Extensions of strict polynomial functors

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Abstract

We compute Ext–groups between Frobenius twists of strict polynomial functors. The main results are calculations of $\mathrm{Ext}^*_{\mathcal{P}}(D^{d(i)}, F^{(i)})$ and $\mathrm{Ext}^*_{\mathcal{P}}(W_\mu^{(i)}, S_\lambda^{d(i)})$, where D^d is the divided power functor, W_μ and S_λ are respectively the Weyl and Schur functors associated to diagrams μ, λ of the same weight, and F is an arbitrary functor.

1 Introduction

Computing Ext–groups between $GL_n(\mathbf{k})$–modules for a field $\mathbf{k}$ of positive characteristic had been known to be a very difficult problem for a long time. Only recently, introducing a suitable category of functors $\mathcal{F}$ in [HLS] has changed the situation significantly. As it was demonstrated in [FLS], [FS], [FFSS], it is possible to make effective computations of Ext–groups in the category $\mathcal{F}$ (and its more sophisticated modification $\mathcal{P}$). In a meantime, in a series of papers ([B2],[K1],[K2],[K3], [FS], [FFSS]) there was established a close relation between the functor categories $\mathcal{F}$, $\mathcal{P}$ and the category of

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$GL_n(\mathbf{k})$–modules. In particular, it was shown in [B2] and [FFSS] that there is an isomorphism of Ext–groups

$$\mathrm{Ext}^*_{\mathcal{P}}(F^{(i)}, G^{(i)}) \simeq \mathrm{Ext}^*_{GL_n(\mathbf{k})-mod}(F(\mathbf{k}^n), G(\mathbf{k}^n)),$$

for a finite field $\mathbf{k}$ and integers i, n large enough ($F^{(i)}$ means the i–th Frobenius twist of a functor F (cf. [FS], sect. 1)). Thus in order to get calculations for GL_n–modules one should compute Ext–groups in the category $\mathcal{P}$ between twists of functors. Such calculations were started in [FS] where (adapting ideas of [FLS] to the context of the category $\mathcal{P}$) the groups $\mathrm{Ext}^*_{\mathcal{P}}(I^{(i)}, I^{(i)})$ were computed. Already this result has a valuable application to the $GL_n(\mathbf{k})$–modules, for after some additional work ([FS], sect. 7) it leads to a computation of $H^*(GL_n(\mathbf{F}_p), M_n(\mathbf{F}_p))$ for large n (with action of $GL_n(\mathbf{F}_p)$ on matrices by conjugation), which is a difficult still unpublished result of Bökstedt ([Bö]). These computations were extended in ([Fr],[FFSS],[PS]).

The aim of the present paper is to generalize and systematize computations of Ext–groups in the functor category by using methods of representation theory. We obtain complete description of the Ext–groups for a large class of functors strongly generalizing and putting into a uniform context known computations.

A direct inspiration for this work was a computation of $\mathrm{Ext}^*_{\mathcal{P}}(D^{d(i)}, S^{d(i)})$ obtained in [FFSS]. Since the tensor products of divided powers form a family of projective generators of $\mathcal{P}$ and the products of symmetric powers — of injective ones, one can hope for computations of Ext–groups for the Frobenius twists of functors of a more general form.

The main results are: the computation of $\mathrm{Ext}^*(D^{d(i)}, F^{(i)})$ for arbitrary F (Th. 4.3) and the computation of $\mathrm{Ext}^*(F^{(i)}, G^{d(i)})$ for F, G satisfying certain simple abstract condition (Th. 4.4). The most important instance of Th. 4.4 is that for functors $F = W_\mu$ and $G = S_\lambda$ (resp. Weyl and Schur functors) for diagrams μ, λ of the same weight. The language in which results of computations are given utilizes a concept of "symmetrization of a functor" (see sect. 3). This notion exploits a strong interplay between representations of the general linear group and the symmetric group coming from the action of these groups on the tensor power of a space, and may be thought of as a generalization of a classical notion of symmetrization of a representation.

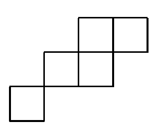
This paper is a first part of my work on homological algebra in the category of functors. In the next article ([C2]) I partially expand computations of

Ext–groups between twisted Weyl and Schur functors to the case of diagrams of different weights. As it is not surprising for a reader of [FLS], [FS], [FFSS], the essential role in that work is played by the De–Rham complex. Its appropriate generalization to the case of an arbitrary Young diagram turned out to be an object complicated and interesting for its own. I investigate it in detail in a separate article ([C1]).

2 Recollections of diagrams and functors

We start by collecting some basic facts concerning Young diagrams and functors one can associate to them. A Young diagram λ of weight d is just a weakly decreasing sequence of positive integers $(\lambda_1, \ldots, \lambda_l)$ with $\sum_{j=1}^{l} \lambda_j = d =: |\lambda|$. We can associate to a Young diagram λ the conjugate diagram $\widetilde{\lambda}$ whose rows are columns of λ (formally: $\widetilde{\lambda}_k = \#\{j : \lambda_j \leq k\}$). We will consider the partial order of dominance on the set of Young diagrams. We say that λ dominates μ ($\mu \trianglelefteq \lambda$) if for all j we have $\sum_{i \leq j} \lambda_i \leq \sum_{i \leq j} \mu_i$. This partial order may be enriched to the total lexicographic order: $\mu \leq \lambda$ if for the least i such that $\mu_i \neq \lambda_i$, we have $\mu_i > \lambda_i$. The direction of dominance and lexicographic relations looks strange, since the lesser diagram is the longer rows it has. The reason is that the terminology in my two main references: [ABW] and [CPS] is not consistent. I decided to follow the conventions of [ABW] when dealing with Schur functors etc, but I follow [CPS] with respect to the direction of orders.

Given two diagrams $\mu \subseteq \lambda$ (ie. $\mu_j \leq \lambda_j$ for all j), we may form a skew diagram λ/μ which should be imagined as a diagram λ with deleted boxes belonging to μ. Here is a picture for $(4,3,1)/(2,1)$



Throughout this paper $\mathbf{k}$ will be a field of positive characteristic p and $\mathcal{P}_d$ will denote the category of homogeneous strict polynomial functors of degree d over $\mathbf{k}$ (see [FS], sect. 2). All Ext–groups will be computed in $\mathcal{P}_d$ for appropriate d. We now recall certain important objects in $\mathcal{P}_d$. The most

fundamental are: the d–th tensor power $I^d(V) := V^{\otimes d}$, the d–th symmetric power $S^d(V) := (V^{\otimes d})_{\Sigma_d}$, the d–th divided power $D^d(V) := (V^{\otimes d})^{\Sigma_d}$, (the last two functors are not isomorphic for $d \geq p$) and the d–th exterior power $\Lambda^d(V) := (V^{\otimes d})^{\Sigma_d} \simeq (V^{\otimes d})_{\Sigma_d}$ for the alternating action of Σ_d on the tensor power (this definition needs a modification for $p = 2$, I will discuss it in detail in the next section). There are also well known transformations between these functors eg. the inclusion $c_d : \Lambda^d \longrightarrow I^d$ and the epimorphism $m_d : I^d \longrightarrow S^d$. Given a diagram λ of weight d, we put $\Lambda^\lambda := \Lambda^{\lambda_1} \otimes \ldots \Lambda^{\lambda_l}$ and $c_\lambda := c_{\lambda_1} \otimes \ldots \otimes c_{\lambda_l} : \Lambda^\lambda \longrightarrow I^d$. In the same fashion we define $S^{\widetilde{\lambda}}$ and $m_{\widetilde{\lambda}} : I^d \longrightarrow S^{\widetilde{\lambda}}$, but one should remember that $m_{\widetilde{\lambda}}$ acts in a "conjugate manner" ie. we gather the elements which have indices belonging to the same column (see [ABW], sect. II.1). We are now in a position to introduce a more complicated object. The Schur functor S_λ is defined as the image of the composition $m_{\widetilde{\lambda}} \circ c_\lambda$. It comes with two structural transformations: the epimorphism $\phi_\lambda : \Lambda^\lambda \longrightarrow S_\lambda$ and monomorphism $\psi_\lambda : S_\lambda \longrightarrow S^{\widetilde{\lambda}}$, which in extreme cases give isomorphisms $S_{(d)} \simeq \Lambda^d$, $S_{(1^d)} \simeq S^d$.
There is a useful contravariant duality in the category $\mathcal{P}_d$ called the Kuhn duality: $F^\#(V) = (F(V^*))^*$ where V^* means the $\mathbf{k}$–dual space. It is easy to check that $(D^d)^\# \simeq S^d$ while Λ^d is selfdual. We will also consider the Kuhn duals of Schur functors which are called Weyl functors and denoted by W_λ. The independent definition of Weyl functor is, of course, as the image of the composition $D^{\widetilde{\lambda}} \longrightarrow I^d \longrightarrow \Lambda^\lambda$.

All these constructions may also be applied to skew diagrams. Although skew Schur and Weyl functors play less important role in the theory (in fact the Littlewood–Richardson rule ([Bo]) says that any skew Schur functor has a filtration with a graded object being a sum of Schur functors), they are often useful in inductive arguments.

We need two important homological properties of Weyl and Schur functors.

Fact 2.1 *For any skew diagrams $\mu/\mu', \lambda/\lambda'$, $\mathrm{Ext}^n(W_{\mu/\mu'}, S_{\lambda/\lambda'}) = 0$, for $n > 0$.*

Fact 2.2 *If for some $n > 0$, $\mathrm{Ext}^n(S_\mu, S_\lambda) \neq 0$, then $\mu \rhd \lambda$.*

Fact 2.1 for solid (ie. not skew) diagrams is mentioned in the proof of ([CPS], Th. 3.11). The general case follows immediately from the Littlewood–Richardson rule.

Fact 2.2 is a part of ([CPS], Lemma 3.2). Both facts are purely formal consequences of the axioms for a "highest weight category" ([CPS], Def. 3.1). ∎

I would like to finish this section by introducing the main technical tool, which will be used repeatedly in the next sections. This tool is the Decomposition Formula. Let $\mathcal{P}^n$ denote the category of strict polynomial functors in n variables. The Decomposition Formula ([ABW], Th. II.4.11) provides an extremely useful filtration of a functor in two variables $S_{\lambda/\mu}(V \oplus W)$.

Fact 2.3 (Decomposition Formula) *The bifunctor $S_{\lambda/\mu}(V \oplus W)$ has a filtration $M_\alpha(V,W)$ (for α satisfying $\mu \subseteq \alpha \subseteq \lambda$) and an order in the filtration comes from the lexicographic order among α. Its associated graded object is*

$$\bigoplus_{\mu \subseteq \alpha \subseteq \lambda} S_{\alpha/\mu}(V) \otimes S_{\lambda/\alpha}(W).$$

Iterating this procedure we get a filtration of the n–functor $S_{\lambda/\mu}(V_1 \oplus \ldots \oplus V_n)$.

Corollary 2.4 *The functor in n variables $S_{\lambda/\mu}(V_1 \oplus \ldots \oplus V_n)$ has a filtration $M_{\mu \subseteq \alpha^1 \subseteq \ldots \subseteq \alpha^{n-1} \subseteq \lambda}$, with ordering coming from the n–fold lexicographic order*

(ie. to compare sequences $(\alpha^1, \ldots, \alpha^{n-1})$ i $(\alpha'^1, \ldots, \alpha'^{n-1})$ we pick the smallest i such that $\alpha^i \neq \alpha'^i$ and compare lexicographically α^i and α'^i). Its graded object is

$$\bigoplus_{\mu \subseteq \alpha^1 \subseteq \ldots \subseteq \alpha^{n-1} \subseteq \lambda} S_{\alpha^1/\mu}(V_1) \otimes \ldots \otimes S_{\alpha^{n-1}/\alpha^{n-2}}(V_{n-1}) \otimes S_{\lambda/\alpha^{n-1}}(V_n).$$

Of course, we get an analogous decomposition for twisted Schur functors and for Weyl functors. This filtration is a powerful tool in computations of Ext–groups, since as it was observed in ([FFSS], pp. 671–672), the evident adjoint functors between $\mathcal{P}$ and $\mathcal{P}^n$ yield an isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}}(F_1 \otimes \ldots \otimes F_n, S^{(i)}_{\lambda/\mu}) = \mathrm{Ext}^*_{\mathcal{P}^n}(F_1(V_1) \otimes \ldots \otimes F_n(V_n), S^{(i)}_{\lambda/\mu}(V_1 \oplus \ldots \oplus V_n)),$$

I write down spaces $V_1, \ldots, V_n$ in the right–hand side of the formula to emphasize the dependence of a functor on all n variables. We recall from ([FFSS], pp. 672), that the "Kunneth formula" gives an isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}^n}(F_1(V_1) \otimes \ldots \otimes F_n(V_n), S^{(i)}_{\alpha^1/\mu}(V_1) \otimes \ldots \otimes S^{(i)}_{\lambda/\alpha^{n-1}}(V_n)) =$$

$$= \mathrm{Ext}^*_{\mathcal{P}}(F_1, S^{(i)}_{\alpha^1/\mu}) \otimes \ldots \otimes \mathrm{Ext}^*_{\mathcal{P}}(F_n, S^{(i)}_{\lambda/\alpha^{n-1}}).$$

Thus the Decomposition Formula leads to a spectral sequence, which we will call the Decomposition Spectral Sequence.

Corollary 2.5 *There exists a spectral sequence converging to* $\mathrm{Ext}^*_{\mathcal{P}}(F_1 \otimes \ldots \otimes F_n, S^{(i)}_{\lambda/\mu})$*, whose* E^1*–term has the form*

$$E^1_{ij} = \bigoplus_{i_1+\ldots+i_n=i+j} \mathrm{Ext}^{i_1}_{\mathcal{P}}(F_1, S^{(i)}_{\alpha^1/\mu}) \otimes \ldots \otimes \mathrm{Ext}^{i_n}_{\mathcal{P}}(F_n, S^{(i)}_{\lambda/\alpha^{n-1}}),$$

where j *stands for a place of* $(\alpha^1, \ldots, \alpha^{n-1})$ *in the* $(n-1)$*–fold lexicographic order.*

Analogous sequences also exist for $\mathrm{Ext}^*_{\mathcal{P}}(F_1 \otimes \ldots \otimes F_n, W^{(i)}_{\lambda/\mu})$, $\mathrm{Ext}^*_{\mathcal{P}}(S^{(i)}_{\lambda/\mu}, F_1 \otimes \ldots \otimes F_n)$ *etc.*

In the present paper we will mainly deal with a very special case of the Decomposition Formula (already considered in [FFSS]), namely the one for the diagram (1^d). The Decomposition Formula in this case splits and takes the form of the known formula

$$S^{d(i)}(V \oplus W) = \bigoplus_{j+k=d} S^{j(i)}(V) \otimes S^{k(i)}(W).$$

Hence the Decomposition Spectral Sequence also splits and gives the formula:

$$\mathrm{Ext}^*_{\mathcal{P}}(F_1 \otimes \ldots \otimes F_n, S^{d(i)}) = \mathrm{Ext}^*_{\mathcal{P}}(F_1, S^{|F_1|/p^i(i)}) \otimes \ldots \otimes \mathrm{Ext}^*_{\mathcal{P}}(F_n, S^{|F_n|/p^i(i)}),$$

for any homogeneous functors $F_1, \ldots, F_n$. We get analogous formulae for the divided and exterior powers and for products of homogeneous functors on the second variable instead of the first. The Schur functors for which the Decomposition Formula takes this simplest form where investigate in detail in [FFSS] where they were called "exponential functors" (see [FFSS], p. 670). These particular instances of the Decomposition Formula and Decomposition Spectral Sequence will be referred to as the Exponential Formula. Some more advanced applications of the Decomposition Formula and Decomposition Spectral Sequence will appear in [C1] and [C2].

3 Symmetrization of a functor

Let $\mathcal{F}_{\Sigma_d}$ denote the category of additive $\mathbf{k}$–linear functors from the category of graded finitely generated Σ_d–modules to the category of graded finitely generated $\mathbf{k}$–spaces ($\mathbf{k}$–linearity of a functor means that the structural map $\mathrm{Hom}_{\mathbf{k}[\Sigma_d]}(M,N) \longrightarrow \mathrm{Hom}_{\mathbf{k}}(f(M),f(N))$ is $\mathbf{k}$–linear; all morphisms, actions etc. are assumed to preserve grading). We will call the objects of this category Σ_d–functors and the morphisms Σ_d–transformations. Observe that for any Σ_d–functor f, an assignment $V \mapsto f(V^{\otimes d})$ (we regard V as concentrated in degree 0) defines a homogeneous strict polynomial functor of degree d. If so happens, we say that a Σ_d–functor is a symmetrization of the respective strict polynomial functor. In fact, we often define strict polynomial functors just giving their symmetrizations, eg. $S^d = f(V^{\otimes d})$, for the Σ_d–functor $f(M) = (M)_{\Sigma_d}$. Usually (if it causes no confusion) we will denote the symmetrization of a functor by the same letter but small. For example it is clear what we mean by $s^\lambda, d^\lambda, s_\lambda, w_\lambda$, eg: $s_\lambda(M) := \mathrm{im}((M^{alt})^{\Sigma_\lambda} \longrightarrow M \longrightarrow (M)_{\Sigma_{\tilde{\lambda}}})$ (for any Σ_d–module M, M^{alt} stands for $M \otimes \mathrm{sgn}$). It is also self–evident how these functors behave with respect to the grading: degree of a tensor product is just a sum of degrees of factors. One should be more cautious in the case of the exterior power for two reasons. The first is that the invariants and coinvariants of the alternating action are not isomorphic Σ_d–functors in general. So we should distinct between $\lambda^\lambda_{inv}(M) = (M^{alt})^{\Sigma_\lambda}$, and $\lambda^\lambda_{coinv}(M) = (M^{alt})_{\Sigma_\lambda}$, (by the way, both Σ_d–functors are symmetrizations of the strict polynomial functor Λ^d). The second reason is a pathology which happens for $p=2$, when we cannot define the exterior power as the (co)invariants of the alternating action. We will briefly discuss a modification which is needed in definition of λ^d_{inv} (the argument for λ^d_{coinv} is similar). We start with $d=2$. Then we may define λ^2_{inv} as the kernel of the Σ_2–epimorphism $id \longrightarrow s^2$. For an arbitrary d we define λ^d_{inv} to be $\bigcap_{\Sigma_2 \subset \Sigma_d} \ker(id \longrightarrow id_{\Sigma_2})$. The definition meets our expectations because Σ_d is generated by the set of transpositions. Its main advantage is that it referres only to a given action of the symmetric group. Therefore from now on we will need not to consider the case $p=2$ separately.

It is worth mentioning that the idea of symmetrization is present in many constructions in representation theory. For example, applying certain Σ_d–functors to the Σ_d–bimodule $\mathbf{k}[\Sigma_d]$ we obtain some important Σ_d–modules (eg. $s_\lambda(\mathbf{k}[\Sigma_d])$ is so–called Specht module Sp_λ). Finally, observe that we still have the Kuhn duality. Namely, for a Σ_d–functor f, we put $f^{\#}(M) := (f(M^*))^*$, where * at the right–hand side means the $\mathbf{k}$–linear duality. Now it is easy to check that $s_\lambda^{\#} = w_\lambda$ and in particular $\lambda_{inv}^{\lambda\#} = \lambda_{coinv}^{\lambda}$.

Later on we will focus on symmetrizations satisfying some additional technical condition.

Definition 3.1 *A Σ_d–functor f^{in} is called an injective symmetrization of a functor $F \in \mathcal{P}_d$ if $f^{in}(V^{\otimes d}) = F(V)$, and there exists a Σ_d–transformation $\psi : f^{in} \longrightarrow \oplus_k s^{\lambda^k}$ such that $\psi(V^{\otimes d})$ is an inclusion.*
Similarly, we say that a symmetrization f^{pr} is a projective symmetrization if there exists a Σ_d–transformation $\phi : \oplus_k d^{\lambda^k} \longrightarrow f^{pr}$ whose evaluation on $V^{\otimes d}$ is onto.

The importance of this class of symmetrizations comes from the fact that the family $\{S^\lambda\}$ (resp. $\{D^\lambda\}$) forms a set of injective (resp. projective) generators of $\mathcal{P}_d$ ([FS], Th. 2.10). In order to express concisely another important property of injective symmetrizations we need the following definition.

Definition 3.2 *We say that a Σ_d–module M is a Y–permutative module if $M \simeq \bigoplus_{i=1}^n M_i$, where $M_i = \mathbf{k} \otimes_{\mathbf{k}[H_i]} \mathbf{k}[\Sigma_d]$ for some Young subgroups H_i (cf. [JK], sect. 1.3).*

The most important example of a Y–permutative module is a Σ_d–module $V^{\otimes d}$ for any space V. Now we can go back to symmetrizations.

Fact 3.3

1. *Any strict polynomial functor has an injective and a projective symmetrization.*
2. *Let $f^{in} \xrightarrow{\psi_0} s^{\lambda^0}$ be an injective symmetrization of F (I adopt the convention: $s^{\lambda^0} := \oplus_k s^{\lambda^{0k}}$). Then ψ_0 may be extended to a sequence of Σ_d–transformations*

   $$f^{in} \xrightarrow{\psi_0} s^{\lambda^0} \xrightarrow{\psi_1} s^{\lambda^1} \xrightarrow{\psi_2} \dots \xrightarrow{\psi_l} s^{\lambda^l},$$

   such that for any Y–permutative Σ_d–module M, the sequence

   $$0 \longrightarrow f^{in}(M) \xrightarrow{\psi_0} s^{\lambda^0}(M) \xrightarrow{\psi_1} s^{\lambda^1}(M) \xrightarrow{\psi_2} \dots \xrightarrow{\psi_l} s^{\lambda^l}(M) \longrightarrow 0$$

   is exact.
   An analogous fact holds for a projective symmetrization.

Proof: We start with comparing transformations and Σ_d–transformations in a very special case.

Lemma 3.4 *For any diagrams λ, λ' of weight d*

$$\mathrm{Hom}_{\mathcal{P}_d}(S^{\lambda}, S^{\lambda'}) = \mathrm{Hom}_{\mathcal{F}_{\Sigma_d}}(s^{\lambda}, s^{\lambda'}).$$

Proof: Since $\mathrm{Hom}_{\mathcal{P}}(S^k, S^k) = \mathbf{k}$, then applying the Exponential Formula to both variables we get a description of $\mathrm{Hom}(S^{\lambda}, S^{\lambda'})$. From a purely combinatorial point of view we may describe it as a space having basis labeled by matrices consisting of positive integers satisfying the following conditions: each row is weakly decreasing, the sum of numbers in the ith row equals λ_i, the sum of numbers in the ith column equals λ'_j (cf. [FFSS], Cor. 1.8). Looking at the construction of the Decomposition Formula it is easy to find the transformation corresponding to a given element of the basis. Namely, to a matrix $[a_{ij}]$ we associate a composition

$$S^{\lambda} \longrightarrow \bigotimes_{ij} S^{a_{ij}} \simeq \bigotimes_{ij} S^{a_{ji}} \longrightarrow S^{\lambda'},$$

where the first and third arrows are respectively tensor products of iterated comultiplication and multiplication in the symmetric power, while the second arrow interchanges factors which on the left–hand side are ordered with respect to rows and on the right–hand side with respect to columns (cf. [FFSS], pp. 673–676). Thus we see that any transformation is a composition of transformations of three simple types (possibly tensored with identities): the multiplication $S^a \otimes S^b \longrightarrow S^{a+b}$, the comultiplication $S^{a+b} \longrightarrow S^a \otimes S^b$, and the transposition $S^a \otimes S^b \longrightarrow S^b \otimes S^a$. These transformations, of course, come from Σ_d–transformations, respectively from the induction, the restriction and the homomorphism of the groups. Thus we have shown that any transformation comes from some Σ_d–transformation.

It remains to show that a nontrivial Σ_d–transformation $\psi : s^{\lambda} \longrightarrow s^{\lambda'}$ has a nontrivial evaluation $\psi(V^{\otimes d}) : S^{\lambda} \longrightarrow S^{\lambda'}$. It will be more convenient to work with the Kuhn dual of ψ, which is a Σ_d–transformation $\psi^{\#} : d^{\lambda'} \longrightarrow d^{\lambda}$. Suppose that $\mathrm{im}(\psi^{\#})(V^{\otimes d}) = 0$. Then $\mathrm{im}(\psi^{\#})$ is a left exact functor vanishing on all Σ_d–modules $V^{\otimes d}$. But if $\dim(V) = d$ then $V^{\otimes d}$ contains $\mathbf{k}[\Sigma_d]$ as a direct summand. Thus $\mathrm{im}(\psi^{\#})(\mathbf{k}[\Sigma_d]) = 0$. But since any finitely generated Σ_d–module embeds into a free module and $\mathrm{im}(\psi^{\#})$ preserves monomorphisms, it must be the trivial functor. ∎

In order to construct an injective symmetrization of a strict polynomial functor F we consider the beginning of a finite injective resolution of F by the sums of products of symmetric powers

$$0 \longrightarrow F \xrightarrow{\psi_0'} S^{\lambda^0} \xrightarrow{\psi_1'} S^{\lambda^1} \longrightarrow \dots .$$

The existence of such a finite resolution follows easily from the axioms for a highest weight category ([CPS], Def. 3.1) and the Littlewood–Richardson rule. Of course $F = \ker(\psi_1')$. Thanks to Lemma 3.4 we know that the transformation ψ_1' comes from the Σ_d–transformation $\psi_1 : s^{\lambda^0} \longrightarrow s^{\lambda^1}$. Therefore the Σ_d–functor $f^{in} := \ker(\psi_1)$ is an injective symmetrization of F. This finishes the proof of the first part of Fact 3.3.

To obtain the second part we take the whole resolution

$$0 \longrightarrow F \xrightarrow{\psi_0'} S^{\lambda^0} \xrightarrow{\psi_1'} S^{\lambda^1} \xrightarrow{\psi_2'} \dots \xrightarrow{\psi_l'} S^{\lambda^l} \longrightarrow 0.$$

According to Lemma 3.4 it lifts to the sequence of Σ_d–transformations

$$0 \longrightarrow f^{in} \xrightarrow{\psi_0} s^{\lambda^0} \xrightarrow{\psi_1} s^{\lambda^1} \xrightarrow{\psi_2} \dots \xrightarrow{\psi_l} s^{\lambda^l} \longrightarrow 0,$$

whose evaluation on $V^{\otimes d}$ is exact. The exactness of evaluation on an arbitrary Y–permutative module follows from the fact, that any such a module is a direct summand in a finite sum of $V^{\otimes d}$ for a space V of dimension d. ■

Of course, an injective symmetrization is not unique. The exterior power provides the easiest example since both $\lambda^d_{inv} \longrightarrow id$ and $\lambda^d_{coinv} \longrightarrow id$ are its injective symmetrizations. The point is that however the arrow $\lambda^d_{coinv} \longrightarrow id$ ("averaging to invariants") is not monomorphic but its evaluation on $V^{\otimes d}$ is, which is sufficient. For a similar reason s_λ is not only injective but also projective symmetrization of S_λ (an analogous fact holds for Weyl functors). We finish this section with one more tricky example of an injective symmetrization. We shall find an injective symmetrization of $S^{d(1)}$. To do this we consider the beginning of the De-Rham complex (cf. [FS], Th. 4.1) augmented by its 0th cohomology

$$0 \longrightarrow S^{d(1)} \longrightarrow S^{pd} \xrightarrow{\delta} S^{pd-1} \otimes \Lambda^1$$

and we put $s^{d(1)}$ to be the kernel of the Σ_{pd}–transformation corresponding to the De–Rham differential δ. A point which may be overlooked is that

$s^{d(1)}(V^{\otimes pd}) = S^{d(1)}(V)$ regarded as a graded space has degrees of nontrivial components multiplied by p. Taking into account this phenomenon it is convenient to say that the Frobenius twist regarded as a functor on the graded spaces multiplies grading by p (ie. we put $V_{pi}^{(1)} := V_i$ and 0 elsewhere).

4 The main theorems

We start with introducing some notation. Let $A_i = \mathrm{Ext}^*(I^{(i)}, I^{(i)})$, $B_i = (A_i)^{\otimes d} \otimes \mathbf{k}[\Sigma_d]$ with a grading in A_i coming from the grading on Ext–groups and the group algebra placed in degree 0. We endow B_i with a structure of Σ_d–bimodule given by the formula $\sigma.a_1 \otimes \ldots \otimes a_d \otimes e_\tau.\lambda = a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(d)} \otimes e_{\sigma\tau\lambda}$. Sometimes it will be more convenient to look at B_i as a bimodule with the action: $\sigma.a_1 \otimes \ldots \otimes a_d \otimes e_\tau.\lambda = a_{\lambda^{-1}(1)} \otimes \ldots \otimes a_{\lambda^{-1}(d)} \otimes e_{\sigma\tau\lambda}$. An isomorphism between these two structures is given by the map $a_1 \otimes \ldots \otimes a_d \otimes e_\tau \mapsto a_{\tau^{-1}(1)} \otimes \ldots \otimes a_{\tau^{-1}(d)} \otimes e_\tau$. The main computational result of [FS] was determination of A_i. It is a graded space which is one–dimensional in even degrees smaller than $2p^i$ and trivial elsewhere. Now it follows easily from the Exponential Formula that

$$\mathrm{Ext}^*(I^{d(i)}, I^{d(i)}) = B_i,$$

as a graded Σ_d–bimodule. It is also easy to see that all computations of ([FFSS], sect. V) may be written in the form

$$\mathrm{Ext}^*(F^{(i)}, G^{(i)}) = (g^{in}(f^{pr\#}(B_i)),$$

(ie. we first apply $f^{pr\#}$ to B_i as a left Σ_d–module and then we apply g^{in} to the resulting right Σ_d–module). The main result of this paragraph is determination of a class of functors for which such a description holds.

We start with a slight generalization of the results of [FFSS]

Fact 4.1

1. *For any diagrams $\mu, \mu', \lambda, \lambda'$ of weight d we have*

$$\mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda(i)}) = s^\mu(s^\lambda(B_i)) = s^\lambda(s^\mu(B_i)),$$

where we apply s^μ to the left Σ_d–structure and s^λ to the right one.

2. *Moreover, for any transformation* $\psi : S^\lambda \longrightarrow S^{\lambda'}$ *the induced map*

$$\psi_*^{(i)} : \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda(i)}) \longrightarrow \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda'(i)})$$

under the above isomorphisms, may be described in two ways: either as $\psi(s^\mu(B_i))$ *or as* $s^\mu(\psi(B_i))$. *Similarly, for any transformation* $\phi : D^\mu \longrightarrow D^{\mu'}$ *the induced map* $\phi^{(i)*}$ *may be described either as* $\phi^\#(s^\lambda(B_i))$ *or as* $s^\lambda(\phi^\#(B_i))$.

Proof: The second description in the first part of the fact for $\lambda = \mu = (1^d)$ is just ([FFSS], Th. 4.5). The general case follows from the Exponential Formula. The first description is the Kuhn dual of the second.

We now turn to the proof of the second part of Fact 4.1 To get the first description we lift ψ to some $\widetilde{\psi} : I^d \longrightarrow I^d$ (the existence of such a lift follows from the projectivity of I^d) and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^*(D^{\mu(i)}, I^{d(i)}) & \xrightarrow{\widetilde{\psi}_*^{(i)}} & \mathrm{Ext}^*(D^{\mu(i)}, I^{d(i)}) \\ \downarrow m_{\lambda *}^{(i)} & & \downarrow\ m_{\lambda' *}^{(i)} \\ \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda(i)}) & \xrightarrow{\psi_*^{(i)}} & \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda'(i)}). \end{array}$$

We recall from ([FFSS], sect. V), that the vertical arrows are epimorphic and, according to the first part of the fact, they may be identified respectively with $m_\lambda(s^\mu(B_i))$ and $m_{\lambda'}(s^\mu(B_i))$. Moreover, since $\widetilde{\psi}$ is just multiplication by an element of $\mathbf{k}[\Sigma_d]$, we have $\widetilde{\psi}_*^{(i)} = \widetilde{\psi}(s^\mu(B_i))$. Hence if we replace $\psi_*^{(i)}$ by $\psi(s^\mu(B_i))$, the diagram remains commutative. But since the left vertical arrow is onto, there is at most one bottom arrow making the diagram commutative. Thus $\psi_*^{(i)} = \psi(s^\mu(B_i))$.

In order to obtain the second description we consider the diagram

$$\begin{array}{ccc} \mathrm{Ext}^*(I^{d(i)}, S^{\lambda(i)}) & \xrightarrow{\psi_*^{(i)}} & \mathrm{Ext}^*(I^{d(i)}, S^{\lambda'(i)}) \\ \downarrow \eta^{(i)*} & & \downarrow \eta^{(i)*} \\ \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda(i)}) & \xrightarrow{\psi_*^{(i)}} & \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda'(i)}), \end{array}$$

where $\eta_\mu : D^\mu \longrightarrow I^d$ is the natural inclusion. After identifying known arrows we get

$$\begin{array}{ccc} \mathrm{Ext}^*(I^{d(i)}, S^{\lambda(i)}) & \xrightarrow{\psi(B_i)} & \mathrm{Ext}^*(I^{d(i)}, S^{\lambda'(i)}) \\ \downarrow \eta_\mu^\#(s^\lambda(B_i)) & & \downarrow \eta_\mu^\#(s^{\lambda'}(B_i)) \\ \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda(i)}) & \xrightarrow{\psi_*^{(i)}} & \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda'(i)}). \end{array}$$

(for the vertical arrows we use a description which is Kuhn dual to that from [FFSS] while for the top arrow we use the previous description for $\mu = (1^d)$). By the epimorphicity of the left vertical arrow it suffices to observe that $s^\mu(\psi(B_i))$ makes the diagram commutative. The case of a transformation between divided powers follows from the Kuhn duality. ■

As it was seen in the proof, all the assertions of Fact 4.1 were quite formal consequences of ([FFSS], Th. 4.5) where the groups $\mathrm{Ext}^*(D^{d(i)}, S^{d(i)})$ were computed. But this generalization, technically rather straightforward, will turn out to be extremely useful, for $\{D^\lambda\}$ (resp. $\{S^\lambda\}$) form the set of projective (resp. injective) generators of $\mathcal{P}_d$. Therefore our strategy for computing Ext–groups will be, roughly speaking, as follows. To compute $\mathrm{Ext}^*(F^{(i)}, G^{(i)})$ we take a resolution of F by (sums of products of) divided powers and a resolution of G by symmetric powers, we twist them i times and we compute Ext–groups between the (twisted) resolutions. By Fact 4.1 we know these Ext–groups and also the arrows between them. This, under some additional hypotheses, will enable us to calculate the original Ext–groups.

For some technical reasons we will also need an "additive analogue" of the last fact. Put $A'_j = \mathrm{Hom}(jI, I)$, $B'_j = (A'_j)^{\otimes d} \otimes \mathbf{k}[\Sigma_d]$, where jI denotes $I^{\oplus j}$. Thus, A'_j is just j–dimensional space concentrated in degree 0.

Fact 4.2

1. *For any diagrams $\mu, \mu', \lambda, \lambda'$ of weight d we have*

$$\mathrm{Ext}^*(D^\mu \circ jI, S^\lambda) = s^\mu(s^\lambda(B'_j)) = s^\lambda(s^\mu(B'_j)),$$

where we apply s^μ to the left Σ_d–structure and s^λ to the right one.

2. *Moreover, for any transformation $\psi : S^\lambda \longrightarrow S^{\lambda'}$ the induced map*

$$\psi_* : \mathrm{Ext}^*(D^\mu \circ jI, S^\lambda) \longrightarrow \mathrm{Ext}^*(D^\mu \circ jI, S^{\lambda'})$$

under the above isomorphisms, may be described in two ways: either as $\psi(s^\mu(B'_j))$ or as $s^\mu(\psi(B'_j))$. Similarly, for any transformation $\phi : D^\mu \longrightarrow D^{\mu'}$ the induced map $(\phi \circ jI)^*$ may be described either as $\phi^\#(s^\lambda(B'_j))$ or as $s^\lambda(\phi^\#(B'_j))$.

Proof: First, observe that by the projectivity of $D^\mu \circ jI$ and the injectivity of S^λ the map $s^\mu(s^\lambda(B'_j)) \longrightarrow \mathrm{Hom}(D^\mu \circ jI, S^\lambda)$ is an epimorphism. Hence, it suffices to show that both the spaces have the same dimensions. According to the Exponential Formula it suffices to do this for $\mu = \lambda = (1^d)$. Then $s^d(s^d(B'_j)) = S^d(A'_j)$, while

$$\mathrm{Hom}(D^d \circ jI), S^d) = \bigoplus_{i_1+\ldots+i_j=d} \mathrm{Hom}(D^{i_1}, S^{i_1}) \otimes \ldots \otimes \mathrm{Hom}(D^{i_j}, S^{i_j}) =$$
$$= \bigoplus_{i_1+\ldots+i_j=d} S^{i_1}(A'_1) \otimes \ldots \otimes S^{i_j}(A'_1).$$

The dimensions of these spaces are clearly equal.

The proof of the second part goes in a similar fashion to that of the second part of Fact 4.1. The only difference is that the epimorphicity of vertical arrows in the diagram

$$\begin{array}{ccc} \mathrm{Ext}^*(D^\mu \circ jI, I^d) & \stackrel{\widetilde{\psi}_*}{\longrightarrow} & \mathrm{Ext}^*(D^\mu \circ jI, I^d) \\ \downarrow m_{\lambda_*} & & \downarrow m_{\lambda'_*} \\ \mathrm{Ext}^*(D^\mu \circ jI, S^\lambda) & \stackrel{\psi_*}{\longrightarrow} & \mathrm{Ext}^*(D^\mu \circ jI, S^{\lambda'}) \end{array}$$

immediately follows from the projectivity of $D^\mu \circ jI$. This concludes the proof of Fact 4.2. ■

We are now in a position to state our first main result.

Theorem 4.3

1. *For any $F \in \mathcal{P}_d$ and any diagram μ of weight d:*

$$\mathrm{Ext}^*(D^{\mu(i)}, F^{(i)}) = f^{in}(s^\mu(B_i)),$$

where f^{in} is an arbitrary injective symmetrization of F.

2. *For any transformation $\phi : D^\mu \longrightarrow D^{\mu'}$, the induced map $(\phi^{(i)*})$: $\mathrm{Ext}^*(D^{\mu'(i)}, F^{(i)}) \longrightarrow \mathrm{Ext}^*(D^{\mu(i)}, F^{(i)})$, under the above isomorphisms takes the form $f^{in}(\phi^{\#}(B_i))$.*

Also "additive analogues" of these formulae hold, ie.

$$\mathrm{Hom}(D^\mu \circ jI, F) = f^{in}(s^\mu(B'_j)),$$

and $(\phi \circ jI)^* = f^{in}(\phi^{\#}(B'_j))$.

Proof: We start by proving the additive version of the theorem. In order to get

$$\mathrm{Hom}(D^\mu \circ jI, F) = f^{in}(s^\mu(B'_j)),$$

we extend the map $\psi_0(V^{\otimes d}) : f^{in}(V^{\otimes d}) \longrightarrow s^{\lambda^0}(V^{\otimes d})$ to a resolution

$$0 \longrightarrow F \xrightarrow{\psi_0} S^{\lambda^0} \xrightarrow{\psi_1} S^{\lambda^1} \xrightarrow{\psi_2} \ldots \xrightarrow{\psi_l} S^{\lambda^l} \longrightarrow 0,$$

(from now on we will slightly abuse notation denoting by the same letter a Σ_d–transformation and its evaluation on $V^{\otimes d}$). Since $\mathrm{Ext}^n(D^\mu \circ jI, F) = 0$ for $n > 0$, this complex remains exact after applying $\mathrm{Hom}(D^\mu \circ jI, -)$. Let us consider the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow f^{in}(s^\mu(B'_j)) & \xrightarrow{\psi_0(s^\mu(B'_j))} & s^{\lambda^0}(s^\mu(B'_j)) & \xrightarrow{\psi_1(s^\mu(B'_j))} & s^{\lambda^1}(s^\mu(B'_j)) & \xrightarrow{\psi_2(s^\mu(B'_j))} & \ldots \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow \mathrm{Hom}(D^\mu \circ jI, F) & \xrightarrow{\psi_{0*}} & \mathrm{Hom}(D^\mu \circ jI, S^{\lambda^0}) & \xrightarrow{\psi_{1*}} & \mathrm{Hom}(D^\mu \circ jI, S^{\lambda^1}) & \xrightarrow{\psi_{2*}} & \ldots \end{array}$$

whose bottom row is exact. Note moreover, that thanks to Fact 4.2, the vertical arrows (exist and) are isomorphisms and that the top row is exact by Fact 3.3.2 ($s^\mu(B'_j)$ is a Y–permutative module because it is a tensor product of two Y–permutative modules). Thus, we get an isomorphism $f^{in}(s^\mu(B'_j)) \simeq \mathrm{Hom}(D^\mu \circ jI, F)$ by an easy diagram chasing. Let us notice for the future use that under this identification we have $(\psi_0)_* = \psi_0(s^\mu(B'_j))$. This is important because this time it need not to be true that $\psi_0(s^\mu(B'_j)) = s^\mu(\psi_0(B'_j))$. The easiest example of this pathology is provided by the arrow $\mathrm{Hom}(D^p, D^p) \longrightarrow \mathrm{Hom}(D^p, I^p)$ induced by the inclusion $\psi_0 : D^p \longrightarrow I^p$ which may be thought of as the beginning of an injective resolution of D^p. Indeed: in this case $s^p(\psi_0(B'_1))$ is trivial. The existence of such phenomena will make us to be very careful in the further arguing.

We now turn to the second part of the additive version of the theorem. A transformation $\phi : D^\mu \longrightarrow D^{\mu'}$ induces a commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \longrightarrow f^{in}(s^{\mu'}(B'_j)) & \xrightarrow{\psi_0(s^{\mu'}(B'_j))} & s^{\lambda^0}(s^{\mu'}(B'_j)) & \xrightarrow{\psi_1(s^{\mu'}(B'_j))} & s^{\lambda^1}(s^{\mu'}(B'_j)) \\ \downarrow \phi(jI)^* & & \downarrow s^{\lambda^0}(\phi^\#(B'_j)) & & \downarrow s^{\lambda^1}(\phi^\#(B'_j)) \\ 0 \longrightarrow f^{in}(s^\mu(B'_j)) & \xrightarrow{\psi_0(s^\mu(B'_j))} & s^{\lambda^0}(s^\mu(B'_j)) & \xrightarrow{\psi_1(s^\mu(B'_j))} & s^{\lambda^1}(s^\mu(B'_j)). \end{array}$$

Now it suffices to observe that replacing $\phi(jI)^*$ by $f^{in}(\phi^\#(B'_j))$ does not destroy the commutativity of the diagram. It means that $\phi(jI)^* = f^{in}(\phi^\#(B'_j))$. Again it turns out that we could not take $\phi^\#(f^{in}(B'_j))$ instead of $f^{in}(\phi^\#(B'_j))$. A simple example of the arrow $\mathrm{Hom}(I^p, D^p) \longrightarrow \mathrm{Hom}(D^p, D^p)$ induced by the inclusion $\phi : D^p \longrightarrow I^p$ shows, that the maps $f^{in}(\phi^\#(B'_j))$ and $\phi^\#(f^{in}(B'_j))$ need not to coincide. It looks strange because, as we remember from Fact 4.2, in all further vertical arrows they do coincide. But I recall that we cannot change the order of applying Σ_d–functors also on the left horizontal arrows.

We now turn to the proper version of Theorem 4.3. This time we first twist i times an injective resolution of F and then we apply to it $\mathrm{Ext}^*(D^{\mu(i)}, -)$. According to Fact 4.1 we get the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & f^{in}(s^\mu(B_i)) & \xrightarrow{\psi_0(s^\mu(B_i))} & s^{\lambda^0}(s^\mu(B_i)) & \xrightarrow{\psi_1(s^\mu(B_i))} & s^{\lambda^1}(s^\mu(B_i)) & \xrightarrow{\psi_2(s^\mu(B_i))} & \ldots \\ & & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \mathrm{Ext}^*(D^{\mu(i)}, F^{(i)}) & \xrightarrow{(\psi_0^{(i)})^*} & \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda^0(i)}) & \xrightarrow{(\psi_1^{(i)})^*} & \mathrm{Ext}^*(D^{\mu(i)}, S^{\lambda^1(i)}) & \xrightarrow{(\psi_2^{(i)})^*} & \ldots \end{array}$$

in which all vertical arrows are isomorphisms. In order to finish the proof like in the additive version it is sufficient to show that the bottom row is exact. But we know that it is exact at least starting from the third term, because the top row is exact. It means that in the first hiperExt spectral sequence converging to $\mathrm{hExt}^*(D^{\mu(i)}, \mathbf{C}) = 0$ (where $\mathbf{C}$ stands for the twisted resolution of F), the E^2–term may be nontrivial only at the first two columns. Therefore, by dimension argument, it must be trivial (this argument generalizes (and may be easily derived from) a well known fact that if every third arrow in a long exact sequence is epimorphic then the sequence splits).

The proof of the second part is analogous to the proof of the additive counterpart. This completes the proof of Theorem 4.3 ■

In order to repeat the argument with respect to the second variable we will need an assumption guaranteeing exactness of a complex in the situation when Fact 3.3.2 is not applicable. Moreover, problems with functoriality make the formulation of the result more complicated and make us to introduce another bit of notation. Let $0 \longrightarrow G \xrightarrow{\psi_0} S^{\lambda^0}$ be the beginning of an injective resolution of G and $D^{\mu^0} \xrightarrow{\phi_0} F \longrightarrow 0$ be the beginning of a projective resolution of F. We consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^*(F^{(i)}, G^{(i)}) & \stackrel{(\psi_0^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(F^{(i)}, S^{\lambda^0}) \\ \downarrow (\phi_0^{(i)})^* & & \downarrow (\phi_0^{(i)})^* \\ \mathrm{Ext}^*(D^{\mu^0}, G^{(i)}) & \stackrel{(\psi_0^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(D^{\mu^0(i)}, S^{\lambda^0(i)}). \end{array}$$

According to Theorem 4.3 we may rewrite it as

$$\begin{array}{ccc} \mathrm{Ext}^*(F^{(i)}, G^{(i)}) & \stackrel{(\psi_0^{(i)})^*}{\longrightarrow} & f^{pr\#}(s^{\lambda^0}(B_i)) \\ \downarrow (\phi_0^{(i)})^* & & \downarrow \phi_0^{\#}(s^{\lambda^0}(B_i)) \\ g^{in}(s^{\mu^0}(B_i)) & \stackrel{\psi_0(s^{\mu^0}(B_i))}{\longrightarrow} & s^{\lambda^0}(s^{\mu^0}(B_i)) = s^{\mu^0}(s^{\lambda^0}(B_i)). \end{array}$$

We put $(f^{pr\#}, g^{in})(B_i)$ to be $\mathrm{im}(\psi_0(s^{\mu^0}(B_i))) \cap \mathrm{im}(\phi_0^{\#}(s^{\lambda^0}(B_i)))$. The point of this definition is that in general we cannot identify this space neither with $f^{pr\#}(g^{in}(B_i))$ nor with $g^{in}(f^{pr\#}(B_i))$. Nevertheless, this is certain explicitly defined space which is determined by the symmetrizations f^{in}, g^{pr}. Quite naturally, this space will be our candidate for $\mathrm{Ext}^*(F^{(i)}, G^{(i)})$ in general.

Theorem 4.4 *Assume that* $\mathrm{Ext}^*(F \circ p^i I, G) = 0$ *for* $* > 0$ *(we will call this assumption the "*Ext*–condition"). Then*

$$\mathrm{Ext}^*(F^{(i)}, G^{(i)}) = (f^{pr\#}, g^{in})(B_i).$$

Proof: We take an injective resolution of F, a projective resolution of G and consider the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}^*(F^{(i)}, G^{(i)}) & \stackrel{(\psi_0^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(F^{(i)}, S^{\lambda^0(i)}) & \stackrel{(\psi_1^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(F^{(i)}, S^{\lambda^1(i)}) & \longrightarrow & \\ & & \downarrow (\phi_0^{(i)})^* & & \downarrow (\phi_0^{(i)})^* & & \downarrow (\phi_0^{(i)})^* & & \\ 0 & \longrightarrow & \mathrm{Ext}^*(D^{\mu^0(i)}, G^{(i)}) & \stackrel{(\psi_0^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(D^{\mu^0(i)}, S^{\lambda^0(i)}) & \stackrel{(\psi_1^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(D^{\mu^0(i)}, S^{\lambda^1(i)}) & \longrightarrow & \\ & & \downarrow (\phi_1^{(i)})^* & & \downarrow (\phi_1^{(i)})^* & & \downarrow (\phi_1^{(i)})^* & & \\ 0 & \longrightarrow & \mathrm{Ext}^*(D^{\mu^1(i)}, G^{(i)}) & \stackrel{(\psi_0^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(D^{\mu^1(i)}, S^{\lambda^0(i)}) & \stackrel{(\psi_1^{(i)})^*}{\longrightarrow} & \mathrm{Ext}^*(D^{\mu^1(i)}, S^{\lambda^1(i)}) & \longrightarrow & \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

According to Theorem 4.3 all the rows except perhaps the first and all the columns except perhaps the first are exact. Hence, the prof will be finished by a diagram chasing if we show that the first column is exact. By Theorem 4.3 we have in the first column the sequence

$$0 \longrightarrow \mathrm{Ext}^*(F^{(i)}, G^{(i)}) \stackrel{(\phi_0^{(i)})^*}{\longrightarrow} g^{in}(s^{\mu^0}(B_i)) \stackrel{\phi_1^{\#}(g^{in}(B_i))}{\longrightarrow} g^{in}(s^{\mu^1}(B_i)) \stackrel{\phi_2^{\#}(g^{in}(B_i))}{\longrightarrow} . \ (*)$$

Now we consider the sequence

$$\ldots \stackrel{\phi_2}{\longrightarrow} D^{\mu^1} \circ p^i I \stackrel{\phi_1(p^i I)}{\longrightarrow} D^{\mu^0} \circ p^i I \stackrel{\phi_0(p^i I)}{\longrightarrow} F \circ p^i \longrightarrow 0.$$

Since it is a projective resolution of $F \circ p^i I$ and $\mathrm{Ext}^n(F \circ p^i I, G) = 0$ for $n > 0$, the sequence

$$0 \longrightarrow \mathrm{Hom}^*(F \circ p^i I, G) \stackrel{(\phi_0(p^i I))^*}{\longrightarrow} \mathrm{Hom}(D^{\mu^0} \circ p^i I, G) \stackrel{(\phi_1^{\#}(p^i I))^*}{\longrightarrow} \mathrm{Hom}(D^{\mu^1} \circ p^i I, G) \stackrel{(\phi_2^{\#}(p^i I))^*}{\longrightarrow}$$

is exact. But thanks to Theorem 4.3 we may rewrite it as

$$0 \longrightarrow \mathrm{Hom}^*(F \circ p^i I, G) \stackrel{(\phi_0(p^i I))^*}{\longrightarrow} g^{in}(s^{\mu^0}(B'_{p^i})) \stackrel{\phi_1^{\#}(g^{in}(B'_{p^i}))}{\longrightarrow} g^{in}(s^{\mu^1}(B'_{p^i})) \stackrel{\phi_2^{\#}(g^{in}(B'_{p^i}))}{\longrightarrow} .$$

We now observe that if we neglect the grading then, starting from the second term, our sequence as a sequence of vector spaces is isomorphic to $(*)$, because $\dim(B_i) = \dim(B'_{p^i})$. Therefore our first column is exact starting from the third term (in order to use this argument we have introduced all these "additive analogues"). Thus by the hiperExt–argument which we used at the end of the proof of Theorem 4.3, the whole column must be exact. This completes the proof of Theorem 4.4. ■

5 Applications to classical functors

We would like to find functors satisfying the assumption of Theorem 4.4. The most important example is provided by $F = W_{\mu/\mu'}$ and $G = S_{\lambda/\lambda'}$. Indeed, Fact 2.1 together with the Decomposition Formula show that they satisfy the Ext–condition. Moreover, in this particular case the statement of the theorem may be formulated in a much simpler way.

Theorem 5.1 *For any skew diagrams μ/μ', λ/λ' of weight d we have*

$$\mathrm{Ext}^*(W^{(i)}_{\mu/\mu'}, S^{(i)}_{\lambda/\lambda'}) = s_{\mu/\mu'}(s_{\lambda/\lambda'}(B_i)) = s_{\lambda/\lambda'}(s_{\mu/\mu'}(B_i)).$$

Moreover, for any transformation $\psi : s_{\lambda/\lambda'} \longrightarrow s_{\lambda^1/\lambda^{1'}}$ the induced map ψ_* : $\mathrm{Ext}^*(W^{(i)}_{\mu/\mu'}, S^{(i)}_{\lambda/\lambda'}) \longrightarrow \mathrm{Ext}^*(W^{(i)}_{\mu/\mu'}, S^{(i)}_{\lambda^1/\lambda^{1'}})$ takes the form $\psi(s_{\mu/\mu'}(B_i)) = s_{\mu/\mu'}(\psi(B_i))$. An analogous fact also holds for transformations of the first variable.

Proof: When we look once again at the proof of Theorem 4.4, we see that the reason for which we could not obtain a simpler description of the Ext–groups was that in general the map $\psi : s^{\lambda} \longrightarrow s^{\lambda'}$ induces on $\mathrm{Ext}^*(F^{(i)}, -)$ the map $f^{in\#}(\psi(B_i))$ which may be different from the map $\psi(f^{in\#}(B_i))$. We will show that for $F = W_{\mu/\mu'}$ and $G = S_{\lambda/\lambda'}$ these two maps coincide. By arguments used in the proof of the second part of Fact 4.1 it suffices to show the lemma (which is very specific to Weyl and Schur functors):

Lemma 5.2 *For any diagrams μ/μ' and λ, the map $m_\lambda : id \longrightarrow s^{\lambda}$ induces an epimorphism $\mathrm{Ext}^*(W^{(i)}_{\mu/\mu'}, I^{d(i)}) \longrightarrow \mathrm{Ext}^*(W^{(i)}_{\mu/\mu'}, S^{\lambda(i)})$.*

Proof: Of course, it suffices to show the additive version of the lemma. Applying the Decomposition Formula to $W_{\mu/\mu'} \circ p^i I$, we reduce the proof to showing that there is an epimorphism $\mathrm{Hom}(W_{\mu/\mu'}, I^d) \longrightarrow \mathrm{Hom}(W_{\mu/\mu'}, S^{\lambda})$. To do this it suffices to show that $\mathrm{Ext}^1(W_{\mu/\mu'}, \ker(m_\lambda)) = 0$. By the Littlewood–Richardson rule, any skew Schur functor has a filtration with a graded object being a sum of Schur functors. Thus it suffices to establish the last formula for $\mu' = \emptyset$. Using the Littlewood–Richardson rule again, we observe that the structural inclusion $W_\mu \longrightarrow \Lambda^{\mu}$ has a cokernel with a graded object being a sum of Schur functors for diagrams lexicographically smaller than μ (the reason is that μ is the largest diagram appearing in the Littlewood-Richardson decomposition of Λ^{μ}). Thus using induction on the lexicographic order we reduce our task to showing that $\mathrm{Ext}^1(\Lambda^{\mu}, \ker(m_\lambda)) = 0$. By the Decomposition Formula, the last statement is equivalent to the fact that m_λ induces an epimorphism $\mathrm{Hom}(\Lambda^d, I^d) \longrightarrow \mathrm{Hom}(\Lambda^d, S^{\lambda})$, which is clear by Fact 2.1. This completes the proof of the lemma. ■

Therefore we may choose a more convenient order of applying Σ_d–functors in the main diagram in the proof of Theorem 4.4. In particular, we may identify the morphism $(\psi^{(i)}_{\lambda/\lambda'})_* : \mathrm{Ext}^*(W^{(i)}_{\mu/\mu'}, S^{(i)}_{\lambda/\lambda'}) \longrightarrow \mathrm{Ext}^*(W^{(i)}_{\mu/\mu'}, S^{\widetilde{\lambda/\lambda'}(i)})$ with the map $\psi_{\lambda/\lambda'}(s_{\mu/\mu'}(B_i)) = s_{\mu/\mu'}(\psi_{\lambda/\lambda'}(B_i))$. This enables us to identify the Ext–groups with $s^{pr\#}_{\mu/\mu'}(s^{in}_{\lambda/\lambda'}(B_i))$ or $s^{in}_{\lambda/\lambda'}(s^{pr\#}_{\mu/\mu'}(B_i))$. In a similar fashion we obtain the desired description of induced maps. The second description of Ext–groups is the Kuhn dual of the first. ■

Thus we have obtained a nice description of Ext–groups between twisted Weyl and Schur functors. In particular for $\mu = (1^d)$ and $\mu = (d)$ we get respectively

$$\mathrm{Ext}^*(D^{d(i)}, S^{(i)}_{\lambda/\lambda'}) = s_{\lambda/\lambda'}(s^d(B_i)) = s_{\lambda/\lambda'}(A_i^{\otimes d}) = S_{\lambda/\lambda'}(A_i),$$

and

$$\mathrm{Ext}^*(\Lambda^{d(i)}, S^{(i)}_{\lambda/\lambda'}) = s_{\lambda/\lambda'}(\lambda^d_{coinv}(B_i)) = s_{\lambda/\lambda'}((A_i^{\otimes d})^{alt}) = W_{\widetilde{\lambda/\lambda'}}(A_i),$$

strongly generalizing computations of ([FFSS], sect. V). The only computationin in [FFSS] among those concerning diagrams of the same weight (we deal with diagrams of different weights in [C2]) which does not fit this scheme is a computation of $\mathrm{Ext}^*(D^{d(i)}, D^{d(i)})$. But these (and more general) groups may be computed directly from our Theorem 4.3. Indeed, it yields the formula

$$\mathrm{Ext}^*(D^{d(i)}, W^{(i)}_{\lambda/\lambda'}) = w_{\lambda/\lambda'}(s^d(B_i)) = w_{\lambda/\lambda'}(A_i^{\otimes d}) = W_{\lambda/\lambda'}(A_i).$$

On the other hand, one should be cautious using Theorem 5.1. For example, it is easy to see that the epimorphism $\psi : I^p \longrightarrow \Lambda^p$ induces the trivial map $\mathrm{Hom}(\Lambda^p, I^p) \longrightarrow \mathrm{Hom}(\Lambda^p, \Lambda^p)$ which seems to contradict Theorem 5.1 which says that it should be an epimorphism. The point is that we should consider the second variable as a Schur functor so its appropriate symmetrization is λ^p_{inv} (we cannot take an arbitrary injective symmetrization in Theorem 5.1). Thus the corresponding Σ_d–transformation $\widetilde{\psi} : id \longrightarrow \lambda^p_{inv}$ is the averaging to invariants which is not an epimorphism in general.

Let us now try to look for other functors satisfying the assumption of Theorem 4.4. Taking into account Fact 2.2, it is tempting to consider Schur functors S_μ, S_λ satisfying $\lambda \not\trianglerighteq \mu$, since we have $\mathrm{Ext}^*(S_\mu, S_\lambda) = 0$ for $* > 0$. But in fact, we need the stronger condition $\mathrm{Ext}^*(S_\mu \circ jI, S_\lambda) = 0$ for $j = p^i$. When we apply the Decomposition Formula to $S_\mu \circ jI$ we see that our lexicographic assumption is quickly weakening. A counterexample is very simple:

already for $\mu = \lambda = (2^2), p = 2$ we get $\mathrm{Ext}^2(S_\mu \circ 2I, S_\lambda) \neq 0$. A pathological element comes from the decomposition of μ into $(1^2), (1^2)$ and λ into $(2), (2)$. Also as small lexicographically diagram as $(2k-1, 1)$ and as large as (2^k) may be decomposed to give a nontrivial element in $\mathrm{Ext}^*(S_{(2k-1,1)} \circ 2I, S_{(2^k)})$ for $p = 2$. Slightly more complicated examples can be constructed for $p > 2$, and even for hooks. The only quite general class of Schur functors satisfying the assumption of Theorem 4.4 is provided by diagrams of weight p.

Corollary 5.3 *If $\lambda \not\trianglerighteq \mu$ are diagrams of weight p then*

$$\mathrm{Ext}^*(S_\mu^{(i)}, S_\lambda^{(i)}) = (w_\mu, s_\lambda)(B_i)).$$

Proof: Observe that when we decompose μ into smaller diagrams we get diagrams of weight smaller than p for which Schur functors are projective. This together with the lexicographic assumption gives the Ext–condition. ■

In the above case there is no reason for expecting that the formula will simplify to the form similar to that of Theorem 5.1. But the general formula from Theorem 4.4 is not very convenient in practice. In order to rephrase the result in a more explicite form, we will show one easy general fact.

Fact 5.4 *Let $0 \longrightarrow F_1 \longrightarrow \ldots \longrightarrow F_k \longrightarrow 0$ be an exact sequence whose all objects satisfy the Ext–condition with some G. Then the sequence: $0 \longrightarrow \mathrm{Ext}^*(F_k^{(i)}, G^{(i)}) \longrightarrow \ldots \longrightarrow \mathrm{Ext}^*(F_1^{(i)}, G^{(i)}) \longrightarrow 0$ is exact.*

Proof: The assertion follows immediately from the fact that, according to Theorem 4.4, all Ext–groups under consideration are concentrated in even degrees. ■

Again it seems that the last result is in conflict with the fact that $\psi : I^p \longrightarrow \Lambda^p$ induces the trivial map $\mathrm{Hom}(\Lambda^p, I^p) \longrightarrow \mathrm{Hom}(\Lambda^p, \Lambda^p)$. But the point is that ψ cannot be extended to an exact sequence satisfying the Ext–condition with D^d.

Now we would like to get a more explicit description of Ext–groups appearing in Cor. 5.3. To this end, we will need a resolution of S_μ by exterior powers starting with the structural arrow ϕ_μ. The existence of such a resolution may be derived from some corollary of Theorem 5.1 and Fact 5.4.

Corollary 5.5 *For any resolution $0 \longrightarrow S_\lambda \xrightarrow{\psi_\lambda} S^{\tilde{\lambda}} \longrightarrow \ldots$, there exists the "Koszul dual complex" $0 \longrightarrow W_{\tilde{\lambda}} \xrightarrow{\phi^\#_{\tilde{\lambda}}} \Lambda^{\tilde{\lambda}} \longrightarrow \ldots$, which is exact.*

Proof: We apply the functor $\mathrm{Ext}^*(\Lambda^{d(i)}, -)$ to the complex $0 \longrightarrow S_\lambda^{(i)} \xrightarrow{\psi_\lambda} S^{\widetilde{\lambda}(i)} \longrightarrow \dots$ According to Theorem 5.1 we get the complex $0 \longrightarrow W_{\widetilde{\lambda}}(A_i) \xrightarrow{\phi^{\#}_{\widetilde{\lambda}}(A_i)} \Lambda^{\widetilde{\lambda}}(A_i) \longrightarrow \dots$, whose exactness follows from Fact 5.4. Since the dimension of A_i may be arbitrarily large, the whole complex of functors must be exact. ■

To obtain the desired resolution of S_μ by exterior powers, we take the Kuhn dual of the complex $0 \longrightarrow W_\mu \xrightarrow{\phi^{\#}_\mu} \Lambda^\mu \longrightarrow \dots$ constructed in Cor. 5.5 (for $\lambda := \widetilde{\mu}$). Thus we get the resolution

$$\dots \longrightarrow \Lambda^{\mu^1} \xrightarrow{\phi_1} \Lambda^\mu \xrightarrow{\phi_\mu} S_\mu \longrightarrow 0.$$

Now observe that since exterior powers are also Weyl functors, they satisfy the Ext–condition with S_λ. Since S_μ satisfy it too, the sequence

$$0 \longrightarrow \mathrm{Ext}^*(S_\mu^{(i)}, S_\lambda^{(i)}) \xrightarrow{(\phi_\mu)^*} \mathrm{Ext}^*(\Lambda^{\mu(i)}, S_\lambda^{(i)}) \xrightarrow{(\phi_1)^*} \dots$$

is exact by Fact 5.4. Hence $\mathrm{Ext}^*(\Lambda^{\mu(i)}, S_\lambda^{(i)}) = \ker((\phi_1)^*)$. In order to compute this kernel let us observe that all groups and arrows appearing in the above sequence starting from the second term are known by Theorem 5.1: $\mathrm{Ext}^*(\Lambda^{\mu(i)}, S_\lambda^{(i)}) = \lambda_i^\mu(s_\lambda(B_i))$ etc. Let $\widetilde{\phi}_1 : \lambda_c^{\mu^1} \longrightarrow \lambda_c^\mu$ be a Σ_d–functor such that $\widetilde{\phi}_1(V^{\otimes d}) = \phi_1$. The proof of existence of such a Σ_d–transformation is analogous to the proof of epimorphicity in Lemma 3.4. Therefore, when we put $\gamma := \mathrm{coker}(\widetilde{\phi}_1)$, we get

$$\mathrm{Ext}^*(S_\mu^{(i)}, S_\lambda^{(i)}) = \gamma^{\#}(s_\lambda(B_i)).$$

Thus we obtained a description of the Exts in terms similar to those used in Theorem 5.1. One should remember however, that although γ is quite explicitly defined symmetrization of W_μ, one cannot expect that $\gamma \simeq w_\mu$ and even that $\gamma^{\#}(s_\lambda(B_i)) \simeq w_\mu(s_\lambda(B_i))$, for $s_\lambda(B_i)$ is not a Y–permutative module.

Finally, I would like to discuss some interesting special cases of Theorem 4.3. When we take $\mu = (d)$, the formula takes the nice form

$$\mathrm{Ext}^*(D^{d(i)}, F^{(i)}) = f^{in}(s^d(B_i)) = f^{in}(A_i^{\otimes d}) = F(A_i).$$

This is the first time when we obtained the description clearly independent of the choice of f^{in}. Let us look at a very simple but instructive example. When we take $F = I^{(i)}$, then our formula gives $\mathrm{Ext}^*(D^{p(i-1)}, I^{(i)}) = A_{i-1}^{(1)}$. We recall however, that the decoration $^{(1)}$ indicates that we should multiply by p degrees of nontrivial components of A_{i-1}. After this modification we get the result predicted by ([FS], Th. 4.5).

The second extreme case: $\mu = (1^d)$ is even more interesting. We get

$$\mathrm{Ext}^*(I^{d(i)}, F^{(i)}) = f^{in}(B_i).$$

But these groups can also be computed directly form the Decomposition Formula. Namely, let us put $\widetilde{cr_d}(F)(V) := cr_d(F)(V, \ldots, V)$, where $cr_d(F)$ means the d–th cross–effect of a functor F (see eg. [B1], pp. 74–75). Then, since F has the Eilenberg–MacLane degree at most d, we have $cr_d(F)(V_1, \ldots, V_d) = \bigoplus(V_1 \otimes \ldots \otimes V_d)$. Therefore:

$$\begin{aligned}
&\mathrm{Ext}^*_{\mathcal{P}}(I^{d(i)}, F^{(i)}) = \mathrm{Ext}^*_{\mathcal{P}^d}(V_1^{(i)} \otimes \ldots \otimes V_d^{(i)}, F^{(i)}(V_1 \oplus \ldots \oplus V_d)) = \\
&\mathrm{Ext}^*_{\mathcal{P}^d}(V_1^{(i)} \otimes \ldots \otimes V_d^{(i)}, cr_d(F)^{(i)}(V_1, \ldots, V_d)) = \\
&\mathrm{Ext}^*_{\mathcal{P}^d}(V_1^{(i)} \otimes \ldots \otimes V_d^{(i)}, \textstyle\bigoplus(V_1^{(i)} \otimes \ldots \otimes V_d^{(i)})) = \\
&= \textstyle\bigoplus(\mathrm{Ext}^*_{\mathcal{P}}(I^{(i)}, I^{(i)}) \otimes \ldots \otimes \mathrm{Ext}^*_{\mathcal{P}}(I^{(i)}, I^{(i)})) = \widetilde{cr_d}(F)(A_i),
\end{aligned}$$

and it is easy to see that a Σ_d–structure on $\mathrm{Ext}^*(I^{d(i)}, F^{(i)})$ agrees with a structure on $\widetilde{cr_d}(F)(A_i)$ coming from the permutation of factors in $cr_d(F)$. Thus we get the formula

$$f^{in}(B_i) = \widetilde{cr_d}(F)(A_i),$$

which suggests the existence of a link between the notion of symmetrization and cross–effects.

Let us examine more closely the case $i = 0$. In this case the space A_0 is one–dimensional, hence our computation simplifies to the formula

$$f^{in}(\mathbf{k}[\Sigma_d]) = \mathrm{Hom}_{\mathcal{P}_d}(I^d, F).$$

Regarding the right–hand side of this equality as a functor of the second variable we get the functor $S : \mathcal{P}_d \longrightarrow \mathbf{k}[\Sigma_d]$–mod. This functor known (unfortunately) also as the Schur functor is an important tool used to compare representations of the general group and symmetric group (see eg. [Ma], chap. 4). The last equality may serve as an "explanation" of some good properties of the Schur functor. Namely, our formula says that the functor F and the Σ_d–module $S(F)$ are obtained by applying the same Σ_d–functor to the right Σ_d–structure of: $V^{\otimes d}$ in the first case, and $\mathbf{k}[\Sigma_d]$ in the second.

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References

[ABW] K. Akin, D. Buchsbaum, J. Weyman, *Schur Functors and Schur Complexes*, Advances in Math. **44** (1982), 207–278.

[B1] S. Betley, *Homology of $GL(R)$ with coefficients in a functor of finite degree*, J. Algebra **150** (1992), 73–86.

[B2] S. Betley, *K–theory of finite fields*, K–Theory **17** (1999), no. 2, 103–111.

[Bo] G. Boffi, *Universal Filtration of Schur Complexes*, Proceedings of the AMS, **119** (1993), 351–355.

[Bö] M. Bökstedt, *The topological Hochschild homology of* $\mathbf{Z}$ *and* $\mathbf{Z}/p$, preprint.

[C1] M. Chałupnik, *Schur–De-Rham complex and its cohomology*, preprint.

[C2] M. Chałupnik, *Extensions of Weyl and Schur functors*, in preparation.

[CPS] E. Cline, B. Parshall, L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.

[Fr] V. Franjou, *Extensions entre puissances exterieures et entre puissances symmetriques*, J. Algebra **179** (1996), 501–522.

[FFSS] V. Franjou, E. Friedlander, A. Scorichenko, A. Suslin, *General Linear and Functor Cohomology over Finite Fields*, Annals of Math. **150** no. 2, (1999), 663–728.

[FLS] V. Franjou, J. Lannes, L. Schwartz, *Autour de la cohomologie de MacLane des corps finis,* Inventiones Math. **115**, (1994), 513–538.

[FS] E. Friedlander, A. Suslin, *Cohomology of finite group schemes over a field*, Inventiones Math. **127** (1997), 209–270.

[HLS] H. W. Henn, J. Lannes, L. Schwartz, *The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects*, Amer. J. Math. **115** (1995) no. 5, 1053–1106.

[JK] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia of Mathematics and Its Applications, Addison–Wesley P. C., 1981.

[K1] N. Kuhn, *Generic Representations of the Finite General Linear Groups and the Steenrod Algebra I*, American Journal of Math. **116** (1994), 327–360.

[K2] N. Kuhn, *Generic Representations of the Finite General Linear Groups and the Steenrod Algebra II*, K–Theory **8** (1994) no. 4, 395–428.

[K3] N. Kuhn, *Generic Representations of the Finite General Linear Groups and the Steenrod Algebra III*, K–Theory **9** (1995) no. 3, 273–303.

[PS] L. Piriou, L. Schwartz, *Extensions de foncteurs simples*, K–Theory **15** (1998), 269–291.