# AUSLANDER-REITEN SEQUENCES AS APPETIZERS FOR HOMOTOPISTS AND ARITHMETICIANS

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*Dedicated to Professor John Labute with admiration, respect and friendship.*

ABSTRACT. We introduce Auslander-Reiten sequences for group algebras and give several recent applications. The first part of the paper is devoted to some fundamental problems in Tate cohomology which are motivated by homotopy theory. In the second part of the paper we interpret Auslander-Reiten sequences in the context of Galois theory and connect them to some important arithmetic objects.

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## 1. INTRODUCTION

In the past the second author had an opportunity to discuss with John Labute some beautiful aspects of modular representation theory and they both came to the conclusion that the techniques from modular representation theory could be applied effectively to

study problems in number theory, particularly in field theory and Galois theory. But unfortunately some of these techniques from representation theory are not well-known to people working in number theory.

Several times recently, we ran into Auslander-Reiten sequences in our work and we began to appreciate the power and elegance of these sequences. These sequences are some special non-split short exact sequences which are very close to being split sequences. They were introduced by Auslander and Reiten in the early 70s [AR75], and they have been among the most important tools from modular representation theory.

The goal of this article is to introduce Auslander-Reiten sequences and to give some applications. This article is by no means a historical survey of Auslander-Reiten theory; it focuses only on some selective applications from our recent joint work. The treatment here will be particularly suitable for homotopy theorists and number theorists. We hope that the readers who are not specialists in representation theory might be inspired further to learn more about Auslander-Reiten sequences and investigate possible Galois field theoretic applications.

We would like to thank the referee for his/her valuable suggestions which have helped us to improve the exposition of our paper

## 2. Auslander-Reiten sequences

Throughout we let $G$ denote a finite group and $k$ a field of characteristic $p$ that divides the order of $G$. We denote by $\Lambda$ the group algebra $kG$. Although Auslander-Reiten theory is often studied in the broader context of Artin algebras, for simplicity we restrict ourselves to the most interesting case of group algebras.

**Definition 2.1.** A short exact sequence of $\Lambda$-modules

$$\epsilon: \quad 0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} N \longrightarrow 0$$

is an Auslander-Reiten (AR) sequence if the following three conditions are satisfied:

- $\epsilon$ is a non-split sequence.
- $\alpha$ is left almost split, i.e., any map $\phi \colon M \to M'$ that is not a split monomorphism factors through $E$ via $\alpha$.
- $\beta$ is right almost split, i.e., any map $\psi \colon N' \to N$ that is not a split epimorphism factors through $E$ via $\beta$.



There are several other equivalent definitions of an Auslander-Reiten sequence. The one we have given here is perhaps the most symmetric of all. Note that a split short exact sequence satisfies the last two conditions. This justifies why Auslander-Reiten sequences are also called *almost split sequences*. Further, we note that these conditions also imply that the terms $M$ and $N$ in an AR sequence have to be indecomposable and non-projective.

*Example* 2.2. **(One-up and One-down)** Let us start with a very simple example. Let $\Lambda = k[x]/x^p$ where $p$ is the characteristic of $k$. Observe that $\Lambda$ is a group algebra $kG$ with a $G$ cyclic group of order $p$. Then for each integer $i$ strictly between 0 and $p$, the short exact sequence

$$0 \longrightarrow k[x]/x^i \xrightarrow{(j,-\pi)} k[x]/x^{i+1} \oplus k[x]/x^{i-1} \longrightarrow k[x]/x^i \longrightarrow 0$$

is easily shown (exercise left to the reader) to be an Auslander-Reiten sequence. Here $j$ is the obvious injection into the first summand of the middle term, and $\pi$ is the obvious surjection on the second summand.

The remarkable result is the existence of these sequences:

**Theorem 2.3.** [ARS95] *Given any indecomposable non-projective $\Lambda$-module $N$, there exists a unique (up to isomorphism of short exact sequences) Auslander-Reiten sequence $\epsilon$ ending in $N$. Moreover the first term $M$ of the AR-sequence $\epsilon$ is isomorphic to $\Omega^2 N$.*

In Section 4 we present a non-standard proof due to Krause of the existence of AR sequences. This uses the Brown representability theorem from algebraic topology. For now, we assume existence and show uniqueness.

Suppose we have two AR-sequences ending in an indecomposable non-projective $kG$-module $L$:

$$0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} L \longrightarrow 0$$
$$0 \longrightarrow M' \xrightarrow{\alpha'} E' \xrightarrow{\beta'} L \longrightarrow 0$$

The defining property of AR sequences then gives the following commutative diagram.



The composite $\phi := gf$ cannot be nilpotent, otherwise $\beta$ would be a split epimorphism. (This is an easy diagram-chase exercise.) Now since $M$ is a module of finite length, for sufficiently large values of $n$, we must have

$$M \cong \mathrm{Im}(\phi^n) \oplus \mathrm{Ker}(\phi^n).$$

Since $M$ is indecomposable, and $\phi$ is not nilpotent, it follows that $M \cong \mathrm{Im}(\phi^n)$ and $0 = \mathrm{Ker}(\phi^n)$. This shows that $\phi^n$, and hence $\phi$, is an isomorphism. Therefore $f$ is the left inverse of $g$. Interchanging the roles of the two AR-sequences above, it follows similarly that $f$ is the right inverse of $g$. Thus we have $M \cong M'$. The five lemma now tells us that $E \cong E'$. Thus the two AR-sequences are isomorphic as short exact sequences.

## 3. Stable module category

We now introduce the stable module category. We will work mostly in this category for the first half of the paper. The *stable module category* $\mathrm{StMod}(kG)$ of $G$ is the category obtained from the category $\mathrm{Mod}(kG)$ of (left) $kG$-modules by killing the projective modules. More precisely, it is the category whose objects are $kG$-modules and whose morphisms are equivalence classes of $kG$-module homomorphisms, where two homomorphisms are equivalent if their difference factors through a projective module (such a map is called *projective*). If $M$ and $N$ are two $kG$-modules, then we write $\underline{\mathrm{Hom}}_{kG}(M, N)$ for the $k$-vector space of maps between $M$ and $N$ in the stable module category. It is easy to see that a $kG$-module is projective if and only if it is the zero object in $\mathrm{StMod}(kG)$. The compact stable category $\mathrm{stmod}(kG)$ is the full subcategory of $\mathrm{StMod}(kG)$ consisting of the finitely generated (equivalently, finite-dimensional) $kG$-modules.

The category $\mathrm{StMod}(kG)$ admits a natural tensor triangulated structure which we now explain. If $M$ is a $kG$-module then define its desuspension $\Omega\, M$ to be the kernel of a surjective map from a projective module to $M$. By Schanuel's Lemma, the new module thus obtained is well-defined up to projective summands and therefore defines an endofunctor

$$\Omega \colon \mathrm{StMod}(kG) \longrightarrow \mathrm{StMod}(kG).$$

Using the fact that $kG$ is a Frobenius algebra (injectives and projective coincide) one can show that $\Omega$ is an equivalence. The inverse functor $\Omega^{-1}$ sends $M$ to the cokernel of an injective map from $M$ to an injective module. This allows one to talk about negative powers of $\Omega$.

The importance of the stable module category can be seen in the following useful and well-known fact.

**Proposition 3.1.** *Let $M$ and $N$ be $kG$-modules. If $n$ is any positive integer, then*

$$\underline{\mathrm{Hom}}_{kG}(\Omega^n\, M, N) \cong \mathrm{Ext}^n_{kG}(M, N) \cong \underline{\mathrm{Hom}}_{kG}(M, \Omega^{-n}\, N).$$

In particular, group cohomology is elegantly encoded in the stable module category. In fact, allowing $n$ to be any integer, one recovers the Tate cohomology ring of $G$ in this way as the (graded) ring of self maps of the trivial representation. Under this isomorphism, the multiplication in the Tate ring of $G$ corresponds to composition in $\underline{\mathrm{Hom}}(\Omega^* k, k)$.

Now we describe the triangles in the stable module category. These arise from short exact sequences in the module category in the following way. Suppose

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of $kG$-modules. This short exact sequence determines a unique class in $\mathrm{Ext}^1_{kG}(N, L)$ which under the isomorphism given in the above proposition corresponds to a map $N \to \Omega^{-1}L$. Thus we get a diagram

$$L \longrightarrow M \longrightarrow N \longrightarrow \Omega^{-1}L. \quad (*)$$

A triangle in $\mathrm{StMod}(kG)$ is any such diagram which is isomorphic to a diagram which arises as above $(*)$. These triangles and the suspension functor satisfy the standard axioms for a triangulated category; see [Car96] for example.

The tensor product of two $kG$-modules (with diagonal $G$-action on the tensor product of $k$-vector spaces) descends to a well-behaved tensor product in the stable module

category. (This follows from the fact that if $M$ and $N$ are $kG$-modules with either $M$ or $N$ projective, then $M \otimes_k N$ is projective.) The trivial representation $k$ of $G$ serves as a unit for the tensor product.

## 4. KRAUSE'S PROOF OF THE EXISTENCE OF AR SEQUENCES

The proof of the existence of an Auslander-Reiten sequence can be found in [ARS95]. We present a different avatar of Auslander-Reiten sequences in the stable module category, and prove their existence following Krause [Kra00].

**Definition 4.1.** A triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Omega^{-1}A$ in StMod$(kG)$ is an Auslander-Reiten triangle if the following three conditions hold.

- $\gamma \neq 0$.
- Every map $f\colon A \longrightarrow A'$ which is not a section factors through $\alpha$.
- Every map $g\colon C' \longrightarrow C$ which is not a retraction factors through $\beta$.

The following theorem is crucial in what follows. This is the Brown representability theorem for StMod$(kG)$.

**Theorem 4.2.** *(Brown) A contravariant functor $F\colon \mathrm{StMod}(kG) \longrightarrow \mathrm{Ab}$ that is exact and sends coproducts to products is representable, i.e., there exists a module $M_F$ in* StMod$(kG)$ *such that*

$$F(-) \cong \underline{\mathrm{Hom}}_{kG}(-, M_F).$$

Now fix a finitely generated non-projective $kG$-module $Z$ that has a local endomorphism ring, and set $\Gamma = \underline{\mathrm{Hom}}_{kG}(Z, Z)$. Let $I$ be the injective envelope of $\Gamma/\mathrm{rad}\,\Gamma$. Consider the functor

$$\mathrm{Hom}_\Gamma(\underline{\mathrm{Hom}}_{kG}(Z, -), I).$$

It is not hard to see that this is a contravariant functor that is exact and sends coproducts to products. So by the Brown representability theorem this functor is representable:

$$(4.1) \qquad\qquad \Psi\colon \mathrm{Hom}_\Gamma(\underline{\mathrm{Hom}}_{kG}(Z, -), I) \cong \underline{\mathrm{Hom}}_{kG}(-, T_Z I).$$

Finally consider the composite $\omega\colon \Gamma \xrightarrow{\pi} \Gamma/\mathrm{rad}\,\Gamma \hookrightarrow I$, and denote by $\gamma$ the map $\Psi(\omega)$. Note that $\gamma$ is a map from $Z \longrightarrow T_Z I$. Extending this map to a triangle in StMod$(kG)$, we get

$$\Omega T_Z I \longrightarrow Y \longrightarrow Z \xrightarrow{\gamma} T_Z I.$$

**Theorem 4.3** ([Kra00])**.** *The triangle $\Omega T_Z I \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} T_Z I$ is an Auslander-Reiten triangle.*

*Proof.* $\gamma \neq 0$: Recall that $\gamma = \Psi(\omega)$, where $\Psi$ is an isomorphism of functors, and $\omega$ is a non-zero map. Therefore $\gamma \neq 0$.

$\beta$ has the right almost split property: That is, given a map $\rho\colon Z' \longrightarrow Z$ that is not a retraction, we have to show that there exists a map $\rho'\colon Z' \longrightarrow Y$ making the following diagram commutative.

$$
\begin{array}{ccccccc}
 & & & & Z' & & \\
 & & \swarrow^{\rho'} & & \downarrow^{\rho} & & \\
\Omega T_Z I & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{\gamma} & T_Z I
\end{array}
$$

For this, it is enough to know that $\gamma\rho = 0$, since the bottom row in the above diagram is an exact triangle. Since $\rho$ is not a retraction, $\mathrm{Im}\,\underline{\mathrm{Hom}}_{kG}(Z, Z') \xrightarrow{\rho_*} \underline{\mathrm{Hom}}_{kG}(Z, Z) = \Gamma$ does not contain the identity map. Consequently $\mathrm{Im}(\rho_*)$ is contained in $\mathrm{rad}\,\Gamma$.

The naturality of the isomorphism $\Psi$ yields the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{Hom}_\Gamma(\underline{\mathrm{Hom}}_{kG}(Z, Z), I) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}_{kG}(Z, T_Z I) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_\Gamma(\underline{\mathrm{Hom}}_{kG}(Z, Z'), I) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}_{kG}(Z', T_Z I)
\end{array}
$$

Note that $\omega$ maps to zero under the left vertical map and by the commutativity of the diagram, we see that $\gamma$ also maps to zero under the right vertical map. Thus $\gamma\rho = 0$, as desired.

$\alpha$ has the left almost split property: For this, it turns out that it suffices to show that $\underline{\mathrm{Hom}}_{kG}(T_z I, T_Z I)$ is a local ring; see [Kra00] for details.

$$
\begin{aligned}
\underline{\mathrm{Hom}}_{kG}(T_z I, T_Z I) & \cong \mathrm{Hom}_\Gamma(\underline{\mathrm{Hom}}_{kG}(Z, T_Z I), I) \\
& \cong \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Gamma(\underline{\mathrm{Hom}}_{kG}(Z, Z), I), I) \\
& \cong \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Gamma(\Gamma, I), I) \\
& \cong \mathrm{Hom}_\Gamma(I, I)
\end{aligned}
$$

The injective $\Gamma$-module $I$ is indecomposable since $\Gamma/\mathrm{rad}\,\Gamma$ is simple, and therefore $\mathrm{Hom}_\Gamma(I, I)$ is a local ring. So we are done.                                    □

## 5. Auslander-Reiten duality

The isomorphism given in eq. 4.1 is just another avatar of Auslander-Reiten duality which we now explain. Let $M$ and $L$ be finitely generated $kG$-modules. Then the Auslander-Reiten duality says that there is a non-degenerate bilinear form (functorial in both variables)

$$
\Phi(-, -) \colon \underline{\mathrm{Hom}}_{kG}(L, \Omega M) \times \underline{\mathrm{Hom}}_{kG}(M, L) \longrightarrow k.
$$

To explain this pairing, first recall that $\Omega M$ is defined by the short exact sequence

$$
0 \longrightarrow \Omega M \xrightarrow{j} P \xrightarrow{q} M \longrightarrow 0,
$$

where $P$ is a minimal projective cover of $M$. Given $f \colon L \to \Omega M$ and $g \colon M \to L$, we then get the following diagram.

$$
\begin{array}{ccc}
P & \longrightarrow & P \\
q \downarrow & & \uparrow j \\
M \xrightarrow{g} & L \xrightarrow{f} & \Omega M
\end{array}
$$

The top horizontal self map of $P$ is in the image of the norm map $\eta \colon \mathrm{End}_k P \to \mathrm{End}_{kG} P$ which sends $\phi$ to $\sum_{g \in G} g \cdot \phi$. Now we define $\Phi(f, g) = tr(\theta)$, where $\eta(\theta) = jfgq$.

To see the connection between the Auslander-Reiten sequence and the natural isomorphism 4.1 which was responsible for the existence of Auslander-Reiten triangles, consider the special case of Auslander-Reiten duality when $M = k$. This gives a pairing (natural in $L$):

$$
\underline{\mathrm{Hom}}_{kG}(L, \Omega k) \times \underline{\mathrm{Hom}}_{kG}(k, L) \longrightarrow k.
$$

Or equivalently, a natural isomorphism of functors

$$\mathrm{Hom}_k(\underline{\mathrm{Hom}}_{kG}(k,-),k) \cong \underline{\mathrm{Hom}}_{kG}(-,\Omega k)$$

This is precisely the isomorphism 4.1 when $Z = k$. Since the representing object is unique, we get $T_k I = \Omega k$. Thus the Auslander-Reiten triangle ending in $k$ has the form

$$\Omega^2 k \longrightarrow M \longrightarrow k \longrightarrow \Omega k.$$

In the next three sections we give three different applications of Auslander-Reiten sequences from our recent work.

## 6. Representations of $V_4$-modules

Consider the Klein four group $V_4 = C_2 \oplus C_2$. We use AR sequences to prove the following intriguing result which also plays a crucial role in the proof of the classification of the indecomposable $V_4$ representations.

**Theorem 6.1.** *Let $M$ be a projective-free $V_4$-module. Then we have the following.*

(1) *If $l$ is the smallest positive integer such that $\Omega^l(k)$ is isomorphic to a submodule of $M$, then $\Omega^l(k)$ is a summand of $M$.*

(2) *Dually, if $l$ is the smallest positive integer such that $\Omega^{-l}(k)$ is isomorphic to a quotient module of $M$, then $\Omega^{-l}(k)$ is a summand of $M$.*

*Proof.* First note that the second part of this lemma follows by dualising the first part; here we also use the fact that $(\Omega^l k)^* \cong \Omega^{-l} k$. So it is enough to prove the first part.

The AR sequences in the category of $kV_4$-modules are of the form (see [Ben84, Appendix, p 180]):

$$0 \to \Omega^{l+2} k \to \Omega^{l+1} k \oplus \Omega^{l+1} k \to \Omega^l k \to 0 \qquad l \neq -1$$

$$0 \to \Omega^1 k \to kV_4 \oplus k \oplus k \to \Omega^{-1} k \to 0$$

To prove the first part of the above lemma, let $l$ be the smallest positive integer such that $\Omega^l k$ embeds in a projective-free $V_4$-module $M$. If this embedding does not split, then by the property of an almost split sequence, it should factor through $\Omega^{l-1} k \oplus \Omega^{l-1} k$ as shown in the diagram below.



Now if either $f$ or $g$ is injective, that would contradict the minimality of $l$, so they cannot be injective. So both $f$ and $g$ should factor through $\Omega^{l-2} k \oplus \Omega^{l-2} k$ as shown in the diagrams below.

$$0 \longrightarrow \Omega^{l-1} k \longrightarrow \Omega^{l-2} k \oplus \Omega^{l-2} k \longrightarrow \Omega^{l-3} k \longrightarrow 0$$

with vertical map $g$ from $\Omega^{l-1} k$ to $M$, and diagonal map $(g_1 \oplus g_2)$ from $\Omega^{l-2} k \oplus \Omega^{l-2} k$ to $M$.

Proceeding in this way we can assemble all the lifts obtained using the almost split sequences into one diagram as shown below.

$$\Omega^l k \hookrightarrow M$$
$$\downarrow$$
$$\Omega^{l-1} k \oplus \Omega^{l-1} k$$
$$\downarrow$$
$$(\Omega^{l-2} k \oplus \Omega^{l-2} k) \oplus (\Omega^{l-2} k \oplus \Omega^{l-2} k)$$
$$\vdots$$
$$(\Omega^1 k \oplus \Omega^1 k) \oplus \cdots \oplus (\Omega^1 k \oplus \Omega^1 k)$$
$$\downarrow$$
$$(kV_4 \oplus k \oplus k) \oplus \cdots \oplus (kV_4 \oplus k \oplus k)$$

So it suffices to show that for a projective-free $M$ there cannot exist a factorisation of the form

$$\Omega^l k \hookrightarrow M$$
$$\downarrow \quad \nearrow \phi$$
$$(kV_4)^s \oplus k^t$$

where $l$ is a positive integer. It is not hard to see that the invariant submodule $(\Omega^l k)^G$ of $\Omega^l k$ maps into $((kV_4)^s)^G$. We will arrive at a contradiction by showing $((kV_4)^s)^G$ maps to zero under the map $\phi$. Since $((kV_4)^s)^G \cong ((kV_4)^G)^s$ it is enough to show that $\phi$ maps each $(kV_4)^G$ to zero. $(kV_4)^G$ is a one-dimensional subspace, generated by an element which we denote as $v$. If $v$ maps to a non-zero element, then it is easy to see that the restriction of $\phi$ on the corresponding copy of $kV_4$ is injective, but $M$ is projective-free, so this is impossible. In other words $\phi(v) = 0$ and that completes the proof of the lemma. $\square$

## 7. TATE COHOMOLOGY

Recall that the Tate cohomology of $G$ with coefficients in a module $M$ is given by $\hat{H}^{*}(G, M) = \underline{\mathrm{Hom}}_{kG}(\Omega^* k, M)$. In our recent joint work with Jon Carlson we focused on two fundamental questions ([CCM07b] and [CCM07a]) about Tate cohomology which

we discuss in the next two subsections. As we shall see, Auslander-Reiten sequences play an important role in answering both of these questions.

7.1. **Modules with finitely generated Tate cohomology.** Let $M$ be a finitely generated $kG$-module. A well-known result of Evens-Venkov states that the ordinary cohomology $H^*(G, M)$ is finitely generated as a module over $H^*(G, k)$. So it is a very natural question to investigate whether the same is true for Tate cohomology. That is, whether $\hat{H}^*(G, M)$ is finitely generated as a module over $\hat{H}^*(G, k)$. As shown in [CCM07b] Tate cohomology is seldom finitely generated. However, there is an interesting family of modules arising from AR sequences whose Tate cohomology is finitely generated.

To start, let $N$ be a finitely generated indecomposable non-projective $kG$-module that is not isomorphic to $\Omega^i k$ for any $i$. Consider the Auslander-Reiten sequence

$$0 \longrightarrow \Omega^2 N \longrightarrow M \longrightarrow N \longrightarrow 0$$

ending in $N$. Assume that $N$ has finitely generated Tate cohomology, then we shall see that the middle term $M$ also has finitely generated Tate cohomology.

Consider the connecting map $\phi\colon N \longrightarrow \Omega N$ in the exact triangle

$$\Omega^2 N \longrightarrow M \longrightarrow N \xrightarrow{\phi} \Omega N$$

corresponding to the above Auslander-Reiten sequence. We first argue that $\phi$ induces the zero map in Tate cohomology. To this end, let $f\colon \Omega^i k \to N$ be an arbitrary map. We want to show that the composite $\Omega^i k \xrightarrow{f} N \xrightarrow{\phi} \Omega N$ is zero in the stable category. Now observe that the assumption on $N$ implies that $f$ is not a split retraction, therefore the map $f$ factors through the middle term $M$. Since the composition of any two successive maps in an exact triangle is zero, it follows that $\phi f = 0$.

Since the boundary map $\phi$ induces the zero map in Tate cohomology, the resulting long exact sequence in Tate cohomology breaks into short exact sequences:

$$0 \longrightarrow \hat{H}^*(G, \Omega^2 N) \longrightarrow \hat{H}^*(G, M) \longrightarrow \hat{H}^*(G, N) \longrightarrow 0.$$

It is now clear that if $N$ has finitely generated Tate cohomology, then so does $M$.

In [CCM07b] we have also shown that the middle term of the AR sequence ending in $k$ has finitely generated Tate cohomology.

These results imply that if we have a sequence of finitely generated indecomposable non-projective $kG$-modules

$$N_1, N_2, \cdots, N_s, s \in \mathbb{N}$$

such that $\hat{H}^*(G, N_1)$ is finitely generated as a module over $\hat{H}^*(G, k)$ and $N_{i+1}$ is a summand of a middle term of the Auslander-Reiten sequence which ends in $N_i, 1 \le i < s$, then all modules $\hat{H}^*(G, N_i)$ are finitely generated over $\hat{H}^*(G, k)$.

7.2. **Counterexamples to Freyd's generating hypothesis.** Motivated by the celebrated generating hypothesis (GH) of Peter Freyd in homotopy theory [Fre66] and its analogue in the derived category of a commutative ring [HLP07, Loc07], we have formulated in [CCM07a] the analogue of Freyd's GH in the stable module category stmod($kG$) of a finite group $G$, where $k$ is a field of characteristic $p$. This is the statement that the Tate cohomology functor detects trivial maps (maps that factor through a projective

module) in the thick subcategory generated by $k$. We studied the GH and related questions in special cases in a series of papers: [CCM08a, BCCM07, CCM08b, CCM07c]. We have finally settled the GH for the stable module category in joint work with Jon Carlson. The main result of [CCM07a] is:

**Theorem 7.1.** [CCM07a] *The generating hypothesis holds for $kG$ if and only if the Sylow $p$-subgroup of $G$ is either $C_2$ or $C_3$.*

Maps in the thick subcategory generated by $k$ that induce the zero map in Tate cohomology are called ghosts. Thus non-trivial ghosts give counterexamples to the GH. After much research on this, we were led to a big revelation when we discovered that a good source of non-trivial ghosts come from Auslander-Reiten sequences.

To explain this in more detail, consider an AR sequence

$$0 \longrightarrow \Omega^2 N \longrightarrow B \longrightarrow N \longrightarrow 0$$

ending in $N$. This short exact sequence represents a exact triangle

$$\Omega^2 N \longrightarrow B \longrightarrow N \xrightarrow{\phi} \Omega N$$

in the stable category. We will show that the map $\phi \colon N \longrightarrow \Omega N$ is a non-trivial ghost. First of all, AR sequences are, by definition, non-split short exact sequences, and therefore the boundary map $\phi$ in the above triangle must be a non-trivial map in the stable category. The next thing to be shown is that the map $\phi \colon N \longrightarrow \Omega N$ induces the zero map on the functors $\underline{\mathrm{Hom}}_{kG}(\Omega^i k, -) \cong \widehat{\mathrm{Ext}}^i(k, -)$ for all $i$. Arguing as in section 7.1, consider any map $f \colon \Omega^i k \longrightarrow N$. We have to show that the composite

$$\Omega^i k \xrightarrow{f} N \xrightarrow{\phi} \Omega N$$

is trivial in the stable category. Consider the following diagram

$$
\begin{array}{ccccc}
 & & \Omega^i k & & \\
 & \nearrow\!\!\!\!\!\!\cdot & \downarrow {\scriptstyle f} & & \\
\Omega^2 N \longrightarrow & B \longrightarrow & N & \xrightarrow{\phi} & \Omega N
\end{array}
$$

where the bottom row is our exact triangle. The map $f \colon \Omega^i k \longrightarrow N$ cannot be a split epimorphism if we choose $N$ to be an indecomposable non-projective module in the thick subcategory generated by $k$ that is not isomorphic to $\Omega^i k$ for any $i$. Then by the defining property of an AR sequence, the map $f$ factors through the middle term $B$ as shown in the above diagram. Since the composite of any two successive maps in a exact triangle is zero, the composite $\phi \circ f$ is also zero by commutativity. Thus the moral of the story is that in order to disprove the GH for $kG$, we just have to find an indecomposable non-projective module in the thick subcategory generated by $k$ that is not isomorphic to $\Omega^i k$ for any $i$. This was the strategy we used to disprove the GH for $kG$ when the Sylow $p$-subgroup has order at least 4.

## 8. Galois theoretic connections

In this section we provide the motivation for some of the arithmetic objects which will appear in the later sections when we study Auslander-Reiten sequences in the context of Galois theory. To set the stage, we begin with our notation.

Let $F$ be a field of characteristic not equal to 2 and let $F_{sep}$ denote the separable closure of $F$. We shall introduce several subextensions of $F_{sep}$.

- $F^{(2)}$ = compositum of all quadratic extensions of $F$.
- $F^{(3)}$ = compositum of all quadratic extensions of $F^{(2)}$, which are Galois over $F$.
- $F^{\{3\}}$ = compositum of all quadratic extensions of $F^{(2)}$.
- $F_q$ = compositum of all Galois extensions $K/F$ such that $[K : F] = 2^n$, for some positive integer $n$.

All of these subextensions are Galois and they fit in a tower

$$F \subset F^{(2)} \subset F^{(3)} \subset F^{\{3\}} \subset F_q \subset F_{sep}.$$

We denote their Galois groups (over $F$) as

$$G_F \longrightarrow G_q \longrightarrow G_F^{\{3\}} \longrightarrow G_F^{[3]}(= \mathcal{G}_F) \longrightarrow G_F^{[2]}(= E) \longrightarrow 1.$$

Observe that $G_F^{[2]}$ is just $\prod_{i \in I} C_2$, where $I$ is the dimension of $F^*/F^{*2}$ over $\mathbb{F}_2$. $F^*$ denotes the multiplicative subgroup of $F$. $G_F$ is the absolute Galois group of $F$. The absolute Galois group of fields are in general rather mysterious objects of great interest. (See [BLMS07] for restrictions on the possible structure of absolute Galois groups.) Although the quotients $G_q$ are much simpler we are far from understanding their structure in general. $F^{\{3\}}$ and its Galois group over $F$ are considerably much simpler and yet they already contain substantial arithmetic information of the absolute Galois group.

To illustrate this point, consider $WF$ the Witt ring of quadratic forms; see [Lam05] for the definition. Then we have the following theorem.

**Theorem 8.1.** [MS96] *Let $F$ and $L$ be two fields of characteristic not 2. Then $WF \cong WL$ (as rings) implies that $\mathcal{G}_F \cong \mathcal{G}_L$ as pro-2-groups. Further if we assume additionally in the case when each element of $F$ is a sum of two squares that $\sqrt{-1} \in F$ if and only if $\sqrt{-1} \in L$, then $\mathcal{G}_F \cong \mathcal{G}_L$ implies $WF \cong WL$.*

Thus we see that $\mathcal{G}_F$ essentially controls the Witt ring $WF$ and in fact, $\mathcal{G}_F$ can be viewed as a Galois theoretic analogue of $WF$. In particular, $\mathcal{G}_F$ detects orderings of fields. (Recall that $P$ is an ordering of $F$ if $P$ is an additively closed subgroup of index 2 in $F^*$.) More precisely, we have:

**Theorem 8.2.** [MS90] *There is a 1-1 correspondence between the orderings of a field $F$ and cosets $\{\sigma\Phi(\mathcal{G}_F) \mid \sigma \in \mathcal{G}_F \backslash \Phi(\mathcal{G}_F) \text{ and } \sigma^2 = 1\}$. Here $\Phi(\mathcal{G}_F)$ is the Frattini subgroup of $\mathcal{G}_F$, which is just the closed subgroup of $\mathcal{G}_F$ generated by all squares in $\mathcal{G}_F$. The correspondence is as follows:*

$$\sigma\Phi(\mathcal{G}_F) \longrightarrow P_\sigma = \{f \in F^* \mid \sigma(\sqrt{f}) = \sqrt{f}\}.$$

This theorem was generalised considerably for detecting additive properties of multiplicative subgroups of $F^*$ in [MMS04]. In this paper (see [MMS04, Section 8]) it was shown that $\mathcal{G}_F$ can be used also for detecting valuations on $F$.

Also in [AKM99, Corollary 3.9] it is shown that $\mathcal{G}_F \cong \mathcal{G}_L$ if and only if $k_*(F) \cong k_*(L)$. Here $k_*(A)$ denotes the Milnor $K$-theory (mod 2) of a field $A$. In particular, in [AKM99, Theorem 3.14] it is shown that if $R$ is the subring of $H^*(\mathcal{G}_F, \mathbb{F}_2)$ generated by one dimensional classes, then $R$ is isomorphic to the Galois cohomology $H^*(G_F, \mathbb{F}_2)$ of $F$. Thus we see that $\mathcal{G}_F$ also controls Galois cohomology and in fact $H^*(\mathcal{G}_F, \mathbb{F}_2)$ contains

some further substantial information about $F$ which $H^*(G_F, \mathbb{F}_2)$ does not contain. In summary, $\mathcal{G}_F$ is a very interesting object. On the one hand $\mathcal{G}_F$ is much simpler than $G_F$ or $G_q$, yet it contains substantial information about the arithmetic of $F$. In [BLMS07, Section 2] the case of $p > 2$ was also considered, and the definition of $\mathcal{G}_F$ was extended to fields which contain a primitive $p^{th}$-root of unity. The key reason for restricting to $p = 2$ in earlier papers stems from the interest in quadratic forms. Nevertheless a number of interesting properties also hold for $\mathcal{G}_F$ in the case $p > 2$.

## 9. TOWARDS THE AUSLANDER-REITEN TRANSLATE $\Omega^2 \mathbb{F}_2$

Because $\mathcal{G}_F$ contains considerable information about the arithmetic of $F$, it is a natural question to ask for properties of $G_q$, or more precisely $G_q(F)$, determined by $\mathcal{G}_F$. First of all, it is possible that for two fields $F_1$ and $F_2$, we have

$$\mathcal{G}_{F_1} \cong \mathcal{G}_{F_2} \quad \text{but} \quad G_q(F_1) \not\cong G_q(F_2).$$

Indeed, set $F_1 = \mathbb{C}((t_1))((t_2))$ to be a field of formal power series in $t_2$ over the field of formal power series in $t_1$ over $\mathbb{C}$ and $F_2$ the field $\mathbb{Q}_5$ of 5-adic numbers. Then $\mathcal{G}_{F_1} = \mathcal{G}_{F_2} = C_4 \times C_4$, but $G_q(F_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ ($\mathbb{Z}_2$ is the additive group of 2-adic numbers), while $G_q(F_2) = \langle \sigma, \tau \mid \tau^4 [\tau^{-1}, \sigma^{-1}] = 1 \rangle$ see [MS96, Koc02]. However, in this case $G_{F_1}^{\{3\}} \cong G_{F_2}^{\{3\}}$, as they are both isomorphic to $C_4 \times C_4$; see [AGKM01].

The very interesting question about possible groups $G_F^{\{3\}}$ when $G_F^{[3]}$ is given, is currently investigated by Mináč, Swallow and Topaz [MST]. In order to tackle this question it is important to understand the universal case when $G_q(F)$ is a free pro-2-group. From now on we further assume that $F^*/F^{*2}$ is finite-dimensional. (This is the most important case to understand, as $G_F^{\{3\}}$ in general is the projective limit of its finite quotients.) Let $n$ be the dimension. Then we have $F^*/F^{*2} = 2^n$. In this case we have an extension

$$1 \longrightarrow A \longrightarrow G_F^{\{3\}} \longrightarrow E \to 1$$

where $E = \prod_1^n C_2$ and $A = \mathrm{Gal}(G_F^{\{3\}}/F^{(2)}) \cong \prod_1^m C_2$ with $m = (n-1)2^n + 1$. Set $R = F^{(2)}$ for simplicity. Indeed from Kummer theory we know that

$$A \cong \mathrm{Hom}_{\mathbb{F}_2}(R^*/R^{*2}, \mathbb{F}_2).$$

Then also by Kummer theory the minimal number of topological generators of $G_q^{(2)} := \mathrm{Gal}(F_q/F^{(2)})$ is equal to the dimension of $R^*/R^{*2}$. From the topological version of Schreier's theorem we know that this number $d(G_q^{(2)})$ is given by $d(G_q^{(2)}) = (n-1)2^n + 1$, because $G_q^{(2)}$ is the open subgroup of index $2^n$ of $G_q$ and $d(G_q) = n$. (See [Koc02, Example 6.3]) Thus

$$
\begin{aligned}
\dim_{\mathbb{F}_2} A &= \dim_{\mathbb{F}_2} R^*/R^{*2} \\
&= \dim_{\mathbb{F}_2} H^1(G_q^{(2)}, \mathbb{F}_2) \\
&= d(G_q^{(2)}) \\
&= (n-1)2^n + 1.
\end{aligned}
$$

Moreover, $A$ is a natural $\mathbb{F}_2 E$-module where the $E$-action is induced by conjugation in $G_F^{\{3\}}$. The next result gives a completely arithmetical interpretation of the Auslander-Reiten translate in our case. We write $\Omega_E^2 \mathbb{F}_2$ in order to stress that we are working in the category of $\mathbb{F}_2 E$-modules.

**Theorem 9.1.** [Gas54]
$$\Omega_E^2 \mathbb{F}_2 \cong A.$$

We sketch in the next section a direct proof of this theorem as it also provides some information about the structure of our module $A$. After that we shall consider the AR sequence

$$0 \longrightarrow \Omega^2 \mathbb{F}_2 \longrightarrow M \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

and interpret $M$ and also its associated group $AR(E)$ which will be defined later below as a certain Galois group.

## 10. GENERATORS AND RELATIONS FOR $\Omega^2 \mathbb{F}_2$

Recall that $E = \prod_1^n C_2 = \prod_1^n \langle \sigma_i \rangle$. We shall first use the definition of $\Omega^2 \mathbb{F}_2$ over $\mathbb{F}_2 E$ to describe $\Omega^2 \mathbb{F}_2$ via generators and relations and then we shall see that indeed $A$ has this presentation. This, in turn, will imply that $A \cong \Omega^2 \mathbb{F}_2$.

Let $P_{a_i}$ be a free $\mathbb{F}_2 E$ module of rank 1 generated by $a_i$ for $1 \leq i \leq n$, so $P_{a_i} = \mathbb{F}_2 E a_i$. Then recall that $\Omega^2 \mathbb{F}_2$ is defined by the short exact sequence:

$$0 \longrightarrow \Omega^2 \mathbb{F}_2 \longrightarrow \oplus_{i=1}^n P_{a_i} (= P) \xrightarrow{\psi} \mathbb{F}_2 E \xrightarrow{\epsilon} \mathbb{F}_2 \longrightarrow 0,$$

where $\epsilon \colon \mathbb{F}_2 E \to \mathbb{F}_2$ is the augmentation map $\epsilon(\sigma_i) = 1$ for all $i$, and $\psi(a_i) = \sigma_i - 1$. We shall denote $\sigma_i - 1$ by $\rho_i$ for simplicity. Exactness at $\mathbb{F}_2$ is trivial and exactness at $\mathbb{F}_2 E$ is a standard exercise which we leave to the reader. Observe that $\Omega^2 \mathbb{F}_2$ is determined only up to a projective summand. But since we insist that $\Omega^2 \mathbb{F}_2$ is projective-free, we actually get it up to an isomorphism of $\mathbb{F}_2 E$-modules.

Note that $\Omega^2 \mathbb{F}_2 = \operatorname{Ker} \psi$, and it certainly contains the $\mathbb{F}_2 E$-submodule of $P$ generated by

$$S = \{ \rho_i a_i, \rho_j a_i + \rho_i a_j, 1 \leq i < j \leq n. \}$$

We claim that the $\mathbb{F}_2 E$ submodule $W$ generated by $S$ is the entire module $\Omega^2 \mathbb{F}_2$. First observe that $I = \Omega^1 \mathbb{F}_2 = \operatorname{Ker} \epsilon$ has dimension $2^n - 1$ and $\rho_T = \prod_{i \in T} \rho_i$, where $T$ is a non-empty subset of $\{1, 2, \cdots, n\}$ form a basis of $I$. This allows us to pick a nice $\mathbb{F}_2$-vector space section of $\psi$: $V$ the vector space span of the set

$$U = \{ \rho_T a_i \;\; 1 \leq i \leq n, \text{and } T \subseteq \{i+1, i+2, \cdots, n\} \}.$$

It is not hard to see that $P$ splits as $\operatorname{Ker} \psi \oplus V$ as $\mathbb{F}_2$-vector spaces. Therefore it is enough to show that $W \subseteq \ker \psi$ together with $V$ generate the entire module $P$ over $\mathbb{F}_2$. Set $Q := W \oplus V$, the vector span of $V$ and $W$ over $\mathbb{F}_2$. Since $\{a_1, a_2, \cdots, a_n\} \subseteq Q$ and $\{a_1, a_2, \cdots, a_n\}$ generates $P$ as an $\mathbb{F}_2 E$-module, it is sufficient to show that $Q$ is an $\mathbb{F}_2 E$ submodule of $P$. Because $W$ is an $\mathbb{F}_2 E$-submodule of $Q$ it is enough to show that for each $v \in U$ and $\rho_l$ we also have $\alpha := \rho_l v$ in $Q$. Write $v = \rho_T a_i$ for some $T$ and $i$. If $l = i$, then $\alpha$ belongs to $W$. If $l > i$, then $\alpha$ belongs to $U \cup \{0\}$. In particular, in both cases, $\alpha \in Q$. If $l < i$, then

$$
\begin{aligned}
\alpha &= \rho_T \rho_l a_i \\
&= \rho_T (\rho_l a_i + \rho_i a_l) + \rho_T \rho_i a_l \\
&= w + v
\end{aligned}
$$

where the first summand belongs to $W$ and the second summand belongs to $U \cup \{0\} \subseteq V$. Hence $\alpha$ belongs to $Q$, as desired. This shows that $\Omega^2 \mathbb{F}_2$ is generated over $\mathbb{F}_2 E$ by the set $S$.

Let us denote:

$$[a_i, a_j, a_{t_1}, \cdots, a_{t_r}] := \rho_{t_r} \rho_{t_{r-1}} \cdots \rho_{t_1}(\rho_i a_j + \rho_j a_i), \text{ and}$$

$$[a_i{}^2, a_{t_1}, \cdots, a_{t_r}] := \rho_{t_r} \rho_{t_{r-1}} \cdots \rho_{t_1} \rho_i a_i,$$

where all the subscripts range over the set $\{1, 2, \cdots, n\}$. Then we have the following identities. Below $\sigma$ denotes an arbitrary permutation of the set $\{1, 2, 3, \cdots, s\}$, and $b_i = a_{t_i}$ for any $i$.

(1) $[a_i, a_j] = [a_j, a_i]$
(2) $[a_i, a_j, b_1, \cdots, b_s] = [a_i, a_j, b_{\sigma(1)}, \cdots, b_{\sigma(s)}]$
(3) $[a_i^2, b_1, \cdots, b_s] = [a_i^2, b_{\sigma(1)}, \cdots, b_{\sigma(s)}]$
(4) $[a_i, a_j, a_r] + [a_j, a_r, a_i] + [a_r, a_i, a_j] = 0$
(5) $[a_i^2, a_r] = [a_i, a_r, a_i]$
(6) $[a_i, a_j, b_1, \cdots, b_\alpha, \cdots, b_\alpha, \cdots, b_s] = 0$

Then $\Omega^2 \mathbb{F}_2 \subseteq P$ is a vector span over $\mathbb{F}_2$ of $[a_i, a_j, a_{t_1}, \ldots a_{t_r}]$ and $[a_i^2, a_{t_1}, \cdots a_{t_r}]$ as above. In fact from the short exact sequence

$$0 \longrightarrow \Omega^2 \mathbb{F}_2 \longrightarrow P \xrightarrow{\psi} \mathbb{F}_2 E \xrightarrow{\epsilon} \mathbb{F}_2 \longrightarrow 0,$$

we see that $\dim_{\mathbb{F}_2} \Omega^2 \mathbb{F}_2 = n2^n - 2^n + 1 = 2^n(n-1) + 1$. With what follows using the above identities we see that the set

$$X := \{[c_{ij}, a_{t_1}, \cdots, a_{t_r}] \mid 1 \le i \le j \le n, i < t_1 < \cdots < t_r \le n\}$$

generates $\Omega^2 \mathbb{F}_2$ over $\mathbb{F}_2$. Here $c_{ij} = [a_i, a_j]$ if $i < j$ or $a_i^2$ if $i = j$. Observe that

$$|X| = \sum_{i=1}^{n} (n - i + 1) 2^{n-i}.$$

Indeed for a fixed $i$ there are $n - i + 1$ possibilities for $j$ and there are $2^{n-i}$ possibilities for the number of subsets of $\{i + 1, \cdots, n\}$ for the choices of $t_1, \cdots, t_r$. Also we have

$$\sum_{i=1}^{n} (n - i + 1) Y^{n-i} = \sum_{i=1}^{n} (Y^{n-i+1})' = \frac{(n+1)Y^n(Y-1) - (Y^{n+1} - 1)}{(Y-1)^2}.$$

Plugging $Y = 2$, we see that $|X| = \sum_{i=1}^{n}(n - i + 1)2^{n-i} = 2^n(n-1) + 1$. Hence $X$ is the basis of $\Omega^2 \mathbb{F}_2$ over $\mathbb{F}_2$. On the other hand, a detailed investigation of identities in $G_F^{[3]}$ (see [GT67, MST]) shows that a basis of $A$ over $\mathbb{F}_2$ is

$$Z = \{[d_{ij}, \sigma_{t_1}, \cdots \sigma_{t_r}] \mid 1 \le i \le j \le n, i < t_1 < \cdots < t_r \le n\},$$

$d_{ij} = [\sigma_i, \sigma_j]$ if $i < j$, $d_{ii} = \sigma_i^2$, and $[\sigma_1, \sigma_2, \cdots, \sigma_l] = [\cdots [[[\sigma_1, \sigma_2], \sigma_3], \cdots, \sigma_l]$. In fact, writing $\sigma_i$ in place of $a_i$ above, the same set of 6 identities hold in $G_F^{[3]}$. This shows that the map $\phi \colon \Omega^2 \mathbb{F}_2 \longrightarrow A$ defined by $\phi(a_i^2) = \sigma_i^2$, and $\phi([a_i, a_j]) = [\sigma_i, \sigma_j]$ is a well-defined surjective $\mathbb{F}_2 E$ homomorphism and because of dimensional reasons it has to be an isomorphism.

## 11. The Auslander-Reiten sequence ending in $\mathbb{F}_2$

Finally we want to obtain the Auslander-Reiten sequence ending in $\mathbb{F}_2$:

$$0 \longrightarrow \Omega^2 \mathbb{F}_2 \longrightarrow M \xrightarrow{\eta} \mathbb{F}_2 \longrightarrow 0.$$

We will get a sequence of this form which is not split. Then simply because

$$\mathrm{Ext}^1_{\mathbb{F}_2 E}(\mathbb{F}_2, \Omega^2 \mathbb{F}_2) = \mathbb{F}_2$$

the non-split sequence we obtain has to be an Auslander-Reiten sequence; see [CR87, Proposition (78.28)].

Observe from our proof of our description of $\Omega^2 \mathbb{F}_2 \subseteq P$, that $a := \rho_n \rho_{n-1} \cdots \rho_2 a_1$ belongs to $P - \Omega^2 \mathbb{F}_2$. Let $M := \Omega^2 \mathbb{F}_2 + \{a, 0\} \subseteq P$. (Because $\rho_i a$ belongs to $\Omega^2 \mathbb{F}_2$ for each $i = 1, 2, \cdots, n$, we see that $M$ is indeed an $\mathbb{F}_2 E$-module.) Now, let $\eta \colon M \to \mathbb{F}_2$ be defined by $\eta(\Omega^2 \mathbb{F}_2) = 0$ and $\eta(a) = 1$. Observe that since $\rho_1 a$ belongs to $\Omega^2 \mathbb{F}_2$ and $\rho_i a = 0$ for $i \geq 2$, we obtain that $\eta$ is indeed an $\mathbb{F}_2 E$-homomorphism.

We claim that the short exact sequence

$$0 \longrightarrow \Omega^2 \mathbb{F}_2 \longrightarrow M \xrightarrow{\eta} \mathbb{F}_2 \longrightarrow 0$$

is an Auslander-Reiten sequence. Suppose to the contrary, this sequence splits. Then the coset $a + \Omega^2 \mathbb{F}_2$ would contain a trivial element $t = a + m$, $m \in \Omega^2 \mathbb{F}_2$ (i.e., the action of each $\rho_i$ on $t$ is 0). But $t \in \mathrm{Soc}(P) \subseteq \Omega^2 \mathbb{F}_2$. Hence $t - m = a$ belongs to $\Omega^2 \mathbb{F}_2$, which is a contradiction. (Here $\mathrm{Soc}(P)$ is the socle of $P$ which is the submodule of $E$ consisting of the fixed elements of $P$ under the action of $E$.) So we are done.

We now introduce Auslander-Reiten groups. Let $E = \prod_{i=1}^n C_2 = \prod_{i=1}^n \langle \sigma_i \rangle$ as before. Assume that $n \geq 2$. Consider the Auslander-Reiten sequence

$$0 \longrightarrow \Omega^2 \mathbb{F}_2 \xrightarrow{h} M \longrightarrow \mathbb{F}_2 \longrightarrow 0.$$

Consider the long exact sequence in cohomology associated to the above AR sequence. As observed before, all connecting homomorphisms are zero except the first one. In particular, we have a short exact sequence:

$$0 \longrightarrow H^2(E, \Omega^2 \mathbb{F}_2) \longrightarrow H^2(E, M) \longrightarrow H^2(E, \mathbb{F}_2) \longrightarrow 0.$$

Furthermore, $H^2(E, \Omega^2 \mathbb{F}_2) \cong \mathbb{F}_2$. The unique non-zero element of $H^2(E, \Omega^2 \mathbb{F}_2)$ corresponds to a short exact sequence

$$1 \longrightarrow \Omega^2 \mathbb{F}_2 \longrightarrow G_F^{\{3\}} \longrightarrow E \longrightarrow 1$$

where $|F^*/F^{*2}| = 2^n$ and $\mathrm{Gal}(F_q/F)$ is a free pro-2-group. Note that in our case $G_F^{\{3\}}$ does not depend on $F$, but only on the number $n$ of the rank of the free absolute pro-2-group $G_F$. Therefore we shall denote it by $G^{\{3\}}(n)$. Now consider the following commutative diagram



Here $T(n)$ is the pushout of $G^{\{3\}}(n)$ and $M$ along $\Omega^2 \mathbb{F}_2$. The second row corresponds to the image of the non-trivial element of $H^2(E, \Omega^2 \mathbb{F}_2)$ in $H^2(E, M)$.

**Definition 11.1.** We call $T(n)$ the Auslander-Reiten group associated to $E$.

Observe that we have obtained a Galois theoretic model for our Auslander-Reiten sequence:

$$0 \longrightarrow \Omega^2 \mathbb{F}_2 \longrightarrow M \longrightarrow \mathbb{F}_2 \longrightarrow 0.$$

Indeed, let $K/L$ be a Galois extension as above such that $\mathrm{Gal}(K/F) = T(n)$. Then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow T(n) \longrightarrow G \longrightarrow 0$$

where AR Galois group $T(n)$ is generated by $G^{\{3\}}(n)$ and $\tau$ subject to the relations $\tau^2 = 1$, $[\tau, \sigma_i] = 1$ for $2 \le i \le n$, and $[\tau, \sigma_1] = [\sigma_1^2, \sigma_2, \cdots, \sigma_n]$. Let $L$ be the fixed field of $M$. Then the Galois group $\mathrm{Gal}(K/L) \cong M$ is the middle term of our Auslander-Reiten sequence. Further, let $H$ be a subgroup generated by $\sigma_1, \cdots, \sigma_n$. Then $H \cong G^{\{3\}}(n)$ and $H \cap M \cong \Omega^2 \mathbb{F}_2$. Hence our Auslander-Reiten sequence has the form

$$0 \longrightarrow H \cap \mathrm{Gal}(K/L) \longrightarrow \mathrm{Gal}(K/L) \longrightarrow \mathbb{F}_2 \longrightarrow 0.$$

## 12. Conclusion

We hope that our hot and mild appetizers will inspire our readers in their choices of further delightful, tasty, and mouth-watering main dishes. John Labute's skillful interplay between Lie theoretic and group theoretic methods in his past work (see for example [Lab67]), as well as in his recent work ([Lab06]) provides us with inspiration for exploiting the connections between modular representation theory and Galois theory. We are now lifting our glasses of champagne in honour of John Labute and we are wishing him happy further research, and pleasant further Auslander-Reiten encounters.

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