ON THE FUNCTORS CW_A AND P_A

WOJCIECH CHACHÓLSKI

1. INTRODUCTION

Let A be a pointed and connected space. A pair of spaces (Y, X) is called a relative A–CW–complex if, roughly speaking, Y can be obtained from X by wedging with suspensions of A and attaching cones on suspensions of A (see [5, corollary 3.7]). If $A = S^1$, then a relative S^1–CW–complex is essentially an ordinary relative CW–complex. Any pointed map $f : X \to Y$ can be factored as a composition $(X \to Y' \xrightarrow{p} Y)$, where (Y', X) is a relative A–CW–complex and p induces a weak equivalence of mapping spaces $p_\star : map_\star(A, Y') \to map_\star(A, Y)$.

Let X be a pointed space. By factoring $\star \to X$, we get a map $CW_A X \to X$, where $(CW_A X, \star)$ is a relative A–CW–complex and the induced map $map_\star(A, CW_A X) \to map_\star(A, X)$ is a weak equivalence. The assignment $X \mapsto CW_A X$ can be made functorial, in such a way that the map $CW_A X \to X$ is natural.

By factoring $X \to \star$, we get a map $X \to P_A X$, where $(P_A X, X)$ is a relative A–CW–complex and the space $map_\star(A, P_A X)$ is weakly contractible. The assignment $X \mapsto P_A X$ can be made functorial, in such a way that the map $X \to P_A X$ is natural.

The functors CW_A and P_A are crucial in studying spaces through "the eyes" of A. The functor CW_A assigns to a space X the largest sub–object $CW_A X \to X$, which is totally "visible" by A. While the functor P_A associates with X the largest quotient $X \to P_A X$, which is totally "invisible" by A. The space $CW_A X$ contains all the information about X that can be detected by A, while $P_A X$ contains all the information about X, that can not be detected by A at all.

The purpose of this paper is to study the relationship between the functors CW_A and P_A. We study these functors by looking at their images and kernels. The image of CW_A (respectively of P_A) is the class of all spaces X, for which there exists Y, such that X is weakly equivalent to $CW_A Y$ (respectively X is weakly equivalent to $P_A Y$). The kernel of CW_A (respectively of P_A) is the class of all spaces X, for which $CW_A X$ is weakly contractible (respectively $P_A X$ is weakly contractible).

We investigate to what extent the following sequence is "exact":

$$\cdots \xrightarrow{P_A} cSpaces_\star \xrightarrow{CW_A} cSpaces_\star \xrightarrow{P_A} cSpaces_\star \xrightarrow{CW_A} cSpaces_\star \xrightarrow{P_A} \cdots$$

where $cSpaces_\star$ is the category of pointed and connected spaces.

As the first result, we prove that the image of P_A coincides with the kernel of CW_A. It can be described as the class of all spaces X, such that $map_\star(A, X)$ is weakly contractible (see theorem 15.2).

A more difficult and subtle problem is the correlation between the image of CW_A and the kernel of P_A.

We characterize the image of CW_A as the smallest closed class $C(A)$, which contains A (see example 4.6). That is the image of CW_A consists of all those spaces that can be built from A using certain simple operations. These operations are taking wedges, homotopy push–outs and homotopy sequential colimits (see theorem 8.2). This constructive characterization was originally proven by E. Dror Farjoun [5].

The crucial property of the kernel of P_A is that it is closed under extensions by fibrations. If $(Z \to E \to B)$ is a fibration, where Z and B belong to the kernel of P_A, then so does E. We characterize the kernel of P_A as the smallest closed class $\overline{C(A)}$, which contains A and is closed under extensions by fibrations (see example 4.9.) That is the kernel of P_A consists of all those spaces that can be built from A using certain simple operations. In addition to taking wedges, homotopy push–outs and homotopy sequential colimits, as it is in the case of $C(A)$, we add one extra operation, which is taking extensions by fibrations. As a result we get:

Theorem. *The kernel of P_A is the closure of the image of CW_A under taking extensions by fibrations.*

We actually prove more. We show that the homotopy fiber $Fib(X \to P_A X)$ belongs to the class $\overline{C(A)}$ (see theorem 17.1). We prove this by constructing the functor P_A in terms of CW_A. The process can be described by the following algorithm:

(1) take $X_0 = X$,
(2) take the homotopy cofiber: $X_1 = Cof(CW_A X_0 \to X_0)$,
(3) take the homotopy cofiber: $X_2 = Cof(CW_A X_1 \to X_1)$,
(4) continue the process (possibly a transfinite number of times),
(5) take the colimit: $colim(X_0 \to X_1 \to X_2 \to \cdots)$.

The space that we get is weakly equivalent to $P_A X$ (see the proof of theorem 17.1).

Finally we describe the functor CW_A in terms of P_A. This process can be expressed by the following algorithm:

(1) take the homotopy cofiber: $X' = Cof\Big(\bigvee_{h \in [A,X]} A \to X\Big)$,
(2) apply the functor $P_{\Sigma A}$ to X',
(3) take the homotopy fiber: $Fib(X \to X' \to P_{\Sigma A} X')$.

The space that we get is weakly equivalent to $CW_A X$ (see theorem 20.5). It follows that we can built the functor CW_A in such a way, so the map $CW_A X \to X$ is a principal fibration (see corollary 20.7).

We use this construction of CW_A, to show that the n–dimensional sphere S^n belongs to $C(\Omega S^{n+1})$ if and only if $n = 1, 3, 7$ (see corollary 20.10). The proof of this corollary was suggested by W. Dwyer.

We also show that for every n, S^n belongs to $\overline{C(\Omega S^{n+1})}$ (see corollary 20.13). This gives a non trivial example of a space, for which there is a proper inclusion $C(\Omega S^{n+1}) \subsetneq \overline{C(\Omega S^{n+1})}$.

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2. Organization of the paper

In section 3 we state the notation.

In section 4 we define and list crucial properties of closed classes.

Sections 5 to 7 contain detailed discussion regarding the CW_A construction in the category of simplicial sets. Most of the results in these sections are due to E. Dror Farjoun [5]. The approach presented in this paper is different from the one of E. Dror Farjoun. Instead of giving a constructive description, we start with pointing out the universal property of A–cellular spaces. We also prove that the functor CW_A can not be defined on the category of *unpointed* simplicial sets (see proposition 7.4).

In sections 8 to 10, we characterize the image and the kernel of CW_A. We also prove various strong cellular inequalities, a notion that was introduced by E. Dror Farjoun [6]. We write $X \gg A$, if and only if X belongs to the image of CW_A. Although most of these inequalities were already known (see [6]), the proofs presented in this paper are original and use only basic properties of closed classes. In particular, we show:

- $X \gg \Sigma A$ if and only if X is simply connected and $\Omega X \gg A$ (see theorem 10.8).
- Let $f : X \to Y$ be a map. If Y is a connected space, then $Fib(\Sigma f : \Sigma X \to \Sigma Y) \gg \Sigma Fib(f : X \to Y)$ (see theorem 10.9).

In section 11, as a corollary, we get the result of E. Dror Farjoun regarding CW_A and loop spaces: $CW_A \Omega X \simeq \Omega CW_{\Sigma A} X$ (see corollary 11.2 and [5, section 4.1]).

In sections 12 to 14 we define and list basic properties of the P_A functor. We follow the approach of A. K. Bousfield [1]. In section 13, we prove that A–cellular equivalences are preserved under unpoined and pointed homotopy colimits. Although to show this one could use the universal property of *hocolim* (this is A. K. Bousfield's argument), we are taking advantage of $\oint_K F$ construction.

In sections 15 to 18, we characterize the image and the kernel of P_A. We also prove various weak cellular inequalities, a notion that was introduced by E. Dror Farjoun [6]. We write $X > A$, if and only if X belongs to the kernel of P_A. Although most of the inequalities presented in this section were already known (see [6]), the proofs given in this paper are original and use only basic properties of closed classes and the characterization of the kernel of P_A. In particular, we show that $X > \Sigma A$ if and only if X is simply connected and $\Omega X > A$. As a simple corollary, we get the result of A. K. Bousfield and E. Dror Farjoun regarding P_A and loop spaces: $P_A \Omega X \simeq \Omega P_{\Sigma A} X$ (see corollary 19.2 and [1, theorem 3.1]).

Finally in section 20, we describe the functor CW_A in terms of P_A.

3. Notation

The symbol Δ denotes the simplicial category [9, §2], in which the objects are the ordered sets $[n] = \{0, 1, \ldots, n\}$, and morphisms are weakly monotone maps of sets. The morphisms of Δ are generated by coface maps $d_i : [n-1] \to [n]$ and codegeneracy maps $s_i : [n] \to [n+1]$ $(i = 0, 1, \ldots, n)$, subject to well–known cosimplicial identities [2]. A simplicial set is a functor $K : \Delta^{op} \to Sets$, where $Sets$ denotes the category of sets [9, §2]. The set $K([n])$ is usually denoted by K_n. A map between two simplicial sets is, by definition, a natural transformation of functors. The set of maps between K and L is denoted by $hom(K, L)$.

A simplicial set K can be interpreted as a collection of sets $(K_n)_{n\geq 0}$ together with face maps $d_i : K_n \to K_{n-1}$ and degeneracy maps $s_i : K_{n+1} \to K_n$ $(i = 1, 2, \ldots, n)$, which satisfy the duals of the cosimplicial identities [2, 8§2]. For description of how to do homotopy theory with simplicial sets see [2] and [9].

If $\sigma \in K_n$, then σ is called a n–dimensional simplex of K. The object $\Delta[n]$ is the standard n–simplex given by $\Delta[n]_k = mor_\Delta([k], [n])$ [2, 10§2]. There is a distinguished n–dimensional simplex $\tau \in \Delta[n]_n$, the one that comes from the identity map $[n] \to [n]$. It is easy to check that for any simplicial set K, the assignment $f \to f(\tau)$ gives a bijection between the set of maps $\Delta[n] \to K$ and K_n. If $\sigma \in K_n$ we will denote the corresponding map by $\sigma : \Delta[n] \to K$.

Let $\tau \in (\Delta[n])_n$ be the distinguished simplex. By $\dot{\Delta}[n]$ we denote the simplicial subset of $\Delta[n]$, which is generated by the set of simplices $\{d_i(\tau)\}_{0\leq i\leq n}$. By $\Delta[n, k]$ we denote the simplicial subset of $\Delta[n]$, which is generated by the set of simplices $\{d_i(\tau)\}_{(0\leq i\leq n\,,\, i\neq k)}$.

The function complex between K and L is a simplicial set $map(K, L)$, whose n–dimensional simplices are maps $\Delta[n] \times K \to L$. There is a natural inclusion $L \to map(K, L)$, such that a n–dimensional simplex $\sigma : \Delta[n] \to L$ is sent to the map $(\Delta[n] \times K \overset{pr_1}{\to} \Delta[n] \overset{\sigma}{\to} L)$. If we choose a simplex k of dimension 0 in K, then we can also define the basepoint evaluation map $map(K, L) \to L$. This map is defined by sending a map $f : \Delta[n] \times K \to L$ to a simplex, which is represented by the map: $(\Delta[n] = \Delta[n] \times \{k\} \hookrightarrow \Delta[n] \times K \overset{f}{\to} L)$.

A pointed simplicial set is a pair (K, k), where k is a chosen simplex of dimension zero in K. We will refer to this 0–dimensional simplex as the basepoint of K. A map between pointed simplicial sets is a map of simplicial sets which preserves the basepoints. The set of pointed maps between K and L is denoted by $hom_\star(K, L)$.

If X is a pointed simplicial set and Y is a simplicial set, then $Y \ltimes X$ is a pointed simplicial set defined as: $Y \ltimes X := \Big((Y \times X)/(Y \times \{\star\}), \langle Y \times \{\star\}\rangle\Big)$. By choosing a basepoint in Y, we get an inclusion $X \to Y \ltimes X$.

If X and Y are both pointed, then $Y \wedge X$ is a pointed simplicial set defined as: $Y \wedge X := (Y \ltimes X)/X$. We will regard $\dot{\Delta}[n+1]$ as a pointed simplicial set, where $\{0\}$ is the chosen basepoint. If X is pointed, then $\Sigma^n X := \dot{\Delta}[n+1] \wedge X$ and

$\widetilde{\Sigma}^n X := (\dot{\Delta}[n+1] \times X) \cup (\Delta[n+1] \times \{\star\}) \subset \Delta[n+1] \times X$.

There is an inclusion map $X = \{0\} \times X \hookrightarrow \widetilde{\Sigma}^n X$. It is not difficult to notice, that this map is weakly equivalent to the map $X \to \dot{\Delta}[n+1] \ltimes X$. In particular, we get that $\Sigma^n X$ is weakly equivalent to the homotopy cofiber of the map $X \to \widetilde{\Sigma}^n X$.

The pointed function complex between pointed simplicial sets K and L is a simplicial set $map_\star(K, L)$ whose n–dimensional simplices are pointed maps $\Delta[n] \ltimes K \to L$. If K and L are pointed simplicial sets and L is Kan [9, definition 1.3], then we denote the set of relative–basepoint homotopy classes of pointed maps between K and L by $[K, L]$. If A is a simplicial subset of K, then we denote the set of relative–A homotopy classes of pointed maps between K and L by $[K, L]_A$.

The homotopy cofiber of a map $A \to X$ is denoted by $Cof(A \to X)$. By $Cof(A \to X \to Y)$ we denote the homotopy cofiber of the composition $(A \to X \to Y)$.

Let us choose a basepoint in X. The homotopy fiber of a map $A \to X$ at the chosen basepoint is denoted by $Fib(A \to X)$. By $Fib(A \to X \to Y)$ we denote the homotopy fiber of the composition $(A \to X \to Y)$. If X is connected, then the homotopy type of $Fib(A \to X)$ does not depend on the choice of a basepoint in X. By ΩX we denote the homotopy fiber of the basepoint map $\star \to X$. We call ΩX the loop space of X.

We will use the following notation for some categories which frequently occur:

- *Spaces* denotes the category of simplicial sets.
- *cSpaces* denotes the category of connected simplicial sets.
- $Spaces_\star$ denotes the category of pointed simplicial sets.
- $cSpaces_\star$ denotes the category of pointed and connected simplicial sets.

In some proofs, we will use the notion of the homotopy colimits of diagrams over simplicial sets. By $\oint_K F$ and $\int_K F$ we denote respectively the *unpointed* and the *pointed* homotopy colimit of F. The references are [3] and [4].

4. Closed classes

In this section we state the definition and properties of closed classes. The references for the proofs are [3] and [5]. The notion of a closed class was introduced and studied by E. Dror Farjoun [5]. The following definition is slightly different from the one given in [5]. We think of a closed class as a class of *unpointed* and connected simplicial sets.

Definition 4.1. *A non empty class C of connected simplicial sets is closed if the empty simplicial set does not belong to C and C is closed under weak equivalences and taking* **pointed** *homotopy colimits. If $F : K \to Spaces_\star$ is a pointed diagram such that for every simplex $\sigma \in K$, $F(\sigma) \in C$, then $\int_K F \in C$.*

Notation. Let C be a closed class.

- By $F : K \to C$ we denote an unpointed diagram such that for every simplex $\sigma \in K$, $F(\sigma) \in C$.
- Let $f : X \to Y$ be a map. We say that the homotopy fiber of f belongs to C, if the homotopy fibers of f over every component of Y belongs to C. In particular, the homotopy fibers of f over various components of Y are connected and f induces an isomorphism on π_0. If the homotopy fiber of f belongs to C, we will write: $Fib(f : X \to Y) \in C$.

Proposition 4.2. *A non empty class C of connected simplicial sets, which does not contain the empty simplicial set, is closed if and only if the following properties are satisfied:*

- *Let X and Y be weakly equivalent. If $X \in C$, then $Y \in C$.*
- *Let $(X_i)_{i\in I}$ be a family of simplicial sets. If $X_i \in C$, then for any choice of base-points in X_i, $\bigvee_{i\in I} X_i \in C$. We say that C is closed under taking arbitrary wedges.*
- *Let $X_1 \leftarrow X_2 \to X_3$ be a diagram. If X_i belongs to C, then so does the homotopy push–out: $hocolim(X_1 \leftarrow X_2 \to X_3)$. We say that C is closed under taking homotopy push–outs.*
- *Let $(X_0 \to X_1 \to X_2 \to \cdots)$ be a diagram. If X_i belongs to C, then so does the telescope: $hocolim(X_0 \to X_1 \to X_2 \to \cdots)$. We say that C is closed under taking homotopy sequential colimits or telescopes.*

The following proposition gives some basic examples of elements of closed classes.

Proposition 4.3. *Let C be a closed class.*

- *If X is weakly contractible, then $X \in C$.*
- *Let K and A be simplicial sets. If $A \in C$, then for any choice of a basepoint in A, $K \ltimes A \in C$. In particular, $\tilde{\Sigma}^n A \in C$ (see section 3).*
- *If $A \in C$, then $\Sigma^n A \in C$.*
- *If $X \in C$ and Y is a retract of X, then $Y \in C$.*

The following theorem is the main tool, we are going to use, to study closed classes. It is a generalization of E. Dror Farjoun's theorem [3, theorem 8.1].

Theorem 4.4. *Let $E : K \to Spaces$, $B : K \to Spaces$ be diagrams and $\Psi : E \to B$ be a natural transformation. If for $\sigma \in K$, $Fib\Big(\Psi_\sigma : E(\sigma) \to B(\sigma)\Big) \in C$, then $Fib\Big(\Psi : \oint_K E \to \oint_K B\Big) \in C$.*

Particular cases of this theorem are listed in the following proposition:

Proposition 4.5.

- *Let $E : K \to Spaces$, $B : K \to Spaces$ be pointed diagrams and $\Psi : E \to B$ be a natural transformation. If for every simplex $\sigma \in K$, $Fib\big(\Psi_\sigma : E(\sigma) \to B(\sigma)\big)$ belongs to C, then so does $Fib\big(\Psi : \int_K E \to \int_K B\big)$.*
- *Let $F : K \to Spaces$ be a diagram. If K is contractible and for every simplex $\sigma \in K$, $F(\sigma) \in C$, then $\oint_K F \in C$.*
- *Let the following be a homotopy push–out square:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

  If the homotopy fiber of f belongs to C, then so does the homotopy fiber of g.
- *Let $f : A \to X$ be a map. If $A \in C$, then $Fib\big(X \to Cof(f : A \to X)\big) \in C$.*

Example 4.6. Let A be a connected simplicial set. By $C(A)$ we denote the smallest closed class such that $A \in C(A)$. One can think of $C(A)$ as the class of simplicial sets that are built from A using certain simple operations: taking arbitrary wedges, homotopy push–outs and homotopy sequential colimits. It turns out that there is a universal property that classifies the elements of this class (see section 10).

If $A = M(\mathbb{Z}/p^k, n)$ is $\mathbb{Z}/p^k$–Moore space, then $X \in C(A)$ if X is $(n-1)$–connected, $\pi_n X$ is generated by by elements of order p^k and for $i > n$, $\pi_i X$ is a p–group.

The following theorem lists some properties of closed classes with respect to fibrations.

Theorem 4.7. *Let $(Z \to E \to B)$ be a fibration sequence over a connected simplicial set B.*

- *If $Z \in C$, $B \in C$ and this fibration has a section, then $E \in C$.*
- *If $X \in C$ and $Y \in C$, then $X \times Y \in C$.*
- *If $\Omega B \in C$ and $Z \in C$, then $E \in C$.*

Definition 4.8. *A closed class is closed under extensions by fibrations if for every fibration sequence $(Z \to E \to B)$, where Z and B belong to C, E belongs to C.*

Example 4.9. Let A be a connected simplicial set. By $\overline{C(A)}$ we denote the smallest closed class which contains A and is closed under extensions by fibrations. As in the case of the class $C(A)$ (see example 4.6), one can think of $\overline{C(A)}$ as the class of simplicial sets that are built from A using certain operations. In addition to taking arbitrary wedges, homotopy push–outs and homotopy sequential colimits, as it is in the case of $C(A)$, we add one extra operation which is taking extensions by fibrations.

It turns out that there is a universal property which characterizes the elements of the class $\overline{C(A)}$ (see section 18).

If $A = M(\mathbb{Z}/p, n)$ is $\mathbb{Z}/p$–Moore space, then $X \in \overline{C(A)}$ if X is $(n-1)$–connected and for $i \geq n$, $\pi_i X$ is a p–group. As a consequence we get that $C(A)$ is not closed under extensions by fibration (see example 4.6). Another example of a spaces A for which $C(A)$ is not closed under extensions by fibrations is given in corollary 20.13.

To measure to what extent a closed class C is closed under extensions by fibrations, a new class $D(C)$ is introduced [3, section 6].

Definition 4.10. *Let C be a closed class.*

$$D(C) := \{K \in cSpaces \mid \text{if } F : K \to C \text{ is a diagram, then } \oint_K F \in C\}$$

In case of $C(A)$, $D(C(A))$ will be denoted simply by $D(A)$.

Theorem 4.11.

- *$D(C)$ is a closed class and it is closed under extensions by fibrations.*
- *$D(C) = \{B \in cSpaces \mid$ if $Z \to E \to B$ is a is a fibration sequence for which $Z \in C$, then $E \in C\}$*
- *$D(C) \subset C$.*
- *C is closed under extensions by fibrations if and only if $D(C) = C$.*
- *Let B be a connected simplicial set. If $\Omega B \in C$, then $B \in D(C)$ (see theorem 4.7).*

Throughout sections 5 to 9, A is assumed to be a *pointed* and connected simplicial set. From section 10 on the assumption that A is pointed will be dropped.

5. A–cellular simplicial sets

Let A be a pointed and connected simplicial set. In this section we will define the class of A–cellular simplicial sets, a notion introduced by E. Dror Farjoun [5]. The approach presented in this paper is different from E. Dror Farjoun's one [5, example 2.2]. We define an A–cellular simplicial set by giving its universal property first.

Definition 5.1. *Let X be a connected simplicial set. A simplicial set X is called A–cellular if for any choice of a basepoint $x \in X$ and for any map of pointed Kan simplicial sets $f : Y \to Z$, for which $f_\star : map_\star(A, Y) \to map_\star(A, Z)$ is a weak equivalence, the map $f_\star : map_\star\big((X, x), Y\big) \to map_\star\big((X, x), Z\big)$ is also a weak equivalence.*

Remarks.

- Although A is pointed, the class of A–cellular simplicial sets consists of connected but *not pointed* simplicial sets.

- Let X be a connected simplicial set. Notice that if for some vertex $x_0 \in X$, a map $f : Y \to Z$ between pointed and connected Kan simplicial sets induces a weak equivalence: $f_\star : map_\star\Big((X, x_0), Y\Big) \to map_\star\Big((X, x_0), Z\Big)$, then for any choice of a basepoint $x \in X$, $f_\star : map_\star\Big((X, x), Y\Big) \to map_\star\Big((X, x), Z\Big)$ is a weak equivalence.

As an easy consequence of definition 5.1, we get:

Proposition 5.2.

(1) *A is A–cellular.*
(2) *Let X and Y be weakly equivalent. If X is A–cellular, then so is Y.*
(3) *Let $(X_i)_{i\in I}$ be a family of simplicial sets. If X_i are A–cellular, then for any choice of basepoints in X_i, $\bigvee_I X_i$ is A–cellular.*
(4) *Let $X_1 \leftarrow X_2 \to X_3$ be a diagram. If X_i are A–cellular, then so is the homotopy push–out: $hocolim(X_1 \leftarrow X_2 \to X_3)$.*
(5) *Let $X_0 \to X_1 \to X_2 \to \cdots$ be a diagram. If X_i are A–cellular, then so is the telescope: $hocolim(X_0 \to X_1 \to X_2 \to \cdots)$.*

The properties (2) to (5) of proposition 5.2 say that the class of A–cellular simplicial sets is closed (see proposition 4.2). The property (1) implies that the smallest closed class $C(A)$, which contains A (see example 4.6), is included in the class of A–cellular simplicial sets. We will see that in fact these properties determine the class of A–cellular simplicial sets. Thus the class of A–cellular simplicial sets is equal to $C(A)$ (see theorem 8.2). Since to define $C(A)$ we do not need A to be pointed, it suggests that to define an A–cellular simplicial set we do not have to choose a basepoint in A. We will see that in order to perform some constructions, we need A to be pointed.

Example 5.3. Let $A = \dot{\Delta}[n+1]$. In this case the class of A–cellular simplicial sets is equal to the class of $(n-1)$–connected simplicial sets.

The following two weakly equivalent simplicial sets belong to $C(A)$, therefore they must be A–cellular (see proposition 4.3):

$$\dot{\Delta}[n] \ltimes A = (\dot{\Delta}[n] \times A)\Big/(\dot{\Delta}[n] \times \{\star\})$$

$$\widetilde{\Sigma}^{n-1} A = (\dot{\Delta}[n] \times A) \cup (\Delta[n] \times \{\star\}) \subset \Delta[n] \times A$$

These simplicial sets will play an important role in the following sections.

6. A–cellular equivalences

Definition 6.1 (E. Dror Farjoun [5]**).** *A map $f : X \to Y$ of pointed Kan simplicial sets is called an A–cellular equivalence if $f_\star : map_\star(A, X) \to map_\star(A, Y)$ is a weak equivalence.*

Remark. An A–cellular equivalence is assumed to be a map of *pointed Kan* simplicial sets.

As an immediate consequence of the definition we get:

Proposition 6.2.

- *Let $f : X \to Y$ be a map of pointed Kan simplicial sets and B be A–cellular. If f is an A–cellular equivalence, then f is a B–cellular equivalence.*
- *Let $f : X \to Y$ and $g : Y \to Z$ be maps of pointed Kan simplicial sets. If two out of $(f,\, g,\, g{\circ}f)$ are A–cellular equivalences, then so is the third.*
- *If $f : X \to Y$ and $f' : X' \to Y'$ are A–cellular equivalences, then so is the product $f \times f' : X \times X' \to Y \times Y'$.*

Proposition 6.3 (E. Dror Farjoun [5]). *Let $f : X \to Y$ be a map of pointed and connected A–cellular Kan simplicial sets. Then f is an A–cellular equivalence if and only if f is a weak equivalence.*

Proof. If f is a weak equivalence, then of course it is an A–cellular equivalence.

Let f be an A–cellular equivalence. Since Y is A–cellular, f induces a bijection $f_\star : [Y, X] \to [Y, Y]$. It implies that there is a map $g : Y \to X$, such that $f{\circ}g$ is homotopic to id_Y. As f and $f{\circ}g$ are A–cellular equivalences, then so is g. Thus $g : Y \to X$ induces a bijection $g_\star : [X, Y] \to [X, X]$. It implies that there is a map $h : X \to Y$, such that $g{\circ}h$ is homotopic to id_X. It follows that h is homotopic to f and g is the homotopy inverse of f. □

Corollary 6.4. *Let X be a pointed and connected A–cellular Kan simplicial set. If $map_\star(A, X)$ is weakly contractible, then so is X.*

Let $f : X \to Y$ be a map between pointed Kan simplicial sets. By definition this map induces a weak equivalence $f_\star : map_\star(A, X) \to map_\star(A, Y)$, if for every pointed map $h : A \to X$ and for every $n \geq 0$, $f_\star : \pi_n\Big(map_\star(A, X), h\Big) \to \pi_n\Big(map_\star(A, Y), f{\circ}h\Big)$ is an isomorphism. The set $\pi_n(map_\star(A, X), h)$ consists of relative$-A$ homotopy classes of those maps $g : \dot{\Delta}[n+1] \ltimes A \to X$, such that $g|_A = h$. Since the pair $(\dot{\Delta}[n+1] \ltimes A, A)$ is weakly equivalent to $(\widetilde{\Sigma}^n A, A)$, we can think of $\pi_n(map_\star(A, X), h)$ as the set of relative$-A$ homotopy classes of maps $g : \widetilde{\Sigma}^n A \to X$, such that $g|_A = h$. It is not difficult to see that for $n > 0$, $g : \widetilde{\Sigma}^n A \to X$ defines a zero element in the homotopy group $\pi_n(map_\star(A, X), h)$, if and only if it can be extended along the map $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n{+}1] \times A$. It is also clear that pointed maps $f : A \to X$ and $g : A \to X$ are homotopic relative the basepoint if and only if the composition $(\widetilde{\Sigma}^0 A \xrightarrow{\simeq} A \vee A \xrightarrow{f \vee g} X)$ can be extended along $\widetilde{\Sigma}^0 A \hookrightarrow \Delta[1] \times A$. We have showed:

Corollary 6.5. *Let $f : X \to Y$ be a map between pointed Kan simplicial sets.*

- *The map f is an A–cellular equivalence if and only if for every $n \geq 0$, it induces an isomorphism: $f_\star : [\dot{\Delta}[n] \ltimes A, X]_A \to [\dot{\Delta}[n] \ltimes A, Y]_A$, or equivalently, if and only if it induces an isomorphism: $f_\star : [\widetilde{\Sigma}^n A, X]_A \to [\widetilde{\Sigma}^n A, Y]_A$, where $[\dot{\Delta}[n] \ltimes A, X]_A$ and $[\widetilde{\Sigma}^n A, Y]_A$ are sets of relative–A homotopy classes of pointed maps.*
- *Assume: f has the property that if for a pointed map $g : \widetilde{\Sigma}^n A \to X$, the composition $f \circ g : \widetilde{\Sigma}^n A \to Y$ can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$, then g itself can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$. Under this assumption, $f : X \to Y$ induces a* **monomorphism** *$f_\star : [\widetilde{\Sigma}^n A, X]_A \to [\widetilde{\Sigma}^n A, Y]_A$.*

7. The functor CW_A

The purpose of this section is to construct a functor $CW_A : Spaces_\star \to Spaces_\star$ and a natural transformation $cw_A X : CW_A X \to X$, such that $CW_A X$ is A–cellular and if X is a pointed and connected *Kan* simplicial set, then $cw_A X : CW_A X \to X$ is an A–cellular equivalence.

Construction. The construction of CW_A and cw_A, presented in this paper, is modeled very closely on a construction of E. Dror Farjoun [5, section 3.4].

Let λ be a limit ordinal whose cofinality [7] is bigger then the cardinality of the set of simplices of A. By λ we also denote the category whose objects are all ordinal numbers smaller than λ and for any two ordinal numbers $j \leq i$, there is only one morphism $j \to i$.

Let X be a pointed simplicial set. We are going to construct by induction a functor $F(X) : \lambda \to cSpaces_\star$ and a pointed map $p : F(X) \to X$, such that:

(1) If $i = j+1$, then $F(X)_j \to F(X)_i$ is a cofibration.
(2) For every ordinal $i < \lambda$, $F(X)_i$ belongs to $C(A)$, so it is A–cellular.
(3) For every $n \geq 0$, the map $p_0 : F(X)_0 \to X$ induces an epimorphism:
$$(p_0)_\star : hom_\star\Big(\widetilde{\Sigma}^n A, F(X)_0\Big) \to hom_\star(\widetilde{\Sigma}^n A, X)$$
where $hom_\star(Z, Y)$ is the set of pointed maps between Z and Y.
(4) If $i = j+1$ and $g : \widetilde{\Sigma}^n A \to F(X)_j$ is a pointed map, for which $p_j \circ g : \widetilde{\Sigma}^n A \to X$ can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$, then the composition $\Big(\widetilde{\Sigma}^n A \xrightarrow{g} F(X)_j \to F(X)_i\Big)$ can also be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$.

Step 0. If $i = 0$, then:

- $F(X)_0$ is defined as: $F(X)_0 = \displaystyle\bigvee_{\substack{n \geq 0 \\ h \in hom_\star(\widetilde{\Sigma}^n A, X)}} \widetilde{\Sigma}^n A$,
- $p_0 : F(X)_0 \to X$ is defined as: $p_0 = \displaystyle\bigvee_{\substack{n \geq 0 \\ h \in hom_\star(\widetilde{\Sigma}^n A, X)}} h$.

It is obvious from the definition that the conditions (1) through (4) are satisfied.

Let us assume that the construction has been carried out for all ordinal numbers smaller than i.

Step i. If i is not of the form $j+1$, then:

- $F(X)_i$ is defined as: $F(X)_i \ = colim_{j<i} F(X)_j$,
- $p_i : F(X)_i \to X$ is defined as: $p_i = colim_{j<i} p_j$.

Conditions (1) and (4) hold automatically. Cofibration assumption (1) implies that the natural map $hocolim_{j<i} F(X)_j \to colim_{j<i} F(X)_j$ is a weak equivalence. Since $hocolim_{j<i} F(X)_j$ belongs to $C(A)$, then so does $F(X)_i = colim_{j<i} F(X)_j$ and condition (2) holds.

If $i = j+1$, then let J be the set of all commutative diagrams of the form:

$$\begin{array}{ccc} \widetilde{\Sigma}^n A & \longrightarrow & F(X)_j \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p_i} \\ \Delta[n] \times A & \longrightarrow & X \end{array}$$

where all the maps are pointed and $\widetilde{\Sigma}^n A \to \Delta[n] \times A$ is the canonical inclusion.

- $\widehat{F}(X)_i$ is defined as: $\widehat{F}(X)_i := colim\Big(\bigvee_J \Delta[n] \times A \hookleftarrow \bigvee_J \widetilde{\Sigma}^n A \to F(X)_j \Big)$,
- $\widehat{p}_i : \widehat{F}(X)_i \to X$ is defined to be the push–out of the following maps:

$$p_j : F(X)_j \to X \quad , \quad \vee_J \widetilde{\Sigma}^n A \to X \quad , \quad \vee_J \Delta[n] \times A \to X$$

Let I be the set of all commutative diagrams of the form:

$$\begin{array}{ccc} \Delta[n,k] & \longrightarrow & \widehat{F}(X)_i \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \widehat{p}_i} \\ \Delta[n] & \longrightarrow & X \end{array}$$

where $\Delta[n,k] \to \Delta[n]$ is the natural inclusion.

- $F(X)_i$ is defined as: $F(X)_i := colim\Big(\coprod_I \Delta[n] \hookleftarrow \coprod_I \Delta[n,k] \to \widehat{F}(X)_i \Big)$,
- $p_i : F(X)_i \to X$ is defined to be the push–out of the following maps:

$$\widehat{p}_i : \widehat{F}_i \to X \quad , \quad \textstyle\coprod_J \Delta[n,k] \to X \quad , \quad \coprod_J \Delta[n] \to X$$

- $F(X)_j \to F(X)_i$ is defined as the composition: $\Big(F(X)_j \to \widehat{F}(X)_i \to F(X)_i \Big)$.

It is clear from the construction, that the map $F(X)_j \to F(X)_i$ is a cofibration, so the condition (1) holds.

By the inductive assumption: $F(X)_j \in C(A)$. By construction $\widehat{F}(X)_i$ is a homotopy push–out of elements of $C(A)$, so it belongs to $C(A)$. Since the map $\widehat{F}(X)_i \to F(X)_i$ is a weak equivalence, we get: $F(X)_i \in C(A)$, thus the condition (2) holds.

Let $g : \widetilde{\Sigma}^n A \to F(X)_j$ be a map such that $p_j \circ g : \widetilde{\Sigma}^n A \to X$ can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$. By construction the map $\widetilde{\Sigma}^n A \xrightarrow{g} F(X)_j \to \widehat{F}(X)_i$ can also be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$. It implies that so does the composition: $(\widetilde{\Sigma}^n A \xrightarrow{g} F(X)_j \to \widehat{F}(X)_i \to F(X)_i)$. Thus the condition (4) is satisfied.

The functor CW_A. It is not difficult to notice that $F(X)$ and $p : F(X) \to X$ are natural constructions. They define a functor and a natural transformation:

Definition 7.1.

- *$CW_A : cSpaces_\star \to cSpaces_\star$ is a functor defined as follows:*

$$CW_A(X) = colim_\lambda F(X)$$

- *$cw_A X : CW_A(X) \to X$ is a natural transformation defined as the following map:*

$$cw_A X = colim_\lambda(p)$$

Proposition 7.2. *The simplicial set $CW_A X$ belongs to $C(A)$, so it is A–cellular.*

Proof. Since for every i, $F(X)_i \to F(X)_{i+1}$ is a cofibration, there is a weak equivalence $hocolim_\lambda F(X) \to colim_\lambda F(X)$. By the condition (2) of the construction, for every i, $F(X)_i$ belongs to $C(A)$ and so $CW_A X = colim_\lambda F(X)$ belongs to $C(A)$. □

Proposition 7.3. *Let X be a pointed and connected simplicial set.*

- *$cw_A X : CW_A X \to X$ is a fibration.*
- *If X is Kan, then the map $cw_A X : CW_A X \to X$ is an A–cellular equivalence.*

Proof. The first part of the proposition follows immediately from Quillen's small object argument [10].

If X is Kan, then since $cw_A X : CW_A X \to X$ is a fibration, $CW_A X$ is also Kan. According to corollary 6.5, in order to prove that the map $cw_A X$ is an A–cellular equivalence, we have to show that $(cw_A X)_\star : [\widetilde{\Sigma}^n A, CW_A X]_A \to [\widetilde{\Sigma}^n A, X]_A$ is a bijection. By the condition (3) of the construction, the following composition is an epimorphism: $hom_\star(\widetilde{\Sigma}^n A, F(X)_0) \to hom_\star(\widetilde{\Sigma}^n A, CW_A X) \xrightarrow{(cw_A X)_\star} \mathrm{hom}_\star(\widetilde{\Sigma}^n A, X)$. This implies, that so is $(cw_A X)_\star : hom_\star(\widetilde{\Sigma}^n A, CW_A X) \to \mathrm{hom}_\star(\widetilde{\Sigma}^n A, X)$. Because X and $CW_A X$ are Kan, $(cw_A X)_\star : [\widetilde{\Sigma}^n A, CW_A X]_A \to [\widetilde{\Sigma}^n A, X]_A$ is an epimorphism as well.

We are going to use corollary 6.5 to show that this map is also a monomorphism. Let $g : \widetilde{\Sigma}^n A \to CW_A X$ be a map for which the composition $(\widetilde{\Sigma}^n A \xrightarrow{g} CW_A X \xrightarrow{cw_A X} X)$ can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$. We have to show that g itself can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$. Since λ is a limit ordinal whose cofinality is bigger than the cardinality of the set of simplices of A, it is bigger that the cardinality of the set of simplices of $\widetilde{\Sigma}^n A$. By Quillen's small object argument [10], the map $g : \widetilde{\Sigma}^n A \to CW_A X$ factors through $g' : \widetilde{\Sigma}^n A \to F(X)_i$ for some ordinal number $i < \lambda$. By assumption $p_i \circ g' : \widetilde{\Sigma}^n A \to X$ can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$. It follows from the condition (4) of the construction, that the composition $(\widetilde{\Sigma}^n A \xrightarrow{g'} F(X)_i \to F(X)_{i+1})$ can also be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$ and so can g. According to corollary 6.5, $cw_A X$ induces a monomorphism $(cw_A X)_\star : [\widetilde{\Sigma}^n A, CW_A X]_A \to [\widetilde{\Sigma}^n A, X]_A$. □

Remark. Although the construction of $F(X)$ can be performed for every pointed and connected X, the map $cw_A X$ is not always an A–cellular equivalence. We proved that it is an A–cellular equivalence when X is Kan.

Proposition 7.4. *Over the category of* **unpointed** *and connected Kan simplicial sets, it is not possible to define a functor $CW : (Kan\text{–}cSpaces) \to (Kan\text{–}cSpaces)$ and a natural transformation $cw(X) : CW(X) \to X$, such that $CW(X)$ is A–cellular and for any choice of a basepoint in $CW(X)$, $cw(X) : CW(X) \to X$ is an A–cellular equivalence.*

Proof. Let us assume that such a functor CW exists. Let X be a connected simplicial set. Let us choose a basepoint in X. Consider the fibration $cw_A X : CW_A X \to X$. Let $Z = Fib(cw_A X)$. Assume that Z is connected. Since $cw_A X$ is an A–cellular equivalence, $map_\star(A, Z)$ is weakly contractible and so is $CW(Z)$ (see corollary 6.4).

The map $cw_A X$ is weakly equivalent to a map of the form $\oint_K F \to K$ for some diagram $F : K \to Spaces$, where for every simplex $\sigma \in K$, $F(\sigma)$ is weakly equivalent to Z [3, example 3.12]. Since $CW(Z)$ is weakly contractible, then so is $CW\big(F(\sigma)\big)$. Thus the map $\oint_K CW(F) \to K$ is a weak equivalence. Looking at the following commutative diagram:

$$\begin{array}{ccccc} \oint_K CW(F) & \xrightarrow{cw(F)} & \oint_K F & \xrightarrow{\simeq} & CW_A X \\ \downarrow \simeq & & \downarrow & & \downarrow \\ K & \xrightarrow{id} & K & \xrightarrow{\simeq} & X \end{array}$$

we get that the map $cw_A X : CW_A X \to X$ has a section.

In case of $X = K(G,2)$ and A is 2–connected, since $map_\star\big(A, K(G,2)\big)$ is weakly contractible, then so is $CW_A K(G,2)$. As a consequence $CW_A K(G,2) \to K(G,2)$ can not have a section. This is a contradiction with the previous statement. □

Proposition 7.5 (E. Dror Farjoun [5]**).** *Let $f : Y \to X$ be an A–cellular equivalence. Up to homotopy, there exists a unique map $g : CW_A X \to Y$, such that $f \circ g$ is homotopic to $cw_A X : CW_A X \to X$.*

Proof. Since $CW_A X$ is A–cellular and f is an A–cellular equivalence, f induces a bijection $f_\star : [CW_A X, Y] \to [CW_A X, X]$. It follows, that up to homotopy, there is a unique map $g : CW_A X \to Y$, such that $f \circ g$ is homotopic to $cw_A X : CW_A X \to X$. □

Proposition 7.6 (E. Dror Farjoun [5]). *Let $f : Y \to X$ be a pointed map between an A–cellular simplicial set Y and a connected Kan simplicial sets X. Up to homotopy, there exists a unique map $g : Y \to CW_A X$, such that the composition $(Y \xrightarrow{g} CW_A X \xrightarrow{cw_A X} X)$ is homotopic to $f : Y \to X$.*

Proof. Since Y is A–cellular and $cw_A X : CW_A X \to X$ is an A–cellular equivalence, there is a bijection $(cw_A X)_\star : [Y, CW_A X] \to [Y, X]$. So up to homotopy, there is a unique map $g : Y \to CW_A X$, such that the composition $(Y \xrightarrow{g} CW_A X \xrightarrow{cw_A X} X)$ is homotopic to $f : Y \to X$. □

Remark. Notice that since $cw_A X : CW_A X \to X$ is a fibration, we can choose g in proposition 7.6, so that: $g \circ (cw_A X) = f$.

Proposition 7.7 (E. Dror Farjoun [5]). *A map $f : X \to Y$ between pointed Kan simplicial sets is an A–cellular equivalence if and only if $CW_A f : CW_A X \to CW_A Y$ is a weak equivalence.*

Proof. Since cw_A is a natural transformation, we have $cw_A Y \circ CW_A f = f \circ cw_A X$. Since $cw_A X : CW_A X \to X$ and $cw_A Y : CW_A Y \to Y$ are A–cellular equivalences, f is an A–cellular equivalence if and only if $CW_A f$ is. According to proposition 6.3, $CW_A f$ is an A–cellular equivalence if and only if it is a weak equivalence. □

Corollary 7.8. *Let X be a pointed and connected Kan simplicial set.*

- *Let $f : Y \to X$ be an A–cellular equivalence. If Y is A–cellular, then $f : Y \to X$ is weakly equivalent to $cw_A X : CW_A X \to X$.*
- *X is A–cellular if and only if for any choice of a basepoint in X, the natural map $cw_A X : CW_A X \to X$ is a weak equivalence.*
- *If $f : X \to Y$ is a weak equivalence between pointed and connected Kan simplicial sets, then so is $CW_A f : CW_A X \to CW_A Y$. In particular, if X is a weakly contractible, then so is $CW_A X$.*
- *$CW_A X$ is weakly contractible if and only if $X \to \star$ is an A–cellular equivalence.*

8. The image CW_A

If we forget about the basepoint of $CW_A X$, we can regard CW_A as a functor with values in the category of unpointed simplicial sets.

Definition 8.1. *The image of $CW_A : cSpaces_\star \to Spaces$ is the class of all those simplicial sets X, for which there exists a pointed and connected simplicial set Y, such that X is weakly equivalent to $CW_A Y$.*

Remark. Notice that the image of CW_A consists of *unpointed* and connected simplicial sets.

Theorem 8.2. *The following classes of connected simplicial sets are equal:*

- *The class of A–cellular simplicial sets.*
- *The image of CW_A.*
- *The smallest closed class $C(A)$, which contains A (see example 4.6).*

Proof. Let X be A–cellular. Let Y be a pointed Kan simplicial set, which is weakly equivalent to X, so it is also A–cellular. Since the map $cw_AY : CW_AY \to Y$ is an A–cellular equivalence between A–cellular simplicial sets, it has to be a weak equivalence (see proposition 6.3). It implies that X is in the image of CW_A and the class of A–cellular simplicial sets is included in the image of CW_A.

Let X be in the image of CW_A. Thus there exists a pointed and connected simplicial set Y, such that X is weakly equivalent to CW_AY. According to the construction, CW_AY belongs to $C(A)$ (see proposition 7.2). This proves that the image of CW_A is contained in $C(A)$.

Since the class of A–cellular simplicial sets is a closed class and A is A–cellular, the class $C(A)$ is included in the class of A–cellular simplicial sets. □

9. The kernel of CW_A

Definition 9.1. *The kernel of CW_A is the class of all those simplicial sets X, such that there exists a connected Kan simplicial set Y, which is weakly equivalent to X and for any choice of basepoint in Y, CW_AY is weakly contractible.*

Remarks.

- Notice that the kernel of CW_A consists of *unpointed* and connected simplicial sets.
- Let X be a connected Kan simplicial set. If for some vertex $x_0 \in X$, $CW_A(X, x_0)$ is weakly contractible, then for any vertex $x \in X$, $CW_A(X, x)$ is weakly contractible.

Proposition 9.2. *A connected simplicial set X belongs to the kernel of CW_A if and only if there exists a Kan simplicial set Y, which is weakly equivalent to X and for any choice of a basepoint in Y, $map_\star(A, Y) \simeq \star$.*

Proof. Let X be in the kernel of CW_A. Let Y be a Kan simplicial set, which is weakly equivalent to X and $CW_AY \simeq \star$. According to corollary 7.8, for any choice of a basepoint in Y, $Y \to \star$ is an A–cellular equivalence. Definition 6.1 implies that $map_\star(A, Y) \simeq \star$. □

From the following section on we will drop the assumption that A is a pointed simplicial set. We will still assume that A is connected.

10. Strong cellular inequalities

Let A be a connected simplicial set.

Notation (E. Dror Farjoun [6]). Let X be a simplicial set. If $X \in C(A)$, then we write $X \gg A$ and we say that X is built by A or A builds X. It is an immediate consequence of the definition that if $X \gg B$ and $B \gg A$, then $X \gg A$.

According to the introduced definitions, notation and to proven theorems and propositions, the following statements are equivalent:

- X belongs to $C(A)$.
- X is built by A.
- $X \gg A$
- For any choice of a basepoint in A, X is A–cellular.
- For any choice of a basepoint in A, X is in the image of CW_A.
- For any choice of basepoints in A and X, if $f : Y \to Z$ is a map between pointed Kan simplicial sets, for which $f_\star : map_\star(A, Y) \to map_\star(A, Z)$ is a weak equivalence, then $f_\star : map_\star(X, Y) \to map_\star(X, Z)$ is also a weak equivalence. We will refer to this property of A–cellular simplicial sets as their universal property.

Proposition 10.1. *Let W be a connected simplicial set. The following is a closed class:*
$$C = \{X \in cSpaces \mid \Sigma X \gg W\}$$

Proof. Let $F : K \to Spaces_\star$ be a pointed diagram such that for every simplex $\sigma \in K$, $F(\sigma) \in C$. Notice:

$$\Sigma\int_K F \simeq hocolim(\int_K \star \leftarrow \int_K F \to \int_K \star) \simeq\simeq \int_K hocolim(\star \leftarrow F \to \star) \simeq \int_K \Sigma F$$

where $\int_K F$ denotes the pointed homotopy colimit of F [3, definition 3.10]. Since for every simplex $\sigma \in K$, $\Sigma F(\sigma) \gg W$, we get: $\Sigma\int_K F \simeq \int_K \Sigma F \gg W$. It shows that $\int_K F \in C$ and C is a closed class. □

Corollary 10.2. *If $X \gg A$, then $\Sigma X \gg \Sigma A$.*

Proof. According to proposition 10.1, $C = \{X \in cSpaces \mid \Sigma X \gg \Sigma A\}$ is a closed class. Since obviously A belongs to C, we get: $C(A) \subset C$. This proves that if $X \gg A$, then $\Sigma X \gg \Sigma A$. □

Proposition 10.3. *Let W be a connected simplicial set. The following is a closed class:*
$$C = \{X \mid X \text{ is simply connected and } \Omega X \gg W\}$$

Proof. Let $F : K \to Spaces_\star$ be a pointed diagram such that for every simplex $\sigma \in K$, $F(\sigma) \in C$. Since F is a pointed diagram, there is a natural transformation $\star \to F$, where $\star : K \to Spaces_\star$ is the constant diagram whose value is the trivial simplicial set. By assumption for every simplex $\sigma \in K$, $Fib\big(\star \to F(\sigma)\big) \simeq \Omega F(\sigma)$ belongs to the closed class $C(W)$. According to proposition 4.5, $Fib\Big(\int_K \star \to \int_K F\Big) = \Omega \int_K F$ also belongs to $C(W)$. Thus $\int_K F \in C$ and C is a closed class. □

Corollary 10.4. *Let X and A be simply connected. If $X \gg A$, then $\Omega X \gg \Omega A$.*

Proof. By proposition 10.3, $C = \{X \mid X \textit{ is simply connected and } \Omega X \gg \Omega A\}$ is a closed class. Since obviously A belongs to C, we get $C(A) \subset C$ and thus the corollary is proven. □

Proposition 10.5 (E. Dror Farjoun [6]). *Let $f : X \to Y$ be a map. If Y is connected, then $Cof(f : X \to Y) \gg \Sigma Fib(f : X \to Y)$.*

Proof. $f : X \to Y$ is weakly equivalent to a map of the form $\oint_K F \to K$ for some diagram $F : K \to Spaces$, where for every simplex $\sigma \in K$, $F(\sigma)$ is weakly equivalent to the homotopy fiber $Fib(f : X \to Y)$ [3, example 3.12]. Observe:

$$Cof\Big(\oint_K F \to K\Big) \simeq hocolim\Big(\star \leftarrow K \overset{id}{\to} K \leftarrow \oint_K F \to K\Big) \simeq$$

$$\simeq hocolim\Big(\star \leftarrow K \to \oint_K \star \leftarrow \oint_K F \to \oint_K \star\Big) \simeq hocolim\Big(\star \leftarrow K \to \oint_K \Sigma F\Big)$$

Since $K \to \oint_K \Sigma F$ is a cofibration, the following natural map is a weak equivalence:

$$Cof\Big(\oint_K F \to K\Big) \xrightarrow{\simeq} colim\Big(\star \leftarrow K \to \oint_K \Sigma F\Big) = \int_K \Sigma F$$

This proves the proposition. □

Corollary 10.6. *If X is connected, then $X \gg \Sigma\Omega X$.*

Proof. Notice that $Fib(\star \to X) \simeq \Omega X$ and $Cof(\star \to X) \simeq X$. According to proposition 10.5: $X \simeq Cof(\star \to X) \gg \Sigma Fib(\star \to X) \simeq \Sigma\Omega X$. □

Proposition 10.7 (E. Dror Farjoun [6]). *If X is connected, then $\Omega\Sigma X \gg X$.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\star & \simeq & hocolim & \Big(& \star & \leftarrow & X & \overset{id}{\to} & X & \Big) \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & \\
\Sigma X & \simeq & hocolim & \Big(& \star & \leftarrow & X & \to & \star & \Big)
\end{array}$$

Since the homotopy fibers: $Fib(\star \to \star) \simeq \star$, $Fib(X \overset{id}{\to} X) \simeq \star$ and $Fib(X \to \star) \simeq X$ belong to $C(X)$, then so does $Fib(\star \to \Sigma X) = \Omega\Sigma X$ (see theorem 4.4). □

Theorem 10.8. *$X \gg \Sigma A$ if and only if X is simply connected and $\Omega X \gg A$.*

Proof. Let us assume that $X \gg \Sigma A$. Notice that the following is a closed class:

$$C = \{X \mid X \textit{ is simply connected and } X \gg \Sigma A\}$$

Since obviously $\Sigma A \in C$, we get $C(\Sigma A) \subset C$ and thus $X \in C$. It follows that X is simply connected.

By corollary 10.4, $X \gg \Sigma A$ implies: $\Omega X \gg \Omega\Sigma A$. According to proposition 10.7, $\Omega\Sigma A \gg A$. It follows that $\Omega X \gg A$.

Let X be simply connected and $\Omega X \gg A$. By corollary 10.2, $\Omega X \gg A$ implies: $\Sigma\Omega X \gg \Sigma A$. According to corollary 10.6, $X \gg \Sigma\Omega X$. It follows that $X \gg \Sigma A$. □

Theorem 10.9 (E. Dror Farjoun [6]**).** *Let $f : X \to Y$ be a map. If Y is connected, then $Fib(\Sigma f : \Sigma X \to \Sigma Y) \gg \Sigma Fib(f : X \to Y)$.*

Proof. $f : X \to Y$ is weakly equivalent to a map of the form $\oint_K F \to K$ for some diagram $F : K \to \textit{Spaces}$, such that for every simplex $\sigma \in K$, $F(\sigma)$ is weakly equivalent to the homotopy fiber $Fib(f : X \to Y)$.

Let us consider the following commutative diagram:

$$\begin{array}{ccccccccccccccccc}
\Sigma\oint_K F & \simeq & hocolim & \Big(& \star & \leftarrow & K & \overset{id}{\to} & K & \leftarrow & \oint_K F & \to & K & \overset{id}{\leftarrow} & K & \to & \star \Big) \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma K & \simeq & hocolim & \Big(& \star & \leftarrow & K & \overset{id}{\to} & K & \overset{id}{\leftarrow} & K & \overset{id}{\to} & K & \overset{id}{\leftarrow} & K & \to & \star \Big)
\end{array}$$

Notice:

$$\oint_K \Sigma F \simeq hocolim\Big(\oint_K \star \leftarrow \oint_K F \to \oint_K \star\Big) \simeq hocolim\Big(K \leftarrow \oint_K F \to K\Big)$$

$$\oint_K \Sigma\star \simeq hocolim\Big(\oint_K \star \leftarrow \oint_K \star \to \oint_K \star\Big) \simeq hocolim\Big(K \overset{id}{\leftarrow} K \overset{id}{\to} K\Big)$$

By simplifying the above diagram, we get:

$$\begin{array}{ccccccccccccc}
\Sigma\oint_K F & \simeq & hocolim & \Big(& \star & \leftarrow & K & \to & \oint_K \Sigma F & \leftarrow & K & \to & \star \Big) \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma K & \simeq & hocolim & \Big(& \star & \leftarrow & K & \to & \oint_K \Sigma\star & \leftarrow & K & \to & \star \Big)
\end{array}$$

Since for every simplex $\sigma \in K$, $Fib\Big(\Sigma F(\sigma) \to \Sigma\star\Big) \simeq \Sigma F(\sigma) \simeq \Sigma Fib(f : X \to Y)$, theorem 4.4 implies that the homotopy fiber $Fib\Big(\oint_K \Sigma F \to \oint_K \Sigma\star\Big)$ belongs to the closed class $C\Big(\Sigma Fib(f : X \to Y)\Big)$. Using again theorem 4.4, we can conclude that: $Fib\Big(\Sigma \oint_K F \to \Sigma K\Big) \in C\Big(\Sigma Fib(f : X \to Y)\Big)$. □

11. The functor CW_A, loop spaces and products

Theorem 11.1 (E. Dror Farjoun [5]). *If X is a pointed and simply connected Kan simplicial set, then the loop of the natural map $\Omega(cw_{\Sigma A}X) : \Omega CW_{\Sigma A}X \to \Omega X$ is an A–cellular equivalence and $\Omega CW_{\Sigma A}X$ is A–cellular.*

Proof. By theorem 10.8, $CW_{\Sigma A}X \gg \Sigma A$ implies: $\Omega CW_{\Sigma A}X \gg A$, so $\Omega CW_{\Sigma A}X$ is A–cellular.

Since X is Kan, $cw_{\Sigma A}X : CW_{\Sigma A}X \to X$ is a ΣA–cellular equivalence. By definition it induces a weak equivalence $(cw_{\Sigma A}X)_\star : map_\star(\Sigma A, CW_{\Sigma A}X) \to map_\star(\Sigma A, X)$. It follows that $\Omega(cw_{\Sigma A}X)_\star : map_\star(A, \Omega CW_{\Sigma A}X) \to map_\star(A, \Omega X)$ is also a weak equivalence and so $\Omega(cw_{\Sigma A}X)$ is an A–cellular equivalence. □

Corollary 7.8 and theorem 11.1 implies:

Corollary 11.2. *If X is a pointed and simply connected Kan simplicial set, then $\Omega(cw_{\Sigma A}X) : \Omega CW_{\Sigma A}X \to \Omega X$ is weakly equivalent to $cw_A\Omega X : CW_A\Omega X \to \Omega X$.*

Theorem 11.3 (E. Dror Farjoun [5]). *If X and Y are pointed and connected Kan simplicial sets, then $CW_AX \times CW_AY$ is A–cellular and the product of the natural maps $cw_AX \times cw_AY : CW_AX \times CW_AY \to X \times Y$ is an A–cellular equivalence.*

Proof. According to theorem 4.7, any closed class is closed under products, in particular the class $C(A)$. Thus $CW_AX \times CW_AY$ is A–cellular.

Since X and Y are Kan, the maps $cw_A : CW_AX \to X$ and $cw_A : CW_AY \to Y$ are A–cellular equivalences and according to proposition 6.2, so is the product map: $cw_AX \times cw_AY : CW_AX \times CW_AY \to X \times Y$. □

Corollary 11.4. *Let X and Y be pointed and connected Kan simplicial sets. If $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are the projection maps, then the product map $CW_A(p_1) \times CW_A(p_2) : CW_A(X \times Y) \to CW_AX \times CW_AY$ is a weak equivalence.*

Throughout sections 12 to 14, we are going to review the basic properties of the periodization functor P_A. The functor P_A was introduced by A. K. Bousfield [1].

12. A-null simplicial sets

Let A be a connected simplicial set. By standard manipulations on mapping spaces one can prove:

Proposition 12.1. *Let X be a Kan simplicial set. The following statements are equivalent:*

- *The inclusion map $X \to map(A, X)$ (see section 3), is a weak equivalence.*
- *For any choice of a basepoint in A, the basepoint–evaluation map $map(A, X) \to X$ (see section 3) is a weak equivalence.*
- *For any choice of basepoints in A and X, $map_\star(A, X)$ is weakly contractible.*
- *For any choice of basepoints in A and X, $X \to \star$ is an A–cellular equivalence.*
- *For any connected component X_i of X and for any choice of a basepoint in X_i, $X_i \to \star$ is an A–cellular equivalence.*

Definition 12.2 (A. K. Bousfield [1]). *A Kan simplicial set X is called A–null if one of the statements of proposition 12.1 is true for X.*

Remark. Notice that the class of A–null simplicial sets consists of Kan simplicial sets.

Proposition 12.3.

- *A Kan simplicial set X is A–null if and only if every connected component of X is A–null.*
- *Let B be A–cellular. If X is A–null, then X is B–null.*
- *Let X be a connected Kan simplicial set. Let us choose basepoints in A and X. Then X is A–null if and only if for every $n \geq 0$, $[\Sigma^n A, X] = \star$.*

Example 12.4. If $A = \dot{\Delta}[n+1]$, then X is A–null if and only if for any choice of a basepoint in X and for $i \geq n$, $\pi_i X = 0$.

Example 12.5. Let $A = M(\mathbb{Z}/p, n)$ be $\mathbb{Z}/p$–Moore space. A simplicial set X is A–null if and only if for any choice of a basepoint in X:

- $\pi_n X \to \pi_n X$, $(g \mapsto g^p)$ is a monomorphism.
- For $i > n$, $\pi_i X \to \pi_i X$, $(g \mapsto g^p)$ is an isomorphism.

13. A–periodic equivalences

By standard manipulations on mapping spaces one can show:

Proposition 13.1. *Let $f : X \to Y$ be a map and Z be a Kan simplicial set. The following statements are equivalent:*

- *$f^\star : map(Y, Z) \to map(X, Z)$ is a weak equivalence.*
- *For any choice of basepoints in X and Z, $f^\star : map_\star(Y, Z) \to map_\star(X, Z)$ is a weak equivalence.*

Definition 13.2 (A. K. Bousfield [1]). *A map $f : X \to Y$ is an A–periodic equivalence if for any A–***null** *Kan simplicial set Z, one of the statements of proposition* 13.1 *is true for f and Z.*

As an immediate consequence of the definition we get:

Proposition 13.3.

- *Let $f : X \to Y$ and $g : Y \to Z$ be maps. If two out of $(f,\ g,\ g\circ f)$ are A–periodic equivalences, then so is the third.*
- *Let $f : X \to Y$ be a map between A–null Kan simplicial sets. Then f is an A–periodic equivalence if and only if f is a weak equivalence.*

The following proposition gives some obvious examples of A–periodic equivalences.

Proposition 13.4.

- *If X and Y are A–cellular, then any map $X \to Y$ is an A–periodic equivalence. In particular:*
  - *$A \to \star$ and $\star \to A$ are A–periodic equivalences*
  - *For any choice of a basepoint $a \in A$, the map $\widetilde{\Sigma}^n(A, a) \hookrightarrow \Delta[n+1] \times A$ is an A–periodic equivalence (see proposition* 4.3*).*
- *Let $(f_i : X_i \to Y_i)_{i\in I}$ be a family of maps. If f_i are A–periodic equivalences, then so is: $\coprod_I f_i : \coprod_I X_i \to \coprod_I Y_i$.*
- *If $f : X \to Y$ and $g : Z \to W$ are A–periodic equivalences, then so is the product map $f \times g : X \times Z \to Y \times W$.*
- *Let the following be a commutative diagram:*

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_2 & \longrightarrow & X_3 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ Y_1 & \longleftarrow & Y_2 & \longrightarrow & Y_3 \end{array}$$

  If f_i are A–periodic equivalences, then so is $hocolim(f_1 \leftarrow f_2 \to f_3)$.
- *Let $G : \lambda \to Spaces$ be a diagram over the category associated to an ordinal number λ. If for every ordinal number $i < \lambda$, $G(i) \to G(i+1)$ is an A–periodic equivalence, then so is the natural map $G(0) \to hocolim_\lambda G$.*

Corollary 13.5. *Let the following be a homotopy push–out square:*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Z & \longrightarrow & W \end{array}$$

If f is an A–periodic equivalence, then so is g.

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} Y & \simeq & hocolim & \big(& Y & \leftarrow & X & \stackrel{id}{\rightarrow} & X & \big) \\ \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & \\ W & \simeq & hocolim & \big(& Y & \leftarrow & X & \stackrel{f}{\rightarrow} & Z & \big) \end{array}$$

Since $f : X \to Z$, id_X and id_Y are A–periodic equivalences, then so is $g : Y \to W$. □

Theorem 13.6 (A. K. Bousfield [1]). *Let $F : K \to Spaces$, $G : K \to Spaces$ be diagrams and $\Psi : F \to G$ be a natural transformation. If for every simplex $\sigma \in K$, $\Psi_\sigma : F(\sigma) \to G(\sigma)$ is an A–periodic equivalence, then so is $\Psi : \oint_K F \to \oint_K F$.*

Proof. We can assume that F and G are bounded diagrams [4, proposition A.5].

We are going to prove the theorem by induction on the dimension of K. Since the disjoint union preserves A–periodic equivalences, the theorem is true for any zero dimensional simplicial set K. Let us assume that the theorem is true for any K, whose dimension is less that n. Let L be a simplicial set, whose dimension is less then n and $\dot{\Delta}[n] \to L$ be a map. We are going to prove that the theorem is true for $K = L \cup_{\dot{\Delta}[n]} \Delta[n]$. Let $\tau \in (\Delta[n])_n$ be the distinguished simplex (see section 3). Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} \oint_K F & = & colim & \big(& \oint_L F & \leftarrow & \dot{\Delta}[n] \times F(\tau) & \hookrightarrow & \Delta[n] \times F(\tau) & \big) \\ \downarrow{\scriptstyle \Psi} & & & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle id\times\Psi_\tau} & & \downarrow{\scriptstyle id\times\Psi_\tau} & \\ \oint_K G & = & colim & \big(& \oint_L G & \leftarrow & \dot{\Delta}[n] \times G(\tau) & \hookrightarrow & \Delta[n] \times G(\tau) & \big) \end{array}$$

By the inductive assumption, $\Psi : \oint_L F \to \oint_L G$ is an A–periodic equivalence. Proposition 13.4 implies that $\dot{\Delta}[n] \times F(\tau) \to \dot{\Delta}[n] \times G(\tau)$ and $\Delta[n] \times F(\tau) \to \Delta[n] \times G(\tau)$ are also A–periodic equivalences. Using once again proposition 13.4, we get that $\Psi : \oint_K F \to \oint_K G$ is an A–periodic equivalence. □

Corollary 13.7. *Let $F : K \to Spaces_\star$, $G : K \to Spaces_\star$ be pointed diagrams and $\Psi : F \to G$ be a natural transformation. If for $\sigma \in K$, $\Psi_\sigma : F(\sigma) \to G(\sigma)$ is an A–periodic equivalence, then so is $\Psi : \int_K F \to \int_K F$.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} \int_K F & \simeq & hocolim & \big(& \star & \leftarrow & K & \rightarrow & \oint_K F & \big) \\ \downarrow{\scriptstyle \Psi} & & & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Psi} & \\ \int_K G & \simeq & hocolim & \big(& \star & \leftarrow & K & \rightarrow & \oint_K G & \big) \end{array}$$

Theorem 13.6 implies that $\Psi : \oint_K F \to \oint_K G$ is an A–periodic equivalence. It follows from proposition 13.4, that $\Psi : \int_K F \to \int_K G$ is also an A–periodic equivalence. □

14. The functor P_A

In this section we state the theorem of the existence of the periodization functor P_A and list some of its basic properties.

Theorem 14.1 (A. K. Bousfield [1]). *On the category of simplicial sets there exist a functor $P_A : Spaces \to Spaces$ and a natural transformation $p_A X : X \to P_A X$, such that $P_A X$ is A–null and the map $p_A X : X \to P_A X$ is an A–periodic equivalence.*

Proposition 14.2.

- *If $f : X \to Y$ is an A–periodic equivalence, then up to homotopy, there exist a unique map $g : Y \to P_A X$ such that $g \circ f : X \to P_A X$ is homotopic to the natural map $p_A X : X \to P_A X$.*
- *Let $f : X \to Y$ be a map. If Y is A–null, then up to homotopy, there exist a unique map $g : P_A X \to Y$, such that f and the composition $(X \xrightarrow{p_A X} P_A X \xrightarrow{g} Y)$ are homotopic.*
- *Let $f : X \to Y$ be an A–periodic equivalence. If Y is A–null, then Y is weakly equivalent to $P_A X$ and $f : X \to Y$ is weakly equivalent to $p_A X : X \to P_A X$.*
- *A map $f : X \to Y$ is an A–periodic equivalence if and only if $P_A f : P_A X \to P_A Y$ is a weak equivalence.*
- *If X is Kan, then X is A–null, if and only if $p_A X : X \to P_A X$ is a weak equivalence.*
- *If $f : X \to Y$ is a weak equivalence, then so is $P_A f : P_A X \to P_A Y$. In particular, if X is weakly contractible, then so is $P_A X$.*
- *$P_A X$ is weakly contractible if and only if $X \to \star$ is an A–periodic equivalence.*

15. The image of P_A

In this section we are going to prove that essentially the image of the functor P_A is equal to the kernel of the functor CW_A.

Definition 15.1. *The image of P_A is the class of all those simplicial sets X, for which there exists a simplicial set Y, such that $P_A Y$ is weakly equivalent to X.*

Theorem 15.2. *The following classes of simplicial sets are equal:*

1. *The class of those simplicial sets X, such that every connected component of X is in the kernel of CW_A.*
2. *The image of P_A.*
3. *The class of those simplicial sets X, for which there exist a Kan simplicial set Y, which is weakly equivalent to X and Y is A–null.*

Proof. Definition 15.1 and proposition 14.2 imply immediately that classes (2) and (3) are equal.

Let X be a simplicial set such that every connected component of X is in the kernel of CW_A. Let Y be a Kan simplicial set, which is weakly equivalent to X. Let Y' be any connected component of Y. According to proposition 9.2, for any choice of basepoints in A and Y', $map_\star(A, Y') \simeq \star$, thus Y' is A–null. By proposition 12.3, Y is also A–null, It shows that classes (1) and (2) are equal. □

16. The kernel of P_A

In this section we are going to show that the kernel of P_A is closed under extensions by fibrations.

Definition 16.1. *The kernel of P_A is the class of all those simplicial sets X, such that P_AX is weakly contractible. The elements of the kernel of P_A will be called A–acyclic simplicial sets.*

Remark. The kernel of P_A consists of connected simplicial sets.

Proposition 14.2 implies:

Corollary 16.2.

- *A simplicial set X is A–acyclic if and only if $X \to \star$ is an A–periodic equivalence.*
- *A is A–acyclic.*

Proposition 16.3. *Let $F : K \to Spaces_\star$ be a pointed diagram. If for every simplex $\sigma \in K$, $F(\sigma)$ is A–acyclic, then so is $\int_K F$.*

Proof. Since $F(\sigma)$ is A–acyclic, $F(\sigma) \to \star$ is an A–periodic equivalence. According to corollary 13.7, $\int_K F \to \int_K \star = \star$ is also an A–periodic equivalence and so $\int_K F$ is A–acyclic. □

Corollary 16.4. *The class of A–acyclic simplicial sets is a closed class and A–cellular simplicial sets are A–acyclic.*

Proposition 16.5. *Let $F : K \to Spaces$ be a diagram. If for every simplex $\sigma \in K$, $F(\sigma)$ is A–acyclic, then $\oint_K F \to K$ is an A–periodic equivalence.*

Proof. Since for every simplex $\sigma \in K$, $F(\sigma) \to \star$ is an A–periodic equivalence, theorem 13.6 implies that $\oint_K F \to \oint_K \star = K$ is also an A–periodic equivalence. □

Corollary 16.6. *Let $F : K \to Spaces$ be a diagram. If K is A–acyclic and for every simplex $\sigma \in K$, $F(\sigma)$ is A–acyclic, then $\oint_K F$ is A–acyclic.*

Proof. According to proposition 16.5, $\oint_K F \to K$ is an A–periodic equivalence. Since K is A–acyclic, $K \to \star$ is an A–periodic equivalence. Composition of two A–periodic equivalences is an A–periodic equivalence, so $\oint_K F \to \star$ is an A–periodic equivalence and $\oint_K F$ is A–acyclic. □

Corollary 16.7. *Let $(Z \to E \to B)$ be a fibration sequence, such that B is connected. If Z is A–acyclic, then $E \to B$ is an A–periodic equivalence.*

Proof. The map $E \to B$ is weakly equivalent to a map of the form $\oint_K F \to K$, where $F : K \to Spaces$ is a diagram such that for every simplex $\sigma \in K$, $F(\sigma)$ is weakly equivalent to Z, so it is A–acyclic [3, example 3.12]. Proposition 16.5 implies that $\oint_K F \to K$ is an A–periodic equivalence. It shows that $E \to B$ is also an A–periodic equivalence. □

Using the same argument as in corollary 16.6, we get:

Corollary 16.8. *Let $(Z \to E \to B)$ be a fibration sequence. If F and B are A–acyclic, then so is E. Thus the class of A–acyclic simplicial sets is closed under extensions by fibrations.*

17. A–acyclic simplicial sets

Theorem 17.1. *If X is connected, then the homotopy fiber $Fib(p_A X : X \to P_A X)$ belongs to $\overline{C(A)}$, where $\overline{C(A)}$ is the smallest closed class such that $A \in \overline{C(A)}$ and $\overline{C(A)}$ is closed under extensions by fibrations (see example 4.9).*

Proof. Let Y be a Kan simplicial set, which is weakly equivalent to X. We are going to prove that $Fib(p_A Y : Y \to P_A Y)$ belongs to $\overline{C(A)}$.

Let λ be a limit ordinal number, whose cofinality [7] is bigger then the cardinality of the set of simplices of A. By λ we also denote the category associated to this ordinal number (see section 7–Construction). Let us choose basepoints in A and Y. The idea of the proof is to define a functor $G : \lambda \to Spaces_\star$, such that:

(1) $G_0 = Y$
(2) If i is not of the form $j + 1$, then $G_i = hocolim_{j<i} G_j$.
(3) For every i, G_i is a connected Kan simplicial set.
(4) If $i = j + 1$, then $Fib(G_j \to G_i) \gg A$
(5) If $i = j + 1$, then for any pointed map $g : \Sigma^n A \to G_j$, the composition $(\Sigma^n A \xrightarrow{g} G_j \to G_i)$ can be extended along $\Sigma^n A \hookrightarrow C\Sigma^n A$, where $C\Sigma^n A$ is the cone over $\Sigma^n A$ (this composition is homotopic to the constant map).

Let us assume that we have constructed such a functor $G : \lambda \to Spaces_\star$. We are going to prove:

(a) The map $G_0 \to hocolim_\lambda G$ is an A–periodic equivalence.
(b) $hocolim_\lambda G$ is A–null.
(c) $Fib(Y = G_0 \to hocolim_\lambda G) \in \overline{C(A)}$.

According to proposition 14.2, conditions (a) and (b) imply that the map $G_0 \to hocolim_\lambda G$ is weakly equivalent to $p_AY : Y \to P_AY$. Using condition (c), we get that $Fib(p_AY : Y \to P_AY) \in \overline{C(A)}$. This proves the theorem.

Proof of (a). Since for every i, the homotopy fiber $Fib(G_i \to G_{i+1})$ is A–cellular, so it is A–acyclic (see corollary 16.4). It follows from corollary 16.7 that $G_i \to G_{i+1}$ is an A–periodic equivalence. According to proposition 13.4, $Y = G_0 \to hocolim_\lambda G$ is also an A–periodic equivalence.

Proof of (b). Observe that since λ is an ordinal number, whose cofinality is bigger than the cardinality of the set of simplices of A, then the cofinality of λ is also bigger than the cardinality of the set of simplices of $\Sigma^n A$. Let $g : \Sigma^n A \to hocolim_\lambda G$ be a pointed map. We will show that this map is homotopic to the constant map. By Quillen's small object argument [10], g factors through $h : \Sigma^n A \to G_i$, for some ordinal number $i < \lambda$. According to the condition (5) of the construction, the composition $(\Sigma^n A \xrightarrow{h} G_i \to G_{i+1})$ can be extended along $\Sigma^n A \hookrightarrow C\Sigma^n A$. It shows that $g : \Sigma^n A \to hocolim_\lambda G$ is homotopicaly trivial. It means that the simplicial set $map_\star(A, hocolim_\lambda G)$ is weakly contractible. Since $hocolim_\lambda G$ is connected and $map_\star(A, hocolim_\lambda G)$ is weakly contractible, $hocolim_\lambda G$ is A–null.

Proof of (c). Let $F : \lambda \to Spaces_\star$ be the constant functor, whose value is Y. Let us consider a natural transformation $\Psi : Y \to G$, such that for every i, $\Psi_i : Y \to G_i$ is equal to the map $Y = G_0 \to G_i$. We are going to prove, that for every i, $Fib(\Psi_i : Y \to G_i) \in \overline{C(A)}$. Since $\overline{C(A)}$ is a closed class, theorem 4.4 implies that the homotopy fiber $Fib(hocolim_\lambda \Psi : hocolim_\lambda F \to hocolim_\lambda G)$ belongs to $\overline{C(A)}$. Observe that $Y = F_0 \to hocolim_\lambda F$ is a weak equivalence. It follows, that the homotopy fiber $Fib(Y = G_0 \to hocolim_\lambda G)$ belongs to $\overline{C(A)}$.

Remark. Notice that in fact $Fib(Y \to hocolim_\lambda G) \simeq hocolim_{i<\lambda} Fib(Y \to G_i)$. It clearly implies, that if for every $i < \lambda$, $Fib(Y \to G_i)$ belongs to $\overline{C(A)}$ then so does $Fib(Y \to hocolim_\lambda G)$.

We are going to prove by induction, that $Fib(Y \to G_i) \in \overline{C(A)}$. For $i = 0$, it is obvious. Let us assume that for $j < i$, $Fib(Y \to G_j) \in \overline{C(A)}$.

If i is not of the form $j+1$, then since $G_i = hocolim_{j<i} G_j$, we have:

$$Fib(Y \to G_i) = Fib(hocolim_{j<i}(Y \to G_j)) \simeq hocolim_{j<i} Fib(Y \to G_j)$$

In this case it is clear that $Fib(Y \to G_i) \in \overline{C(A)}$.

If $i = j+1$, then the map $Y \to G_i$ can be factored as $(Y \to G_j \to G_i)$. This gives a fibration sequence: $\big(Fib(Y \to G_j) \to Fib(Y \to G_i) \to Fib(G_j \to G_i)\big)$. By the inductive assumption $Fib(Y \to G_j) \in \overline{C(A)}$. The condition (4) of the construction implies that $Fib(G_j \to G_i)$ is A–cellular, so it belongs to $\overline{C(A)}$. Since $\overline{C(A)}$ is closed under extensions by fibrations, $Fib(Y \to G_i) \in \overline{C(A)}$.

Construction of the functor $G : \lambda \to Spaces_\star$. The construction will be by induction. If $i = 0$, then $G_0 = Y$. Let us assume that the construction has been carried out for all ordinal numbers $j < i$.

If i is not of the form $j + 1$, then $G_i = hocolim_{j<i} G_j$. Clearly, conditions (1) through (5) are satisfied (to prove that G_i is Kan, we use Quillen's small object argument).

If $i = j + 1$, let $\widetilde{G_i} := Cof(cw_A G_j : CW_A G_j \to G_j)$ and G_i be a Kan simplicial set, for which there is a weak equivalence $\widetilde{G_i} \to G_i$. The map $G_j \to G_i$ is define to be the composition: $(G_j \to \widetilde{G_i} \to G_i)$. Since $CW_A G_j$ is A–cellular, proposition 4.5 implies that the homotopy fiber $Fib\Big(G_j \longrightarrow Cof(CW_A G_j \overset{cw_A G_j}{\longrightarrow} G_j)\Big)$ is also A–cellular and the condition (4) is satisfied.

Let $g : \Sigma^n A \to G_j$ be any pointed map. Since by the inductive assumption, G_j is Kan, the map $cw_A G_j : CW_A G_j \to G_j$ is an A–cellular equivalence. It follows from proposition 7.6, that since $\Sigma^n A$ is A–cellular, there is a map $\Sigma^n A \to CW_A G_j$, for which the composition $(\Sigma^n A \to CW_A G_j \overset{cw_A G_j}{\longrightarrow} G_j)$ is homotopic to g. This shows that the map $\Big(\Sigma^n A \overset{g}{\to} G_j \to Cof(CW_A G_j \overset{cw_A G_j}{\longrightarrow} G_j) \to G_i\Big)$ is homotopicaly trial and thus the condition (5) has been proven. □

The construction of the diagram $G : \lambda \to cSpaces_\star$, in the proof of theorem 17.1, suggests that, up to homotopy, the functor P_A can be constructed using CW_A. The process can be described as “killing” the maps from A–cellular simplicial sets by attaching cones. This procedure implies:

Corollary 17.2. *If X is n–connected, then so is $P_A X$.*

As a corollary of theorem 17.1, we get:

Theorem 17.3. *The class of A–acyclic simplicial sets is equal to $\overline{C(A)}$.*

Proof. According to corollaries 16.4 and 16.8, the class of A–acyclic simplicial sets is a closed class and it is closed under extensions by fibrations. Since A is A–acyclic, the smallest closed class $\overline{C(A)}$, that contains A and is closed under extensions by fibrations, is included in the class of A–acyclic simplicial sets.

Let X be A–acyclic, so it is connected. Since $P_A X$ is weakly contractible, the map $Fib(p_A X : X \to P_A X) \to X$ is a weak equivalence. According to theorem 17.1, since $Fib(p_A X : X \to P_A X)$ belongs to $\overline{C(A)}$, then so does X. This proves that the class of A–acyclic simplicial sets is included in $\overline{C(A)}$. □

Example 17.4. Let $A = \dot{\Delta}[n+1]$. The class $C(A)$ consists of $(n-1)$–connected simplicial sets (see example 5.3). Since this class is closed under extensions by fibrations, we get: $C(A) = \overline{C(A)}$.

18. Weak cellular inequalities

Notation (E. Dror Farjoun [6]). Let X be a simplicial set. If $X \in \overline{C(A)}$, then we will write $X > A$ and say that X is killed by A or A kills X.

It follows immediately from the definition that if $X > B$ and $B > A$, then $X > A$. It is also straightforward to see that $X \gg A$ implies $X > A$. This means that if X can be built by A, then it can be killed by A. Since the class $\overline{C(A)}$ is usually bigger than the class $C(A)$ (for a non trivial example see corollary 20.13), the converse to this statement is not true. If X can be killed by A, then in order to built X from A, we have to add an extra tool: taking extensions by fibrations. So the converse would be true if the class $C(A)$ were closed under extensions by fibrations. This is the case, if for example $A = \dot{\Delta}[n+1]$.

According to the introduced definitions, notation and proven theorems and propositions the following statements are equivalent:

- X belongs to $\overline{C(A)}$.
- X is killed by A.
- $X > A$.
- X is A–acyclic.
- X is in the kernel of P_A.
- For any choice of basepoints in A and X, if Y is a pointed Kan simplicial set for which $map_\star(A, Y) \simeq \star$, then $map_\star(X, Y) \simeq \star$. We will refer to this property of A–acyclic simplicial sets as their universal property.

Proposition 18.1. *Let W be connected. The class $C = \{X \in cSpaces \mid \Sigma X > W\}$ is a closed class and it is closed under extensions by fibrations.*

Proof. Let $F : K \to Spaces_\star$ be a pointed diagram such that for every simplex $\sigma \in K$, $F(\sigma) \in C$, so $\Sigma F(\sigma) \in \overline{C(W)}$. Since $\Sigma \int_K F \simeq \int_K \Sigma F$, we get that $\Sigma \int_K F$ belongs to $\overline{C(W)}$. It implies: $\int_K F \in C$ and thus C is a closed class.

Let $(Z \to E \to B)$ be a fibration sequence for which $Z \in C$ and $B \in C$, so ΣZ and ΣB belong to $\overline{C(W)}$. We have to show that $\Sigma E \in \overline{C(W)}$. Consider the fibration sequence $\big(Fib(\Sigma E \to \Sigma B) \to \Sigma E \to \Sigma B\big)$. Theorem 10.9 implies: $Fib(\Sigma E \to \Sigma B) \gg \Sigma Fib(E \to B) \simeq \Sigma Z > W$. As a result, we get: $Fib(\Sigma E \to \Sigma B) \in \overline{C(W)}$. Since $\overline{C(W)}$ is closed under extensions by fibrations, $\Sigma E \in \overline{C(W)}$. $\square$

Corollary 18.2. *If $X > A$, then $\Sigma X > \Sigma A$.*

Proof. According to proposition 18.1, the class $C = \{X \in cSpaces \mid \Sigma X > \Sigma A\}$ is a closed class and it is closed under extensions by fibrations. Since obviously $A \in C$, we get $\overline{C(A)} \subset C$. $\square$

Proposition 18.3. *Let W be connected. The following class is a closed class and it is closed under extensions by fibrations:*

$$C = \{X \mid X \textit{ is simply connected and } \Omega X > W\}$$

Proof. Let $F : K \to Spaces_\star$ be a pointed diagram such that for every simplex $\sigma \in K$, $F(\sigma) \in C$. Since F is a pointed diagram, there is a natural transformation $\star \to F$, where $\star : K \to Spaces_\star$ is the constant diagram whose value is the trivial simplicial set. By assumption, for every simplex $\sigma \in K$, $Fib\Big(\star \to F(\sigma)\Big) \simeq \Omega F(\sigma)$ belongs to $\overline{C(W)}$. According to proposition 4.5, $Fib\Big(\int_K \star \to \int_K F\Big) \simeq \Omega \int_K F$ is also in $\overline{C(W)}$, and so C is a closed class.

Let $(Z \to E \to B)$ be a fibration sequence such that Z and B belong to C, thus ΩZ and ΩB are elements of $\overline{C(W)}$. Since $(\Omega Z \to \Omega E \to \Omega B)$ is a fibration sequence and $\overline{C(W)}$ is closed under extensions by fibrations, we get $\Omega E \in \overline{C(W)}$. It follows that C is closed under extensions by fibrations. $\square$

Corollary 18.4. *Let X and A be simply connected. If $X > A$, then $\Omega X > \Omega A$.*

Proof. By proposition 18.3, $C = \{X \mid X \textit{ is simply connected and } \Omega X > \Omega A\}$ is a closed class and it is closed under extensions by fibrations. Since obviously $A \in C$, we get $\overline{C(A)} \subset C$. $\square$

Theorem 18.5. *$X > \Sigma A$ if and only if X is simply connected and $\Omega X > A$.*

Proof. Let $X > \Sigma A$. Notice that $C = \{X \mid X \textit{is simply connected and } X > \Sigma A\}$ is a closed class and it is closed under extensions by fibrations. Since obviously $\Sigma A \in C$, it follows that $\overline{C(\Sigma A)} \subset C$. This implies that $X \in C$ and thus X is simply connected.

By corollary 18.4, $X > \Sigma A$ implies: $\Omega X > \Omega \Sigma A$. According to proposition 10.7, $\Omega \Sigma A \gg A$. It follows that $\Omega X > A$.

Let X be simply connected such that $\Omega X > A$. By corollary 18.2, we have $\Sigma \Omega X > \Sigma A$. According to corollary 10.6, $X \gg \Sigma \Omega X$, thus $X > \Sigma A$. $\square$

19. The functor P_A and loop spaces

Theorem 19.1 (A.K. Bousfield, E. Dror Farjoun). *Let X be simply connected. The loop of the natural map $\Omega(p_{\Sigma A} X) : \Omega X \to \Omega P_{\Sigma A} X$ is an A–periodic equivalence and $\Omega P_{\Sigma A} X$ is A–null.*

Proof. Since $P_{\Sigma A} X$ is ΣA–null, $map_\star(\Sigma A, P_{\Sigma A} X)$ is weakly contractible. It follows that $map_\star(A, \Omega P_{\Sigma A} X)$ is also weakly contractible and thus $\Omega P_{\Sigma A} X$ is A–null.

By theorem 17.1, $Fib(p_{\Sigma A} X : X \to P_{\Sigma A} X) \in \overline{C(\Sigma A)}$. Using theorem 18.5 we have $\Omega Fib(p_{\Sigma A} X : X \to P_{\Sigma A} X) \in \overline{C(A)}$ and so $Fib(\Omega(p_{\Sigma A} X) : \Omega X \to \Omega P_{\Sigma A} X)$ is

A–acyclic. Corollary 17.2 implies that since X is simply connected, then $P_{\Sigma A}X$ is also simply connected and $\Omega P_{\Sigma A}X$ is connected. Corollary 16.7 indicates that $\Omega X \to \Omega P_{\Sigma A}X$ is an A–periodic equivalence. □

Corollary 19.2. *If X is simply connected, then $\Omega P_{\Sigma A}X$ is weakly equivalent to $P_A\Omega X$ and the loop of the natural map $\Omega(p_{\Sigma A}X) : \Omega X \to \Omega P_{\Sigma A}X$ is weakly equivalent to $(p_A\Omega X) : \Omega X \to P_A\Omega X$*

20. Construction of CW_A "from" P_A

Proposition 20.1. $\overline{C(\Sigma A)} \subset D(A)$ *(see definition* 4.10*).*

Proof. According to proposition 10.7, $\Omega\Sigma A \gg A$. By theorem 4.11, $\Omega\Sigma A \gg A$ implies: $\Sigma A \in D(A)$. Since $\Sigma A \in D(A)$ and $D(A)$ is a closed class, which is closed under extensions by fibrations, we get $\overline{C(\Sigma A)} \subset D(A)$. □

Corollary 20.2 (E. Dror Farjoun [6]**).**

- *If $X > \Sigma A$, then $X \gg A$.*
- *Let $f : X \to Y$ be a map. If $Y > \Sigma A$ and $Fib(f : X \to Y) \gg A$, then $X \gg A$.*

Theorem 20.3. *Let A be* **pointed** *and connected and let $X \to X'$ be a map of pointed and connected Kan simplicial sets. Assume that:*

- *$Fib(X \to X') \gg A$.*
- *The induced map $[A, X] \to [A, X']$ is the trivial map.*

Under these assumptions $Fib(X \to X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X')$ is A–cellular and the map $Fib(X \to X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X') \to X$ is an A–cellular equivalence.

Lemma 20.4. *Let the following be a pull–back square of pointed simplicial sets:*

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ E & \overset{p}{\longrightarrow} & B \end{array}$$

where $p : E \to B$ is a fibration. Let $f : A \to L$ be a map. If $(A \overset{f}{\to} L \to B)$ is homotopic to the constant map, then there exists a lifting $A \to K$, such that $f : A \to L$ is equal to the composition $(A \to K \to L)$.

Proof. Using the fact that $p : E \to B$ is a fibration, we can lift $(A \overset{f}{\to} L \to B)$ to $A \to E$. By the universal property of the pull–back, we can construct $A \to K$, such that $(A \to K \to L)$ is equal to $f : A \to L$. □

Proof of the theorem. We have the following fibration sequence:

$$Fib(X \to X') \to Fib(X \to X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X') \to Fib(X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X')$$

Since $Fib(X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X') > \Sigma A$ and by assumption $Fib(X \to X')$ is A–cellular, corollary 20.2 implies: $Fib(X \to X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X') \gg A$.

Let us consider the following pull–back square:

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ P & \longrightarrow & P_{\Sigma A}X' \end{array}$$

where $P \to P_{\Sigma A}X'$ is a fibration, such that P is contractible and $p : X \to P_{\Sigma A}X'$ is equal to the composition $(X \to X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X')$. We are going to show that the map $Z \to X$ is an A–cellular equivalence. Since $Z \to X$ is homotopic to the map $Fib(p : X \to P_{\Sigma A}X') \to X$, the theorem will be proven.

Step 1. $[A, Z] \to [A, X]$ *is an epimorphism.*

We are going to show that in fact $hom_\star(A, Z) \to hom_\star(A, X)$ is an epimorphism. Let $f : A \to X$ be a pointed map. We want to show that there is $A \to Z$, such that $f : A \to X$ is equal to $(A \to Z \to X)$. By assumption the composition $(A \overset{f}{\to} X \to X')$ is homotopic to the constant map. This implies that so is the following composition $(A \overset{f}{\to} X \to X' \overset{p_{\Sigma A}X'}{\longrightarrow} P_{\Sigma A}X')$. Using lemma 20.4, we can construct $A \to Z$ such that $f : A \to X$ is equal to $(A \to Z \to X)$.

Step 2. *For* $n > 0$, $[\widetilde{\Sigma}^n A, Z]_A \to [\widetilde{\Sigma}^n A, X]_A$ *is an epimorphism.*

We are going to show that in fact $hom_\star(\widetilde{\Sigma}^n A, Z) \to hom_\star(\widetilde{\Sigma}^n A, X)$ is an epimorphism. Let $f : \widetilde{\Sigma}^n A \to X$ be a pointed map. Consider the composition $(A \to \widetilde{\Sigma}^n A \overset{f}{\to} X \to X')$. By assumption this map is homotopic to the constant map and so $(\widetilde{\Sigma}^n A \overset{f}{\to} X \to X')$ factors through some map $Cof(A \to \widetilde{\Sigma}^n A) \to X'$. Notice that $Cof(A \to \widetilde{\Sigma}^n A) \simeq \Sigma^n A$ (see section 3), so it is ΣA–cellular. Since $P_{\Sigma A}X'$ is ΣA–null, the composition $\left(Cof(A \to \widetilde{\Sigma}^n A) \to X' \to P_{\Sigma A}X'\right)$ is homotopic to the constant map. As a result we get that $(\widetilde{\Sigma}^n A \overset{f}{\to} X \to X' \to P_{\Sigma A}X')$ is also homotopic to the constant map. By lemma 20.4, we can find $\widetilde{\Sigma}^n A \to Z$ for which $f : \widetilde{\Sigma}^n A \to X$ is equal to $(\widetilde{\Sigma}^n A \to Z \to X)$.

Step 3. *For every* $n \geq 0$, $[\widetilde{\Sigma}^n A, Z]_A \to [\widetilde{\Sigma}^n A, X]_A$ *is a monomorphism.*

We are going to use corollary 6.5 to prove step 3. Let us assume that $g : \widetilde{\Sigma}^n A \to Z$ is a pointed map such that $(\widetilde{\Sigma}^n A \overset{g}{\to} Z \to X)$ can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$ by $h : \Delta[n+1] \times A \to X$. We are going to show that g itself can be extended along $\widetilde{\Sigma}^n A \hookrightarrow \Delta[n+1] \times A$. Since $\Delta[n+1] \times A$ is weakly equivalent to A, we get that $(\Delta[n+1] \times A \overset{h}{\to} X \to X')$ is homotopic to the constant map. It implies that

$(\Delta[n+1]\times A \xrightarrow{h} X \xrightarrow{p} P_{\Sigma}X')$ is also homotopic to the constant map. According to lemma 20.4, we can find $\Delta[n+1]\times A \to Z$ such that $(\Delta[n+1]\times A \to Z \to X)$ is equal to $h : \Delta[n+1]\times A \to X$. By the universal property of the pull–back, the map $\Delta[n+1]\times A \to Z$ is an extension of $g : \widetilde{\Sigma}^n A \to Z$ and thus step 3 is proven. □

Theorem 20.5. *Let A be a pointed simplicial set and X be a pointed Kan simplicial set. If X' is a Kan simplicial set for which there is a weak equivalence: $Cof\Big(\bigvee_{h\in[A,X]} A \to X\Big) \to X'$, then $Fib(X \to X' \xrightarrow{p_{\Sigma A}X'} P_{\Sigma A}X')$ is A–cellular and the map $Fib(X \to X' \xrightarrow{p_{\Sigma A}X'} P_{\Sigma A}X') \to X$ is an A–cellular equivalence.*

Proof. Let $X \to X'$ be the composition $\Big(X \to Cof\Big(\bigvee_{h\in[A,X]} A \to X\Big) \to X'\Big)$. We are going to prove that this map satisfies the assumptions of theorem 20.3.

By proposition 4.5 the homotopy fiber $Fib\Big(X \to Cof(\bigvee_{h\in[A,X]} A \to X)\Big)$ is A–cellular. Since the map $Cof\Big(\bigvee_{h\in[A,X]} A \to X\Big) \to X'$ is a weak equivalence, thus $Fib(X \to X') \gg A$.

It is clear that $X \to X'$ induces the trivial map $[A,X] \to [A,X']$. □

Corollary 20.6 (E. Dror Farjoun [5]). *Let A be a pointed simplicial set and X be a pointed Kan simplicial set. If $[A,X] = \star$, then $Fib(p_{\Sigma A}X : X \to P_{\Sigma A}X)$ is weakly equivalent to $CW_A X$ and the map $Fib(p_{\Sigma A}X : X \to P_{\Sigma A}X) \to X$ is weakly equivalent to $cw_A X : CW_A X \to X$.*

Corollary 20.7. *If X is pointed and connected Kan simplicial set, then the fibration $cw_A X : CW_A X \to X$ is a principal fibration.*

Corollary 20.8. *Let A be a pointed simplicial set. The following classes are equal:*

- $\{X \in cSpaces_\star \mid X \text{ is Kan}, [A,X] = \star \text{ and } X \in \overline{C(A)}\}$
- $\{X \in cSpaces_\star \mid X \text{ is Kan}, [A,X] = \star \text{ and } X \in C(\Sigma A)\}$
- $\{X \in cSpaces_\star \mid X \text{ is Kan}, [A,X] = \star \text{ and } X \in D(A)\}$

Theorem 20.9. *If X is a pointed and connected Kan simplicial set, then X is A–cellular if and only if $Cof\Big(\bigvee_{h\in[A,X]} A \to X\Big)$ is ΣA–acyclic.*

Proof. Let $Cof\Big(\bigvee_{h\in[A,X]} A \to X\Big) \xrightarrow{\simeq} X'$ be a weak equivalence such that X' is Kan. It follows that $Cof\Big(\bigvee_{h\in[A,X]} A \to X\Big)$ is ΣA–acyclic if and only if X' is.

If X' is ΣA–acyclic, then $P_{\Sigma A}X'$ is a weakly contractible simplicial set and the map $Fib(X \to X' \xrightarrow{p_{\Sigma A}X'} P_{\Sigma A}X') \to X$ is a weak equivalence. Since the homotopy fiber $Fib(X \to X' \xrightarrow{p_{\Sigma A}X'} P_{\Sigma A}X')$ is weakly equivalent to $CW_A X$ (see theorem 20.5), we get that X is A–cellular.

Let us assume that X is A–cellular. Notice that X is connected and so is $P_{\Sigma A}X'$. Since X is an A–cellular Kan simplicial set, $cw_A X : CW_A X \to X$ is a weak equivalence. By theorem 20.5, $Fib(X \to P_{\Sigma A}X') \to X$ is also a weak equivalence. Since $P_{\Sigma A}X'$ is connected, we get that $P_{\Sigma A}X'$ is weakly contractible and thus X' is ΣA–acyclic. □

Corollary 20.10 (W. Dwyer). *Let S^n be a Kan simplicial set which is weakly equivalent to $\dot{\Delta}[n+1]$. Then $S^n \gg \Omega S^{n+1}$ if and only if $n = 1, 3, 7$.*

Lemma 20.11. $\Sigma\Omega S^{n+1} \gg S^{n+1}$

Proof. It is obvious that $S^{n+1} \gg S^{n+1} \simeq \Sigma S^n$. By theorem 10.8, $\Omega S^{n+1} \gg S^n$. It follows from corollary 10.2, that $\Sigma\Omega S^{n+1} \gg \Sigma S^n \simeq S^{n+1}$. □

Proof of the corollary. If $n = 1, 3, 7$, then S^n is an H–space and thus S^n is a retract of ΩS^{n+1}. According to proposition 4.3, S^n is ΩS^{n+1}–cellular.

Let us assume: $S^n \gg \Omega S^{n+1}$. By theorem 20.9, $X' = Cof\Big(\bigvee_{h\in[\Omega S^{n+1},S^n]} \Omega S^{n+1} \to S^n\Big)$ is $\Sigma\Omega S^{n+1}$–acyclic. According the lemma, X' is also S^{n+1}–acyclic. It follows that X' is n–connected (see example 17.5). As a consequence, we get that the map $\bigvee_{h\in[\Omega S^{n+1},S^n]} \Omega S^{n+1} \to S^n$ induces an epimorphism on n–dimensional integral homology. This means that there is $h : \Omega S^{n+1} \to S^n$ which also induces an epimorphism on n–dimensional integral homology. By Hurewicz theorem, h has to be an epimorphism on π_n. This implies that there is $S^n \to \Omega S^{n+1}$ for which the composition $(S^n \to \Omega S^{n+1} \xrightarrow{h} S^n)$ is a weak equivalence. It shows that S^n is a retract of an H–space, so it is also an H–space. This can happen only for $n = 1, 3, 7$. □

Proposition 20.12. *Let S^n be a pointed Kan simplicial set which is weakly equivalent to $\dot{\Delta}[n+1]$. For every $n \geq 1$, $S^n > \Omega S^{n+1}$.*

Proof. For $n = 1$, since S^1 is an H–space, S^1 is a retract of $\Omega\Sigma S^1$ and so $S^1 \gg \Omega S^2$.

Let $n > 1$. Let us consider the canonical map $S^n \to \Omega\Sigma S^n$. By Freudenthal suspension theorem the homotopy fiber $Fib(S^n \to \Omega\Sigma S^n)$ is $(2n-2)$–connected and so it is n–connected. This implies: $Fib(S^n \to \Omega\Sigma S^n) \gg S^{n+1}$ (see example 5.3). According to corollary 10.6, $S^{n+1} \gg \Sigma\Omega S^{n+1}$ and so $S^{n+1} \gg \Omega S^{n+1}$. It follows that $Fib(S^n \to \Omega\Sigma S^n) \gg \Omega S^{n+1}$. Since $Fib(S^n \to \Omega\Sigma S^n) \to S^n \to \Omega\Sigma S^n$ is a fibration sequence, where the base and the homotopy fiber are ΩS^{n+1}–cellular, the total space S^n is ΩS^{n+1}–acyclic. □

Corollary 20.13. *If S^n is a pointed Kan simplicial set which is weakly equivalent to $\dot{\Delta}[n+1]$, then:*

- $C(S^n) = \overline{C(S^n)} = \overline{C(\Omega S^{n+1})}$ *for $n \geq 1$,*
- $C(S^n) = C(\Omega S^{n+1})$ *if and only if $n = 1, 3, 7$.*

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Department of Mathematics, University of Notre Dame, Mail Distribution Center, Notre Dame, Indiana 46556-5683, USA
E-mail address: wchacho1@math.nd.edu

The Fields Institute, 222 Colage St., University of Toronto, Toronto, Ontario M5S 1A1, Canada