CLOSED CLASSES

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1. Introduction

A non empty class C of connected spaces is said to be a **closed class** if it is closed under weak equivalences and **pointed** homotopy colimits. A closed class can be characterized as a non empty class of connected spaces which is closed under weak equivalences and is closed under certain simple operations: arbitrary wedges, homotopy push-outs and homotopy sequential colimits. The notion of a closed class was introduced by E. Dror Farjoun [6].

Two important constructions give rise to examples of closed classes. The first one is the Bousfield-Dror periodization functor P_A [2]. The class of those spaces X, such that P_AX is weakly contractible, forms a closed class. By looking just at the properties of this class we can prove, for example, that $P_A\Omega X$ is weakly equivalent to $\Omega P_{\Sigma A}X$ (see [2], [4]). The second construction is E. Dror Farjoun's colocalization functor CW_A. The class of those spaces X, for which there exists a space Y, such that X is weakly equivalent to CW_AY, forms a closed class. This class is denoted by $C(A)$ and is called the class of A-cellular spaces. By looking just at the properties of the class $C(A)$ we can prove, for example, that $CW_A\Omega X$ is weakly equivalent to $\Omega CW_{\Sigma A}X$ (see [4], [6]).

We say that a closed class C is closed under extensions by fibrations, if for every fibration sequence $(Z \to E \to B)$, such that Z and B belong to C, E belongs to C. A closed class C is closed under extensions by fibrations if and only if for every diagram $F : I \to C$, such that the classifying space BI belongs to C, the unpointed homotopy colimit $hocolim_I F$ belongs to C.

The purpose of this paper is to understand to what extent a closed class is closed under extensions by fibrations and under taking unpointed homotopy colimits. We start with proving a theorem that, in particular, implies:

- *Let $F : I \to Spaces_\star$ be a pointed diagram, such that the classifying space BI belongs to C. If for every $i \in I$, $F(i)$ belongs to C, then so does the unpointed homotopy colimit $hocolim_I F$.*
- *Let $(Z \to E \to B)$ be a fibration sequence with a section. If Z and B belong to C, then so does E.*
- *Let $F : I \to C$ and $G : I \to C$ be diagrams and $\Psi : F \to G$ be a natural transformation. If $hocolim_I F$ belongs to C, then so does $hocolim_I G$.*

Surprisingly these and many other results are the consequences of just one statement, see theorem 5.1.

We continue with investigating the properties of a base space B (respectively of the classifying space BI), which will guarantee that a closed class C is closed under extensions by fibrations with base B (respectively C is closed under taking the unpointed homotopy colimit of diagrams $F : I \to C$). We study the following class:

$$D(C) = \{BI \mid \text{if } F : I \to C \text{ is a diagram, then } hocolim_I F \in C\}$$

The main result of this paper is:

Theorem. *The class $D(C)$ is a closed class and it is closed under extensions by fibrations.*

Using this theorem, we can characterize the class $D(C)$ as follows:

$$D(C) = \{B \mid \text{if } Z \to E \to B \text{ is a fibration sequence and } Z \in C, \text{ then } E \in C\}$$

Since $D(C)$ is a closed class, it is closed under weak equivalences. This is a very non trivial fact itself. It is obvious that $\star$ belongs to $D(C)$. What is not clear at all is that for any diagram $F : I \to C$ over a contractible category I, the homotopy colimit $hocolim_I F$ belongs to C. As a result we get a new characterization of a closed class:

A non empty class C of connected spaces is a closed class if and only if it is closed under weak equivalences and for any unpointed diagram $F : I \to C$ over a contractible category I, the homotopy colimit $hocolim_I F$ belongs to C.

The techniques that are used to prove the main theorem involve studying the homotopy fiber of a map $f : X \to Y$ through inverse images of simplices in Y (a simplicial analogues of point inverse images). We prove, roughly, that if point inverse images belong to a closed class, then so does the homotopy fiber, see corollary 7.9. One consequence of this is that if the point inverse images of f are acyclic with respect to some homology theory, then so is the homotopy fiber of f.

Techniques, we have introduced, and properties of $D(C)$ are applied to prove a generalization of a theorem of E. Dror Farjoun (see 9.1 and [7, theorem I]):

Theorem. *Let $\Psi : E \to B$ be a natural transformation between* **unpointed** *diagrams $E : I \to Spaces$ and $B : I \to Spaces$. If for every $i \in I$, the homotopy fiber $Fib(\Psi_i : E(i) \to B(i))$ belongs to a closed class C, then so does $Fib(\Psi : hocolim_I E \to hocolim_I B)$.*

2. Notation

The symbol Δ denotes the simplicial category [9, §2], in which the objects are the ordered sets $[n] = \{0, 1, \dots, n\}$, and morphisms are weakly monotone maps of sets. The morphisms of Δ are generated by codegeneracy maps $s_i : [n] \to [n+1]$ $(i = 0, 1, \dots, n)$ and coface maps $d_i : [n-1] \to [n]$, subject to well-known cosimplicial identities (see [3]). A simplicial set is a functor $K : \Delta^{op} \to Sets$, where $Sets$ denotes the category of sets [9, §2]. The set $K([n])$ is usually denoted by K_n. A map between two simplicial sets is by definition a natural transformation of functors. A simplicial set K can be interpreted as a collection of sets K_n together with face maps $d_i : K_n \to K_{n-1}$ and degeneracy maps $s_i : K_{n+1} \to K_n$ $(i = 1, 2, \dots, n)$ which satisfy the duals of the cosimplicial identities (see [3]). For description of haw to do homotopy theory with simplicial sets see [3] and [9].

If $\sigma \in K_n$, then σ is called an n-dimensional simplex of K. The dimension of σ will be denoted by $dim(\sigma)$. The object $\Delta[n]$ is the standard n-simplex, given by $\Delta[n]_k = mor_\Delta([k], [n])$ (see [3]). There is a distinguish n-dimensional simplex $\tau \in \Delta[n]_n$, the one that comes from the identity map $[n] \to [n]$. It is easy to check that for any simplicial set K, the assignment $f \to f(\tau)$ gives a bijection between the set of maps $\Delta[n] \to K$ and K_n. If $\sigma \in K_n$, we will denote the corresponding map by $\sigma : \Delta[n] \to K$.

A pointed simplicial set is a pair (K, k), where k is a chosen simplex of dimension zero in K. We will refer to this 0-dimensional simplex as the basepoint of (K, k). A map between pointed simplicial sets is a map of simplicial sets which preserves the basepoints. We will use the following notation for some categories which frequently occur:

- $Spaces$ denotes the category of simplicial sets.
- $cSpaces$ denotes the category of connected simplicial sets.
- $Spaces_\star$ denotes the category of pointed simplicial sets.
- $cSpaces_\star$ denotes the category of pointed and connected simplicial sets.

If I is a small category, the nerve of I, denoted by $N(I)$, is the simplicial set whose n-simplices are n-tuples $(i_0 \to i_1 \to \cdots i_n)$ of composable morphisms in I (see [3]).

If C is a category and K is an object in C, then by C/K we denote the category of objects of C over K [8, 1§6]. This is the category whose objects are morphisms $f : X \to K$ in C and maps from $f : X \to K$ to $g : Y \to K$ are those morphisms $h : X \to Y$ in C such that $g \circ h = f$.

3. The homotopy colimit

In this section we describe the notion of a diagram indexed by a simplicial set and define the homotopy colimit of such a diagram. The particular form of the homotopy colimit which is going to be used was introduced by E. Dror

Farjoun [7]. Various properties of the homotopy colimit are also listed in the appendix. The reference for the proofs is [5].

The motivation for these constructions comes from the fact that any map $f : X \to K$ of simplicial sets, can be reconstructed (up to a weak equivalence) from the homotopy colimit of a diagram indexed essentially by the range of K. The constituents of this diagram are the analogues (in the simplicial category) of the point inverse images of f. The ability to build a map in a homotopically meaningful way from its range and its point inverse images is going to be explored in the following sections.

3.1. Definition. *Let $K : \Delta^{op} \to Sets$ be a simplicial set. The category associated to K, sometimes called the transport category of K or Grothendieck construction on K, is the category whose objects are pairs $([n], \sigma)$ where $[n]$ is an object of Δ^{op} and $\sigma \in K_n$. A morphism $([n], \sigma) \to ([m], \tau)$ is a map $\varphi : [m] \to [n]$ in Δ (or $\varphi : [n] \to [m]$ in Δ^{op}) such that $K(\varphi)(\sigma) = \tau$.*

To avoid introducing to many notation we will denote the category associated to a simplicial set K by the same symbol K, and speak of "functors with domain K". One can think about the category K as having as objects the simplices of K and morphisms generated by the arrows $d_i : \sigma \to d_i\sigma$ $s_i : \sigma \to s_i\sigma$, subject to some relations that come from the simplicial structure of K. The morphisms $s_i : \sigma \to s_i\sigma$ is called the degeneracy morphisms and $d_i : \sigma \to d_i\sigma$ the boundary morphisms.

The notion of the category associated to a simplicial set can be used to define the subdivision of K (see [11]):

3.2. Definition. *If K is a simplicial set, the subdivision of K is the nerve (see section 2) of the category associated to K. The subdivision of K will be denoted by sdK.*

3.3. Example. The category associated to $\Delta[0] = \star$ is isomorphic to Δ^{op}. Diagrams over $\star$ are simplicial spaces.

If K is a simplicial set, a functor $F : K \to Spaces$ over the category associated to K will be called a diagram with shape K. Diagrams are our main object of interest. As example 3.3 suggests, the category associated to a simplicial set can be quite complicated. Since we care about diagrams, in order to simplify the situation, we will distinguish the class of bounded diagram which are technically more manageable. It turns out that all the examples of diagrams, we are going to consider, are bounded.

3.4. Definition. *$F : K \to Spaces$ is a bounded diagram if for any degeneracy morphism $s_i : \sigma \to s_i\sigma$, $F(\sigma) = F(s_i\sigma)$ and $F(s_i) = id_{F(\sigma)}$*

3.5. Example. A bounded diagram with shape $\star$ is determined by the value on the only zero dimensional simplex of $\star$. The category of bounded diagrams with shape $\star$ is equivalent to the category of simplicial sets.

3.6. Example. The category of bounded diagrams over $\Delta[1]$ is equivalent to the category of diagrams of the form $A \leftarrow B \rightarrow C$, so called push-out diagrams. Out of a bounded diagram $F : \Delta[1] \rightarrow Spaces$, a push-out diagram can be extracted in the following way:

$$F(0) \leftarrow F(0,1) \rightarrow F(1)$$

3.7. Example. Let $f : X \rightarrow K$ be a map. For every simplex $\sigma \in K$, let $F_f(\sigma)$ be the space that fits into the following pull-back square:

$$\begin{array}{ccc} F_f(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta[dim(\sigma)] & \xrightarrow{\sigma} & K \end{array}$$

Roughly speaking $F_f(\sigma)$ is the inverse image in X of the simplex σ of K. This construction clearly defines a functor $F_f : K^{op} \rightarrow Spaces$. Out of F_f we can built a bounded diagram $df : sdK \rightarrow Spaces$ such that for an n-dimensional simplex $v = \sigma_0 \stackrel{\varphi_0}{\rightarrow} \sigma_1 \stackrel{\varphi_1}{\rightarrow} ... \stackrel{\varphi_{n-1}}{\rightarrow} \sigma_n$ in sdK:

$$df(v) = F_f(\sigma_n)$$

$$df(d_i : v \rightarrow d_i v) = \begin{cases} id : F_f(\sigma_n) \rightarrow F_f(\sigma_n) & \text{if } i < n \\ F_f(\varphi_{n-1}) : F_f(\sigma_n) \rightarrow F_f(\sigma_{n-1}) & \text{if } i = n \end{cases}$$

The construction d is natural. If $f : X \rightarrow K$, $g : Y \rightarrow K$ and $h : X \rightarrow Y$ are maps such that $g \circ h = f$, then there is a natural transformation $df \rightarrow dg$. In particular there is a natural transformation $\Psi : df \rightarrow d(id)$ induced by $f : X \rightarrow K$ itself. It turns out that:

$$\big(colim_{sdK}\Psi : colim_{sdK} df \rightarrow colim_{sdK} d(id)\big) = (f : X \rightarrow K)$$

We will see in example 3.12 that the diagram df has nice homotopic properties with respect to the map f.

If $f : X \rightarrow K$ is a fibration, then the functor $F_f : K^{op} \rightarrow Spaces$ has the property that for every morphism φ, $F_f(\varphi)$ is a weak equivalence. In this case $F_f(\sigma)$ is weakly equivalent to the homotopy fiber of f. It is clear from the definition that the diagram $df : sdK \rightarrow Spaces$ inherits the same properties. This motivates the following definition:

3.8. Definition. *$F : K \rightarrow Spaces$ is a good diagram if it is a bounded diagram and for every morphism $\varphi \in K$, $F(\varphi)$ is a weak equivalence.*

The construction d defines a functor:

$$d : Spaces/K \rightarrow \{\text{bounded diagrams over } sdK\}$$

$$(f : X \rightarrow K) \mapsto (df : sdK \rightarrow Spaces)$$

such that fibrations are carried out to good diagrams.

We will now introduce a construction which will allow us to recover, up to homotopy, a map f from the diagram df. One can think about this construction as sort of an "inverse" of d:

3.9. Definition. *The homotopy colimit is the following functor:*

$$\oint_K : \{\textit{diagrams over } K\} \to \textit{Spaces}$$

$$\oint_K F = \Big(\coprod_{\sigma\in K} \Delta[dim(\sigma)] \times F(\sigma) \Big) \Big/ \sim$$

where $\sim$ is an equivalence relation generated by:

$$\textit{let } \varphi \in mor_\Delta([n],[m]) \ , \ \tau \in K_m \ , \ x \in F(\tau) \ , \ t \in \Delta[n]$$
$$(\Delta[\varphi](t), x) \sim (t, F(\varphi)(x))$$

3.10. Definition. *The pointed homotopy colimit is the following functor:*

$$\int_K : \{\textit{pointed diagrams over } K\} \to \textit{Spaces}_\star$$

$$\int_K F = \Big(\bigvee_{\sigma\in K} \big(\Delta[dim(\sigma)] \times F(\sigma)\big) / \big(\Delta[dim(\sigma)] \times \{\star\}\big) \Big) \Big/ \sim$$

where $\sim$ is an equivalence relation generated by:

$$\textit{let } \varphi \in mor_\Delta([n],[m]) \ , \ \tau \in K_m \ , \ x \in F(\tau) \ , \ t \in \Delta[n]$$
$$(\Delta[\varphi](t), x) \sim (t, F(\varphi)(x))$$

3.11. Example. If $F : K \to Spaces_\star$ is a constant diagram $F(\sigma) = X$ and $F(\varphi) = id_X$, then:

$$\oint_K F = K \times X \ , \ \int_K F = (K \times X)/(K \times \{\star\}) \ = K \ltimes X$$

In case $X = \star$, we get that $\oint_K \star = K$, where $\star : K \to Spaces$ denotes the constant diagram whose value is $\star$.

Let $F : K \to Spaces$ be a diagram. There is a natural transformation $F \to \star$ between F and the constant diagram $\star$. This natural transformation induces the following map:

$$\oint_K F \to \oint_K \star = K$$

Let $F : K \to Spaces_\star$ be a pointed diagram. This means that there is a natural transformation $\star \to F$ between the constant diagram $\star$ and F. This transformation induces the map $K = \oint_K \star \to \oint_K F$, which is a section of the map $\oint_K F \to K$.

As a consequence we get that the homotopy colimit can be seen as a functor with values in $Spaces/K$:

$$\oint_K : \{\text{diagrams over } K\} \to Spaces/K$$

$$(F : K \to Spaces) \mapsto \big(\oint_K F \to K\big)$$

which carries out pointed diagrams into maps with a section.

If $F : K \to Spaces_\star$ is a pointed diagram, then the following is a cofibration sequence:

$$K \to \oint_K F \to \int_K F$$

As a consequence we get that if $F : K \to Spaces_\star$ is a pointed diagram over weakly contractible simplicial set, then the unpointed and the pointed homotopy colimits of F are weakly equivalent.

3.12. Example. Let $f : X \to K$ be a map. Out of f we have constructed a diagram $df : sdK \to Spaces$ (see example 3.7). The main property of df is that in the following commutative diagram, all the horizontal arrows are weak equivalences:

$$\begin{array}{ccccccc}
\oint_{sdK} df & \xleftarrow{id} & \oint_{sdK} df & \longrightarrow & colim_{sdK} df & = & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow f \\
sdK & \longleftarrow & \oint_{sdK} d(id) & \longrightarrow & colim_{sdK} d(id) & = & K
\end{array}$$

It implies that every map $f : X \to K$ is weakly equivalent to a map of the form $\oint_L F \to L$ for some bounded diagram $F : L \to Spaces$. Since every map is weakly equivalent to a fibration, we can assume that $F : L \to Spaces$ is a good diagram whose values are weakly equivalent to the homotopy fiber of f (see example 3.7).

3.13. Example. Let $\dot{\Delta}[n] \to L$ be a map, where $\dot{\Delta}[n]$ is the boundary of $\Delta[n]$. We are going to consider diagrams over $L \cup_{\dot{\Delta}[n]} \Delta[n]$.

- Let $F : L \cup_{\dot{\Delta}[n]} \Delta[n] \to Spaces$ be a diagram. One can show that:

$$\oint_{L\cup_{\dot{\Delta}[n]}\Delta[n]} F = colim\big(\oint_L F \leftarrow \oint_{\dot{\Delta}[n]} F \hookrightarrow \oint_{\Delta[n]} F\big)$$

- Let $F : L \cup_{\dot{\Delta}[n]} \Delta[n] \to Spaces$ be a bounded diagram. If $\tau \in (\Delta[n])_n$ is the only non degenerate simplex and $F(\tau) = X$, one can conclude that in this case:

$$\oint_{L\cup_{\dot{\Delta}[n]}\Delta[n]} F = colim\Big(\oint_L F \leftarrow \dot{\Delta}[n] \times X \hookrightarrow \Delta[n] \times X\Big)$$

3.14. Example. A functor $F : I \to Spaces$ over a small category I defines a bounded diagram $F_{sd} : N(I) \to Spaces$ over the nerve of I. Let:

$$\sigma = (a_0 \overset{\varphi_0}{\to} a_1 \overset{\varphi_1}{\to} \dots \overset{\varphi_{n-1}}{\to} a_n) \in N(I)_n$$

F_{sd} is defined as follows:

$$F_{sd}(\sigma) = F(a_0)$$

$$F_{sd}(d_i : \sigma \to d_i\sigma) = \begin{cases} F(\varphi_0) : F(a_0) \to F(a_1) & \text{if } i = 0 \\ id : F(a_0) \to F(a_0) & \text{if } i > 0 \end{cases}$$

It can be shown that in this case:

$$\oint_{N(I)} F_{sd} = hocolim_I F$$

where $hocolim_I F$ denotes the homotopy colimit of F in the sense of Bousfield and Kan [3].

In case of a pointed functor $F : I \to Spaces_\star$ there is a similar equality:

$$\int_{N(I)} F_{sd} = phocolim_I F$$

where $phocolim_I F$ denotes the **pointed** homotopy colimit of F in the sense of Bousfield and Kan [3].

4. Closed Classes

In this section we state the definition and give some examples and basic properties of closed classes. The notion of a closed class was introduced by E. Dror Farjoun [6], [7]. The definition presented in this papers is slightly different from the one given by E. Dror Farjoun [6, definition 2.1]. We think about a closed class as a class of **unpointed** and connected simplicial sets. A good example, to keep in mind, is the class of acyclic spaces with respect to some homology theory.

4.1. Definition. *(E. Dror Farjoun [6]). A non empty class C of connected simplicial sets is said to be a closed class if it is closed under weak equivalences and taking* **pointed** *homotopy colimits. If $F : K \to Spaces_\star$ is a pointed diagram such that for every simplex $\sigma \in K$, $F(\sigma) \in C$, then $\int_K F \in C$.*

Observe that a closed class is assumed to be non-empty. Notice also that since the empty space is not connected, it does not belong to any closed class.

4.2. Notation.

- Throughout this article C always denotes a closed class.
- By $F : K \to C$ we denote a diagram such that for every simplex $\sigma \in K$, $F(\sigma)$ belongs to C.
- Let $f : X \to Y$ be a map. We say that the homotopy fiber $Fib(X \xrightarrow{f} Y)$ belongs to a closed class C, if the homotopy fibers of f over every component of Y belong to C. In particular, the homotopy fibers of f, over various components, are connected and f induces an isomorphism on π_0. If $Fib(X \xrightarrow{f} Y)$ belongs to C, then we will write $Fib(X \xrightarrow{f} Y) \in C$.

4.3. Remark. The definition of the pointed homotopy colimit 3.10 implies that a class C is closed if and only if:

- C is non-empty.
- Let X and Y be weakly equivalent simplicial sets. If $X \in C$, then $Y \in C$.
- Let $(X_i)_{i \in I}$ be a family of simplicial sets. If $X_i \in C$, then for any choice of basepoints in X_i, $\bigvee_{i \in I} X_i \in C$.
- Let $X_\star$ be a simplicial space. If for every $n \geq 0$, $X_n \in C$, then the realization $|X_\star| \in C$.

It follows that a closed class can be characterized as a class of connected simplicial sets, such that:

- C is non-empty.
- C is closed under weak equivalences.
- C is closed under taking arbitrary wedges.
- Let $X_1 \leftarrow X_2 \to X_3$ be a diagram. If $X_i \in C$, then the following simplicial set belongs to C:

$$hocolim(X_1 \leftarrow X_2 \to X_3)$$

- Let $(X_0 \to X_1 \to X_2 \to \cdots)$ be a diagram. If $X_i \in C$, then the following simplicial set belongs to C:

$$hocolim(X_0 \to X_1 \to X_2 \to \cdots)$$

4.4. Examples. Here is a list of some examples of closed classes:

- Let A be a connected simplicial set. The smallest closed class $C(A)$ such that $A \in C(A)$. This class is called the class of A-cellular spaces. This class was introduced by E. Dror Farjoun, see [4],[6] and [7].
- The class of acyclic spaces with respect to some homology theory.
- The class $C(\star)$ of weakly contractible spaces.
- The class $C(S^{n+1})$ of n-connected spaces.
- $\{X \in cSpaces \mid \tilde{H}^i(X, G) \text{ is trivial for } i \leq n\}$.

- Let A be a pointed and connected Kan simplicial set.
  $\{X \in cSpaces \mid \text{for any choice of basepoints in } X,\ map_{\star}(X,A) \simeq \star\}$.
- Let A be a connected simplicial set.
  $\{X \in cSpaces \mid$ if Y is Kan and the basepoint evaluation map $map(A,Y) \to Y$ is a weak equivalence, then $map(X,Y) \to Y$ is also a weak equivalence$\}$.
  This class is called the class A-acyclic spaces, see [4, definition 16.1].

The following two propositions give some examples of elements of a closed class.

4.5. Proposition. (E. Dror Farjoun [6, section 2.3]). *If C is a closed class, then $\star \in C$.*

Proof. Let $X \in C$ and $X \xrightarrow{\star} X$ be the constant map. Notice that the following space is contractible and belongs to C:

$$hocolim(X \xrightarrow{\star} X \xrightarrow{\star} X \xrightarrow{\star} X \cdots)$$

It implies that $\star \in C$. □

4.6. Proposition. (E. Dror Farjoun [6, theorem 2.8]). *Let K and X be simplicial sets. If $X \in C$, then for any choice of basepoint in X, $K \ltimes X \in C$ (see example 3.11).*

Proof. Lets consider the constant diagram $X : K \to C$, $X(\sigma) = X$ and $X(\varphi) = id_X$. Since C is a closed class, $K \ltimes X = \int_K X \in C$. □

5. Closed classes and unpointed homotopy colimits

Closed classes are not usually closed under unpointed homotopy colimits (see [6, section 2.3]). This section contains the first approach to this question, to what extent a closed class is closed under unpointed homotopy colimits.

The motivation for the following theorem can be found in the next section. This theorem is going to be applied to prove various properties of closed classes with respect to fibrations. Surprisingly all those properties are just particular cases of this one statement.

5.1. Theorem. *Let $F : K \to C$, $G : K \to C$ be diagrams and $\Psi : F \to G$ be a natural transformation. If $h : \oint_K F \to Y$ is a map such that $Y \in C$, then:*

$$hocolim\Big(\oint_K G \leftarrow \oint_K F \xrightarrow{h} Y\Big) \in C$$

5.2. Lemma. *For every diagram $F : \Delta[n] \to C$, $\oint_{\Delta[n]} F$ belongs to C.*

Proof. Let $\tau \in (\Delta[n])_n$ be the non-degenerate simplex. Observe that by choosing a vertex in $F(\tau)$, we can think about $F : \Delta[n] \to C$ as a pointed diagram (this chosen vertex determines a basepoint in $F(\sigma)$, for all $\sigma \in \Delta[n]$). Since $\Delta[n]$ is contractible, pointed and unpointed homotopy colimits are weakly equivalent. It proves the lemma. $\square$

Proof of the theorem. the proof will be by the induction on the dimension of K. It is obvious that the theorem is true for K such that $dim(K) = 0$. Lets assume that the theorem is true for K such that $dim(K) < n$. Let $dim(L) < n$ and $\dot{\Delta}[n] \to L$ be a map. We prove that the theorem holds for $K = L \cup_{\dot{\Delta}[n]} \Delta[n]$.

Lets consider the following commutative diagram:

$$\begin{array}{ccccc}
Y & \xleftarrow{id} & Y & \xrightarrow{id} & Y \\
\uparrow{\scriptstyle h} & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle h} \\
\oint_L F & \longleftarrow & \oint_{\dot{\Delta}[n]} F & \longrightarrow & \oint_{\Delta[n]} F \\
\downarrow & & \downarrow & & \downarrow \\
\oint_L G & \longleftarrow & \oint_{\dot{\Delta}[n]} G & \longrightarrow & \oint_{\Delta[n]} G
\end{array}$$

By the inductive assumption:

$$hocolim\Big(\oint_L G \leftarrow \oint_L F \xrightarrow{h} Y\Big) \in C$$

$$hocolim\Big(\oint_{\dot{\Delta}[n]} G \leftarrow \oint_{\dot{\Delta}[n]} F \xrightarrow{h} Y\Big) \in C$$

According to lemma 5.2, $\oint_{\Delta[n]} F$ and $\oint_{\Delta[n]} G$ belong to C. Because C is closed under homotopy push-outs we get:

$$hocolim\Big(\oint_K G \leftarrow \oint_K F \xrightarrow{h} Y\Big) \in C$$

$\square$

5.3. Corollary. *Let $F : K \to C$, $G : K \to C$ be diagrams and $\Psi : F \to G$ be a natural transformation. If $\oint_K F \in C$, then $\oint_K G \in C$.*

Proof. Apply theorem 5.1 to the case when $Y = \oint_K F$ and $h = id$. $\square$

5.4. Corollary. *Let K be a simplicial set. If $F : K \to C$ is a diagram such that $\oint_K F \in C$, then $K \in C$.*

Proof. Since there is a natural transformation $F \to \star$ between F and the constant diagram $\star : K \to Spaces$, whose value is $\star$, corollary 5.3 implies that $K = \oint_K \star$ belongs to C. □

6. Closed classes and fibrations

The behavior of a closed class with respect to fibrations has been studied by E. Dror Farjoun [6],[7]. This section contains generalizations of his results.

6.1. Definition. We say that a closed class C is closed under extensions by fibrations if for every fibration sequence $Z \to E \to B$ such that Z and B belong to C, E belongs to C. 2mm

Closed classes are not usually closed under extensions by fibrations [7].

6.2. Examples. Here is a list of some examples of closed classes that are closed under extensions by fibrations:

- The class of acyclic spaces with respect to some homology theory.
- The class $C(\star)$ of weakly contractible spaces.
- The class $C(S^{n+1})$ of n-connected spaces.
- Let A be a connected simplicial set.
  $\{X \in cSpaces \mid$ if Y is Kan and the basepoint evaluation map $map(A, Y) \to Y$ is a weak equivalence, then $map(X, Y) \to Y$ is also a weak equivalence$\}$.

The next theorem is a geometric interpretation of theorem 5.1.

6.3. Theorem. *Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ and $s : E_1 \to E_2$ be maps such that $p_1 = p_2 \circ s$ and the homotopy fibers $Fib(E_1 \xrightarrow{p_1} B)$, $Fib(E_2 \xrightarrow{p_2} B)$ belong to C. For any map $h : E_1 \to Y$, where $Y \in C$:*

$$hocolim(E_2 \xleftarrow{s} E_1 \xrightarrow{h} Y) \in C$$

Proof. Example 3.12 implies that p_1 and p_2 are weakly equivalent, respectively to maps of the form:

$$\oint_{sdB} F \to sdB \ , \ \oint_{sdB} G \to sdB$$

where F has values weakly equivalent to the homotopy fibers of p_1 and G has values weakly equivalent to the homotopy fibers of p_2. Since the definitions were natural, $s : E_1 \to E_2$ induces a natural transformation $\Psi : F \to G$. Theorem 5.1 implies then:

$$hocolim(E_2 \leftarrow E_1 \to Y) \simeq hocolim(\oint_{sdB} G \leftarrow \oint_{sdB} F \to Y) \in C$$

□

The following corollaries are particular cases of theorem 6.3.

6.4. Corollary. *Let $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ and $s : E_1 \to E_2$ be maps such that $p_1 = p_2 \circ s$. If the homotopy fibers $Fib(E_1 \xrightarrow{p_1} B)$, $Fib(E_2 \xrightarrow{p_2} B)$ and E_1 belong to C, then so does E_2.*

Proof. Apply theorem 6.3 to the case when $Y = E_1$ and $h = id$. □

6.5. Corollary. (E. Dror Farjoun [7]). *Let $p : E \to B$ be a map such that the homotopy fiber $Fib(E \xrightarrow{p} B)$ belongs to C. If $E \in C$, then $B \in C$.*

Proof. Apply corollary 6.4 to the case when $p_1 = p$, $p_2 = id_B$ and $s = p$. □

6.6. Corollary. (E. Dror Farjoun [7]). *Let $F \to E \xrightarrow{p} B$ be a fibration sequence. If ΩB and F belong to C, then so does E.*

Proof. Since $\Omega B \to F \to E$ is a fibration sequence such that ΩB and F belong to C, corollary 6.5 implies that E belongs to C. □

6.7. Corollary. *Let $F \to E \xrightarrow{p} B$ be a fibration sequence and $B \xrightarrow{s} E$ be a section of p. If $h : B \to Y$ is a map such that $Y \in C$, then:*

$$colim(E \xleftarrow{s} B \xrightarrow{h} Y) \in C$$

Proof. Apply theorem 6.3 to the case when $p_1 = id_B$, $p_2 = p$. □

6.8. Corollary. *Closed classes are closed under split extensions. Let $F \to E \xrightarrow{p} B$ be a fibration sequence such that p has a section. If $F \in C$ and $B \in C$, then $E \in C$.*

Proof. Apply corollary 6.7 to the case when $Y = B$ and $h = id_B$ and s is a section of p. □

6.9. Corollary. (W. Dwyer [6]). *Closed classes are closed under products. If $X \in C$ and $Y \in C$, then $X \times Y \in C$.*

Proof. Notice that $X \to X \times Y \to Y$ is a fibration sequence with a section. According to corollary 6.8, $X \times Y$ belongs to C. □

7. Homotopy properties of shapes of diagrams

In this section the behavior of a closed class under unpointed homotopy colimits is going to be investigated further (see section 5).

A diagram $F : K \to Spaces$ consist of bunch of spaces which are related to each other by various maps. Those relations are coming from the geometry of K. We will try to understand how the geometry of K effects the homotopy colimit functor of diagrams over K.

7.1. Definition. $D(C) = \{K \mid \textit{if } F : K \to C \textit{ is a diagram, then } \oint_K F \in C\}$

Class $D(C)$ consists of those simplicial sets that carry enough information so by gluing elements of class C, according to K, we get back a space in C.

7.2. Proposition. $D(C) \subseteq C$

Proof. Let $K \in D(C)$. Since $\star \in C$, according to the definition, $K = \oint_K \star$ belongs to C.

Notice that corollary 5.4 is stronger than this proposition. It says that if there exist a diagram $F : K \to C$ such that $\oint_K F \in C$, then automatically $K \in C$. □

Observe that lemma 5.2 implies:

7.3. Proposition. *For every n, $\Delta[n] \in D(C)$.*

It turns out that $D(C)$ has nice homotopic properties. The next theorem suggests that to some extant, not geometry but the homotopy type of a simplicial set K plays a crucial role toward the homotopic properties of the homotopy colimit functor of diagrams over K.

7.4. Theorem. *Class $D(C)$ is closed under weak equivalences.*

7.5. Lemma.

- *Let $K \leftarrow L \hookrightarrow M$ be a push-out diagram such that $L \hookrightarrow M$ is a cofibration. If K, L and M belong to $D(C)$, then so does:*
$$K \cup_L M = colim(K \leftarrow L \hookrightarrow M)$$
- *Let Θ be the category associated with an ordinal number Θ (see* [8, page 11]*). Let $G : \Theta \to Spaces$ be a functor such that for every morphism $\varphi \in \Theta$, $G(\varphi)$ is a cofibration. If for every $\theta \in \Theta$, $G(\theta)$ belongs to $D(C)$, then so does $colim_\Theta G$.*

Proof. We will prove only the first part of the lemma. The second part can be proven in the same way.

Let $F : K \cup_L M \to C$ be a diagram. According to example 3.13:

$$\oint_{K\cup_L M} F = colim\Big(\oint_K F \leftarrow \oint_L F \hookrightarrow \oint_M F\Big)$$

By the assumption $\oint_K F$, $\oint_L F$ and $\oint_M F$ belong to C. Since any closed class is closed under taking homotopy push-outs:

$$hocolim\Big(\oint_K F \leftarrow \oint_L F \hookrightarrow \oint_M F\Big) \in C$$

Notice that the cofibration assumption implies that the following map is a weak equivalence:

$$hocolim\Big(\oint_K F \leftarrow \oint_L F \hookrightarrow \oint_M F\Big) \to colim\Big(\oint_K F \leftarrow \oint_L F \hookrightarrow \oint_M F\Big)$$

It implies that $\oint_{K\cup_L M} F$ belongs to C. □

7.6. Lemma. *Let $0 \le k \le n$. $\Delta[n,k]$ belongs to $D(C)$, where if $\tau \in \Delta[n]_n$ is the non degenerate simplex, $\Delta[n,k]$ is the simplicial subset of $\Delta[n]$, generated by simplices $\{d_i\tau\}_{i\neq k}$.*

Proof. We are going to present $\Delta[n,k]$ as a sequence of push-outs of standard simplices. In order to do this we have to introduce some notation:

- Let $i \in \{0,1,\cdots,n\}$. Δ_i denotes the simplicial subset of $\Delta[n]$ generated by the simplex $d_i\tau$.
- $\{i\}$ denotes the simplicial subset of $\Delta[n]$ generated by the vertex $\{i\}$.
- Let $\{i,j\} \in \{0,1,\cdots,n\}$. $\Delta_{i,j}$ denotes the simplicial subset of $\Delta[n]$ generated by the simplex $d_{i-1}d_j\tau$ if $i > j$, or by $d_{j-1}d_i\tau$ if $i < j$.

There are obvious inclusions $\Delta_{i,j} \to \Delta_i \to \Delta[n]$. Let X be the colimit of the following diagram:

$$\begin{array}{ccccccccc}
\Delta_{k+2} & \xrightarrow{id} & \Delta_{k+2} & & \overset{id}{\to} \; \Delta_n & & \Delta_1 & \overset{id}{\to} & \Delta_{k-1} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Delta_{k+1,k+2} & & \Delta_{k+2,k+3} & \cdots & \Delta_{n,0} & & \Delta_{0,1} & \cdots & \Delta_{k-2,k-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Delta_{k+1} & & \Delta_{k+3} \; \overset{id}{\to} & & \Delta_0 & \xrightarrow{id} & \Delta_0 & \overset{id}{\to} & \Delta_{k-2}
\end{array}$$

Out of the construction of X, we have two natural inclusions $\Delta_{k-1} \to X$, $\Delta_{k+1} \to X$. Notice that there is a cofibration map $\Delta_{k-1,k+1} \vee_{\{k\}} \Delta_{k-1,k+1} \to X$ which is the wedge of the following maps:

$$\Delta_{k-1,k+1} \to \Delta_{k-1} \to X$$

$$\{k\} \to \Delta_{k-1} \to X$$

$$\Delta_{k-1,k+1} \to \Delta_{k+1} \to X$$

By laborious but straightforward calculation one can show that:

$$\Delta[n,k] = colim\big(\Delta_{k-1,k+1} \overset{id\vee id}{\longleftarrow} \Delta_{k-1,k+1} \vee_{\{k\}} \Delta_{k-1,k+1} \to X\big)$$

Since $\Delta[n,k]$ is built from standard simplices by push-out process, where the maps involved are cofibrations, according to the lemma 7.5, $\Delta[n,k]$ belongs to $D(C)$. □

Proof of the Theorem. The proof will be divided into several steps.

Step 1. *Let $E \overset{\simeq}{\twoheadrightarrow} B$ be a fibration and a weak equivalence. If $E \in D(C)$, then $B \in D(C)$.*

Proof. Let $F : B \to C$ be a diagram. We have to show that $\oint_B F \in C$. According to section A.1, the following is a pull-back square:

$$\begin{array}{ccc} \oint_E F\circ p & \longrightarrow & \oint_B F \\ \downarrow & & \downarrow \\ E & \xrightarrow{p} & B \end{array}$$

Since p is a fibration and a weak equivalence, $\oint_E F\circ p \to \oint_B F$ is a weak equivalence as well. Because $E \in D(C)$, $\oint_E F\circ p$ belongs to C. It implies that $\oint_B F \in C$.

Step 2. *Let $f : X \to Y$ be a weak equivalence. If $X \in D(C)$, then $Y \in D(C)$.*

Proof. We are going to construct by the induction a sequence of spaces and inclusions:

$$(X^0 \to X^1 \to X^2 \to \cdots)$$

together with a sequence of maps:

$$\{i_l : X \to X^l\}_{l\geq 0}\ ,\ \{p_l : X^l \to Y\}_{l\geq 0}$$

such that $X^l \in D(C)$, i_l is a cofibration and a weak equivalence, $i_l\circ p_l = f$ and the map $colimit(p_l) : colimit(X^l) \to Y$ is a fibration. We denote $colimit(p_l)$ by $p : \bar{X} \to Y$, in particular $\bar{X} = colimit(X^l)$.

Let $X_0 = X$, $i_0 = id_X$ and $p_0 = f$. Lets assume that the construction has been carried out for $i < l$. Let J be the set of all commutative diagrams of the form:

$$\begin{array}{ccc} \Delta[n,k] & \longrightarrow & X^{l-1} \\ \downarrow & & \downarrow{\scriptstyle p_{l-1}} \\ \Delta[n] & \longrightarrow & Y \end{array}$$

where $\Delta[n,k] \to \Delta[n]$ is the canonical inclusion. X_l is defined to be the simplicial set that fits into the following push-out square:

$$\begin{array}{ccc} \coprod_J \Delta[n,k] & \longrightarrow & X^{l-1} \\ \downarrow & & \downarrow \\ \coprod_J \Delta[n] & \longrightarrow & X^l \end{array}$$

i_l is defined to be the following composition:

$$X \overset{i_{l-1}}{\to} X^{l-1} \to X^l$$

p_l is defined to be the push-out of the following maps:

$$X^{l-1} \overset{p_{l-1}}{\to} Y \quad , \quad \coprod_J \Delta[n,k] \to Y \quad , \quad \coprod_J \Delta[n] \to Y$$

By the inductive assumption $X^{l-1} \in D(C)$. Since X^l is built by gluing lots of $\Delta[n]$ along $\Delta[n,k]$ to X^{l-1}, according to lemma 7.5, $X^l \in D(C)$. Observe that the natural map $i : X = X^0 \to \bar{X}$ is a weak equivalence. Notice also that $p \circ i = f$. By Quillen's small object argument (see [10]), p is a fibration. Since f and i are weak equivalences, so is p.

Lemma 7.5 implies that $\bar{X} \in D(C)$. Since $p : \bar{X} \to Y$ is a fibration and a weak equivalence, according to step 1, $Y \in D(C)$.

Step 3. *If X is contractible, then $X \in D(C)$.*

Proof. Since $\star \in D(C)$ and $\star \to X$ is a weak equivalence, step 2 implies that $X \in D(C)$.

Step 4. *Let $F : K \to C$ be a diagram. The homotopy fiber $Fib(\oint_K F \to K)$ belongs to C.*

Proof. Lets choose a connected component of K and a fibration $PK \to K$ such that PK is contractible and the image of PK is in the chosen component. According to corollary A.2, the homotopy fiber of $\oint_K F \to K$ over the chosen component is weakly equivalent to $\oint_{PK} F$. Since PK is contractible, according to step 3, $\oint_{PK} F$ belongs to C.

Step 5. *Let $Z \to E \to B$ be a fibration sequence. If $B \in D(C)$ and $Z \in C$, then $E \in C$.*

Proof. We can assume that p is a fibration. Example 3.12 implies that $E \to B$ is weakly equivalent to $\oint_{sdB} dp \to sdB$, where $dp : sdB \to Spaces$ is a good diagram whose values are weakly equivalent to Z. Proposition A.5 gives the following weak equivalence:

$$\oint_B \oint_{N(B/\sigma)} (dp)\circ l_\sigma \to \oint_{sdB} dp$$

Since $N(B/\sigma)$ is contractible $\oint_{N(B/\sigma)} (dp)\circ l_\sigma \in C$. The assumption $B \in D(C)$ implies:

$$E \simeq \oint_B \oint_{N(B/\sigma)} (dp)\circ l_\sigma \in C$$

Step 6. *Let $f : X \to Y$ be a weak equivalence. If $Y \in D(C)$, then $X \in D(C)$.*

Proof. Let $F : X \to C$ be a diagram. Notice that the homotopy fiber of the composition $\oint_X F \to X \xrightarrow{f} Y$ is weakly equivalent to the homotopy fiber $Fib(\oint_X F \to X)$. According to step 4, it belongs to C. Since $Y \in D(C)$, Step 5 implies that $\oint_X F \in C$. This proves that $X \in D(C)$.
□

Theorem 7.4 implies an interesting characterization of a closed class (see also[1]):

7.7. Corollary. *Non empty class C of connected simplicial sets is closed if it is closed under weak equivalences and for every, not necessarily pointed diagram, $F : K \to C$ over a contractible simplicial set K, $\oint_K F \in C$.*

The definition of a closed class says that it is closed under pointed homotopy colimits. It means that for any pointed diagram $F : K \to Spaces_\star$ the homotopy cofiber:

$$Cof(K \to \oint_K F) = \int_K F$$

belongs to C. The next corollary implies that the dual statement is also true (see [6] and [7] for discussion of similar statements).

7.8. Corollary. *Let $F : K \to C$ be a diagram. $Fib(\oint_K F \to K) \in C$.*

The following corollary says that if the the pre-images of simplices have certain properties (belong to a closed class), then so does the homotopy fiber (see also [7]).

7.9. Corollary. *Let $f : X \to K$ be a map. If for every simplex $\sigma \in K$:*

$$pullback\big(X \overset{f}{\to} K \leftarrow \Delta[dim(\sigma)]\big) \in C$$

then $Fib(X \overset{f}{\to} Y) \in C$.

Proof. According to example 3.12, $f : X \to K$ is weakly equivalent to a map of the form $\oint_{sdK} df \to sdK$, where for $v = (\sigma_0 \to \cdots \to \sigma_n) \in sdK$, df is a diagram such that:

$$df(v) = pullback\big(X \overset{f}{\to} K \leftarrow \Delta[dim(\sigma_n)]\big)$$

Corollary 7.8 implies $Fib\big(\oint_{sdK} df \to sdK\big) \in C$. □

8. Class $D(C)$

In this section we present other characterizations of the class $D(C)$. We will restrict the class of diagrams on which a simplicial set should be tested in order to find out if it belongs to $D(C)$. We will show also that the class $D(C)$ is a closed class and is closed under extensions fibrations.

8.1. Proposition.

$$D(C) = \{K \mid \textit{if } F : K \to C \textit{ is a bounded diagram, then } \oint_K F \in C\}$$

Proof. Let:

$$D' = \{K \mid \text{if } F : K \to C \text{ is a bounded diagram, then } \oint_K F \in C\}$$

Inclusion $D(C) \subset D$ is obvious.

By the same arguments as in theorem 7.4, we can show that the class D' is closed under weak equivalences. Let $K \in D'$ and $F : K \to C$ be a diagram. According to remark A.6, $\oint_K F$ is weakly equivalent to $\oint_{sdK} F_{sd}$. Since sdK is weakly equivalent to K, it belongs to D'. Notice that F_{sd} is a bounded diagram, therefore $\oint_{sdK} F_{sd} \in C$. It implies that $\oint_K F \in C$ and $K \in D(C)$. □

8.2. Proposition.

$$D(C) = \{B \mid \textit{ if } Z \to E \to B \textit{ is a fibration sequence and } Z \in C, \textit{ then } E \in C\}$$

Proof. Let:

$$D' = \{B \mid \text{ if } Z \to E \to B \text{ is a fibration sequence and } Z \in C, \text{ then } E \in C\}$$

Let $B \in D$ and $F : B \to C$ be a diagram. Since $\oint_{PB} F \to \oint_B F \to B$ is a fibration sequence (see A.2) and $\oint_{PB} F \in C$, we get $\oint_B F \in C$. It implies the inclusion $D' \subset D(C)$.

Let $K \in D(C)$ and $Z \to E \xrightarrow{p} K$ be a fibration sequence. According to example 3.12, $E \to K$ is weakly equivalent to a map of the form $\oint_L F \to L$, where the values of F are weakly equivalent to Z. Since L is weakly equivalent to K, it belongs to $D(C)$. As a consequence we get $E \simeq \oint_L F \in C$. It proves that $K \in D'$ and $D \subset D'$. □

8.3. Corollary. *A closed class C is closed under extensions by fibrations if and only if $C = D(C)$.*

The next corollary says that class $D(C)$ is usually quite big.

8.4. Corollary. *If B is such that $\Omega B \in C$, then $B \in D(C)$.*

Proof. See corollary 6.6. □

8.5. Proposition.

$$D(C) = \{K \mid \textit{if } F : K \to C \textit{ is a good diagram, then } \oint_K F \in C\}$$

Proof. Let:

$$D' = \{K \mid \text{if } F : K \to C \text{ is a good diagram, then } \oint_K F \in C\}$$

Inclusion $D(C) \subset D'$ is obvious.

By the same arguments as in the theorem 7.4, we can show that the class D' is closed under weak equivalences. Let $B \in D'$ and $p : E \to B$ be a fibration such that the fiber of p belongs to C. According to example 3.12, $p : E \to B$ is weakly equivalent to a map of the form $\oint_{sdB} dp \to sdB$, where dp is a good diagram whose values are weakly equivalent to the fiber of p. It implies that $E \simeq \oint_{sdB} dp \in C$. This proves the proposition. □

8.6. Theorem. *Let $G : K \to D(C)$ be a diagram. If K belongs to $D(C)$, then so does $\oint_K G$.*

Proof. According to remark A.6, $\oint_K G$ is weakly equivalent to $\oint_{sdK} G_{sd}$. Theorem 7.4 implies that $\oint_K G$ belongs to $D(C)$ if and only if $\oint_{sdK} G_{sd}$ does. Since

G_{sd} is a bounded diagram, without loss of generality, it is enough to prove the theorem for a bounded diagram $G : K \to D(C)$.

Let $G : D(C) \to b$e a bounded diagram and $F : \oint_K G \to C$ be a diagram. Theorem A.9 implies that $\oint_{\oint_K G} F$ is weakly equivalent to $\oint_{sdK} \oint_{\Delta G(v)} F$. Since G has values in $D(C)$, then so does ΔG, therefore $\oint_{\Delta G(v)} F$ belongs to C. Because $K \in D(C)$, $sdK \in D(C)$ and it follows that $\oint_{sdK} \oint_{\Delta G(v)} F$ belongs to C. This proves $\oint_{\oint_K G} F \in C$. □

8.7. Corollary. *$D(C)$ is a closed class and $D(D(C)) = D(C)$, therefore $D(C)$ is closed under extensions by fibrations.*

9. Theorem of E. Dror Farjoun

9.1. Theorem. *Let $\Psi : E \to B$ be a natural transformation between diagrams $E : K \to Spaces$ and $B : K \to Spaces$. If for every simplex $\sigma \in K$ the homotopy fiber $Fib(E(\sigma) \xrightarrow{\Psi_\sigma} B(\sigma))$ belongs to C, then:*

$$Fib(\oint_K E \xrightarrow{\Psi} \oint_K B) \in C$$

9.2. Lemma. *Lets consider the following commutative diagram:*

$$\begin{array}{ccccc} E_1 & \longleftarrow & E_2 & \longrightarrow & E_3 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} \\ B_1 & \xleftarrow{f} & B_2 & \xrightarrow{g} & B_3 \end{array}$$

where the maps $E_2 \to E_3$, $B_2 \xrightarrow{g} B_3$ are cofibrations. If $Fib(p_1)$, $Fib(p_2)$ and $Fib(p_3)$ belong to C, then:

$$Fib(E_1 \cup_{E_2} E_3 \to B_1 \cup_{B_2} B_3) \in C$$

Proof. Without loss of generality we can assume that p_1, p_2 and p_3 are fibrations. Let $p = colim(p_1 \leftarrow p_2 \to p_3)$. According to corollary 7.9, it is enough to prove that for every simplex, $\sigma \in B_1 \cup_{B_2} B_3$:

$$F(\sigma) = pullback(E_1 \cup_{E_2} E_3 \xrightarrow{p} B_1 \cup_{B_2} B_3 \leftarrow \Delta[dim(\sigma)]) \in C$$

Let $\sigma \in B_1 \cup_{B_2} B_3$. Either σ lies in the image of B_1 or B_2. Lets assume that it belongs to the image of B_1. Let $K = pullback(\Delta[dim(\sigma)] \to B_1 \xleftarrow{f} B_2)$. There is a natural map $K \to B_2$. Let X_1, X_2 and X_3 be simplicial sets that fit into the following pull-back squares:

$$\begin{array}{ccc} X_1 & \longrightarrow & E_1 \\ \downarrow & & \downarrow p_1 \\ \Delta[dim(\sigma)] & \longrightarrow & B_1 \end{array} \qquad \begin{array}{ccc} X_2 & \longrightarrow & E_2 \\ \downarrow & & \downarrow p_2 \\ K & \longrightarrow & B_2 \end{array} \qquad \begin{array}{ccc} X_3 & \longrightarrow & E_3 \\ \downarrow & & \downarrow p_3 \\ K & \longrightarrow & B_3 \end{array}$$

Observe that the definition gives natural maps $X_2 \to X_1$ and $X_2 \to X_3$. By straightforward combinatorial calculation one can show:

$$F(\sigma) = colim(X_3 \leftarrow X_2 \to X_1)$$

Notice that the maps $X_3 \to K$, $X_2 \to K$ and $X_2 \to X_1$ satisfy the assumptions of theorem 6.3, therefore $hocolim(X_3 \leftarrow X_2 \to X_1) \in C$. Cofibration assumption of the lemma implies:

$$hocolim(X_3 \leftarrow X_2 \to X_1) \simeq colim(X_3 \leftarrow X_2 \to X_1)$$

It proves the lemma. □

Proof of the theorem. Instead of $E : K \to Spaces$, $B : K \to Spaces$ we can consider bounded diagrams $E_{sd} : sdK \to Spaces$, $B_{sd} : sdK \to Spaces$. Since the homotopy fibers $Fib(\oint_K E \to \oint_K B)$ and $Fib(\oint_{sdK} E_{sd} \to \oint_{sdK} B_{sd})$ are weakly equivalent, it is enough to prove the theorem for bounded diagrams.

The proof will be by the induction on the dimension of K. If $dim(K) = 0$, the theorem is obvious. Lets assume that the theorem is true for K such that $dim(K) < n$. Let L be a simplicial of dimension less than n and $\dot{\Delta}[n] \to L$ be a map. We prove that the theorem holds for $K = L \cup_{\dot{\Delta}[n]} \Delta[n]$.

Let $\tau \in (\Delta[n])_n$ be the only non degenerate simplex. Lets consider the following diagram:

$$\begin{array}{ccccccccc} \oint_K E & = & colim & (& \oint_L E & \leftarrow & \dot{\Delta}[n] \times E(\tau) & \hookrightarrow & \Delta[n] \times E(\tau) &) \\ \downarrow \Psi & & & & \downarrow \Psi & & \downarrow id\times\Psi_\tau & & \downarrow id\times\Psi_\tau & \\ \oint_K B & = & colim & (& \oint_L B & \leftarrow & \dot{\Delta}[n] \times B(\tau) & \hookrightarrow & \Delta[n] \times B(\tau) &) \end{array}$$

By the inductive assumption $Fib(\oint_L E \xrightarrow{\Psi} \oint_L B)$ belongs to C. Since the homotopy fiber $Fib(E(\tau) \xrightarrow{\Psi_\tau} B(\tau))$ also belongs to C, according to lemma 9.2:

$$Fib(\oint_K E \xrightarrow{\Psi} \oint_K B) \in C$$

□

As a corollary we get the theorem of E. Dror Farjoun

9.3. Corollary. *(E. Dror Farjoun [7, theorem I]). Let $E : K \to Spaces_\star$ and $B : K \to Spaces_\star$ be pointed diagrams and $\Psi : E \to B$ be a natural transformation. If for every simplex $\sigma \in K$, $Fib(E(\sigma) \xrightarrow{\Psi_\sigma} B(\sigma)) \in C$, then:*

$$Fib\Big(\int_K E \xrightarrow{\Psi} \int_K B\Big) \in C$$

Proof. Lets consider the following diagram:

$$\begin{array}{ccccccccccc}
\int_K E & \simeq & hocolim & (& \star & \leftarrow & K & \to & \oint_K E &) \\
\downarrow \Psi & & & & \downarrow & & \downarrow id & & \downarrow \Psi & \\
\int_K B & \simeq & hocolim & (& \star & \leftarrow & K & \to & \oint_K B &)
\end{array}$$

According to theorem 9.1, $Fib\big(\oint_K E \stackrel{\Psi}{\to} \oint_K B\big)$ belongs to C. Since the homotopy fiber $Fib(K \stackrel{id}{\to} K)$ belongs to C, applying once again theorem 9.1 we get:

$$Fib\Big(\int_K E \stackrel{\Psi}{\to} \int_K B\Big) \in C$$

□

9.4. Theorem. *Let the following be a homotopy push-out square:*

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow i & & \downarrow \\
X & \xrightarrow{g} & Y
\end{array}$$

If the homotopy fiber $Fib(A \stackrel{f}{\to} B)$ belongs to C, then so does the homotopy fiber $Fib(X \stackrel{g}{\to} Y)$.

Proof. Lets consider the following diagram:

$$\begin{array}{cccccccccc}
X & \simeq & hocolim & (& X & \stackrel{i}{\leftarrow} & A & \stackrel{id}{\to} & A &) \\
\downarrow g & & & & \downarrow id & & \downarrow id & & \downarrow f & \\
Y & \simeq & hocolim & (& X & \stackrel{i}{\leftarrow} & A & \stackrel{f}{\to} & B &)
\end{array}$$

Since $Fib(A \stackrel{f}{\to} B)$ belongs to C, theorem 9.1 implies that $Fib(X \stackrel{g}{\to} Y) \in C$. □

9.5. Corollary. *Let $f : X \to Y$ be a map. If X belongs to C, then so does the homotopy fiber $Fib\big(Y \to Cof(X \xrightarrow{f} Y)\big)$.*

Proof. Apply theorem 9.4 to the following homotopy push-out square:

$$\begin{array}{ccc} X & \longrightarrow & \star \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Cof(X \xrightarrow{f} Y) \end{array}$$

□

Appendix A. The homotopy colimit

The reference for the proofs of the statements, listed in the appendix, is [5].

A.1. Pulling-back of diagrams. Let $F : K \to Spaces$ be a diagram over K and $f : L \to K$ be a map. We can pull-back F into a diagram $F\circ f$ over L ($F\circ f$ will be often denoted simply by F). Let τ and σ be simplices in L and $\varphi : \tau \to \sigma$ be a morphism in the category associated with L. $F\circ f : L \to Spaces$ is defined as follows:

$$F\circ f(\sigma) = F(f(\sigma))$$
$$F\circ f(\varphi) = F(\varphi)$$

The basic property of the pull-back diagram $F\circ f$ is that the following is a pull-back square:

$$\begin{array}{ccc} \oint_L F\circ f & \longrightarrow & \oint_K F \\ \downarrow & & \downarrow \\ L & \xrightarrow{f} & K \end{array}$$

As corollary of the this property we get:

A.2. Corollary. *Let $F : K \to Spaces$ be a diagram and $PK \to K$ be a fibration such that PK is contractible. The following is a fibration sequence:*

$$\oint_{PK} F \to \oint_K F \to K$$

A.3. Diagrams over $colim_I G$**.** Let $G : I \to Space,\ F : colim_I G \to Spaces$ be diagrams. There is a family of maps $\{l_i : G(i) \to colim_I G\}_{i\in I}$ which satisfies the universal property of the colimit of the diagram $G : I \to Space$ (see [8, 3§3]). Out of this data we can construct a functor:

$$I \longrightarrow Spaces$$

$$i \longmapsto \oint_{G(i)} F\circ l_i$$

$$(a \xrightarrow{\varphi} b) \longmapsto (\oint_{G(a)} F\circ l_a \xrightarrow{\oint_{G(\varphi)} F\circ l_b} \oint_{G(b)} F\circ l_b)$$

This functor has the following property:

$$\oint_{colim_I G} F = colim_G \oint_{G(i)} F\circ l_i$$

A.4. Diagrams over sdK**.** Let σ be a simplex in K. By K/σ we denote the over category of K (see section 2). There is a functor:

$$K/\sigma \to K$$

$$(\tau \to \sigma) \mapsto \tau\ ,\ (\tau_0 \to \tau_1 \to \sigma) \mapsto (\tau_0 \to \tau_1)$$

This functor induces a map between simplicial sets:

$$l_\sigma : N(K/\sigma) \to N(K) = sdK$$

$$(\tau_0 \to \cdots \to \tau_n \to \sigma) \overset{l_\sigma}{\mapsto} (\tau_0 \to \cdots \to \tau_n)$$

One can verify that the family of maps $\{l_\sigma : N(K/\sigma) \to sdK\}_{\sigma\in K}$ satisfies the universal property of the colimit of the functor:

$$K \to Spaces\ ,\ \sigma \mapsto N(K/\sigma)$$

It implies:

$$sdK = colim_K N(K/\sigma)$$

Let $F : sdK \to Spaces$ be a diagram. According to subsection A.3:

$$\oint_{sdK} F = \oint_{colim_K N(K/\sigma)} F = colim_K \oint_{N(K/\sigma)} F\circ l_\sigma$$

A.5. Proposition. *The natural map:*

$$\oint_K \oint_{N(K/\sigma)} F\circ l_\sigma \to colim_K \oint_{N(K/\sigma)} F\circ l_\sigma = \oint_{sdK} F$$

is a weak equivalence.

A.6. Remark. Let $F : K \to Spaces$ be a diagram. This means that F is a functor over the category associated to K. According to example 3.14, it defines a bounded diagram $F_{sd} : sdK \to Spaces$. It turns out that $\oint_K F$ is weakly equivalent to $\oint_{sdK} F_{sd}$ and $\oint_{sdK} F_{sd} = hocolim_K F$, where $hocolim_K F$ is the homotopy colimit of $F : K \to Spaces$ in the sense of Bousfield-Kan.

A.7. Diagrams over $\oint_K G$. Let $G : K \to Spaces$ be a diagram. Out of G we can construct a new diagram $\Delta G : sdK \to Spaces$. Let:

$$u = (\tau_0 \overset{\psi_0}{\to} \tau_1 \overset{\psi_1}{\to} \dots \overset{\psi_{m-1}}{\to} \tau_m) \in (sdK)_m$$

$$v = (\sigma_0 \overset{\varphi_0}{\to} \sigma_1 \overset{\varphi_1}{\to} \dots \overset{\varphi_{n-1}}{\to} \sigma_n) \in (sdK)_n$$

$$\eta : u \to v \text{ be a morphism in } sdK$$

By definition 3.1, η is a morphism in Δ such that $\eta : [n] \to [m]$ and $sdK(\eta)(u) = v$. $\Delta G : sdK \to Spaces$ is a diagram defined as follows:

$$\Delta G(v) = \Delta[dim(\sigma_n)] \times G(\sigma_0)$$

$$\Delta G(\eta) = \begin{cases} id \times id & \text{if } \eta(0) = 0\ ,\ \eta(n) = m \\ id \times G(\psi_{\eta(0)-1} \circ \cdots \circ \psi_0) & \text{if } \eta(0) > 0\ ,\ \eta(n) = m \\ \Delta[\psi_{m-1} \circ \cdots \circ \psi_{\eta(n)}] \times id & \text{if } \eta(0) = 0\ ,\ \eta(n) < m \\ \Delta[\psi_{m-1} \circ \cdots \circ \psi_{\eta(n)}] \times G(\psi_{\eta(0)-1} \circ \cdots \circ \psi_0) & \text{if } \eta(0) > 0\ ,\ \eta(n) < m \end{cases}$$

Observe that ΔG has the values weakly equivalent to the values of G.

A.8. Proposition.

$$\oint_K G = colim_{sdK} \Delta G$$

A.9. Theorem. *Let $G : K \to Spaces$ and $F : \oint_K G \to Spaces$ be diagrams. If G is a bounded diagram, then the following natural map is a weak equivalence:*

$$\oint_{sdK} \oint_{\Delta G(v)} F \to colim_{sdK} \big(\oint_{\Delta G(v)} F \big) = \oint_{\oint_K G} F$$

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