

BROWN REPRESENTABILITY FOR SPACE-VALUED FUNCTORS

BORIS CHORNY

ABSTRACT. In this paper we prove two theorems which resemble the classical cohomological and homological Brown representability theorems. The main difference is that our results classify small contravariant functors from spaces to spaces up to weak equivalence of functors.

In more detail, we show that every small contravariant functor from spaces to spaces which takes coproducts to products up to homotopy and takes homotopy pushouts to homotopy pullbacks is naturally weakly equivalent to a representable functor.

The second representability theorem states: every contravariant continuous functor from the category of finite simplicial sets to simplicial sets taking homotopy pushouts to homotopy pullbacks is equivalent to the restriction of a representable functor. This theorem may be considered as a contravariant analog of Goodwillie's classification of linear functors [12].

1. INTRODUCTION

The classical Brown representability theorem [3] classifies contravariant homotopy functors $F: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}ets$ from the category of spaces to the category of sets satisfying Milnor's wedge axiom (W) and Mayer-Vietoris property (MV).

$$(W): F(\coprod X_i) = \prod F(X_i);$$

$$(MV): F(D) \rightarrow F(B) \times_{F(A)} F(C) \text{ is surjective for every homotopy pushout square } \begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & D \end{array}.$$

In this paper we address a similar classification problem, but the functors we want to classify are homotopy functors from spaces to spaces, satisfying (hW) and (hMV), the higher homotopy versions of (W) and (MV).

$$(hW): F(\coprod X_i) \simeq \prod F(X_i);$$

$$(hMV): \begin{array}{ccc} F(D) & \rightarrow & F(B) \\ \downarrow & & \downarrow \\ F(C) & \rightarrow & F(A) \end{array} \text{ is a homotopy pullback for every homotopy pushout square } \begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & D \end{array}.$$

Functors $F: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$ satisfying (hW) and (hMV) are called *cohomological* in this paper (note that any functor satisfying (hMV) is automatically a homotopy functor; cf 3.1). Perhaps this terminology is not quite adequate, but we consider it as a temporary notation, since we are going to prove that these functors are equivalent to representable functors, so no alternative name is required.

Date: July 6, 2007.

We should mention right away, that by *spaces* we always mean simplicial set in this paper. The results are formulated for unpointed spaces, but they remain valid in the pointed situation too.

The classical cohomological Brown representability theorem ensures that every contravariant homotopy functor satisfying (W) and (MV) is representable on the homotopy category. In this work we show that every space-valued functor satisfying (hW) and (hMV) is naturally weakly equivalent to a functor representable in the enriched sense on the category of spaces.

Note however, that neither our theorem implies Brown representability, nor the converse. We assume stronger (higher homotopy) conditions about the functor, but we also obtain an enriched representability result.

Nevertheless, our result has a natural predecessor from the Calculus of homotopy functors. Goodwillie's classification of linear functors [12] is related to the homological Brown representability in the same way as our representability theorem related to the cohomological Brown representability.

The second classification result proven in this paper is "essentially equivalent" to Goodwillie's classification of finitary linear functors. The difference is that we prove a higher homotopy version of the homological Brown representability in its contravariant form: every cohomological functor from the compact spectra to abelian groups is a restriction of a representable functor. We consider a non-stable version of this statement: every contravariant homotopy functor from finite spaces to spaces satisfying (hMV) is equivalent to a restriction of a representable functor. Such functors are called *homological*.

Although there is no direct implications between our theorem and Goodwillie's classification of linear functors, there is an additional feature that our results share. In both cases every small functor may be approximated by an initial, up to homotopy, representable/linear functor. Small functors form a considerable subcollection of all functors, so that the full subcategory of small functors is locally small. We discuss them in more detail in Section 2. This property allows us to hope that the representability theorem for contravariant functors may be interpreted in terms of the Calculus of Functors. Homotopy Calculus is suited well for the study of covariant functors, while Embedding Calculus provides a similar machinery for the study of contravariant functors with small domains (the category of open sets of a manifold). We expect that there exists a calculus machine for contravariant functors, which generalizes Embedding Calculus.

The method of proof of our results deserves a comment. Contravariant functors satisfying (hW) and (hMV) are represented first as local objects with respect to certain class of maps in the category of small functors. Then the classification of objects with this local property in our model category is performed in terms comparison with a Quillen equivalent model, in which it is easier to perform the required localization.

Goodwillie has certainly used similar considerations (at least implicitly) in [13] in order to classify homogeneous functors. But he has worked on the level of homotopy categories. For the first time the idea that linear functors (or more generally n -excisive functors) may be viewed as local objects was spelled out by W. Dwyer [8]. In [1] we have tried (with

G. Biedermann and O. Röndigs) to enhance Goodwillie's proof by the model-categorical machinery. The goal was partially fulfilled, but we have still used many of Goodwillie's results in order to complete the classification on the level of model categories.

In this work we develop tools which allow for the independent approach to classification theorems on the level of models. We prove a statement, which is very similar to the classification of finitary linear functors and prove a new result, not related immediately to the Calculus of Functors.

1.1. Acknowledgment. We thank Amnon Neeman for numerous helpful conversation, which led to the results in this paper. We also thank Tom Goodwillie for helpful remarks about the early version of this paper.

2. PRELIMINARIES ON SMALL FUNCTORS

The object of study of this paper is homotopy theory of contravariant functors from the category of spaces \mathcal{S} to \mathcal{S} . The totality of these functors does not form a category in the usual sense, since the natural transformations between two functors need not form a set in general, but rather a proper class. We are willing to be satisfied with a treatment of a reasonable subcollection of functors, a subcollection which does form a category. The purpose of this section is to describe such a subcollection.

Definition 2.1. Let \mathcal{D} be a (not necessarily small) simplicial category. A functor $\underline{X} : \mathcal{D} \rightarrow \mathcal{S}$ is *representable* if there is an object $D \in \mathcal{D}$ such that \underline{X} is naturally equivalent to R^D , where $R^D(D') = \text{hom}_{\mathcal{D}}(D, D')$. A functor $\underline{X} : \mathcal{C} \rightarrow \mathcal{S}$ is called *small* if \underline{X} is a small weighted colimit of representables.

Remark 2.2. Since the category of small functors is tensored over simplicial sets, the *small weighted colimit* above may be expressed as a coend of the form

$$(1) \quad R^F \otimes_{\mathcal{J}} G = \int^{I \in \mathcal{J}} R^{F(I)} \otimes G(I),$$

where \mathcal{J} is a small category and $F : \mathcal{J} \rightarrow \mathcal{D}$, $G : \mathcal{J} \rightarrow \mathcal{S}$ are functors. Here $R^F : \mathcal{J}^{\text{op}} \rightarrow \mathcal{S}^{\mathcal{D}}$ assigns to $I \in \mathcal{J}$ the representable functor $R^{F(I)} : \mathcal{D} \rightarrow \mathcal{S}$. For the general treatment of weighted limits and colimits see [16].

Since the simplicial tensor structure on the category of small functors $\mathcal{S}^{\mathcal{D}}$ is given by the objectwise direct product, we will use $\underline{X} \times K$ to denote tensor product of $\underline{X} \in \mathcal{S}^{\mathcal{D}}$ with $K \in \mathcal{S}$.

The above coend is the (enriched) left Kan extension of the functor G over the functor F . Using the transitivity of left Kan extensions, it is easy to see that the following four conditions are equivalent [16, Prop. 4.83]:

- $\underline{X} : \mathcal{D} \rightarrow \mathcal{S}$ is a small functor,
- there is a small simplicial category \mathcal{J} and a functor $G : \mathcal{J} \rightarrow \mathcal{S}$, such that \underline{X} is isomorphic to the left Kan extension of G over some functor $\mathcal{J} \rightarrow \mathcal{D}$,
- there a small simplicial subcategory $i : \mathcal{D}' \rightarrow \mathcal{D}$ and a functor $G : \mathcal{D}' \rightarrow \mathcal{S}$, such that \underline{X} is isomorphic to the left Kan extension of G over i , and

- there is a small full simplicial subcategory $i: \mathcal{D}_{\underline{X}} \rightarrow \mathcal{D}$ such that \underline{X} is isomorphic to the left Kan extension of $i^*(\underline{X})$ over i .

If $D \in \mathcal{D}$ and \underline{Y} is a functor $\mathcal{D} \rightarrow \mathcal{S}$, then by Yoneda's lemma the simplicial class of natural transformations $R^D \rightarrow \underline{Y}$ is $\underline{Y}(D)$; in particular, this simplicial class is a simplicial set. It follows easily that if \underline{X} is a small functor $\mathcal{D} \rightarrow \mathcal{S}$, then the natural transformations $\underline{X} \rightarrow \underline{Y}$ also form a simplicial set (this also follows from 2.2 above and the adjointness property of the left Kan extension). In particular, the collection of all small functors is a simplicial category.

Remark 2.3. M.G. Kelly [16] calls small functors *accessible* and weighted colimits *indexed*. He proves that small functors form a simplicial category which is closed under small (weighted) colimits [16, Prop. 5.34].

In order to do homotopy theory we need to work in a category which is not only cocomplete, but also complete (at least under finite limits). Fortunately, there is a simple sufficient condition in the situation of small functors.

Theorem 2.4. *If \mathcal{D} is cocomplete, then the category $\mathcal{S}^{\mathcal{D}}$ of small functors $\mathcal{D} \rightarrow \mathcal{S}$ is complete.*

Remark 2.5. There is a long story behind this theorem. P. Freyd [11] introduced the notion of *petty* and *lucid* set-valued functors. A set-valued functor is called *petty* if it is a quotient of a small sum of representable functors. Any small functor is clearly petty. A functor $F: \mathcal{A} \rightarrow \mathcal{S}ets$ is called *lucid* if it is petty and for any functor $G: \mathcal{A} \rightarrow \mathcal{S}ets$ and any pair of natural transformations $\alpha, \beta: G \rightrightarrows F$, the equalizer of α and β is petty. Freyd proved [11, 1.12] that the category of lucid functors from \mathcal{A}^{op} to $\mathcal{S}ets$ is complete if and only if \mathcal{A} is *approximately* complete (that means that the category of cones over any small diagram in \mathcal{A} has a weakly initial set). J. Rosický then proved [18, Lemma 1] that if the category \mathcal{A} is approximately complete, a functor $F: \mathcal{A}^{op} \rightarrow \mathcal{S}ets$ is small if and only if it is lucid. Finally, these results were partially generalized by B. Day and S. Lack [17] to the enriched setting. They show, in particular, that the category of small \mathcal{V} -enriched functors $\mathcal{K}^{op} \rightarrow \mathcal{V}$ is complete if \mathcal{K} is complete and \mathcal{V} is a symmetric monoidal closed category which is locally finitely presentable as a closed category. This last condition is certainly satisfied if $\mathcal{V} = \mathcal{S}$.

3. MODEL CATEGORIES AND THEIR LOCALIZATION

The main technical tool used in the prove of the classification theorem is the theory of homotopy localizations. More specifically, we apply certain homotopy localizations in the category of small contravariant functors $\mathcal{S}^{\mathcal{S}^{op}}$, or in a Quillen equivalent model category of maps of spaces with the equivariant model structure [6, 4].

Before proving the main classification result, we suggest the following alternative characterization of functors satisfying (hW) and (hMV) as local objects with respect to some class of maps.

3.1. Homotopy functors as local objects. First of all every cohomological functor F is a homotopy functor, i.e., $F(f): F(B) \rightarrow F(A)$ is a weak equivalence for every weak equivalence $f: A \rightarrow B$. Denote by \mathcal{F}_1 the class of maps between representable functors induced by weak equivalences:

$$\mathcal{F}_1 = \{f^*: R_A \rightarrow R_B \mid f: A \rightarrow B \text{ is a w.e.}\},$$

where R_A denotes the representable functor $R_A = \mathcal{S}(-, A)$.

Yoneda's lemma implies that \mathcal{F}_1 -local functors are precisely the fibrant homotopy functors.

3.2. Cohomology functors as local objects. Given a homotopy functor F , it suffices to demand two additional properties for the functor F to be cohomological: F must convert coproducts to products up to homotopy and it also must convert homotopy pushouts to homotopy pullbacks. Yoneda's lemma and the standard commutation rules of various (ho)(co)limits with $\text{hom}(-, -)$ implies that both properties are local with respect to the following classes of maps:

$$\mathcal{F}_2 = \left\{ \coprod R_{X_i} \rightarrow R_{\coprod X_i} \mid \forall \{X_i\}_{i \in I} \in \mathcal{S}^I \right\}$$

and

$$\mathcal{F}_3 = \left\{ \text{hocolim} \begin{pmatrix} R_A \rightarrow R_C \\ \downarrow \\ R_B \end{pmatrix} \rightarrow R_D \mid \begin{array}{ccc} A \rightarrow C \\ \downarrow & & \downarrow \\ B \rightarrow D \end{array} \text{ - homotopy pushout in } \mathcal{S} \right\}.$$

Objects which are local with respect to $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ are precisely the fibrant homotopy functors.

Remark 3.1. For any map $f: A \rightarrow B$ the following commutative square is a homotopy pushout and/or homotopy pullback iff f is a weak equivalence:

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \parallel & & \downarrow f \\ A & \xrightarrow{\quad f \quad} & B. \end{array}$$

Therefore, any functor satisfying (hMV) is automatically a homotopy functor, so it suffices to invert $\mathcal{F}' = \mathcal{F}_2 \cup \mathcal{F}_3$, but it does not simplify anything in our proof.

Remark 3.2. The indexing category I used to describe \mathcal{F}_2 is a completely arbitrary small discrete category. In particular I can be empty. This implies that the map $\emptyset \rightarrow R_\emptyset$ is in \mathcal{F}_2 . In other words, if F is a cohomological functor, then $F(\emptyset) = *$. This property is analogous to the requirement that every linear functor is reduced in homotopy calculus.

Remark 3.3. Since homological functors are defined on the category of finite simplicial sets, we need to adjust the definition of \mathcal{F}_3 .

$$\mathcal{F}'_3 = \left\{ \text{hocolim} \left(\begin{array}{c} R_A \rightarrow R_C \\ \downarrow \\ R_B \end{array} \right) \longrightarrow R_D \left| \begin{array}{c} A \rightarrow C \\ \downarrow \quad \downarrow \\ B \rightarrow D \end{array} \right. - \text{homotopy pushout in } \mathcal{S}_{\text{fin}} \right\}.$$

Then the reduced homological functors in $\mathcal{S}_{\text{fin}}^{\text{op}}$ (with the projective model structure) are precisely the functors which are local with respect to $\mathcal{F}'_3 \cup \{\emptyset \rightarrow R_\emptyset\}$

3.3. Localization. Of course, in order to classify objects of a model category with certain property it is not enough to nominate a convenient class of maps \mathcal{F} , so that our objects will be \mathcal{F} -local. One must also make sure that there exists a localization of the model structure with respect to \mathcal{F} and the class of objects we are willing to classify will be represented, up to homotopy, by the elements of the homotopy category of the localized model category.

The localization procedure is not always a routine. For example, the class \mathcal{F} of maps is so ‘big’ that we are unable to perform the localization in the original model category at the moment. However, there exists a Quillen equivalent model for our model category in which we are able to perform the localization with respect to the class of maps corresponding to \mathcal{F} under the Quillen equivalence. This Quillen equivalence [4] is

$$(2) \quad | - |_2: \mathcal{S}^{\text{op}} \rightleftarrows \mathcal{S}_{\text{eq}}^2 : (-)^{\mathcal{O}}.$$

We hope to learn, in the future, how to localize with respect to \mathcal{F} in the original category of small functors, since this will allow for generalizations to other functor categories, which do not have a more convenient model, e.g., the category of covariant functors from spaces to spaces.

We need to localize the model category $\mathcal{S}_{\text{eq}}^2$ with respect to the class of maps $|\mathcal{F}|_2 = |\mathcal{F}_1|_2 \cup |\mathcal{F}_2|_2 \cup |\mathcal{F}_3|_2$, where

$$|\mathcal{F}_1|_2 = \left\{ \begin{array}{c} A \rightarrow B \\ \downarrow \rightarrow \downarrow \\ * \quad * \end{array} \left| A \rightarrow B \text{ is a w.e. in } \mathcal{S} \right. \right\},$$

$$|\mathcal{F}_2|_2 = \left\{ \prod_* X_i \rightarrow \prod_* X_i \left| \forall \{X_i\}_{i \in I} \in \mathcal{S}^I \right. \right\},$$

and

$$|\mathcal{F}_3|_2 = \left\{ \text{hocolim} \left(\begin{array}{c} A \rightarrow C \\ \downarrow \quad \downarrow \\ * \quad * \\ \downarrow \\ B \\ \downarrow \\ * \end{array} \right) \longrightarrow \begin{array}{c} D \\ \downarrow \\ * \end{array} \left| \begin{array}{c} A \rightarrow C \\ \downarrow \quad \downarrow \\ B \rightarrow D \end{array} \right. \text{ is a homotopy pushout in } \mathcal{S} \right\}.$$

Remark 3.4. The realization functor $| - |_2$ may be viewed as a coend $\text{Inc} \otimes_{\mathcal{S}} -$, where $\text{Inc}: \mathcal{S} = \mathcal{O}_2 \hookrightarrow \mathcal{S}^2$ is the fully-faithful embedding of the subcategory of orbits [4]. Therefore, computing the realization of the representable functors is just the evaluation of Inc at the representing object, since the dual of the Yoneda lemma applies.

We construct the localization of $\mathcal{S}_{\text{eq}}^2$ with respect to \mathcal{F} using the Bousfield-Friedlander localization technique. The role of the Q -construction will be taken by the counit of the adjunction

$$(3) \quad L: \mathcal{S}_{\text{eq}}^2 \rightleftarrows \mathcal{S} : R,$$

where $L\left(\begin{smallmatrix} A \\ \downarrow \\ B \end{smallmatrix}\right) = A$ and $R(A) = \begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix}$. We could have just said that $Q\left(\begin{smallmatrix} A \\ \downarrow \\ B \end{smallmatrix}\right) = \begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix}$, but we would like the reader to notice adjunction (3), which forms also a Quillen pair and will become a Quillen equivalence after we perform the localization with respect to Q .

In order to apply the Bousfield-Friedlander theorem [2, A.7] we have to verify that

- (1) Q preserves weak equivalences;
- (2) Q is a coaugmented $\left(\eta: \begin{smallmatrix} A & A \\ \downarrow & \rightarrow \downarrow \\ B & * \end{smallmatrix}\right)$, homotopy idempotent functor;
- (3) The resulting localized category must be right proper.

The verifications are routine.

Remark 3.5. This localization of the model category structure on $\mathcal{S}_{\text{eq}}^2$ has the following remarkable property: every orbit is \aleph_0 -small with respect to the cofibrations [6]. In other words, we have found a model of the category of spaces, in which every homotopy type is represented by an \aleph_0 -small, with respect to cofibrations, object. This conclusion seems contra-intuitive in view of Hovey's proof that every cofibrant and finite relative to cofibrations object in a pointed finitely generated model category \mathcal{C} is small in $\text{Ho}(\mathcal{C})$ [15, 7.4.3]. However, there is no contradiction with our result, since the localized model category $\mathcal{S}_{\text{eq}}^2$ is very far from being finitely generated (even though the original model category was pretty close to be finitely generated – it could be classified as class-finitely generated model category, but the localization process significantly changes the class of trivial cofibrations)

We have obtained so far a new model structure on $\mathcal{S}_{\text{eq}}^2$ that is Quillen equivalent to the usual model structure on \mathcal{S} . It is easy to verify that all the maps in $|\mathcal{F}|_2$ are weak equivalences in the new model structure. It remains to show that the class of $|\mathcal{F}|_2$ -local objects coincides with the class of Q -local objects. We will show it in the next section.

4. TECHNICAL PRELIMINARIES

Recall that we are going to prove two theorems in this paper. Theorem 5.1 classifies cohomological functors and Theorem 5.8 classifies homological functors. However the technicalities behind the proofs are very similar. Therefore, while we are heading towards the proof of Theorem 5.1 first, we indicate little adjustments required to adapt the argument for the proof of Theorem 5.8. Let us denote by \mathcal{E} the class of maps $|\mathcal{F}|_2$, as required for classification of cohomological functors, and by \mathcal{E}' the union $|\mathcal{F}'|_2 \cup \left(\emptyset \rightarrow \begin{smallmatrix} \emptyset \\ \downarrow \\ * \end{smallmatrix}\right)$ needed to classify the reduced homological functors.

The Q -local objects are precisely the orbits \downarrow_{*}^A with A fibrant. We need to show that every object in $\mathcal{S}_{\text{eq}}^2$ is \mathcal{E} -local equivalent to an orbit.

Every 2-diagram may be approximated by an I -cellular diagram [5], up to an equivariant weak equivalence [7], where

$$I = \left\{ \begin{array}{c} \partial\Delta^n \\ \downarrow \\ \Delta^n \end{array} \otimes \underline{T} \mid \underline{T} = \downarrow_{*}^A, A \in \mathcal{S} \right\}.$$

Therefore, it suffices to show that every I -cellular diagram is \mathcal{E} -equivalent to an orbit. We are going to prove it by cellular induction, but we precede the proof with the following lemma, which says that the basic building blocks of cellular complexes are \mathcal{E} -equivalent to orbits.

Lemma 4.1. *For every $A \in \mathcal{S}$, $n \geq 0$, there exists $A' \in \mathcal{S}$ such that $\left(\partial\Delta^n \otimes \downarrow_{*}^A \right) \stackrel{\mathcal{E}}{\simeq} \left(\downarrow_{*}^{A'} \right)$.*

Proof. We will show that $A' \simeq \partial\Delta^n \otimes A$. The proof is by induction on n . For $n = 0$ we have $\left(\partial\Delta^0 \otimes \downarrow_{*}^A \right) = \left(\emptyset \otimes \downarrow_{*}^A \right) = \emptyset \stackrel{\mathcal{E}}{\simeq} \left(\downarrow_{*}^{\emptyset} \right) = \left(\downarrow_{*}^{\partial\Delta^0 \otimes A} \right)$, since the map $\emptyset \rightarrow \left(\downarrow_{*}^{\emptyset} \right)$ is in $|\mathcal{F}_2|_2 \subset \mathcal{E}$. Alternatively, if one is willing to exclude \mathcal{F}_2 from \mathcal{F} , then for the base of induction it suffices to assume that the cohomology functor F is reduced, i.e., $F(\emptyset) = *$; cf. Remark 3.2. In other words the basis for induction holds for \mathcal{E}' equivalences as well.

Suppose the statement is true for n , i.e., $\left(\partial\Delta^n \otimes \downarrow_{*}^A \right) \stackrel{\mathcal{E}}{\simeq} \left(\downarrow_{*}^{\partial\Delta^n \otimes A} \right)$; we need to show it for $n + 1$.

$$\begin{aligned} \left(\partial\Delta^{n+1} \otimes \downarrow_{*}^A \right) &\simeq \text{colim} \left(\begin{array}{ccc} \partial\Delta^n \otimes \downarrow_{*}^A & \hookrightarrow & \Delta^n \otimes \downarrow_{*}^A \\ \downarrow & & \downarrow \\ \Delta^n \otimes \downarrow_{*}^A & & \downarrow_{*}^A \end{array} \right) \stackrel{\mathcal{E}}{\simeq} \text{hocolim} \left(\begin{array}{ccc} \partial\Delta^n \otimes A & \longrightarrow & A \\ \downarrow_{*} & & \downarrow_{*} \\ \downarrow & & \downarrow \\ A & & \downarrow_{*} \end{array} \right) \simeq \\ \text{hocolim} \left(\begin{array}{ccc} \partial\Delta^n \otimes A & \longrightarrow & \Delta^n \otimes A \\ \downarrow_{*} & & \downarrow_{*} \\ \downarrow & & \downarrow \\ \Delta^n \otimes A & & \downarrow_{*} \end{array} \right) &\stackrel{\mathcal{E}}{\simeq} \text{colim} \left(\begin{array}{ccc} \partial\Delta^n \otimes A & \hookrightarrow & \Delta^n \otimes A \\ \downarrow & & \downarrow \\ \Delta^n \otimes A & & \downarrow_{*} \end{array} \right) \simeq \left((\Delta^n \amalg_{\partial\Delta^n} \Delta^n) \otimes A \right) \simeq \left(\downarrow_{*}^{\partial\Delta^{n+1} \otimes A} \right), \end{aligned}$$

where the first \mathcal{E} -equivalence is induced by \mathcal{E} -equivalences of all vertices of respective homotopy pushouts. (If we will map both homotopy pushouts into an arbitrary \mathcal{E} -local object \underline{W} , we will obtain a levelwise weak equivalence of homotopy pullback squares of spaces). The

last \mathcal{E} -equivalence is induced by the map from $|\mathcal{F}_3|_2 \subset \mathcal{E}$ corresponding to the homotopy pushout square:

$$\begin{array}{ccc} \partial\Delta^n \otimes A & \hookrightarrow & \Delta^n \otimes A \\ \downarrow & & \downarrow \\ \Delta^n \otimes A & \longrightarrow & (\Delta^n \amalg_{\partial\Delta^n} \Delta^n) \otimes A. \end{array}$$

The above argument does not change if we consider \mathcal{E}' instead of \mathcal{E} . \square

Proposition 4.2. *Every I -cellular complex $\underline{X} \in \mathcal{S}^2$ is \mathcal{E} -equivalent to an orbit $\left(\begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix}\right)$ for some A .*

Proof. Every I -cellular complex \underline{X} has a decomposition into a colimit indexed by a cardinal λ :

$$\underline{X} = \operatorname{colim}_{a < \lambda} (\underline{X}_0 \rightarrow \cdots \rightarrow \underline{X}_a \rightarrow \underline{X}_{a+1} \rightarrow \cdots),$$

where \underline{X}_{a+1} is obtained from \underline{X}_a by attaching a cell:

$$\begin{array}{ccc} \partial\Delta^n \otimes \begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix} & \longrightarrow & \underline{X}_a \\ \downarrow & & \downarrow \\ \Delta^n \otimes \begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix} & \longrightarrow & \underline{X}_{a+1} \end{array} .$$

Assuming, by cellular induction, that \underline{X}_a is \mathcal{E} -equivalent to an orbit $\begin{smallmatrix} C_a \\ \downarrow \\ * \end{smallmatrix}$, we notice, by Lemma 4.1, that $\partial\Delta^n \otimes \begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix}$ is \mathcal{E} -equivalent to $\begin{smallmatrix} \partial\Delta^n \otimes A \\ \downarrow \\ * \end{smallmatrix}$, and $\left(\Delta^n \otimes \begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix}\right) \simeq \left(\begin{smallmatrix} A \\ \downarrow \\ * \end{smallmatrix}\right)$, so all the vertices of the homotopy pushout above are \mathcal{E} -equivalent to orbits $\begin{smallmatrix} A' \\ \downarrow \\ * \end{smallmatrix}$ for some A' . We conclude that \underline{X}_{a+1} is \mathcal{E} -equivalent to an orbit $\begin{smallmatrix} C_{a+1} \\ \downarrow \\ * \end{smallmatrix}$, where C_{a+1} is the homotopy pushout $(A \leftarrow \partial\Delta^n \otimes A \rightarrow B_a)$, similarly to the argument of Lemma 4.1.

We obtain the following commutative ladder:

$$\begin{array}{ccccccc} \underline{X}_0 & \hookrightarrow & \cdots & \hookrightarrow & \underline{X}_a & \hookrightarrow & \underline{X}_{a+1} & \hookrightarrow & \cdots \\ \downarrow \wr_{\mathcal{E}} & & & & \downarrow \wr_{\mathcal{E}} & & \downarrow \wr_{\mathcal{E}} & & \\ C_0 & \longrightarrow & \cdots & \longrightarrow & C_a & \longrightarrow & C_{a+1} & \longrightarrow & \cdots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ * & & & & * & & * & & \end{array}$$

Taking homotopy colimit of the upper and the lower rows we find that $\underline{X} \stackrel{\mathcal{E}}{\simeq} \operatorname{hocolim}_{a < \lambda} \begin{smallmatrix} C_a \\ \downarrow \\ * \end{smallmatrix}$, since if we will map both homotopy colimits into an arbitrary \mathcal{E} -local diagram \underline{W} , we will obtain a weak equivalence between the homotopy inverse limits.

Finally, $\text{hocolim}_{a < \lambda} \downarrow_*^{C_a} = \text{hocolim}_{a < \lambda} \left(\downarrow_*^{C_0} \xrightarrow{f_0} \dots \rightarrow \downarrow_*^{C_a} \xrightarrow{f_a} \downarrow_*^{C_{a+1}} \xrightarrow{f_{a+1}} \dots \right)$ may be represented as a homotopy pushout as follows:

$$\text{hocolim}_{a < \lambda} \downarrow_*^{C_a} \simeq \text{hocolim} \left(\begin{array}{c} \left(\prod \downarrow_*^{C_a} \right) \prod \left(\prod \downarrow_*^{C_a} \right) \xrightarrow{1 \prod f} \prod \downarrow_*^{C_a} \\ \downarrow \nabla \\ \prod \downarrow_*^{C_a} \end{array} \right),$$

where $f = \prod_{a < \lambda} f_a$ is the shift map and ∇ is the codiagonal. Observe that the homotopy pushout above is weakly equivalent to the infinite telescope construction.

All vertices of the homotopy pushout above are \mathcal{E} -equivalent to certain orbits through the respective \mathcal{E} -equivalences from $|\mathcal{F}_2|_2$. Testing by mapping into an arbitrary \mathcal{E} -local diagram \underline{W} , we find that the homotopy pushout above is \mathcal{E} -equivalent to the homotopy pushout of the respective orbits.

The latter pushout is \mathcal{E} -equivalent to an orbit \downarrow_*^A through an \mathcal{E} -equivalence from $|\mathcal{F}_3|_2$. \square

Remark 4.3. The only place in the proof where we use the property that infinite coproducts are converted into products by cohomological functors is the last argument. Alternatively, for $\mathcal{E}' = |\mathcal{F}'_3|_2 \cup \left(\emptyset \rightarrow \downarrow_*^\emptyset \right)$, as required by classification of homological functors, then we may

conclude that \underline{X} is \mathcal{E}' -equivalent to $\left(\text{hocolim}_{a < \lambda} C_a \right) \downarrow_*^A$ in the model category on \mathcal{S}^2 generated

by the orbits $\left\{ \downarrow_*^A \mid A \in \mathcal{S}_{\text{fin}} \right\}$, since these orbits are \aleph_0 -small. In the case of homological functors the same conclusion is easier to make in the Quillen-equivalent model category $\mathcal{S}_{\text{fin}}^{\text{op}}$, but we would like to stress that the proofs differ only in few places and also to make the exposition shorter.

5. MAIN RESULTS

We are ready now to prove the representability theorems.

Theorem 5.1. *Let $F: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$ be a small, homotopy functor converting coproducts to products, up to homotopy, and homotopy pushouts to homotopy pullbacks. Then there exists a fibrant simplicial set Y , such that $F(-) \simeq \mathcal{S}(-, Y)$. Moreover, for every functor $G: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$ there exists an approximation of G by a universal, up to homotopy, cohomological functor, which is not necessarily representable but is equivalent to one, i.e., there exists a natural transformation $\gamma: G \rightarrow \hat{G} \simeq R_A$, such that any other map $G \rightarrow R_B$ factors through γ and the factorization is unique up to simplicial homotopy for every representable functor R_B with fibrant simplicial sets A and B .*

Proof. We have proven so far that that the Q -localization constructed in 3.3 is essentially the localization with respect to \mathcal{E} : \mathcal{E} -equivalences are obviously Q -equivalences, and the inverse inclusion follows from Proposition 4.2, which says, in particular, that every \mathcal{E} -local object is also Q -local, hence any Q -equivalence is also an \mathcal{E} -equivalence.

The Quillen equivalence (2) gives rise to the equivalence of homotopy function complexes of the model categories [9], therefore the realization of a cohomological functor F (i.e., \mathcal{F} -local functor) after a fibrant replacement in $\mathcal{S}_{\text{eq}}^2$ becomes \mathcal{E} -local. Proposition 4.2 implies that $|F|_2 \xrightarrow{\sim} \widehat{|F|_2} \simeq \downarrow_*^A$ for some $A \in \mathcal{S}$. Considering the adjoint map we obtain the weak equivalence:

$$F \xrightarrow{\sim} \left(\widehat{|F|_2} \right)^{\mathcal{O}} \simeq \left(\downarrow_*^A \right)^{\mathcal{O}} = R_A = \mathcal{S}(-, A).$$

Let us point out that this conclusion could have been obtained without the use of the Quillen equivalence (2): Proposition 4.2 could be proven in the category of small contravariant functors by exactly the same argument, than the conclusion of the theorem would follow from the \mathcal{F} -local Whitehead theorem [14].

Now we will take advantage of the localized model structure constructed in 3.3. Given $G: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$, apply first the realization functor and than the fibrant replacement in the localized model category $|G|_2 \xrightarrow{\hookrightarrow} \widehat{|G|_2} \simeq \left(\downarrow_*^A \right)^{\mathcal{O}}$. Passing to the adjoint map we obtain the required approximation of G by a cohomological functor:

$$G \longrightarrow \left(\widehat{|G|_2} \right)^{\mathcal{O}} \simeq \left(\downarrow_*^A \right)^{\mathcal{O}} = R_A.$$

The verification of the universal property of the constructed map is performed by passing through adjunction to the category of maps of spaces and constructing the lift of the trivial cofibration to the Q -local object:

$$\begin{array}{ccc} G & \longrightarrow & R_B = \left(\downarrow_*^B \right)^{\mathcal{O}} \\ \downarrow & \nearrow l^{\mathcal{O}} & \leftarrow \text{wavy arrow} \\ \left(\widehat{|G|_2} \right)^{\mathcal{O}} & & \left\{ \begin{array}{ccc} |G|_2 & \longrightarrow & \downarrow_*^B \\ \downarrow & \nearrow l & \vdots \\ \widehat{|G|_2} & \dashrightarrow & * \end{array} \right. \end{array}$$

□

Remark 5.2. It is possible to formulate a stronger universal property of the cohomological approximation map: it should be initial with respect to maps into all \mathcal{F} -local, i.e., cohomological functors. In order to prove it, however, we would need to consider the \mathcal{F} -localized model structure on the category of small contravariant functors. We indicate in Remark 5.4 a way to construct this localizations.

Remark 5.3. There is a different, simpler, approach to the classification of cohomological functors: given a fibrant simplicial cohomological functor $G: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$ (no need to assume

that G is small), consider the natural map $q: G(X) \rightarrow \mathcal{S}(X, G(*))$ obtained by adjunction from the natural map $X = \mathcal{S}(*, X) \rightarrow \mathcal{S}(G(X), G(*))$, which exists, in turn, since G is simplicial. The map q is an equivalence if $X = *$, which gives a basis for induction on the cellular structure of X similarly to Proposition 4.2. This approach is simpler, and seemingly more general (works for all functors, not necessarily small), but it does not allow for generalizations, where the representing object is not so obvious. Another advantage of using model categories is that they help to prove the most interesting part of our result that any functor may be approximated by a cohomological functor. We owe this remark to T. Goodwillie.

Remark 5.4. There is also a different way to prove our main result. Consider the projective model structure on the category of small contravariant functors and let $Q(G) = q(\widehat{G})$ be a functorial fibrant replacement of G composed with q from Remark 5.3. Since the factorizations are not functorial in \mathcal{S}^{Sop} , the existence of the functorial fibrant replacement requires an explanation. One way to construct it is to apply the realization functor and then to take the fibrant replacement in the category of maps of spaces with the equivariant model structure followed by the orbit-points functor, so $Q(G) = q(|G|_2^{\mathcal{O}})$. This Q satisfies the conditions of Bousfield-Friedlander localization theorem, therefore we could prove the same result using the localized category of small contravariant functors. We obtain another interesting model for the category of spaces, where every object has a representative, which is \aleph_0 -small. However the proof of Proposition ?? is easier in $\mathcal{S}_{\text{eq}}^2$ and we can not avoid using the category of maps completely, so we have chosen to present this approach.

Homological Brown representability for spaces is essentially Goodwillie's classification of linear functors. We choose, however, to discuss the contravariant version of this theorem in our work. Even though philosophically the two versions are the same, we are doubtful if such an implication exists, since there are several significant differences. We repeat the basic definitions first:

Definition 5.5. A small functor $F \in \mathcal{S}_{\text{fin}}^{\text{Sop}}$ is called *homological* if F converts homotopy pushouts of (finite) simplicial sets to homotopy pullbacks.

Let $i: \mathcal{S}_{\text{fin}} \hookrightarrow \mathcal{S}$ be the fully faithful embedding.

Example 5.6. Any functor of the form $X \times i^* \mathcal{S}(-, Y)$ is homological, hence the next definition.

Definition 5.7. A homological functor F is *reduced* if $F(\emptyset) = *$.

Theorem 5.8. *Any reduced homological functor $F: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$ is weakly equivalent to the restriction a representable functor $i^* \mathcal{S}(-, Y)$ for some fibrant simplicial set Y , unique up to homotopy. Moreover, there exists for every functor $G \in \mathcal{S}_{\text{fin}}^{\text{Sop}}$ an approximation $G \rightarrow \widehat{G}$ by a reduced homological functor, which is initial beneath all maps into homological functors.*

Proof. The proof of this theorem is essentially the same as of Theorem 5.1. We give a sketch of the proof, pointing out the differences.

Consider the projective model structure on the category of functors $\mathcal{S}_{\text{fin}}^{\mathcal{S}^{\text{op}}}$. If we will localize this model category with respect to $\mathcal{F}' = \mathcal{F}'_3 \cup \{\emptyset \rightarrow \mathcal{S}_{\text{fin}}(-, \emptyset)\}$, then we will obtain a classification of reduced homological functors as local objects. We have to show that every local object is equivalent to the restriction of a representable functor.

Consider the model structure on \mathcal{S}^2 generated by the set of orbits $\mathcal{O}' = \left\{ \begin{array}{c} A \\ \downarrow \\ * \end{array} \middle| A \in \mathcal{S}_{\text{fin}} \right\}$ in the sense of Dwyer and Kan [10]. This model category is Quillen equivalent to the projective model structure on $\mathcal{S}_{\text{fin}}^{\mathcal{S}^{\text{op}}}$ [10].

Next, we perform the localization of the model structure on \mathcal{S}^2 generated by \mathcal{O}' with respect to the class of maps $\mathcal{E}' = \mathcal{F}'$. We could do it by the same method as in the previous theorem, but the class of maps \mathcal{E}' is a small set in this situation, so nothing prevents us from applying one of the standard localization machines in the cofibrantly generated model category \mathcal{S}^2 . Proposition 4.2 together with its adaptation to the model category generated by \mathcal{O}' in Remark 4.3, shows that the fibrant objects of this localization have the form

$$\left\{ \begin{array}{c} A \\ \downarrow \\ * \end{array} \middle| \mathcal{S} \ni A - \text{fibrant} \right\}.$$

The conclusion of Theorem 5.1 is derived in exactly the same way. \square

Remark 5.9. Remarks 5.2, 5.3 and 5.4 apply without change.

If required, the same result may be reformulated for small contravariant functors from spaces to spaces. We only need to introduce the concept of finitary functor:

Definition 5.10. A small functor $F \in \mathcal{S}^{\mathcal{S}^{\text{op}}}$ is called *finitary* if F is a left Kan extension from its restriction onto the full subcategory of finite simplicial sets. The category of small functors $\mathcal{S}^{\mathcal{S}^{\text{op}}}$ supports the *finitary model structure*: a natural transformation is a weak equivalence or a fibration if it induces a weak equivalence or a fibration between values of the functors in finite simplicial sets. Cofibrant objects in the finitary model structure are finitary functors. The finitary model structure is Quillen equivalent to the projective model structure on the category of contravariant functors from finite simplicial sets to simplicial sets.

Remark 5.11. Origin of this terminology and the discussion of the elementary properties of the finitary model structure on the category of small covariant functors may be found in [1, Section 9].

The definition of a small homological contravariant functor is the same: it takes homotopy pushouts of finite simplicial sets to homotopy pullbacks. In the finitary model structure any reduced homological functor is equivalent to a representable functor. Relevant approximation result also applies. The details are left to the interested reader.

Finally we would like to point out one crucial difference between the classification of contravariant homological functors in this paper and Goodwillie's classification of linear functors: it is usually not a simple task to compute the linear approximation of a functor, whether in our framework the answer is ready – the functor represented in the value of the original functor in $*$.

REFERENCES

- [1] G. Biedermann, B. Chorny, and O. Röndigs. Calculus of functors and model categories. *Adv. in Math.*, 214(1):92–115, 2007.
- [2] Bousfield and Friedlander. Homotopy theory of Γ -spaces, spectra, and bisimplicial sets. In *Geometric Applications of Homotopy Theory II*, number 658 in Lecture Notes in Mathematics. Springer, 1978.
- [3] E. H. Brown, Jr. Cohomology theories. *Ann. of Math. (2)*, 75:467–484, 1962.
- [4] B. Chorny and W. G. Dwyer. Homotopy theory of small diagrams over large categories. Preprint, 2006.
- [5] E. Dror Farjoun. Homotopy and homology of diagrams of spaces. In *Algebraic topology (Seattle, Wash., 1985)*, Lecture Notes in Math. 1286, pages 93–134. Springer, Berlin, 1987.
- [6] E. Dror Farjoun. Homotopy theories for diagrams of spaces. *Proc. Amer. Math. Soc.* 101, pages 181–189, 1987.
- [7] E. Dror Farjoun and A. Zabrodsky. Homotopy equivalence between diagrams of spaces. *J. Pure Appl. Algebra* 41(2-3), pages 169–182, 1986.
- [8] W. G. Dwyer. Localizations. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 3–28. Kluwer Acad. Publ., Dordrecht, 2004.
- [9] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. *J. Pure Appl. Algebra*, 18(1):17–35, 1980.
- [10] W. G. Dwyer and D. M. Kan. Singular functors and realization functors. *Proc. Kon. Nederl. Acad. 87(2); Indag. Math.* 46(2), pages 147–153, 1984.
- [11] P. Freyd. Several new concepts: Lucid and concordant functors, pre-limits, pre-completeness, the continuous and concordant completions of categories. In *Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, pages 196–241. Springer, Berlin, 1969.
- [12] T. G. Goodwillie. Calculus. II. Analytic functors. *K-Theory*, 5(4):295–332, 1991/92.
- [13] T. G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711 (electronic), 2003.
- [14] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
- [15] M. Hovey. *Model categories*. Mathematical Surveys and Monographs 63. American Mathematical Society, Providence, RI, 1999.
- [16] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982.
- [17] S. Lack. Limits of small functors. Preprint, 2002.
- [18] J. Rosický. Cartesian closed exact completions. *J. Pure Appl. Algebra*, 142(3):261–270, 1999.

CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 0200, AUSTRALIA
E-mail address: Boris.Chorny@maths.anu.edu.au