} G \times F \xrightarrow{p_2} F)$. It follows that $hocolim(G \leftarrow H \to F)$ is connected and by theorem 5.1:

$$Fib(E/Y \to B/X) >> hocolim(F \leftarrow H \to G) = F \star G$$

□

As a corollary we get Serre's theorem (see [9, theorem 6.1]):

Corollary 5.3 (Serre). *Let B be a connected space. Let the following be a homotopy pull–back square:*

$$\begin{array}{ccc} Y & \xrightarrow{l} & E \\ \Big\downarrow f & & \Big\downarrow p \\ X & \xrightarrow{g} & B \end{array}$$

Let $G = Fib(g : X \to B)$ and $F = Fib(p : E \to B)$. If $\widetilde{H}_i(G) = \star$ for $i \leq n$ and $\widetilde{H}_i(F) = \star$ for $i \leq m$, then $H_i(p, f) : H_i(E, Y) \to H_i(B, X)$ is an isomorphism for $i \leq n + m + 2$ and an epimorphism for $i = n + m + 3$.

Proof. According to theorem 5.2, $Fib(E/Y \to B/X) >> G \star F$. Since $\widetilde{H}_i(G \star F) = \star$ for $i \leq m + n + 2$ (see corollary 4.3), it follows that also $\widetilde{H}_i\Big(Fib(E/Y \to B/X)\Big) = \star$ for $i \leq m + n + 2$ (see example 3.3). By corollary 3, $\widetilde{H}_i\Big(Cof(E/Y \to B/X)\Big) = \star$ for $i \leq n + m + 3$. This is equivalent to the map $H_i(p, f) : H_i(E, Y) \to H_i(B, X)$ being an isomorphism for $i \leq n + m + 2$ and an epimorphism for $i = n + m + 3$. □

6. The case of a cofibration

Let $A \to X \to X/A$ be a cofibration sequence for which A and X are connected. We will try to find out how far this cofibration sequence is from being a fibration sequence. Let $F \to X$ be the homotopy fiber of the cofiber map $X \to X/A$. By proposition 3.6, we already know that $F >> A$, in particular F is connected. Let $q : A \to F$ be a map such that the composition $(A \xrightarrow{q} F \to X)$ equals to $A \to X$. The map $q : A \to F$ measures the difference between the original cofibration sequence and the fibration sequence $F \to X \to X/A$. The purpose of this section is to give some cellular estimation for the homotopy fiber and the homotopy cofiber of the map $\Sigma q : \Sigma A \to \Sigma F$.

The above data can be put together by assuming that we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[drr, "q"] \arrow[dd] & P \arrow[drr, "id"] \arrow[dd] & & \\
& & F \arrow[r] \arrow[dd] & P \arrow[dd] \\
X \arrow[r] \arrow[drr, "id"] & X/A \arrow[drr, "id"] & & \\
& & X \arrow[r] & X/A
\end{tikzcd}$$

where:

- P is contractible,
- the following are respectively a homotopy push–out square and a homotopy pull–back square:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & P \arrow[d] \\
X \arrow[r] & X/A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F \arrow[r] \arrow[d] & P \arrow[d] \\
X \arrow[r] & X/A
\end{tikzcd}$$

Proposition 6.1. *$\Sigma q : \Sigma A \to \Sigma F$ has a right homotopy inverse.*

Proof. Commutativity of the following diagram proves clearly the proposition:

$$\begin{tikzcd}
\Sigma A \arrow[d, "\Sigma q"] \arrow[dd, bend right=60, "id"'] & \simeq & hocolim & \Big(& \star \arrow[d] & A \arrow[l] \arrow[r] \arrow[d, "q"] & \star \arrow[d] & \Big) \\
\Sigma F \arrow[d, "r"] & \simeq & hocolim & \Big(& \star \arrow[d] & F \arrow[l] \arrow[r] \arrow[d] & P \arrow[d] & \Big) \\
\Sigma A & \simeq & hocolim & \Big(& \star & X \arrow[l] \arrow[r] & X/A & \Big)
\end{tikzcd}$$

□

Notation. By $r : \Sigma F \to \Sigma A$ we denote the homotopy inverse of $\Sigma q : \Sigma A \to \Sigma F$ constructed in proposition 6.1.

Remark. Notice that since $\Sigma q : \Sigma A \to \Sigma F$ has a right homotopy inverse, $q : A \to F$ induces a monomorphism on homology. This observation is due to William Dwyer.

Corollary 6.2. *(E. Dror Farjoun, see* [6]*). Closed classes $C(\Sigma A)$ and $C(\Sigma F)$ are equal.*

Proof. According to proposition 3.6, $F >> A$. By suspending this inequality we get: $\Sigma F >> \Sigma A$ (see corollary 3.6). This proves $C(\Sigma F) \subset C(\Sigma A)$.

Since the map $\Sigma q : \Sigma A \to \Sigma F$ has a right homotopy inverse, proposition 3.7 implies $\Sigma A >> \Sigma F$. This shows $C(\Sigma A) \subset C(\Sigma F)$. $\square$

We start investigating the map $\Sigma q : \Sigma A \to \Sigma F$ by looking at the homotopy fiber of its right homotopy inverse $r : \Sigma F \to \Sigma A$.

Proposition 6.3. *$Fib(r : \Sigma F \to \Sigma A) >> F \star \Omega(X/A)$.*

Proof. Let us consider the following homotopy pull–back square:

$$\begin{array}{ccc} F & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & X/A \end{array}$$

The homotopy fiber of $P \to X/A$ is weakly equivalent to $\Omega(X/A)$. By applying theorem 5.2, we get: $Fib(r : \Sigma F \to \Sigma A) >> F \star \Omega(X/A)$. $\square$

The following proposition gives a cellular estimation for $Fib(\Sigma q : \Sigma A \to \Sigma F)$. Although this inequality is in terms of F, out of it we will be able to extract more useful and calculable inequalities. Those inequalities are going to be in terms of A and either the homotopy fiber $Fib(A \to X)$ or the homotopy cofiber X/A.

Proposition 6.4. *$Fib(\Sigma q : \Sigma A \to \Sigma F) >> F \wedge \Omega(X/A)$.*

Proof. The composition $(\Sigma A \xrightarrow{\Sigma q} \Sigma F \xrightarrow{r} \Sigma A)$ is a weak equivalence, thus:

$$Fib(\Sigma q : \Sigma A \to \Sigma F) \simeq \Omega Fib(r : \Sigma F \to \Sigma A)$$

Since F is connected (it is A–cellular), so $F \star \Omega(X/A)$ is simply connected and according to proposition 3.7, we can loop the inequality of proposition 6.3. As a result we get:

$$Fib(\Sigma q : \Sigma A \to \Sigma F) \simeq \Omega Fib(r : \Sigma F \to \Sigma A) >> \Omega\Big(F \star \Omega(X/A)\Big)$$

Since $\Omega\Big(F \star \Omega(X/A)\Big) \simeq \Omega\Sigma\Big(F \wedge \Omega(X/A)\Big)$, it follows from proposition 3.7 that:

$$Fib(\Sigma q : \Sigma A \to \Sigma F) >> \Omega\Sigma\Big(F \wedge \Omega(X/A)\Big) >> F \wedge \Omega(X/A)$$

□

The following estimation of the homotopy fiber $Fib(\Sigma q : \Sigma A \to \Sigma F)$ is entirely in terms of A and the homotopy cofiber X/A:

Theorem 6.5. *$Fib(\Sigma q : \Sigma A \to \Sigma F) >> A \wedge \Omega(X/A)$.*

Proof. Since $F >> A$ (see proposition 3.6), using corollary 3.10, we get:

$$Fib(\Sigma q : \Sigma A \to \Sigma F) >> F \wedge \Omega(X/A) >> A \wedge \Omega(X/A)$$

□

Corollary 6.6. *If $Fib(A \to X)$ is connected, then:*

$$Fib(\Sigma q : \Sigma A \to \Sigma F) >> A \wedge Fib(A \to X)$$

Proof. Since $Fib(A \to X)$ is connected and $X/A >> \Sigma Fib(A \to X)$ (see proposition 3.7), according again to proposition 3.7, $\Omega(X/A) >> Fib(A \to X)$. Therefore we can write:

$$Fib(\Sigma q : \Sigma A \to \Sigma F) >> A \wedge \Omega(X/A) >> A \wedge Fib(A \to X)$$

□

We continue investigating $\Sigma q : \Sigma A \to \Sigma F$ by giving a cellular estimation for its homotopy cofiber:

Theorem 6.7. *$\Sigma(F/A) >> F \star \Omega(X/A)$.*

Proof. According to proposition 3.7:

$$\Sigma(F/A) \simeq Cof(\Sigma q : \Sigma A \to \Sigma F) >> \Sigma Fib(\Sigma q : \Sigma A \to \Sigma F)$$

Proposition 6.4 gives $Fib(\Sigma q : \Sigma A \to \Sigma F) >> F \wedge \Omega(X/A)$. By suspending this inequality we get:

$$\Sigma(F/A) >> \Sigma Fib(\Sigma q : \Sigma A \to \Sigma F) >> \Sigma\Big(F \wedge \Omega(X/A)\Big) \simeq F \star \Omega(X/A)$$

□

Remark. The inequality $\Sigma(F/A) >> \Sigma Fib(\Sigma q : \Sigma A \to \Sigma F)$, that appears in the proof of theorem 6.7, can not be usually de–suspended.

By the same argument as in the theorem 6.5 we can get a cellular estimation of the homotopy cofiber of $\Sigma q : \Sigma A \to \Sigma F$ in terms of A and the homotopy cofiber X/A:

Corollary 6.8. $\Sigma(F/A) >> A \star \Omega(X/A)$

Under an extra assumption we can get even more explicit formula:

Corollary 6.9. *If X/A is weakly equivalent to the suspension of a connected space, then $\Sigma(F/A) >> A \wedge X/A$.*

Proof. Let $X/A \simeq \Sigma Y$. According to proposition 3.7: $\Omega(X/A) \simeq \Omega\Sigma Y >> Y$. By suspending this inequality, we get: $\Sigma\Omega(X/A) >> \Sigma Y \simeq X/A$. Since:

$$\Sigma(F/A) >> A \star \Omega(X/A) \simeq \Sigma\Big(A \wedge \Omega(X/A)\Big) \simeq A \wedge \Sigma\Omega(X/A)$$

corollary 3.10 implies: $\Sigma(F/A) >> A \wedge X/A$. □

Using the same arguments as in corollary 6.6 one can easily prove the following cellular estimation of the homotopy cofiber of Σq. This inequality is in terms of A and the homotopy fiber of $A \to X$:

Corollary 6.10. *If $Fib(A \to X)$ is connected, then:*

$$\Sigma(F/A) >> A \star Fib(A \to X)$$

Out of proven inequalities we can recover Serre's result (see [9, corollary 6.3]):

Corollary 6.11. *Let A be simply connected. If:*

- $\widetilde{H}_i(A) = \star$ *for* $i \leq n$,
- $\widetilde{H}_j\Big(Fib(A \to X)\Big) = \star$ *or* $\widetilde{H}_j\Big(\Omega(X/A)\Big) = \star$ *for* $j \leq m$,

then $\pi_i(q) : \pi_i(A) \to \pi_i(F)$ is an isomorphism for $i \leq n+m$ and an epimorphism for $i = n+m+1$.

Proof. Since $\Sigma(F/A) >> A\star\Omega(X/A)$ and $\Sigma(F/A) >> A\star Fib(A \to X)$, according to corollary 4.3, the assumptions imply: $\widetilde{H}_i\Big(\Sigma(F/A)\Big) = \star$ for $i \leq n+m+2$. It follows that $\widetilde{H}_i(F/A) = \star$ for $i \leq n+m+1$.

Since F is A–cellular and A is simply connected, F is also simply connected. It implies: $\pi_1(F, A) = \star$. Using relative Hurewicz theorem (see [8, proposition 1.7]), we get: $\pi_i(F, A) = \star$ for $i \leq n+m+1$. It proves that $\pi_i(q) : \pi_i(A) \to \pi_i(F)$ is an isomorphism for $i \leq n+m$ and an epimorphism for $i = n+m+1$. □

7. The case of a push–out

In this section we are going to investigate how far a homotopy push–out square is from being a homotopy pull–back square. Let B be a connected space. Let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & & & F \arrow[d] \\
 & A \arrow[r] \arrow[dd] \arrow[drr, "q"] & E \arrow[dd] \arrow[drr, "id"] & & \\
 & & & Y \arrow[r] \arrow[dd] & E \arrow[dd] \\
G \arrow[r] & X \arrow[r] \arrow[drr, "id"'] & B \arrow[drr, "id"] & & \\
 & & & X \arrow[r] & B
\end{tikzcd}$$

where:

- the following are respectively a homotopy push–out square and a homotopy pull–back square:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

- $F \to E$ is the homotopy fiber of $E \to B$,
- $G \to X$ is the homotopy fiber of $X \to B$,

The map $q : A \to Y$ measures how far the above homotopy push–out square is from being homotopy pull–back. The purpose of this section is to give some cellular estimation for the homotopy fiber and the homotopy cofiber of the map $\Sigma q : \Sigma A \to \Sigma Y$.

Theorem 7.1. *If F or G is connected, then $Fib(\Sigma q : \Sigma A \to \Sigma Y) >> F \wedge G$*

Lemma 7.2. *Let $A \xrightarrow{q} Y \to E$ be two composable maps. The following is a homotopy push–out square:*

$$\begin{tikzcd} E/A \arrow[r] \arrow[d] & \Sigma A \arrow[d, "\Sigma q"] \\ E/Y \arrow[r] & \Sigma Y \end{tikzcd}$$

where

- *$E/A \to \Sigma A$ and $E/Y \to \Sigma Y$ are the homotopy cofibers respectively of $E \to E/A$ and $E \to E/Y$,*
- *the map $E/A \to E/Y$ is induced by $id : E \to E$ and $q : A \to Y$.*

Proof. We have to show that there is a weak equivalence:

$$hocolim(\Sigma A \leftarrow E/A \to E/Y) \longrightarrow \Sigma Y$$

whose restriction to ΣA is $\Sigma q : \Sigma A \to \Sigma Y$. Let us consider the following commutative diagram:

$$\begin{tikzcd} \star & \star \arrow[l] \arrow[r] & \star & \star \\ A \arrow[u] \arrow[d] & A \arrow[l, "id"'] \arrow[r, "q"] \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d] \\ \star & E \arrow[l] \arrow[r, "id"] & E & \star \\ \Sigma A & E/A \arrow[l] \arrow[r] & E/Y & \end{tikzcd}$$

Applying the homotopy colimit functor first to the columns of this diagram and then to the obtained row, we get: $hocolim(\Sigma A \leftarrow E/Y \to E/Y)$. Changing the order of this procedure and applying the homotopy colimit functor first to the rows of the diagram and then to the obtained column, we get: ΣY. This proves the lemma. □

According to proposition 3.6, lemma 7.2 implies:

Corollary 7.3. *Let $A \xrightarrow{q} Y \to E$ be two composable maps. If $Fib(E/A \to E/Y)$ is connected, then: $Fib(\Sigma q : \Sigma A \to \Sigma Y) >> Fib(E/A \to E/Y)$*

Proof of the theorem. Corollary 7.3 implies that in order to prove the theorem, it is enough to show that if F or G is connected, then $Fib(E/A \to E/Y) >> F \wedge G$.

Let us consider the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow[q]{} & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{id} & E & \longrightarrow & B \end{array}$$

This diagram induces maps between the homotopy cofibers $E/A \to E/Y \to B/X$. Since the appropriate square is homotopy push–out, the map $E/A \to B/X$ is a weak equivalence. It implies:

$$Fib(E/A \to E/Y) \simeq \Omega Fib(E/Y \to B/X)$$

Since the appropriate square is homotopy pull–back, theorem 5.2 gives:

$$Fib(E/Y \to B/X) >> F \star G \simeq \Sigma(F \wedge G)$$

If F or G is connected, then $F \wedge G$ is also connected and proposition 3.7 implies:

$$Fib(E/A \to E/Y) \simeq \Omega Fib(E/Y \to B/X) >> F \wedge G$$

□

If $Fib(A \to E)$ and $Fib(A \to X)$ are connected, then $F >> Fib(A \to X)$ and $G >> Fib(A \to E)$ (see proposition 3.6). As a consequence we get:

Corollary 7.4. *If $Fib(A \to E)$ and $Fib(A \to X)$ are connected, then:*

$$Fib(\Sigma q : \Sigma A \to \Sigma Y) >> Fib(A \to X) \wedge Fib(A \to E)$$

As it was in the case of a cofibration, we can use cellular estimations for the homotopy fiber $Fib(\Sigma q : \Sigma A \to \Sigma Y)$ to get some cellular estimations for the homotopy cofiber $\Sigma(Y/A)$:

Theorem 7.5. *If $Fib(A \to E)$ and $Fib(A \to X)$ are connected then:*

$$\Sigma(Y/A) >> Fib(A \to E) \star Fib(A \to X)$$

As a corollary we get:

Corollary 7.6. *Let p and s be distinct prime numbers. If $Fib(A \to E)$ has p–torsion reduced homology and $Fib(A \to X)$ has s–torsion reduced homology, then $q : A \to Y$ is a homology isomorphism.*

Proof. Since $Fib(A \to E)$ and $Fib(A \to X)$ have respectively p and s–torsion homology, Künneth theorem (see [8, theorem 10.3]) implies that $Fib(A \to E) \wedge Fib(A \to X)$ is acyclic. Thus $Fib(\Sigma q : \Sigma A \to \Sigma B)$ is also acyclic, so $\Sigma q : \Sigma A \to \Sigma B$ induces an isomorphism on homology. It follows that $q : A \to Y$ is a homology isomorphism. □

Out of proven inequalities we can recover the following result of Blakers–Massey (see [9, theorem 7.14]):

Corollary 7.7. *Let the following be a homotopy push–out of simply connected spaces:*

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

If $H_i(X, A) = \star$, for $i \leq n$ $(n \geq 1)$ and $H_i(E, A) = \star$, for $i \leq m$ $(m \geq 1)$, then $\pi_i(E, A) \to \pi_i(B, X)$ is an isomorphism for $i < n + m$ and an epimorphism for $i \leq n + m$.

Proof. Since A, X and E are simply connected, according to Hurewicz theorem (see [8, proposition 7.1]), $\pi_i(X, A) = \star$ for $i \leq n$ and $\pi_i(E, A) = \star$ for $i \leq m$. It follows:

- $\pi_i\Big(Fib(A \to X)\Big) = \star$ for $i < n$,
- $\pi_i\Big(Fib(A \to E)\Big) = \star$ for $i < m$.

Let Y be the homotopy pull–back: $holim(X \to B \leftarrow E)$ and $q : A \to Y$ be the natural map. Inequality $\Sigma(Y/A) >> Fib(A \to E) \star Fib(A \to X)$ implies that: $\widetilde{H}_i\Big(\Sigma(Y/A)\Big) = \star$ for $i \leq n + m$. Therefore: $\widetilde{H}_i(Y/A) = \star$ for $i < n + m$.

The assumption: E and B are simply–connected, implies that there is an epimorphism: $\pi_2(B) \to \pi_1\Big(Fib(E \to B)\Big)$, so $\pi_1\Big(Fib(E \to B)\Big)$ is abelian. Thus $\pi_1(Y)$ is also abelian and Hurewicz map $\pi_1(Y, A) \to \widetilde{H}_1(Y, A)$ is an isomorphism. As a consequence we get: $\pi_i(Y, A) = \star$ for $i < n + m$. This implies that $\pi_i(A) \to \pi_i(Y)$ is an isomorphism for $i < n + m - 1$ and an epimorphism for $i < n + m$.

By looking at the long exact sequences of homotopy groups of pairs $(A \to E)$ and $(Y \to E)$, we get that $\pi_i(E, A) \to \pi_i(E, Y)$ is an isomorphism for $i < n + m$ and an epimorphism for $i \leq n + m$.

Since $\pi_i(E, Y) \to \pi_i(B, X)$ is an isomorphism for all i, $\pi_i(E, A) \to \pi_i(B, X)$ is an isomorphism for $i < n + m$ and an epimorphism for $i \leq n + m$. □

References

1. A. L. Blakers and W. S. Massey, *The homotopy group of a triad 1*, Ann. of Math. 53 1951, 161-205.
2. W. Chachólski, *Functors CW_A and P_A*, Ph.D. thesis, Univ. of Notre Dame 1995.
3. W. Chachólski, *Closed classes*, Proc. to the conf. in Alg. Top. Barcelona, Summer 1994.
4. W. Chachólski, *Homotopy properties of shapes of diagrams*, report No. 6, 1993/94, Institut Mittag-Leffler.
5. E. Dror Farjoun, *Cellular spaces*, preprint.
6. E. Dror Farjoun, *Cellular inequalities*, Proc. to the conf. in Alg. Top. Northeastern Univ. June 1993, Springer Verlag.
7. I. Namioka, *Maps of pairs in homotopy theory*, Proc. London Math. Soc. 12 1962, 725-738.
8. E. H. Spanier, *Algebraic Topology*, McGraw-Hill 1964.
9. G. W. Whitehead, *Elements of Homotopy Theory*, Grad. Texts in Math. 61, Springer 1978.

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