Extensions of Weyl and Schur functors

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Abstract

We study Ext–groups between twisted Weyl and Schur functors in the category of strict polynomial functors. We give complete description of the groups $\mathrm{Ext}^*_{\mathcal{P}}(W_\mu^{(i+j)}, S^{(i)}_{F_k^j(\lambda)})$ for $|\mu| = |\lambda|$ and we obtain some partial results in the general case.

1 Introduction

This paper is a third part of my work on homological algebra in the category of functors. We continue here the calculations of Ext–groups started in [C1], armed with the Schur–De-Rham complex studied in [C2].

The main result of the present article (Th. 3.4) is the calculation of the groups

$$\mathrm{Ext}^*_{\mathcal{P}}(W_\mu^{(i+j)}, S^{(i)}_{F_k^j(\lambda)}),$$

for diagrams μ, λ of equal weights (F_k^j means the j–fold iteration of the operation F_k introduced in [C2]). This completes the process of generalizing results of [FFSS] but also opens new prospects for computing Ext–groups in the functor category. The main challenge is to understand the groups

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$\mathrm{Ext}^*_{\mathcal{P}}(W_\mu^{(*)}, S_\lambda^{(*)})$ for arbitrary Young diagrams μ, λ. Section 4 contains some partial results related to this problem, which suggest further connections of the subject with the combinatorics of hooks.

2 Acyclic functors and localization

Throughout this article we work in the category $\mathcal{P}$ of strict polynomial functors over a fixed field $\mathbf{k}$ of characteristic $p > 0$ (cf. [FS, Sect. 2]). We will use freely all definitions and conventions made in [C1, Sect. 2,3] and [C2, Sect. 2,3,5,6]. In particular, we recall that λ may denote a *skew* Young diagram. All other references to [C1] and [C2] (on which the present article depends heavily) will be explicitly mentioned.

Our main goal is generalization of [C1,Th. 5.1] to the case of diagrams of different weights. Taking into account the role played by the De–Rham complex in the proof of [FFSS, Th. 4.5] and the fact that we computed the cohomology of the Schur–De-Rham complex for diagrams of the form $F_k(\lambda)$ in [C2, Th. 5.3], we can expect that the groups $\mathrm{Ext}^*_{\mathcal{P}}(W_\mu^{(i+j)}, S^{(i)}_{F^j_k(\lambda)})$ admit description similar to that of [C1, Th. 5.1]. One should remember however, that the pair $W_\mu^{(j)}, S_{F^j_k(\lambda)}$ does not satisfy the "Ext–condition" of [C1, Th. 4.4]. Also, when we try to repeat the proof of [C1, Th. 4.4], we face the problem of constructing maps $S_{F^j_k(\lambda)} \longrightarrow S^{F^j_k(\lambda)}$ or $\Lambda^{F^j_k(\lambda)} \longrightarrow S_{F^j_k(\lambda)}$ which could serve as analogues of structural arrows. However lexicographic properties of diagrams do not guarantee us the existence of such maps any more, we will manage to perform the calculations in a fashion similar to that of [C1]. The point is that we need not maps between functors but only between their Ext–groups. But the Schur functors associated to diagrams which form combinatorial obstacles for constructing the analogues of structural maps will turn out to have trivial Ext–groups. The situation resembles that in [C2] where only after neglecting some acyclic complexes we were able to construct maps on cohomology (in fact, as we will see, the combinatorics in both contexts is similar). In order to make considerations of this sort precise and functorial we will use a formalism of localization of the derived category.

Definition 2.1 *Given $A \in \mathcal{P}$, we say that a triangulated category $\mathbf{D}\mathcal{P}_A$ is an A–localization of the derived category $\mathbf{D}\mathcal{P}$ if there exists a functor $L : \mathcal{P} \longrightarrow \mathbf{D}\mathcal{P}_A$ such that*

- *L takes short exact sequences to distinguished triangles.*
- *For all $X \in \mathcal{P}$,*

$$\mathrm{Ext}^n_{\mathcal{P}}(A, X) = \mathrm{Hom}_{\mathbf{D}\mathcal{P}_A}(L(A)[n], L(X)).$$

- *If $f : X \longrightarrow Y$ induces an isomorphism $f_* : \mathrm{Ext}^*_{\mathcal{P}}(A, X) \simeq \mathrm{Ext}^*_{\mathcal{P}}(A, Y)$ then $L(f)$ is an isomorphism.*

The existence of such a localization is well known (see eg. [Ne, Chap. 2.1]). The second condition allows us to carry over all the calculations to the localized category, hence from now on we will not differ between $\mathrm{Hom}_{\mathbf{D}\mathcal{P}_A}(L(A)[n], L(X))$ and $\mathrm{Ext}^n_{\mathcal{P}}(A, X)$ usually denoting both just by $\mathrm{Ext}^n(A, X)$. The most important is the third condition which makes constructing maps much easier.

We start with quite formal lemma which enables us to formulate further results in a concise manner.

Lemma 2.2 *For any diagram μ of weight d, we have the following isomorphisms in the category $\mathbf{D}\mathcal{P}_{W_\mu^{(i+j)}}$:*

$$S^{(i)}_{F^j_k(\lambda)} \simeq \Lambda^{dp^j(i)}[h^j_k(\lambda)],$$

for an arbitrary diagram λ of weight d consisting of a single row, and

$$S^{(i)}_{F^j_k(\lambda')} \simeq S^{dp^j(i)}[h^j_k(\lambda')],$$

for an arbitrary diagram λ' of weight d consisting of a single column. The shift is given by the formula $h^j_k(\alpha) = k\frac{p^j-1}{p-1} + (p^j - 1)f_\alpha$, where f_α stands for the number of boxes lying above the principal diagonal.

Proof: Since the proofs for λ and λ' are analogous, we focus on the case of one–rowed diagram. The idea of proof is similar to that of [C2, Lemma 7.1], the difference is that this time we deal with p^j–hooks instead of p–hooks. The proof goes by induction on the number of rows in $\alpha = F^j_k(\lambda)$. We prove the assertion for a slightly wider class of diagrams, namely for skew hooks α whose "p^j–slices" (definition of p^j–slices is analogous to that of p–slices given in [C2, Sect. 5]; see also discussion after Fact 2.4 in the present article) are placed horizontally (ie. the foot of the next slice lies to the right of the hand of the previous one). In order to get the assertion for a diagram α, we consider the exact sequence

$$0 \longrightarrow S^{(i)}_{\alpha/(\alpha_1)|_h(\alpha_1)} \longrightarrow S^{(i)}_{\alpha/(\alpha_1)} \otimes \Lambda^{\alpha_1(i)} \longrightarrow S^{(i)}_{\alpha} \longrightarrow 0,$$

whose exactness may be proved by the argument used in [C2, Lemma 7.1]. Since the p^j–slices of α are placed horizontally, the number α_1 cannot be divisible by p^j. Hence, by the Decomposition Spectral Sequence (cf. [C1, Sect. 2]), we get $\mathrm{Ext}^*(W^{(i+j)}_{\mu}, S^{(i)}_{\alpha/(\alpha_1)} \otimes \Lambda^{\alpha_1(i)}) = 0$. this leads to an isomorphism $S^{(i)}_{\alpha/(\alpha_1)|_h(\alpha_1)} \simeq S^{(i)}_{\alpha}[-1]$ in the localized derived category. But $\alpha/(\alpha_1)|_h(\alpha_1)$ has less rows than α, so we can apply the induction hypothesis. The final formula for shift comes from the fact that $F^j_k((1)) = (k\frac{p^j-1}{p-1}+1, 1^{p^j-1-k\frac{p^j-1}{p-1}})$. ■

Now, we focus on the problem of finding functors which are trivial in the localized category.

Lemma 2.3 *Assume that for some $A \in \mathcal{P}$, s, i, j satisfying $s > i+j$, and for some diagram λ of weight d, we have $\mathrm{Ext}^*(A^{(s)}, S^{(i)}_{F^j_k(\lambda)}) \neq 0$. Then λ has a trivial p–core.*

Proof: The proof falls naturally into three parts.

1: If the assertion holds for A_1 and A_2, then it does so for $A_1 \otimes A_2$.
Assume that $\mathrm{Ext}^*(A^{(s)}_1 \otimes A^{(s)}_2, S^{(i)}_{F^j_k(\lambda)}) \neq 0$. By the Decomposition Spectral Sequence, there must exist $\alpha \subset F^j_k(\lambda)$ such that $\mathrm{Ext}^*(A^{(s)}_1, S^{(i)}_{\alpha})$ and $\mathrm{Ext}^*(A^{(s)}_2, S^{(i)}_{F^j_k(\lambda)/\alpha})$ are nontrivial. Hence, according to the assertion for A_2 (and $j=0$), $F^j_k(\lambda)/\alpha$ has a trivial core. Then, by [C2, Fact 6.1], there exists $\alpha' \subset F^{j-1}_k(\lambda)$ such that $\alpha = F_k(\alpha')$. Using our assertion again, this time for $j=1$, we conclude that $F^{j-1}_k(\lambda)/\alpha'$ has a trivial core, hence there exists $\alpha'' \subset F^{j-2}_k(\lambda)$ such that $\alpha' = F_k(\alpha'')$ etc. till we get that $\alpha = F^j_k(\alpha^j)$ and $\alpha^j, \lambda/\alpha^j$ have trivial cores. Therefore, also λ has a trivial core.

2: $A = D^{d(t)}$.
We proceed by induction on d. Let $d=1$ and assume that $c(\lambda) \neq \emptyset$. Since we will consider both the classical Schur complex and the Schur–De-Rham complex introduced in [C2] we will call the former one the Schur-Koszul complex or we will say that we consider the Schur complex equipped with

the Koszul or De–Rham differential (but we denote both the complexes by $\mathbf{S}_\lambda$). We start with taking the twisted Schur–De-Rham complex $\mathbf{S}_\lambda^{(i+j)}$. Since by ([C2], Fact 4.3) it is acyclic and $\mathrm{Ext}^*(I^{(s+t)}, B\otimes C) = 0$ for all homogeneous functors of positive degree (eg. by the Exponential Formula [C1, Sect. 2]), the first spectral sequence converging to $\mathrm{hExt}^*(I^{(s+t)}, \mathbf{S}_\lambda^{(i+j)})$ (by this we mean the hyperExt groups of $I^{(s+t)}$ with coefficients in $\mathbf{S}_\lambda^{(i+j)}$ (see eg. [CE, Chap. XVII])) converges to 0 and may have nontrivial the first or last column only. Thus we obtain $\mathrm{Ext}^*(I^{(s+t)}, S_\lambda^{(i+j)}) = \mathrm{Ext}^{*+|\lambda|-1}(I^{(s+t)}, W_{\tilde{\lambda}}^{(i+j)})$. Applying the same argument to the Schur–Koszul complex we get shift into another direction $\mathrm{Ext}^*(I^{(s+t)}, S_\lambda^{(i+j)}) = \mathrm{Ext}^{*-|\lambda|+1}(I^{(s+t)}, W_{\tilde{\lambda}}^{(i+j)})$. But by the hypothesis $s > i+j$ we know that $|\lambda| > 1$. This means that $\mathrm{Ext}^*(I^{(s+t)}, S_\lambda^{(i+j)}) = 0 = \mathrm{Ext}^*(I^{(s+t)}, W_{\tilde{\lambda}}^{(i+j)})$.
Now, we show in a similar manner that $\mathrm{Ext}^*(I^{(s+t)}, S_{F_k(\lambda)}^{(i+j-1)}) = 0$. This time the Schur–De-Rham complex $\mathbf{S}_{F_k(\lambda)}^{(i+j-1)}$ is not acyclic, but since by [C2, Th. 5.3] the second spectral sequence converging to $\mathrm{hExt}^*(I^{(s+t)}, \mathbf{S}_{F_k(\lambda)}^{(i+j-1)})$ is trivial, we still have the shift in grading between $\mathrm{Ext}^*(I^{(s)}, S_{F_k(\lambda)}^{(i+j-1)})$ and $\mathrm{Ext}^*(I^{(s)}, W_{F_k(\tilde{\lambda})}^{(i+j-1)})$, which gives the desired vanishing of the Ext–groups. Repeating this argument we get $\mathrm{Ext}^*(I^{(s+t)}, S_{F_k^q(\lambda)}^{(i+j-q)}) = 0$ for larger and larger q. At last for $q = j$ we obtain our assertion.
The proof of the induction step on d is similar. We assume the assertion for all $d < d_0$. In order to get it for d_0 we look at the spectral sequences converging to $\mathrm{hExt}^*(\mathbf{\Lambda}^{d_0(s+t)}, S_\lambda^{(i+j)})$ (equipped with the De-Rham differential). By the induction hypothesis and part **1**, the second spectral sequence is trivial. Since for the same reason the first sequence has at most two nontrivial columns, we get the shift in grading. The shift into another direction is provided by the Koszul complex.

3: The general case.
For arbitrary A we take a resolution by products of divided powers. By parts **1** and **2**, the assertion holds for all functors in the resolution. Therefore it also holds for A. ■

Remark: Observe that even for $s = 1, i = j = 0$, Lemma 2.3 is not obvious. For solid λ it follows from the fact that each twisted functor belongs to the trivial block and the Nakayama Conjecture for $\mathcal{P}$ [Do]. But this argument fails for skew diagrams.

We use this lemma to derive a powerful criterion for detecting functors with trivial Ext–groups.

Fact 2.4 *If for some* $\beta \subseteq F_k^j(\lambda)$ *and some* $A \in \mathcal{P}$, $\mathrm{Ext}^*(A^{(i+j)}, S^{(i)}_{F_k^j(\lambda)/\beta})$ *is nontrivial, then* $\beta = F_k^j(\alpha)$ *for some* $\alpha \subseteq \lambda$.

Proof: Assume that the above Ext–group is nontrivial . Then, by Lemma 2.3 (for $j = 0$), $c(\beta) = \emptyset$. Hence, by [C2, Fact 6.1] we get $\beta = F_k(\beta')$. Iterating this argument we obtain our assertion (we have used a similar trick in the proof of part **1** of Lemma 2.3). ■

It seems that we have been doing the same work many times in the last proofs. Let us look more closely at the situation from the point of view of combinatorics. Since we are interested in multi–twisted functors, we should consider (in contrast to [C2]) the operation F_k^j of j–fold enlargement of a diagram. It is completely analogous to F_k, the only difference is that we replace boxes by p^j–hooks instead of p–hooks. Namely, to build $F_k^j(\lambda)$ out of λ, we replace the boxes in λ lying above the diagonal by the horizontal p^j–hooks, those below it by the vertical ones, and we replace the boxes lying on the diagonal by the hooks of shape $F_k^j((1)) = (k\frac{p^j-1}{p-1}+1, 1^{p^j-1-k\frac{p^j-1}{p-1}})$. Thus it would be tempting to derive Fact 2.2 from the "p^j–analogue" of [C2, Fact 6.1] which obviously holds. Unfortunately it not true that if $\mathrm{Ext}^*(A^{(j)}, S_\lambda)$ is nontrivial, then λ has a trivial p^j–core (we will come back to this problem in Section 4).

Now, we shall construct the arrows $\Lambda^{F_k^j(\lambda)} \longrightarrow S_{F_k^j(\lambda)}$, $S_{F_k^j(\lambda)} \longrightarrow S^{F_k^j(\widetilde{\lambda})}$ which could play roles of the structural arrows for enlarged diagrams. It will turn out that the construction which fails in the category $\mathcal{P}$ is possible in the localized category. To see this, let us investigate the combinatorics of the situation. An attempt to construct an arrow $\Lambda^{F_k^j(\lambda)} \longrightarrow S_{F_k^j(\lambda)}$ in the category $\mathcal{P}$ breaks down because the diagram $F_k^j((\lambda_1, \ldots, \lambda_{l-1}))$ is not the lexicographically smallest subdiagram of a given weight in $F_k^j((\lambda_1, \ldots, \lambda_l))$ etc. We will show however, that all smaller diagrams give trivial objects in $\mathbf{D}\mathcal{P}_{W_\mu^{(i+j)}}$. To see this, let us take $\beta \subseteq F_k^j(\lambda)$ such that $\mathrm{Ext}^*(W_\mu^{(i+j)}, S_\beta^{(i)} \otimes S^{(i)}_{F_k^j(\lambda)/\beta}) \neq 0$. Then, by the Decomposition Spectral Sequence, there exists

$\gamma \subseteq \mu$ of weight $|\beta|/p^j$ such that $\mathrm{Ext}^*(W_\gamma^{(i+j)}, S_\beta^{(i)}) \otimes \mathrm{Ext}^*(W_{\mu/\gamma}^{(i+j)}, S_{F_k^j(\lambda)/\beta})$ is nontrivial. Hence, by Fact 2.4, $\beta = F_k^j(\alpha)$. But among diagrams of the form $F_k^j(\alpha')$, our diagram is the smallest (of a given weight). This observation (note that the underlying combinatorics is completely analogous to that of the "Homological Decomposition Formula" in [C2, Sect. 6]) enables us to construct the arrow

$$F_k^{j(i)}(\phi_\lambda) : \Lambda^{F_k^j(\lambda)(i)} \longrightarrow S^{(i)}_{F_k^j(\lambda)}$$

in the category $\mathbf{D}\mathcal{P}_{W_\mu^{(i+j)}}$, and by a similar reasoning, the map

$$F_k^{j(i)}(\psi_\lambda) : S^{(i)}_{F_k^j(\lambda)} \longrightarrow S^{F_k^j(\widetilde{\lambda})(i)}.$$

Next, observe an interesting fact, that the composition $F_k^{j(i)}(\psi_\lambda) \circ F_k^{j(i)}(\phi_\lambda)$ exists already in $\mathcal{P}$, for it is equal to the composition of the "comultiplication" and "multiplication": $\Lambda^{F_k^j(\lambda)(i)} \longrightarrow I^{F_k^j(\lambda)(i)} \longrightarrow S^{F_k^j(\widetilde{\lambda})(i)}$. In the last formula $I^{F_k^j(\lambda)(i)}$ stands for the tensor product of twisted Schur functors corresponding to the p^j–slices in $F_k^j(\lambda)$ (we recall the interpretation of the operation F_k^j in terms of p^j–hooks given after the proof of Fact 2.4). The first arrow is the tensor product of $F_k^{j(i)}(\phi_{(\lambda_s)})$ for all rows of λ, while the second is the product of $F_k^{j(i)}(\psi_{(\widetilde{\lambda}_s)})$ for all columns of λ. It is easy to see (and we have taken advantage of this in [C2, Sect. 7]) that for one–rowed (or one–columned) diagrams the combinatorial obstacles for the existence of maps in $\mathcal{P}$ disappear.

Similarly, in a dual situation we define the "structural maps": $F_k^{j(i)}(\phi_\lambda^\#) : W^{(i)}_{F_k^j(\lambda)} \longrightarrow \Lambda^{F_k^j(\lambda)(i)}$, $F_k^{j(i)}(\psi_\lambda^\#) : D^{F_k^j(\lambda)(i)} \longrightarrow W_{F_k^j(\widetilde{\lambda})(i)}$, whose composition exists in $\mathcal{P}$. Moreover, thanks to Lemma 2.2 and [FFSS, Th. 4.5], we are able to describe the maps induced on Ext–groups by these compositions. In order to express them concisely, we shall introduce notation analogous to that of [C1, Sect. 4]. Let $A_{ij} = \mathrm{Ext}^*(I^{(i+j)}, S^{p^j(i)})$ (this is a one–dimensional space in degrees divided by $2p^j$, less than $2p^{i+j}$, and trivial elsewhere [FS, Th. 4.5]), and let $B_{ij} = A_{ij}^{\otimes d} \otimes \mathbf{k}[\Sigma_d]$. The space B_{ij} is endowed with a structure of a Σ_d–bimodule defined by the formula known from [C1, Sect. 4]: $\sigma.a_1 \otimes \ldots \otimes a_d \otimes e_\tau.\lambda = a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(d)} \otimes e_{\sigma\tau\lambda}$. Slightly generalizing the results of [FFSS, Sect. 4, 5] by means of the Exponential Formula and Lemma 2.2, we obtain

Fact 2.5 *The following isomorphisms hold:*

$$\mathrm{Ext}^*(D^{d(i+j)}, \Lambda^{F_k^j(\lambda)(i)}) = \lambda_{inv}^{\lambda}(s^d(B_{ij}))[h_k^j(\lambda)] = \Lambda^{\lambda}(A_{ij})[h_k^j(\lambda)],$$

$$\mathrm{Ext}^*(D^{d(i+j)}, S^{F_k^j(\lambda)(i)}) = s^{\lambda}(s^d(B_{ij}))[h_k^j(\lambda)] = S^{\lambda}(A_{ij})[h_k^j(\lambda)],$$

$$\mathrm{Ext}^*(D^{d(i+j)}, D^{F_k^j(\lambda)(i)}) = d^{\lambda}(s^d(B_{ij}))[h_k'^j(\lambda)] = D^{\lambda}(A_{ij})[h_k'^j(\lambda)],$$

$$\mathrm{Ext}^*(\Lambda^{d(i+j)}, \Lambda^{F_k^j(\lambda)(i)}) = \lambda_{inv}^{\lambda}(\lambda_{inv}^d(B_{ij}))[h_k^j(\lambda)] = D^{\lambda}(A_{ij})[h_k^j(\lambda)],$$

$$\mathrm{Ext}^*(\Lambda^{d(i+j)}, S^{F_k^j(\lambda)(i)}) = s^{\lambda}(\lambda_{inv}^d(B_{ij}))[h_k^j(\lambda)] = \Lambda^{\lambda}(A_{ij})[h_k^j(\lambda)],$$

the shifts are given by the formulae:

$$h_k^j(\lambda) = f_\lambda(p^j - 1) + e_\lambda k \frac{p^j - 1}{p - 1},$$

$$h_k'^j(\lambda) = (2d - f_\lambda)(p^j - 1) - e_\lambda k \frac{p^j - 1}{p - 1},$$

where e_λ, f_λ denote respectively the number of boxes lying on and above the main diagonal.

Under these identifications the map induced on $Ext^*(D^{d(i+j)}, -)$ by the map $F_k^{j(i)}(\psi_\lambda) \circ F_k^{j(i)}(\phi_\lambda)$ is equal to $\psi_\lambda \circ \phi_\lambda(s^d(B_{ij}))[h_k^j(\lambda)] = \psi_\lambda \circ \phi_\lambda(A_{ij})[h_k^j(\lambda)]$, and the one induced by $F_k^{j(i)}(\phi_\lambda^\#) \circ F_k^{j(i)}(\psi_\lambda^\#)$ is equal to $\phi_\lambda^\# \circ \psi_\lambda^\#(s^d(B_{ij}))[h_k'^j(\lambda)] = \phi_\lambda^\# \circ \psi_\lambda^\#(A_{ij})[h_k'^j(\lambda)]$.

Similarly, $F_k^{j(i)}(\psi_\lambda) \circ F_k^{j(i)}(\phi_\lambda)$ induces on $Ext^*(\Lambda^{d(i+j)}, -)$ the map $\phi_\lambda^\# \circ \psi_\lambda^\#(s^d(B_{ij}))[h_k^j(\lambda)] = \phi_\lambda^\# \circ \psi_\lambda^\#(A_{ij})[h_k^j(\lambda)]$.

3 The main theorem

Let us now sketch our strategy of computing $\mathrm{Ext}^*(W_\mu^{(i+j)}, S^{(i)}_{F_k^j(\lambda)})$. First we compute $\mathrm{Ext}^*(D^{\mu(i+j)}, S^{(i)}_{F_k^j(\lambda)})$ by manipulating with the second variable and using Fact 2.5. Then, by considering the resolution of the first variable we get the general formula. Of course, the first step is more difficult because the operation F_k^j is involved here. However we have succeeded in constructing of the "structural arrow" $F_k^j(\psi_\lambda)$, I was not able to conduct the proof along the lines of the proof of [C1, Th. 4.3]. The problem was, that after lifting the resolution to the level of F_k^j it was difficult to show that we obtained an exact complex in the sense of the triangulated structure in $\mathbf{D}\mathcal{P}_{D^{\mu(i+j)}}$. Thus I was forced to come back to ideas of [FLS], [FS], [FFSS], and prove the formula inductively using the (Schur)–De-Rham complex. The difference with [FFSS] is that we perform induction decreasing the number of twists (the starting point is provided by [C1, Th. 5.1]).

Since the Schur–De-Rham complex will play a crucial role in the proof, we need an analogue for complexes of some constructions and computations we achieved in the previous section. We start by observing that we have the "enlarged structural arrows" also for the Schur complexes. In order to show that there exists an arrow $\mathbf{S}^{(i)}_{F_k^j(\lambda)} \longrightarrow \mathbf{S}^{F_k^j(\widetilde{\lambda})(i)}$ in $\mathbf{D}\mathcal{P}_{W_\mu^{(i+j)}}$ we recall that by [ABW, Cor. V.1.14] the s–th degree component of $\mathbf{S}_{F_k^j(\lambda)}$ has a filtration with the graded object

$$\bigoplus_{|\beta|=s} W_{\widetilde{\beta}} \otimes S_{F_k^j(\lambda)/\beta}.$$

Now, by Fact 2.4 and the obvious remark that a diagram has a trivial p–core if and only if its conjugate has, we conclude that all factors in the filtration for β which are not of the form $F_k^j(\alpha)$ are trivial in the localized category. Hence, the obstacle to the existence of the arrow (which will also be denoted by $F_k^{j(i)}(\psi_\lambda)$) disappears. In a similar fashion we show the existence of an arrow $F_k^{j(i)}(\phi_\lambda) : \mathbf{\Lambda}^{F_k^j(\lambda)(i)} \longrightarrow \mathbf{S}^{(i)}_{F_k^j(\lambda)}$. Thanks to the Exponential Formula we get the following "graded analogue" of some formulae of Fact 2.5

Fact 3.1 *The following isomorphisms hold:*

$$\mathrm{Ext}^*(D^{d(i+j)}, \mathbf{\Lambda}^{F_k^j(\lambda)(i)})) = \boldsymbol{\lambda}^\lambda(s^d(B_{ij}))[h_k^j(\lambda)] = \mathbf{\Lambda}^\lambda(A_{ij})[h_k'^j(\lambda)],$$

$$\mathrm{Ext}^*(D^{d(i+j)}, \mathbf{S}^{F_k^j(\lambda)(i)}) = \mathbf{s}^\lambda(s^d(B_{ij}))[h_k^j(\lambda)] = \mathbf{S}^\lambda(A_{ij})[h_k^j(\lambda)].$$

Under these isomorphisms, the map induced on $Ext^*(D^{d(i+j)}, -)$ by the composition $F_k^{j(i)}(\psi_\lambda) \circ F_k^{j(i)}(\phi_\lambda)$ is equal to $\psi_\lambda \circ \phi_\lambda(A_{ij})$.

These formulae are straightforward consequences of Fact 2.5 ($\mathrm{Ext}^*(F, \mathbf{C})$ stands for the sum $\bigoplus_s \mathrm{Ext}^*(F, \mathbf{C}^s)$; $\boldsymbol{\lambda}^\lambda$, $\mathbf{s}^\lambda$ mean self–evident graded Σ_d–functors).

Now we are in a position to complete the first part of our program.

Theorem 3.2 *For any diagram λ of weight d, the map $F_k^{j(i)}(\psi_\lambda) : \mathbf{S}^{(i)}_{F_k^j(\lambda)} \longrightarrow \mathbf{S}^{F_k^j(\widetilde{\lambda})(i)}$ induces a monomorphism $Ext^*(D^{d(i+j)}, \mathbf{S}^{(i)}_{F_k^j(\lambda)}) \longrightarrow Ext^*(D^{d(i+j)}, \mathbf{S}^{F_k^j(\widetilde{\lambda})(i)})$, whose image is $\mathbf{S}_\lambda(A_{ij})[h_k^j(\lambda)]$.*
Under this identification, the map induced on $\mathrm{Ext}^*(D^{d(i+j)}), -)$ by $F_k^{j(i)}(\phi_\lambda)$ is equal to $\phi_\lambda(A_{ij})[h_k^j(\lambda)] : \mathbf{\Lambda}^\lambda(A_{ij})[h_k^j(\lambda)] \longrightarrow \mathbf{S}_\lambda(A_{ij})[h_k^j(\lambda)]$.

Proof: We proceed by a double induction: the external on d and internal on j. Let first $j = 0$. In order to determine the group $\mathrm{Ext}^*(D^{d(i)}, (\mathbf{S}^{(i)}_\lambda)^s)$ (s refers to the grading in the Schur complex) we consider a filtration on $(\mathbf{S}^{(i)}_\lambda)^s$ with the graded object $\bigoplus_\alpha W^{(i)}_{\widetilde{\alpha}} \otimes S^{(i)}_{\lambda/\alpha}$ (cf. [ABW, Cor. V.1.14]). The spectral sequence of this filtration has $E^1_{st} = \bigoplus_{u+v=s+t} \mathrm{Ext}^u(D^{s(i)}, W^{(i)}_{\widetilde{\alpha}}) \otimes \mathrm{Ext}^v(D^{d-s(i)}, S^{(i)}_{\lambda/\alpha})$, where t is a position of α in the lexicographic ordering of subdiagrams in λ of weight s. Thanks to [C1, Th. 4.3] we know the groups E^1_{**}. In particular, they are concentrated in even total degrees (because $\mathrm{Ext}^{odd}(D^{d(i)}, F^{(i)}) = 0$). Therefore, the differentials in this spectral sequence are trivial, and we obtain

$$\mathrm{Ext}^*(D^{d(i)}, (\mathbf{S}^{(i)}_\lambda)^s) = \bigoplus_{|\alpha|=s} \mathrm{Ext}^*(D^{s(i)}, W^{(i)}_{\widetilde{\alpha}}) \otimes \mathrm{Ext}^*(D^{d-s(i)}, W^{(i)}_{\lambda/\alpha})$$

$$= \bigoplus_{|\alpha|=s} (W_{\widetilde{\alpha}} \otimes S_{\lambda/\alpha})(A_{i0}) = \mathbf{S}^s_\lambda(A_{i0}).$$

To get part of the theorem concerning the maps $\phi^{(i)}_\lambda$ and $\psi^{(i)}_\lambda$, we observe that it follows immediately from the construction of the filtration that it is compatible with these maps (in fact this filtration is a special case of the filtration giving the Decomposition Formula for Schur complexes (cf. [ABW, Cor. V.1.14])), hence we may apply the induction hypothesis. Thus we may restrict attention to the cases $s = 0$ and $s = d$. In the first case the description of the maps $\phi^{(i)}_{\lambda *}$ and $\psi^{(i)}_{\lambda *}$ easily follows from [C1, Th. 5.1]. But the case $s = d$ requires some work. Namely, the fact that $\psi^{\#(i)}_{\lambda *} : \mathrm{Ext}^*(D^{d(i)}, D^{\widetilde{\lambda}(i)}) \longrightarrow \mathrm{Ext}^*(D^{d(i)}, W^{(i)}_\lambda)$ and $\phi^{\#(i)}_{\lambda *} : \mathrm{Ext}^*(D^{d(i)}, W^{(i)}_\lambda) \longrightarrow \mathrm{Ext}^*(D^{d(i)}, \Lambda^{\lambda(i)})$ are respectively epic and monic follows only from ([C1], Fact 5.4). But once we already know this, the required description follows from the fact that $(\phi^{\#(i)}_\lambda \circ \psi^{\#(i)}_\lambda)_* : \mathrm{Ext}^*(D^{d(i)}, D^{\widetilde{\lambda}(i)}) \longrightarrow \mathrm{Ext}^*(D^{d(i)}, \Lambda^{\lambda(i)})$ may be, by [C1, Th. 5.1], identified with $\phi^{\#(i)}_\lambda \circ \psi^{\#(i)}_\lambda(A_{i0})$. This completes the proof for $j = 0$.

Thus we may start our induction step. We will show our assertion for given j,d assuming it for d',j' such that $d' < d, j' \leq j$ and such that $d' \leq d, j' < j$. Let E_I, E_{II} be respectively the first and the second spectral sequence of hyperExt of $D^{d(i+j)}$ with coefficients in the Schur–De-Rham complex $\mathbf{S}^{(i)}_{F^j_k(\lambda)}$, and let D be the spectral sequence of hyperExt of $D^{d(i+j)}$ with coefficients in the Schur–Koszul complex $\mathbf{S}^{(i)}_{F^j_k(\lambda)}$. We will compare these spectral sequences with the sequences E'_I, E'_{II}, D' defined analogously for the complex $\mathbf{S}^{F^{j(i)}_k(\widetilde{\lambda})}$. Let us first examine the sequence E_{II}. Thanks to [C2, Th. 5.3] and the induction hypothesis, we have the group $\mathbf{S}_\lambda(A_{i+1,j-1})$ in the second term of this sequence, and we know that $F^{j-1(i+1)}_k(\psi_\lambda)$ induces a monomorphism $E^2_{II} \longrightarrow E'^2_{II}$, which may be identified with the map $\psi_\lambda(A_{i+1,j-1}) : \mathbf{S}_\lambda(A_{i+1,j-1}) \longrightarrow \mathbf{S}^{\widetilde{\lambda}}(A_{i+1,j-1})$. Moreover, as we remember from the proof of [FFSS,Th. 4.5], there is only one nontrivial differential in the sequence E'_{II}. Hence E'^2_{II} may be viewed as a complex. This complex is the Schur–Koszul complex $\mathbf{S}^{\widetilde{\lambda}}(\delta)$ associated to a sequence $\mathbf{A}_{i+1,j-1}$:

$$0 \longrightarrow A_{i+1,j-1} \xrightarrow{\delta} A_{i+1,j-1} \longrightarrow 0,$$

whose differential δ is a differential in the sequence E_{II} for $d = 1$. It easily follows from the acyclicity of a Schur–Koszul complex associated to an identity map [ABW, Cor. V.1.5] that for a general $f : V \longrightarrow W$ we have $H_*(\mathbf{S}_\lambda(f)) = \mathbf{S}_\lambda(f')$ as graded functors where $f' : \ker(f) \longrightarrow \operatorname{coker}(f)$ is an arbitrary map (we recall that the vector spaces in a Schur complex depend merely on the source and target of the map). Thus, since it was shown in [FS, Th. 4.5] that $H_*(\mathbf{A}_{i+1,j-1}) = \mathbf{A}_{ij}$, we obtain $\mathrm{hExt}^*(D^{d(i+j)}, \mathbf{S}^{F^j_k(\widetilde{\lambda})(i)}) = \mathbf{S}^{\widetilde{\lambda}}(A_{ij})$, which is yet another translation of the calculations of [FFSS] into a more invariant language. The most important consequence of this point of view is that we still have $\mathrm{hExt}^*(D^{d(i+j)}, \mathbf{S}^{(i)}_{F^j_k(\lambda)}) = \mathbf{S}_\lambda(A_{ij})$ and we get an inclusion $\mathrm{hExt}^*(D^{d(i+j)}, \mathbf{S}^{(i)}_{F^j_k(\lambda)}) \longrightarrow \mathrm{hExt}^*(D^{d(i+j)}, \mathbf{S}^{F^j_k(\widetilde{\lambda})(i)})$.

We now turn to the sequences E_I, D. Let us look at the first terms of these sequences. In the sth column we have the group $\mathrm{Ext}^*(D^{d(i+j)}, (\mathbf{S}^{(i)}_{F^j_k(\lambda)})^s)$. Applying to the second variable the filtration described in [ABW, Cor. V.1.14] we get a spectral sequence converging to the Ext–group under consideration. The first term of this sequence is $E^1_{*q} = \mathrm{Ext}^{*+q}(D^{s(i+j)}, W^{(i)}_{\widetilde{F^j_k(\alpha)}}) \otimes$

$\mathrm{Ext}^*(D^{d-s(i+j)}, S^{(i)}_{F^j_k(\lambda/\alpha)})$, where q is a position of α in the lexicographic ordering of subdiagrams in λ of weight s/p^j (the columns with numbers nondivisible by p^j are trivial by Fact 2.4). Let us note that the groups appearing in this spectral sequence are known by the induction hypothesis (for F_{p-1-k}), unless $s \neq 0, dp^j$. Our task will be to show that the differentials in this sequence are trivial. The situation is slightly more complicated than that for $j = 0$, because this time it does not suffice to observe that A_{ij} is concentrated in even degrees. We should also show that the lowest Ext–degrees of nontrivial elements in all columns have the same parity. To this end, let us compute the smallest $u + v$ for which there exists a nontrivial element in $\mathrm{Ext}^u(D^{s(i+j)}, W^{(i)}_{\widetilde{F^j_k(\alpha)}}) \otimes \mathrm{Ext}^v(D^{d-s(i+j)}, W^{(i)}_{F^j_k(\lambda/\alpha)})$. This degree is equal to

$$(s - f_\alpha - e_\alpha)(p^j - 1) + e_\alpha(2(p^j - 1) - (p - 1 - k)\frac{p^j - 1}{p - 1})+$$

$$+(f_\lambda - f_\alpha)(p^j - 1) + (e_\lambda - e_\alpha)k\frac{p^j - 1}{p - 1}.$$

Now we observe that since the multiplicities of e_α and f_α are always even, the parity of the whole expression does not depend on α. Therefore, we get for all $s \neq 0, dp^j$

$$\mathrm{Ext}^*(D^{d(i+j)}, (\mathbf{S}^{(i)}_{F^j_k(\lambda)})^s) = \bigoplus_{|\alpha|=s} \mathrm{Ext}^*(D^{s(i+j)}, W^{(i)}_{\widetilde{F^j_k(\alpha)}}) \otimes \mathrm{Ext}^*(D^{d-s(i+j)}, W^{(i)}_{F^j_k(\lambda/\alpha)})$$

$$= \bigoplus_{|\alpha|=s} (W_{\widetilde{\alpha}} \otimes S_{\lambda/\alpha})(A_{ij}) = \mathbf{S}^s_\lambda(A_{ij}).$$

In the further part of the proof, the shifts of gradings will not play an essential role, so we will skip them in order to simplify notation.

Let us put $X = \ker(\mathrm{Ext}^*(D^{d(i+j)}, \mathbf{S}^{(i)}_{F^j_k(\lambda)}) \longrightarrow \mathrm{Ext}^*(D^{d(i+j)}, \mathbf{S}^{F^j_k(\widetilde{\lambda})(i)}))$. By the induction hypothesis and the previous considerations, X is concentrated in the two extreme columns of E_I and D. We will denote by X_0 its part contained in the 0th column and by X_1 the part contained in the (dp^j)th column. Let ∂ means the differential in D. Since the sequence D converges to 0, $\partial^{dp^j}(X_1) \subset X_0$. Thus we see that X_1 must be trivial up to Ext–degree $dp^j - 2$. We now turn to the sequence E_I. We denote by Y the subset of the 0th column of E^1_I consisting of elements surviving to infinity. Since, as we know from [FFSS, Th. 4.5], the differentials in E'_I are trivial; there are no elements outside Y up to Ext–degree dp^j-2. It means that there are no differentials in the sequence E_I up to the total degree dp^j-2. Hence, for $s \leq dp^j-2$ we have $\mathrm{hExt}^s(D^{d(i+j)}, \mathbf{S}^{(i)}_{F^j_k(\lambda)}) = \bigoplus_{q=0}^{s} \mathrm{Ext}^q(D^{d(i+j)}, (\mathbf{S}^{(i)}_{F^j_k(\lambda)})^{s-q}) = \bigoplus_{q=0}^{s-1}(\mathbf{S}^{s-q}_\lambda(A_{ij}))^q \oplus \mathrm{Ext}^s(D^{d(i+j)}, S^{(i)}_{F^j_k(\lambda)})$. But on the other hand, as we remember from the analysis of the sequence E_{II}, $\mathrm{hExt}^s(D^{d(i+j)}, \mathbf{S}^{(i)}_{F^j_k(\lambda)}) = \bigoplus_{q=0}^{s}(\mathbf{S}^{s-q}_\lambda(A_{ij}))^q$. Hence we get $\dim(\mathrm{Ext}^s(D^{d(i+j)}, S^{(i)}_{F^j_k(\lambda)})) = \dim((\mathbf{S}^0_\lambda(A_{ij}))^s = \dim((S_\lambda(A_{ij}))^s)$. But since the last space is a subquotient of the first, we obtain that $\mathrm{Ext}^s(D^{d(i+j)}, S^{(i)}_{F^j_k(\lambda)}) = (S_\lambda(A_{ij}))^s$. It also means that X_0 is trivial up to degree dp^j-2. Let us come back to the sequence D. On account of the last sentence we get that X_1 is trivial up to degree $2(dp^j-2)$. This, when we turn again to E_I, enables us to enlarge the range of degrees in which Y is trivial to $2(dp^j-2)$. As a result we obtain the required computation of $\mathrm{Ext}^s(D^{d(i+j)}, S^{(i)}_{F^j_k(\lambda)})$ and the triviality of X_0 up to degree $2(dp^j-2)$. Iterating this argument (ie. strictly speaking applying the third induction, this time on Ext–degree) we conclude that $X=0$, $\mathrm{Ext}^*(D^{d(i+j)}, S^{(i)}_{F^j_k(\lambda)}) = S_\lambda(A_{ij})$ and that the differentials in E_I are trivial. The last two facts also show, by dimension counting, that $\mathrm{Ext}^*(D^{d(i+j)}, W^{(i)}_{\widetilde{F^j_k(\lambda)}}) = W_{\widetilde{\lambda}}(A_{ij})$. This completes the proof of the description of the groups $\mathrm{Ext}^*(D^{d(i+j)}, \mathbf{S}^{(i)}_{F^j_k(\lambda)})$. The last part of the theorem (concerning the arrow $F^j_k(\phi_\lambda)$) easily follows from the facts we have already proved. ■

Remark: Thanks to the Exponential Formula one can immediately generalize Theorem 3.2 to the formula $\mathrm{Ext}^*(D^{\mu(i+j)}, S^{(i)}_{F^j_k(\lambda)}) = s_\lambda(s_\mu(B_{ij}))[h^j_k(\lambda)] = s_\mu(s_\lambda(B_{ij}))[h^j_k(\lambda)]$ for an arbitrary diagram μ. We remind the reader that the case of Weyl and Schur funcors is very special (see discussion after [C1, Lemma 5.2]), and we have two alternative descriptions of the Ext–groups here (in general, Σ_d–functors applied to left and right Σ_d–structures need not commute (see an example given in the proof of [C1, Th. 4.3])). We will use both descriptions: the first in the proof of Th. 3.4, the second in the proof of Fact 3.3.

Now, we would like to generalize the computation of Ext–groups to the case of an arbitrary Weyl functor in the first variable. Observe however, that the method of the proof of Th. 3.2 does not work in this case, because it would require the computation of Ext–groups between two Weyl functors which does not fit our scheme. Luckily, this time we can apply the machinery developed in [C1, Sect. 3], since there are no problems with transformations of the first variable.

Therefore, we should start with understanding the functoriality of the computations achieved in Th. 3.2.

Fact 3.3 *For any transformation* $\phi : D^{\mu} \longrightarrow D^{\mu'}$ *the induced morphism* $\phi^{(i+j)*} : Ext^*(D^{\mu'(i+j)}, S^{(i)}_{F^j_k(\lambda)}) \longrightarrow Ext^*(D^{\mu(i+j)}, S^{(i)}_{F^j_k(\lambda)})$ *is equal to* $\phi^{\#}(s_\lambda(B_{ij}))$.

Proof: Let us consider a commutative diagram (strictly speaking coming from the morphisms in the category $\mathbf{D}\mathcal{P}_{D^{\mu(i+j)} \oplus D^{\mu'(i+j)}})$

$$\begin{array}{ccc}
\mathrm{Ext}^*(D^{\mu'(i+j)}, S^{(i)}_{F^j_k(\lambda)}) & \stackrel{\phi^{(i+j)*}}{\longrightarrow} & \mathrm{Ext}^*(D^{\mu(i+j)}, S^{(i)}_{F^j_k(\lambda)}) \\
\downarrow F^{j(i)}_k(\psi_\lambda)_* & & \downarrow F^{j(i)}(\psi_\lambda)_* \\
\mathrm{Ext}^*(D^{\mu'(i+j)}, S^{F^j_k(\widetilde{\lambda})(i)}) & \stackrel{\phi^{(i+j)*}}{\longrightarrow} & \mathrm{Ext}^*(D^{\mu(i+j)}, S^{F^j_k(\widetilde{\lambda})(i)}).
\end{array}$$

Identifying known groups and arrows we get (up to shift) the diagram

$$\begin{array}{ccc}
s^{\mu'}(s_\lambda(B_{ij})) & \stackrel{\phi^{(i+j)*}}{\longrightarrow} & s^{\mu}(s_\lambda(B_{ij})) \\
\Big\downarrow s^{\mu'}(\psi_\lambda(B_{ij})) & & \Big\downarrow s^{\mu}(\psi_\lambda(B_{ij})) \\
s^{\mu'}(s^\lambda(B_{ij})) & \stackrel{\phi^{\#}(s^\lambda(B_{ij}))}{\longrightarrow} & s^{\mu}(s^\lambda(B_{ij})).
\end{array}$$

Of course, replacing of the top arrow by $\phi^{\#}(s_\lambda(B_{ij}))$ does not affect the commutativity of the diagram. This, thanks to the monomorphicity of the right vertical arrow, gives our assertion. ■

We have now all ingredients we need for the proof of our main result.

Theorem 3.4 *For any diagrams* μ, λ *of weight* d, *and any* i, j, k, *we have*

$$\mathrm{Ext}^*(W^{(i+j)}_\mu, S^{(i)}_{F^j_k(\lambda)}) = s_\lambda(s_\mu(B_{ij}))[h^j_k(\lambda)] = s_\mu(s_\lambda(B_{ij}))[h^j_k(\lambda)].$$

Moreover, for any transformation $\phi : w_\mu \longrightarrow w_{\mu'}$, *the induced map* $\phi^{*(i+j)}$: $\mathrm{Ext}^*(W^{(i+j)}_{\mu'}, S^{(i)}_{F^j_k(\lambda)}) \longrightarrow \mathrm{Ext}^*(W^{(i+j)}_\mu, S^{(i)}_{F^j_k(\lambda)})$ *is equal to* $\phi^{\#}(s_\lambda(B_{ij}))$.

Proof: In fact, the proof consists of slightly rearranged elements of the proofs of [C1, Th. 4.4] and [C1, Th. 5.1].

We apply the functor $\mathrm{Ext}^*(-, S^{(i)}_{F^j_k(\lambda)})$ to the $(i+j)$–th twisted resolution of W_μ by divided powers starting with the structural arrow. In the resulting complex

$$0 \longrightarrow \mathrm{Ext}^*(W^{(i+j)}_\mu, S_{F^j_k(\lambda)}) \xrightarrow{\psi^{\#*}_\mu} \mathrm{Ext}^*(D^{\widetilde{\mu}(i+j)}, S_{F^j_k(\lambda)}) \xrightarrow{\phi^*_1} \mathrm{Ext}^*(D^{\mu^1(i+j)}, S_{F^j_k(\lambda)}) \longrightarrow$$

all the groups and arrows starting with the second spot are known by Theorem 3.2 and Fact 3.3. Thus, let us consider the commutative diagram (in which we omit shifts)

$$\begin{array}{ccccccc} 0 \longrightarrow s_\lambda(s_\mu(B_{ij})) & \xrightarrow{\psi_\mu(s_\lambda(B_{ij}))} & s_\lambda(s^{\widetilde{\mu}}(B_{ij})) & \xrightarrow{\phi^{\#}_1(s_\lambda(B_{ij}))} & s_\lambda(s^{\mu^1}(B_{ij})) \longrightarrow \\ & & \downarrow & & \downarrow \\ 0 \longrightarrow \mathrm{Ext}^*(W^{(i+j)}_\mu, S_{F^j_k(\lambda)}) & \xrightarrow{\psi^{\#*}_\mu} & \mathrm{Ext}^*(D^{\widetilde{\mu}(i+j)}, S_{F^j_k(\lambda)}) & \xrightarrow{\phi^*_1} & \mathrm{Ext}^*(D^{\mu^1(i+j)}, S_{F^j_k(\lambda)}) \longrightarrow \end{array}$$

where the vertical arrows are isomorphisms. The proof will be finished if we complete this diagram by the left vertical arrow. For this, it suffices to show that the top row is exact. But this complex, if we neglect internal grading, is isomorphic to the sequence (*) from the proof of ([C1], Th. 4.4) for $F = W_\mu, G = S_\lambda, f = w_\mu, g = s_\lambda$.

We get the assertion concerning functoriality in a similar fashion to that in ([C1], Th. 4.4). I leave details to the reader, for we just apply the same Σ_d–transformations to the same Σ_d–functors as in ([C1], Sect. 4). ■

Remark: We say nothing about the functoriality with respect to the second variable in Th. 3.4. The reason is that, of course, one cannot expect good properties of any morphism $S_{F^j_k(\lambda)} \longrightarrow S_{F^j_k(\lambda')}$. It seems reasonable to consider only maps somehow "induced" by maps $S_\lambda \longrightarrow S_{\lambda'}$, but then we encounter a problem caused by the fact that the shape of a slice which determines the shift in Ext–grading depends on the position of a box with respect to the main diagonal. Let us illustrate this by a simple example. For $p = 2$ we have a nontrivial transformation $\rho : S^2 \longrightarrow \Lambda^2$. Thus we would expect that the induced map $\mathrm{Ext}^*(D^{2(i+1)}, S^{(i)}_{F_0((1^2))}) \longrightarrow \mathrm{Ext}^*(D^{2(i+1)}, S^{(i)}_{F_0((2))})$ is equal to $\rho(A_{i1})$. But this cannot happen because $h^1_0((1^2)) \neq h^1_0((2))$.

4 Toward the general case

We have studied in [C1] diagrams of the same weight, and we have got the calculations of Ext–groups in terms of $\mathrm{Ext}^*(I^{(i)}, I^{(i)})$. Thus we can say that the Ext–groups "were localized in the boxes of diagrams" (we recall how strong functoriality we obtained in [C1, Th. 5.1]). The situation considered in the previous section was more complicated. For example, the epimorphism $I^p \longrightarrow S^p$ preserves no information about $\mathrm{Ext}^*(I^{(i)}, -)$. Nevertheless, when we deal with a diagram $F_k^j(\lambda)$ then the information about $\mathrm{Ext}^*(W_\mu^{(i+j)}, S_{F^j(\lambda)}^{(i)})$ for $|\mu| = |\lambda|$ seems to be localized in p^j–slices of $F_k^j(\lambda)$ (we express these Ext–groups by $\mathrm{Ext}^*(I^{(i+j)}, S^{p^j(i)})$). The reason for which the map $I^p \longrightarrow S^p$ looses information is that it tears the p–slice apart. We had to introduce the "maps" $F_k^j(\phi_\lambda), F_k^j(\psi_\lambda)$ in order to preserve the structure of p^j–slices.

Now, we turn to the general situation. Our ultimate goal is to compute the groups $\mathrm{Ext}^*(W_\mu^{(*)}, S_\lambda^{(*)})$ for arbitrary diagrams μ, λ. But it turns out to be a difficult problem. The example which indicates the nature of difficulties in a very striking way is just $\mu = (1)$. It seems impossible to describe the groups $\mathrm{Ext}^*(I^{(a+i)}, S_\lambda^{(i)})$ (for a diagram λ of weight p^a) in terms of some other already known Ext–groups, for all structural arrows loose information since $\mathrm{Ext}^*(I^{(i)}, F \otimes G) = 0$ if $|F|, |G| > 0$. One can rather suppose that such groups form another elementary "block" by which other Ext–groups can be expressed. According to this point of view, $\mathrm{Ext}^*(I^{(i)}, I^{(i)})$ is just the simplest example of these blocks. But the groups $\mathrm{Ext}^*(I^{(i)}, I^{(i)})$ were computed in [FS, Th. 4.5], what made all further calculations possible. So, let us check whether the method used in [FS] can be applied to computing $\mathrm{Ext}^*(I^{(a+i)}, S_\lambda^{(i)})$ in general. The proof of [FS, Th. 4.5] consisted of two steps. The first was computation of $\mathrm{Ext}^*(I^{(i)}, S^{p^i})$, which was easy, because of the injectivity of S^{p^i}. The second (and main) was induction on j which allowed to compute $\mathrm{Ext}^*(I^{(i)}, S^{p^{i-j}(j)})$ for larger and larger j (till $j = i$). Here, the crucial role was played by analysis of hyperExt–groups with coefficients in the Koszul and De–Rham complexes. It turns out that the second part of this procedure carries over to the general situation.

Fact 4.1 *Let λ be a diagram of weight p^a and let $G = \mathrm{Ext}^*(I^{(a+i)}, S_{F_k^i(\lambda)})$.*

Then

$$\mathrm{Ext}^*(I^{(a+i)}, S_\lambda^{(i)}) = G \otimes \mathbf{k}[x]/(x^{p^{a+i}})[-h_k^i(\lambda)],$$

where $|x| = 2p^a$.

Proof: Let $G_j = \mathrm{Ext}^*(I^{(a+i)}, S^{(i-j)}_{F_k^j(\lambda)})$. It suffices if we show that $G_{j-1} = G_j \otimes \mathbf{k}[y]/(y^p)[-h_k^1(F_k^j(\lambda)]$, where $|y| = 2p^{a+j-1}$.

Let $\mathbf{K}_j^{(i-j)}$ denote the quotient of $\mathbf{S}^{(i-j)}_{F_k^j(\lambda)}$ consisting of boundaries of the differential in the Schur–Koszul complex (the reader of [FLS] and [FS] recognizes the familiar strategy). By [C2,Th. 5.3] and the arguments of [FLS, proof of Prop. 3.5], $H^*(\mathbf{K}_j^{(i-j)}) = \mathbf{K}_{j-1}^{(i-j+1)}$. We denote by E_I, E_{II} respectively the first and second spectral sequence converging to $\mathrm{hExt}^*(I^{(a+i)}, \mathbf{K}_j)$. Since the Schur–Koszul complex is exact, we have a short exact sequence of complexes

$$0 \longrightarrow \mathbf{K}_j^{(i-j)}[-1] \longrightarrow \mathbf{S}^{(i-j)}_{F_k^j(\lambda)} \longrightarrow \mathbf{K}_j^{(i-j)} \longrightarrow 0.$$

Thus we see that in the sth column in the first term of E_I we have $G_j[s-1]$, while in the sth column in the second term of E_{II} for $s \geq h_k^1(F_k^j(\lambda))$ we have $G_{j-1}[s-1-h_k^1(F_k^j(\lambda))]$. Dimension counting shows that the proof will be finished if we show that the differentials in these spectral sequences are trivial. Indeed, if we would show this, then, since both sequences have a common limit, we would get

$$G_j \otimes \mathbf{k}[z]/(z^{p^{a+j}}) = G_{j-1} \otimes \mathbf{k}[z]/(z^{p^{a+j-1}})[h_k^1(F_k^j(\lambda))],$$

where $|z| = 2$, which gives our assertion. Thus it remains to show the triviality of the differentials.

We first show it for E_{II}. Let $A_q : E^2_{II,*q} \longrightarrow E^2_{II,*+1,q+1}$ be the connecting homomorphism in the long exact sequence of Ext–groups induced by $\mathbf{K}_{j-1}^{(i-j+1)}[-1] \longrightarrow \mathbf{S}^{(i-j+1)}_{F_k^{j-1}(\lambda)} \longrightarrow \mathbf{K}_{j-1}^{(i-j+1)}$ (this map gives an isomorphism (shifting degree) between columns of E^2_{II}). Since the maps inducing A_* sum up to the morphism of complexes, the maps A_* commute (up to sign) with differentials in E_{II} (when we look at the hyperExt spectral sequences as sequences of suitable filtrations, we may derive this fact from the classical observation in [CE, Prop. IV.2.1]). But since these differentials are of type $(r, -r+1)$, we immediately conclude that they must be trivial, because there is only a finite number of nonzero columns in E_{II}.

For the sequence E_I this argument is unsufficient because the differentials are of type $(-r+1,r)$. Using it we can only conclude that if there is some nontrivial differential then there must be a nontrivial differential arriving to the last column. But to show the triviality of differentials coming to the last column we will use a different argument. We take some $x \in E^1_{I,n,p^{a+j}-1}$ and assume that $0 \neq y = \partial^1(x)$ (∂ stands for the differential in E_I), where $y \in E^1_{I,n,p^{a+j}} = \mathrm{Ext}^n(I^{(a+i)}, W^{(i-j)}_{\widetilde{F^j_k(\lambda)}})$.

Let us consider a twisted "resolution" of $\mathbf{S}^{(i-j)}_{F^j_k(\lambda)}$ by symmetric powers

$$0 \longrightarrow \mathbf{S}^{(i-j)}_{F^j_k(\lambda)} \longrightarrow \mathbf{S}^{\lambda^0(i-j)} \longrightarrow \dots$$

We get it by extension of a resolution of $S^{(i-j)}_{F^j_k(\lambda)}$ to whole Schur complexes, which is possible because each transformation between products of symmetric powers is a composition of transformations of three simple types (cf. [C1, proof of Lemma 3.4]), and these simplest transformations are obviously extendable. I put the word resolution into quotation marks, because I do not claim that the resulting sequence of complexes is exact (which is probably true, but we need not this fact). The important thing is that we do have an exact sequence in the highest degree component of this "resolution". Indeed, as we remember from [C1, Cor. 5.5], the sequence $(\mathbf{S}^{\lambda^*(i-j)})^{p^{a+j}}$ of the highest degree components in the "resolution" is the "Koszul dual" of the sequence $0 \longrightarrow S^{(i-j)}_{F^j_k(\lambda)} \longrightarrow S^{\lambda^0(i-j)} \longrightarrow$. Thus it is a resolution of $W^{(i-j)}_{\widetilde{F^j_k(\lambda)}}$ by twisted exterior powers. Now, when we consider the spectral sequence converging to hyperExt of $I^{(a+i)}$ with coefficients in the complex of boundaries of the Schur–Koszul complex $(\mathbf{S}^{\lambda^*(i-j)})^{p^{a+j}}$ (this sequence remains exact by the arguments of [FLS, proof of Prop. 3.5]), then, of course, there exists $0 \neq z \in \mathrm{Ext}^*(I^{(a+i)}, \mathbf{S}^{\lambda^s(i-j)})$ such that $z = \delta^s(y)$, where δ denotes the differential in this spectral sequence. Let ∂' denote the differential in the first spectral sequence converging to $\mathrm{hExt}^*(I^{(a+i)}, -)$ with coefficients in the complex of boundaries of the differential in the Schur–Koszul complex $\mathbf{S}^{\lambda^s}$. Since the maps in the"resolution" of $\mathbf{S}^{(i-j)}_{F^j_k(\lambda)}$ are morphisms of complexes, we have $\delta \circ \partial = \pm \partial' \circ \delta$ (we use again [CE, Prop. IV.2.1]). But according to the calculations of [FS, Th. 4.5], $\delta' = 0$ by dimension argument. Hence we get $z = 0$ which leads to a contradiction finishing the proof of the triviality of differential ∂^1 in E_I. We prove the triviality of higher differentials in the same manner. ■

Remark: Comparing this proof with that of [FS, Th. 4.5], we find a new ingredient: use of maps A_*. This trick would also allow to simplify the proof of that theorem.

Thus our task is reduced to computing $\mathrm{Ext}^*(I^{(a+i)}, S_{F_k^i(\lambda)})$. Unfortunately, in general $S_{F_k^i(\lambda)}$ is not injective and, surprisingly enough, the computation of these groups becomes the main problem. In the remainder of this section I will sketch some approach to this problem, which at least suggests the language in which the answer should be given. But first of all we shall show that the groups under consideration may be nontrivial for λ not being a p^a–hook (for p^a hooks the computation is easy). This vanishing would be quite reasonable in light of the correspondence between the twisting of the first variable and enlarging a diagram in the second (eg. this is true for $a = 1$ where $\mathrm{Ext}^*(I^{(1)}, S_\lambda) = 0$ for λ not being a p–hook). But the simplest possible example shows that this is not the case

Fact 4.2 *For $p = 2$,*

$$\dim(\mathrm{Ext}^n(I^{(2)}, S_{(2,2)})) = \begin{cases} 1 & n = 1, 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: We consider the spectral sequence converging to $\mathrm{Ext}^*(I^{(2)}, S_{(2,1)} \otimes S_{(1)}) = 0$ provided by the Littlewood–Richardson decomposition of the second variable [Bo]. In the columns of E_1 we have $\mathrm{Ext}^*(I^{(2)}, S_{(3,1)}) = \mathrm{Ext}^2(I^{(2)}, S_{(3,1)}) = \mathbf{k}$, $\mathrm{Ext}^*(I^{(2)}, S_{(2,2)})$ and $\mathrm{Ext}^*(I^{(2)}, S_{(2,1^2)}) = \mathrm{Ext}^1(I^{(2)}, S_{(2,1^2)}) = \mathbf{k}$. Thus we see, that there are two possibilities: either the groups we are interested in have the description predicted by our assertion, or they are trivial. To rule out the second possibility, we consider the spectral sequences converging to $\mathrm{hExt}^*(I^{(2)}, \mathbf{S}_{(2,2)})$ (we take the Schur–de-Rham complex here). If our groups were trivial, then the whole first term of the first spectral sequence would be trivial (the triviality of all columns except the first and last one follows from $\mathrm{Ext}^*(I^{(2))}, F \otimes G) = 0$ for $|F|, |G| > 0$; hence the triviality of the last column follows from the exactness of the Schur–Koszul complex). Therefore the second spectral sequence would converge to 0. But according to [C2, Fact 8.1], the second term of this sequence has four nontrivial columns in which we have the groups $\mathrm{Ext}^*(I^{(2)}, \Lambda^{2(1)})$. This gives a contradiction which finishes the proof. ■

In fact, it is possible to compute groups $\mathrm{Ext}^*(I^{(a)}, S_\lambda)$ also for some other small diagrams λ using such ad hoc methods. But I think it will be more interesting to sketch the general approach to the problem.

The main idea is that since one can hope that $\mathrm{Ext}^*(D^{p^j(a-j)}, S_\lambda) = \mathrm{Ext}^*(I^{p^j(a-j)}, S_\lambda)_{\Sigma_{p^j}}$, one can try to compute $\mathrm{Ext}^*(D^{p^j(a-j)}, S_\lambda)$ by induction on j using the Koszul and De–Rham complexes. But to start this program we should first compute the groups $\mathrm{Ext}^*(I^{p^j(a-j)}, S_\lambda)$. Already this problem turns out to be highly nontrivial. Let us focus on the case $j = a-1$. It will be convenient to consider a slightly more general problem, namely that of computing $\mathrm{Ext}^*(I^{d(1)}, S_\lambda)$ for a diagram λ of weight dp. By the earlier considerations this group is trivial if λ has a nontrivial p–core; and if $\lambda = F_k(\lambda')$ then, according to Th. 3.4, $\mathrm{Ext}^*(I^{d(1)}, S_\lambda) = s_{\lambda'}(B_{01})[h^1_k(\lambda')] = s_{\lambda'}(\mathbf{k}[\Sigma_d])[h^1_k(\lambda')] = Sp_{\lambda'}[h^1_k(\lambda')]$ (we recall that Sp_λ means the Specht module associated to the diagram λ (cf. [C1, Sect. 3]). In order to understand the situation of a diagram with a trivial core but the quotient consisting of several diagrams we shall develop notation introduced in [C2, Sect. 4]. We say that $R = \{\lambda = \alpha^0 \supset \alpha^1 \supset \ldots \supset \alpha^d = \emptyset\}$ is a decomposition of λ into slices if for every $1 \leq s \leq d$, the diagram α^s is obtained from α^{s-1} by removing a rim p–hook. In such a situation we call skew hooks $\chi^s = \alpha^{s-1} \setminus \alpha^s$, the slices of this decomposition. We recall from [C2, Sect. 4] that it may happen that different decompositions produce the same set of slices (in fact this is always the case if $\lambda = F_k(\lambda')$). Let us come back to Ext–groups. We consider the Decomposition Spectral Sequence [C1, Sect. 2] converging to $\mathrm{Ext}^*(I^{d(1)}, S_\lambda)$. Columns of the first term of this sequence are labeled by decompositions of λ into slices. In the column corresponding to a decomposition R with the set of slices $\{\chi^1, \ldots, \chi^d\}$ we have

$$\mathrm{Ext}^*(I^{(1)}, S_{\chi^1}) \otimes \ldots \otimes \mathrm{Ext}^*(I^{(1)}, S_{\chi^d}) = \mathbf{k}[h(R)],$$

where the shift $h(R)$ is given by the formula $h(R) = \sum_i h(\chi^i)$, and $h(\chi^i)$ is equal to the number of columns in χ^i minus 1. We shall show that the differentials in this spectral sequence are trivial. To do this, it suffices to show that all the numbers $h(R)$ have the same parity. But the last statement follows form the fact that the number $(-1)^{h(R)}$ is equal to "the sign of the permutation taking the natural ordering of beads before moving them to the configuration corresponding to the core and after it" [JK, p. 81]. For the reader who does not like beads and runners we can can give a less elementary argument. Namely, it follows from the Muranghan–Nakayama formula [JK, p. 60] that the value of the character of Sp_λ on a permutation being a sum of d cycles of length p equals $\sum_R (-1)^{h(R)}$. On the other hand this value is computed on p. 83 as $\pm f(\lambda)$ where $f(\lambda)$ is the number of decompositions of λ into slices. Therefore all the numbers $(-1)^{h(R)}$ must be equal (of course, beads are hidden in the proof of the formula on p. 83). Thus we have shown that

$$\dim(\mathrm{Ext}^*(I^{d(1)}, S_\lambda)) = f(\lambda).$$

Moreover, it immediately follows from [JK, Th. 7.27] that for λ with a trivial p–core and the p–quotient $(q^0(\lambda), \ldots, q^{p-1}(\lambda))$ we have

$$f(\lambda) = \dim((Sp_{q^0(\lambda)} \otimes \ldots \otimes Sp_{q^{p-1}(\lambda)}) \uparrow \Sigma_d).$$

This formula is attractive, because it suggests the structure of a Σ_d–module on $\mathrm{Ext}^*(I^{d(1)}, S_\lambda)$, which we should understand, since the next step in our program will be taking the coinvariants. Unfortunately, it is unlikely that there is an isomorphism of Σ_d–modules $\mathrm{Ext}^*(I^{d(1)}, S_\lambda) \simeq (Sp_{q^0(\lambda)} \otimes \ldots \otimes Sp_{q^{p-1}(\lambda)}) \uparrow \Sigma_d$, because the left–hand side is a sum of its homogeneous components, while (it seems that) there is no such a decomposition of the right–hand side. But some numerical experiments suggest that, at least in some special cases, the situation is even simpler.

Conjecture 4.3 *There is an isomorphism of Σ_d–modules*

$$\mathrm{Ext}^*(I^{d(1)}, S_\lambda) \simeq \bigoplus_\alpha N_{\alpha; q^0(\lambda), \ldots, q^{p-1}(\lambda)} Sp_\alpha,$$

where $N_{\alpha; q^0(\lambda), \ldots, q^{p-1}(\lambda)}$ is the multiplicity of Sp_α in the Littlewood–Richardson decomposition of $(Sp_{q^0(\lambda)} \otimes \ldots \otimes Sp_{q^{p-1}(\lambda)}) \uparrow \Sigma_d$.

The proof of this conjecture would require a thorough understanding of an interplay between the Decomposition Formula and the Littlewood–Richardson rule.

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