

# Fibrewise nullification and the cube theorem

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## Abstract

In this paper we explain when it is possible to construct fibrewise localizations in model categories. For pointed spaces, the general idea is to decompose the total space of a fibration as a diagram over the category of simplices of the base and replace it by the localized diagram. This of course is not possible in an arbitrary category. We have thus to adapt another construction which heavily depends on Mather's cube theorem. Working with model categories in which the cube theorem holds, we characterize completely those who admit a fibrewise nullification.

## Introduction

Mather's cube theorem states that the top face of a cube of spaces whose bottom face is a homotopy push-out and all vertical faces are homotopy pull-backs is again a homotopy push-out ([Mat76, Theorem 25]). This theorem is one of the very few occurrences of a situation where homotopy limits and colimits commute. It is actually related to a theorem of Puppe about commuting fibers and push-outs ([Pup74]), and also to Quillen's Theorem B in [Qui73]. Doeraene's work on  $J$ -categories has incorporated the cube theorem as an axiom in pointed model categories and allowed him to study the L.S.-category in an abstract setting ([Doe93]). Roughly speaking a  $J$ -category is a model category in which the cube theorem holds. Such a model category is very suitable for studying the relationship between a localization functor (constructed by means of certain homotopy colimits) and fibrations.

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Recall that a localization functor in a model category  $\mathcal{M}$  is any coaugmented idempotent functor  $L : \mathcal{M} \rightarrow \mathcal{M}$ . The coaugmentation is a natural transformation  $\eta : Id \rightarrow L$ . We will only deal with nullification functors  $P_A$ . In this context the image of  $P_A$  is characterized by the property that  $map(A, P_A X) \simeq *$ . We are looking for an existence theorem of fibrewise nullification, i.e. a construction which associates to any fibration  $F \rightarrow E \rightarrow B$  another fibration together with a natural transformation

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow \eta & & \downarrow & & \downarrow \\ P_A F & \longrightarrow & \bar{E} & \longrightarrow & B \end{array}$$

where  $E \rightarrow \bar{E}$  is a  $P_A$ -equivalence. This is achieved by imposing the join axiom for the object  $A$ : We require the join  $X * A$  to be killed by  $P_A$ , i.e.  $P_A(X * A) \simeq *$ , for any object  $X$ .

For pointed spaces, the most elegant construction of fibrewise localization is due to E. Dror Farjoun (in [DF96, Theorem F.3]). His idea is to decompose the total space of a fibration as a diagram over the category of simplices of the base and replace it by the corresponding localized diagram. In certain particular settings, some authors used other constructions (P. May [May80], W. Dwyer, H. Miller, and J. Neisendorfer in [DMN89] for completions, C. Casacuberta and A. Descheemaker in [CD02] in the category of groups), but none of these can be adapted in model categories. We prove the following:

**Theorem 3.3** *Let  $\mathcal{M}$  be a model category which is pointed, left proper, cellular and in which the cube and the join axiom hold. Then the nullification functor  $P_A$  admits a fibrewise version.*

This condition is actually necessary and we characterize completely the model categories for which fibrewise nullifications exist. This is closely related to the property of preserving products: A nullification functor  $P_A$  preserves (finite) products if  $P_A(X \times Y) \simeq P_A X \times P_A Y$ .

**Theorem 3.5** *Let  $\mathcal{M}$  be a model category which is pointed, left proper, cellular and in which the cube axiom holds. Then the following conditions are equivalent:*

- (i) *The nullification functor  $P_A$  admits a fibrewise version.*
- (ii) *The nullification functor  $P_A$  preserves finite products.*

(iii) The canonical projection  $X \times A \rightarrow X$  is a  $P_A$ -equivalence for any  $X \in \mathcal{M}$ .

(iv) The join axiom for  $A$  is satisfied.

We show in the last part of the paper that the category of algebras over an admissible operad satisfies the cube axiom. Therefore the plus-construction developed in [CRS] has a fibrewise analogue. Let us only say that the plus-construction performed on a  $\mathcal{O}$ -algebra  $B$  kills the maximal  $\mathcal{O}$ -perfect ideal in  $\pi_0 B$  and preserves Quillen homology. As a direct consequence we get the following result which is classical for spaces.

**Theorem 4.4** *Let  $\mathcal{O}$  – alg be the category of algebras over an admissible operad  $\mathcal{O}$ . For any  $\mathcal{O}$ -algebra  $B$ , denote by  $B \rightarrow B^+$  the plus construction. The homotopy fiber  $AB = \text{Fib}(B \rightarrow B^+)$  is then acyclic with respect to Quillen homology.*

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## 1 The cube axiom

We work in a model category  $\mathcal{M}$  which is *pointed*, i.e. the terminal object coincides with the initial one and is denoted by  $*$ . In such a category the homotopy fiber  $\text{Fib}(p)$  of a map  $p : E \rightarrow B$  is defined as the homotopy pull-back of the diagram  $* \rightarrow B \leftarrow E$ . We also assume the category is *left proper*, meaning that the push-out of a weak equivalence along a cofibration is again a weak equivalence. Finally we require  $\mathcal{M}$  to be *cellular* as defined in [Hir, Definition 14.1.1]. Basically the small object argument applies in a cellular model category, as one has  $I$ -cells which replace the usual spheres. There exists a cardinal  $\kappa$  such that any morphism from an  $I$ -cell to a telescope of length  $\lambda \geq \kappa$  factorizes through an object of this telescope. Moreover every object has a cofibrant replacement by an  $I$ -cell complex by [Hir, Theorem 13.3.7]. Localization functors exist in this setting, see [Hir, Theorem 4.1.1], but in general we do not know if it is possible to localize fibrewise in any (pointed, left proper, cellular) model category. We will thus work in model categories satisfying an extra-condition.

**Definition 1.1** A model category  $\mathcal{M}$  satisfies the *cube axiom* if for every commutative cubical diagram in  $\mathcal{M}$  in which the bottom face is a homotopy push-out square and all vertical faces are homotopy pull-back squares, then the top face is a homotopy push-out square as well.

M. Mather proved the cube Theorem for spaces in [Mat76, Theorem 25] and J.-P. Doraene introduced it as an axiom for model categories. His paper [Doe93] contains a very useful appendix with several examples of model categories satisfying this rather strong axiom.

**Example 1.2** Any stable model category satisfies the cube axiom. Indeed homotopy push-outs coincide with homotopy pull-backs, so that this axiom is a tautology. On the other hand the category of groups does not satisfy the cube axiom. Let us give an easy counter-example by considering the push-out of  $(\mathbb{Z} \leftarrow * \rightarrow \mathbb{Z})$ , which is a free group on two generators  $a$  and  $b$ . The pull-back along the inclusion  $\mathbb{Z} \langle ab \rangle \hookrightarrow \mathbb{Z} \langle a \rangle * \mathbb{Z} \langle b \rangle$  is obviously not a push-out diagram. However fibrewise localizations exist in the category of groups as shown by the recent work of Casacuberta and Descheemaker [CD02].

The following proposition claims that under very special circumstances the push-out of the fibers coincides with the fiber of the push-outs. In the category of spaces this is originally due to V. Puppe, see [Pup74]. The close link between the cube Theorem and Puppe's theorem was already well-known to M. Mather and M. Walker, as can be seen in [MW80].

**Proposition 1.3** *Let  $\mathcal{M}$  be a pointed model category in which the cube axiom holds. Consider natural transformations between push-out diagrams:*

$$\begin{array}{ccc}
 F & = & \text{hocolim} \left( \begin{array}{ccccc} F_1 & \longleftarrow & F_0 & \longrightarrow & F_2 \end{array} \right) \\
 j \downarrow & & \begin{array}{ccc} j_1 \downarrow & & j_0 \downarrow \\ & & & & \downarrow j_2 \end{array} \\
 E & = & \text{hocolim} \left( \begin{array}{ccccc} E_1 & \longleftarrow & E_0 & \longrightarrow & E_2 \end{array} \right) \\
 p \downarrow & & \begin{array}{ccc} p_1 \downarrow & & p_0 \downarrow \\ & & & & \downarrow p_2 \end{array} \\
 B & = & \text{hocolim} \left( \begin{array}{ccccc} B & \longleftarrow & B & \longrightarrow & B \end{array} \right)
 \end{array}$$

Assume that  $F_i = \text{Fib}(p_i)$  for any  $0 \leq i \leq 2$ . Then  $F = \text{Fib}(p)$ .

PROOF. Denote by  $k : G \rightarrow E$  the homotopy fiber of  $p$ . We show that  $G$  and  $F$  are weakly equivalent. Let us construct a cube by pulling-back  $E_i \rightarrow E$  along  $k$ . The bottom face consists thus in the middle row of the above diagram and the top face consists in the homotopy pull-backs of  $E_i \rightarrow E \leftarrow G$ , which are the same as the homotopy pull-backs of  $E_i \rightarrow B \leftarrow *$ , i.e.  $F_i$ . The cube axiom now states that the top face is a homotopy push-out and we are done.  $\square$

This result will be the main tool in constructing fiberwise localization in  $\mathcal{M}$ . In his paper [Doe93] on L.S.-category, J.-P. Doeraene used the cube axiom in a very similar fashion to study fiberwise joins. Indeed Ganea’s characterization of the L.S.-category uses iterated fibers of push-outs over a fixed base space. The same ideas have also been used in [DT95].

**Lemma 1.4** *Let  $\mathcal{M}$  be a model category in which the cube axiom holds. Let  $D$  be the homotopy push-out in  $\mathcal{M}$  of the diagram  $A \leftarrow B \rightarrow C$ . Then, for any object  $X \in \mathcal{M}$ ,  $X \times D$  is the homotopy push-out of the diagram  $X \times A \leftarrow X \times B \rightarrow X \times C$ .*

PROOF. It suffices to consider the cube obtained by pulling back the mentioned push-out square along the canonical projection  $X \times D \rightarrow D$ .  $\square$

## 2 The join

We check here that we can use all the classical facts about the join in any model category and introduce the join axiom. Most proofs here are not new, but probably folklore. Recall that the join  $A * B$  of two objects  $A, B \in \mathcal{M}$  is the homotopy push-out of  $A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$ . First notice that the induced maps  $A \rightarrow A * B$  and  $B \rightarrow A * B$  are trivial. Indeed the map  $A \rightarrow A * B$  can be seen as the composite  $A \xrightarrow{i_1} A \times B \xrightarrow{p_1} A \rightarrow A * B$  which by definition coincides with the obviously trivial map  $A \xrightarrow{i_1} A \times B \xrightarrow{p_2} B \rightarrow A * B$ .

**Lemma 2.1** *For any objects  $A, B \in \mathcal{M}$ , we have  $A * B \simeq \Sigma(A \wedge B)$ .*

PROOF. We use a “classical” Fubini argument (homotopy colimit commute with itself, cf. for example [CS02, Theorem 24.9]). Let  $P$  be the homotopy push-out of  $A \leftarrow A \vee B \rightarrow A \times B$  and consider first the commutative diagram

$$\begin{array}{ccccc}
 A & \longleftarrow & A \vee B & \longrightarrow & B \\
 \parallel & & \parallel & & \uparrow \\
 A & \longleftarrow & A \vee B & \longrightarrow & A \times B \\
 \parallel & & \downarrow & & \parallel \\
 A & \longleftarrow & A \times B & \xlongequal{\quad} & A \times B
 \end{array}$$

Its homotopy colimit can be computed in two different ways. By taking first vertical homotopy push-outs and next the resulting horizontal homotopy push-out one gets  $A * B$ .

By taking first horizontal homotopy push-outs one gets the homotopy cofiber of  $P \rightarrow A$ . Consider finally the commutative diagram

$$\begin{array}{ccccc}
 * & \xlongequal{\quad} & * & \xlongequal{\quad} & * \\
 \uparrow & & \uparrow & & \uparrow \\
 A & \longleftarrow & A \vee B & \longrightarrow & A \times B \\
 \parallel & & \downarrow & & \parallel \\
 A & \longleftarrow & A \times B & \xlongequal{\quad} & A \times B
 \end{array}$$

The same process as above shows that  $Cof(P \rightarrow A)$  is homotopy equivalent to  $\Sigma(A \wedge B)$ .  $\square$

**Lemma 2.2** *For any objects  $A, B \in \mathcal{M}$ , we have  $\Sigma A \wedge B \simeq \Sigma(A \wedge B)$ .*

PROOF. Apply again the Fubini commutation rule to the following diagram

$$\begin{array}{ccccc}
 * & \xlongequal{\quad} & * & \xlongequal{\quad} & * \\
 \uparrow & & \uparrow & & \uparrow \\
 B & \longleftarrow & A \vee B & \longrightarrow & B \\
 \parallel & & \downarrow & & \parallel \\
 B & \longleftarrow & A \times B & \longrightarrow & B
 \end{array}$$

where one uses Lemma 1.4 to identify the push-out of the bottom line.  $\square$

For a fibration  $F \rightarrow E \twoheadrightarrow B$ , the *holonomy action* is the map  $m : \Omega B \times F \rightarrow F$  induced on the pull-backs by the natural transformation from  $\Omega B \rightarrow * \leftarrow F$  to  $PB \twoheadrightarrow B \leftarrow E$ .

**Corollary 2.3** *For any fibration  $F \rightarrow E \twoheadrightarrow B$ , the homotopy push-out of  $\Omega B \leftarrow \Omega B \times F \xrightarrow{m} F$  is weakly equivalent to  $\Omega B * F$ .*

PROOF. Copy the proof above to compare this homotopy push-out to  $\Sigma(\Omega B \wedge F)$ .  $\square$

When working with a nullification functor  $P_A$  for some object  $A \in \mathcal{M}$ , we say that  $X$  is *A-acyclic* or *killed by A* if  $P_A X \simeq *$ . By universality this is equivalent to  $map(X, Z) \simeq *$  for any  $A$ -local object  $Z$ , or even better to the fact that any morphism  $X \rightarrow Z$  to an  $A$ -local object is homotopically trivial.

**Definition 2.4** A cellular model category  $\mathcal{M}$  satisfies the *join axiom* for the nullification functor  $P_A$  if the join of  $A$  with any  $I$ -cell is  $A$ -acyclic.

**Example 2.5** Any stable model category satisfies trivially the join axiom, as push-outs coincide with pull-backs. In such a category the join is always trivial. The category of groups satisfies the join axiom for a similar reason (but we saw in Example 1.2 that the cube axiom does not hold).

**Proposition 2.6** *Let  $\mathcal{M}$  be a cellular model category in which the join axiom and the cube axiom hold. Then  $\Sigma^i A * Z$  is  $A$ -acyclic for any  $i \geq 0$  and any object  $Z$ .*

PROOF. The join is a homotopy colimit and thus commutes with other homotopy colimits. Since any object in  $\mathcal{M}$  has a cofibrant approximation which can be constructed as a telescope by attaching  $I$ -cells, the lemma will be proven if we show that  $\Sigma^i A * Z$  is acyclic for any  $I$ -cell  $Z$ . By assumption we know that  $A * Z$  is acyclic and we conclude by Lemma 2.2 since  $\Sigma^i A * Z \simeq \Sigma^i(A * Z)$  is  $P_A$ -acyclic.  $\square$

**Remark 2.7** Given a family  $S$  of  $I$ -cells, we say  $\mathcal{M}$  satisfies the *restricted join axiom* if the join of  $A$  with any  $I$ -cell in  $S$  is  $A$ -acyclic. One refines then the above proposition to cellular model categories in which the restricted join axiom holds. Here  $\Sigma^i A * Z$  is  $A$ -acyclic for any  $i \geq 0$  and any  $S$ -cellular object  $Z$ , i.e. any object weakly equivalent to one which can be built by attaching only  $I$ -cells in  $S$ .

### 3 Fibrewise nullification

Let  $A$  be any object in  $\mathcal{M}$ . Recall that it is always possible to construct mapping spaces up to homotopy in  $\mathcal{M}$  eventhough we do not assume  $\mathcal{M}$  is a simplicial model category (see [CS02]). Thus we can define an object  $Z \in \mathcal{M}$  to be  $A$ -local if there is a weak equivalences  $map(A, Z) \simeq *$ . A map  $g : X \rightarrow Y$  is a  $P_A$ -equivalence if it induces a weak equivalences on mapping spaces  $g^* : map(Y, Z) \rightarrow map(X, Z)$  for any  $A$ -local object  $Z$ . Hirschhorn shows that there exists a coaugmented functor  $P_A : \mathcal{M} \rightarrow \mathcal{M}$  such that the coaugmentation  $\eta : X \rightarrow P_A X$  is a  $P_A$ -equivalence to an  $A$ -local object. This functor is called nullification or periodization.

The nullification  $X \rightarrow P_A X$  can be constructed up to homotopy by imitating the topological construction 2.8 in [Bou94]. One must iterate (possibly transfinitely, for a cardinal given by the smallness of any cofibrant object in  $\mathcal{M}$ , see [Hir, Theorem 14.4.4]) the process of gluing  $A$ -cells, i.e. take the homotopy cofiber of a map  $\Sigma^i A \rightarrow X$ . We

assume throughout this section that the model category  $\mathcal{M}$  satisfies both the join axiom and the cube axiom.

Let us explain now how to adapt the fibrewise construction [DF96, F.7] in a model category. The following lemma is the step we will iterate on and on so as to construct the space  $\bar{E}$  (in Theorem 3.3).

**Proposition 3.1** *Consider a commutative diagram*

$$\begin{array}{ccccc}
 F & \xhookrightarrow{\eta} & P_A F & \longrightarrow & F_1 \\
 j \downarrow & & \downarrow & & \downarrow j_2 \\
 E & \xhookrightarrow{\quad} & E' & \xrightarrow{\simeq} & E_1 \\
 p \downarrow & & p' \downarrow & & \downarrow p_1 \\
 B & \xlongequal{\quad} & B & \xlongequal{\quad} & B
 \end{array}$$

where the left column is a fibration sequence, the upper left square is a homotopy push-out square,  $p' : E' \rightarrow B$  is the unique map extending  $p$  such that the composite  $P_A F \rightarrow E' \rightarrow B$  is trivial,  $p_1$  is a fibration, and  $F_1$  is the homotopy fiber of  $p_1$ . Then the composites  $E \rightarrow E' \rightarrow E_1$  and  $F \rightarrow P_A F \rightarrow F_1$  are both  $P_A$ -equivalences.

PROOF. We can assume that the map  $\eta : F \hookrightarrow P_A F$  is a cofibration as indicated in the diagram, so that  $E'$  is obtained as a push-out, not only a homotopy push-out. Since  $\eta$  is a  $P_A$ -equivalence, so is its push-out along  $j$  by left properness (see [Hir, Proposition 3.5.4]). To prove that  $F \rightarrow F_1$  is a  $P_A$ -equivalence, it suffices to analyze the map  $P_A F \rightarrow F_1$ . We use Puppe's Proposition 1.3 to compute  $F_1$  as homotopy push-out of the homotopy fibers of  $P_A F \leftarrow F \rightarrow E$  over the fixed base  $B$ . This yields the diagram  $P_A F \times \Omega B \leftarrow F \times \Omega B \rightarrow F$  whose homotopy push-out is  $F_1$ . We investigate more closely the map  $F \rightarrow F_1$  by decomposing the map  $F \rightarrow P_A F$  into several steps obtained by gluing  $A$ -cells.

Consider a cofibration of the form  $\Sigma^i A \xrightarrow{f} F \rightarrow C_f$ . Let  $E_f$  be the homotopy push-out of  $C_f \leftarrow F \rightarrow E$  and compute as above the homotopy fiber  $F_f$  of  $E_f \rightarrow B$ . It is weakly equivalent to the homotopy push-out of  $C_f \times \Omega B \leftarrow F \times \Omega B \rightarrow F$ . Hence  $F_f$  is also weakly equivalent to the homotopy push-out of  $\Omega B \leftarrow \Sigma^i A \times \Omega B \rightarrow F$ , using the definition of  $C_f$ . Decompose this push-out as follows

$$\begin{array}{ccccc}
 \Sigma^i A \times \Omega B & \longrightarrow & \Sigma^i A & \longrightarrow & F \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega B & \longrightarrow & \Sigma^i A * \Omega B & \longrightarrow & F_f
 \end{array}$$

The right-hand square must be a homotopy push-out square as well. But both  $\Sigma^i A$  and  $\Sigma^i A * \Omega B$  are  $A$ -acyclic (by Proposition 2.6), so that the map  $\Sigma^i A \rightarrow \Sigma^i A * \Omega B$  is a  $P_A$ -equivalence. Thus so is  $F \rightarrow F_f$  by left properness. Iterating this process of gluing  $A$ -cells shows that  $F \rightarrow F_1$  is a telescope of  $P_A$ -equivalences, hence a  $P_A$ -equivalence.  $\square$

**Remark 3.2** In the category of spaces it is of course true that  $\Omega B \times F \rightarrow \Omega B \times P_A F$  is a  $P_A$ -equivalence, because localization commutes with finite products. In general we will see in Theorem 3.5 that the join axiom is actually equivalent to the commutation of  $P_A$  with products. With the restricted join axiom we would have to impose the additional restriction on  $B$  that  $\Omega B$  be  $S$ -cellular.

**Theorem 3.3** *Let  $\mathcal{M}$  be a model category which is pointed, left proper, cellular and in which the cube axiom and the join axiom hold. Let  $P_A : \mathcal{M} \rightarrow \mathcal{M}$  be a nullification functor. Then there exists a fibrewise nullification, i.e. a construction which associates to any fibration  $F \rightarrow E \rightarrow B$  another fibration together with a natural transformation*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow \eta & & \downarrow & & \downarrow \\ P_A F & \longrightarrow & \bar{E} & \longrightarrow & B \end{array}$$

where  $E \rightarrow \bar{E}$  is a  $P_A$ -equivalence.

PROOF. We construct first by the method provided in Lemma 3.1 a natural transformation to the fibration  $F_1 \rightarrow E_1 \rightarrow B$ . We iterate then this construction and get a fibration  $\bar{F} \rightarrow \bar{E} \rightarrow B$  where  $\bar{F} = \text{hocolim}(F \rightarrow F_1 \rightarrow F_2 \rightarrow \dots)$  and  $\bar{E} = \text{hocolim}(E \rightarrow E_1 \rightarrow E_2 \rightarrow \dots)$ . All maps in these telescopes are  $P_A$ -equivalences by the lemma, hence so are  $E \rightarrow \bar{E}$  and  $F \rightarrow \bar{F}$ . Moreover any map  $F_n \rightarrow F_{n+1}$  factorizes as  $F_n \rightarrow P_A F_n \simeq P_A F \rightarrow F_{n+1}$  so that  $\bar{F} \simeq P_A F$ . We obtain thus the desired fibration  $P_A F \rightarrow \bar{E} \rightarrow B$ .  $\square$

Define  $\bar{P}_A X = \text{Fib}(X \rightarrow P_A X)$ , the fiber of the nullification. As in the case of spaces we get:

**Corollary 3.4** *For any object  $X$  in  $\mathcal{M}$  we have  $P_A \bar{P}_A X \simeq *$ .*

PROOF. Apply the fiberwise localization to the fibration  $\bar{P}_A X \rightarrow X \rightarrow P_A X$ . This yields a fibration  $P_A \bar{P}_A X \rightarrow \bar{X} \rightarrow P_A X$  in which the base and the fiber are  $A$ -local. Therefore  $\bar{X}$  is  $A$ -local as well. But then  $\bar{X} \simeq P_A X$  and so  $P_A \bar{P}_A X \simeq *$ .  $\square$

We end this section with a complete characterization of the model categories which admit fibrewise nullifications.

**Theorem 3.5** *Let  $\mathcal{M}$  be a model category which is pointed, left proper, cellular and in which the cube axiom holds. Then the following conditions are equivalent:*

- (i) *The nullification functor  $P_A$  admits a fibrewise version.*
- (ii) *The nullification functor  $P_A$  preserves finite products.*
- (iii) *The canonical projection  $X \times A \rightarrow X$  is a  $P_A$ -equivalence for any  $X \in \mathcal{M}$ .*
- (iv) *The join axiom for  $A$  is satisfied.*

PROOF. We prove first that (i) implies (ii). Consider the trivial fibration  $X \rightarrow X \times Y \rightarrow Y$  and apply the fibrewise nullification to get a new fibration  $P_A X \rightarrow E \rightarrow Y$ . The inclusion of the fiber admits a retraction  $E \rightarrow P_A X$ , i.e.  $E \simeq P_A X \times Y$ . Applying once again the fibrewise nullification to  $Y \rightarrow Y \times P_A X \rightarrow P_A X$ , we see that the map  $X \times Y \rightarrow P_A X \times P_A Y$  is a  $P_A$ -equivalence. As a product of local objects is local, this means precisely that  $P_A(X \times Y) \simeq P_A X \times P_A Y$ .

Property (iii) is a particular case of (ii). We show now that (iii) implies (iv). If the canonical projection  $X \times A \rightarrow X$  is a  $P_A$ -equivalence, the push-out of it along the other projection yields another  $P_A$ -equivalence, namely  $A \rightarrow X * A$ . Therefore the join  $X * A$  is  $P_A$ -acyclic. Finally (iv) implies (i) as shown in Theorem 3.3.  $\square$

The construction we propose for fibrewise nullification does not translate to the setting of general localization functors. We do not know if the cube and join axioms are sufficient conditions for the existence of fibrewise localizations.

## 4 Algebras over an operad

In this section we provide the motivating example for which this theory has been developed. For a fixed field  $k$ , we work with  $\mathbb{Z}$ -graded differential  $k$ -vector spaces ( $k$ -dgm) and consider the category of algebras in  $k$ -dgm over an admissible operad. This is indeed a pointed, left proper and cellular category. Weak equivalences are quasi-isomorphisms and fibrations are epimorphisms.

We do not know if the join axiom holds in full generality for any object  $A$ . It does so however when  $A$  is acyclic with respect to Quillen homology, which is the case we are most interested in, or when  $A$  is a free algebra. We check that the cube axiom always holds, following the strategy of [Doe93, Proposition A.15], which guarantees the existence of fibrewise versions of the plus-construction and Postnikov sections. In the case of  $\mathbb{N}$ -graded  $\mathcal{O}$ -algebras (the case  $\mathcal{O} = \mathcal{A}s$  is treated by Doeraene) one has to restrict to a particular set of fibrations (the so-called  $J$ -maps), because they must be surjective in each degree in order to compute pull-backs. In our context all fibrations are epimorphisms, so that the cube axiom holds in full generality.

**Theorem 4.1** *The cube axiom holds in the category of  $\mathcal{O}$ -algebras.*

PROOF. Let us briefly recall the key steps in Doeraene's strategy. We consider a push-out square of  $\mathcal{O}$ -algebras (along a generic cofibration  $B \hookrightarrow B \amalg \mathcal{O}(V)$ ):

$$\begin{array}{ccc} B & \hookrightarrow & C = B \amalg \mathcal{O}(V) \\ \downarrow & & \downarrow \\ A & \hookrightarrow & D = A \amalg \mathcal{O}(V) \end{array}$$

We need to compute the pull-back of this square along a fibration  $p : E \twoheadrightarrow D$  (which is hence an epimorphism of chain complexes). We have thus the following isomorphism of chain complexes:

$$E \cong A \amalg \mathcal{O}(V) \oplus \ker(p).$$

This allows to compute the successive pull-backs  $A \times_D E$ ,  $C \times_D E$ , and  $B \times_D E$ . In order to construct the homotopy push-out  $P$  of these pull-backs (which must coincide with  $E$ ) we factorize the morphism  $B \times_D E \rightarrow C \times_D E$  as

$$B \times_D E \hookrightarrow (B \oplus \ker p) \amalg \mathcal{O}(V \oplus W) \xrightarrow{\sim} C \times_D E$$

Thus  $P$  is identified with  $(A \oplus \ker(p)) \amalg \mathcal{O}(V \oplus W)$ , which allows us to build finally a quasi-isomorphism to  $E$ .  $\square$

The plus-construction for an  $\mathcal{O}$ -algebra is a nullification with respect to a universal acyclic algebra  $\mathcal{U}$ . We refer to [CRS] for an explicit construction and nice applications.

**Proposition 4.2** *The join axiom holds for any acyclic  $\mathcal{O}$ -algebra  $A$ . It holds in particular for the universal acyclic algebra  $\mathcal{U}$  constructed in [CRS], so that the fibrewise plus-construction exists.*

PROOF. The join  $A * X$  is weakly equivalent to  $\Sigma A \wedge X$  by Lemmas 2.1 and 2.2. Since  $\Sigma A$  is 0-connected and acyclic, it is trivial by the Hurewicz Theorem [CRS, Theorem 1.1]. Thus  $A * X \simeq *$  is always  $P_A$ -acyclic.  $\square$

We consider next the case of Postnikov sections  $P_{\mathcal{O}(x)}$ , where  $x$  is a generator of arbitrary degree  $n \in \mathbb{Z}$ . Because  $[\mathcal{O}(x), X] \cong \pi_n X$  for any  $\mathcal{O}$ -algebra  $X$ , the nullification functor  $P_{\mathcal{O}(x)}$  is really a Postnikov section, i.e.  $P_{\mathcal{O}(x)}X \simeq X[n-1]$ . Let us also recall that  $\pi_n(X \times Y) \cong \pi_n X \times \pi_n Y$ .

**Proposition 4.3** *The join axiom holds for any free  $\mathcal{O}$ -algebra  $\mathcal{O}(x)$  on one generator of degree  $n \in \mathbb{Z}$ . Therefore fibrewise Postnikov sections exist.*

PROOF. By Theorem 3.5 we might as well check that the map  $X \times \mathcal{O}(x) \rightarrow X$  is a  $P_{\mathcal{O}(x)}$ -equivalence for any  $\mathcal{O}$ -algebra  $X$ . Clearly the  $n$ -th Postnikov section of the product  $X \times \mathcal{O}(x)$  is equivalent to  $X[n-1]$  and we are done.  $\square$

Our final result is a particular case of Corollary 3.4. A direct proof (without fibrewise techniques) seems out of reach.

**Theorem 4.4** *Let  $\mathcal{O}$ -alg be the category of algebras over an admissible operad  $\mathcal{O}$ . For any  $\mathcal{O}$ -algebra  $B$ , denote by  $B \rightarrow B^+$  the plus construction. The homotopy fiber  $AB = \text{Fib}(B \rightarrow B^+)$  is then acyclic with respect to Quillen homology.  $\square$*

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