AN EXAMPLE OF A NON-COFIBRANTLY GENERATED MODEL CATEGORY

BORIS CHORNY

Abstract. We show that the model category of diagrams of spaces generated by a proper class of orbits is not cofibrantly generated. In particular the category of maps between spaces may be given a non-cofibrantly generated model structure.

1. Introduction and formulation of results

Several examples of non-cofibrantly generated model categories have appeared recently (see [1], [2], [6]) in response to a question stated by Mark Hovey on his home page. In this note we introduce another family of such examples.

By the *category of spaces*, denoted by $\mathcal{S}$, we mean the category of simplicial sets (or compactly generated topological spaces). There are plenty of model structures on categories of diagrams of spaces, with different notions of weak equivalences. Some of them are cofibrantly generated, e.g. for weak equivalences and fibrations being objectwise and cofibrations obtained by the left lifting property with respect to trivial fibrations, the corresponding model category is cofibrantly generated.

Let us remind (from [3], [4], [5]) that a diagram O of spaces is called an *orbit* if $\operatorname{colim} O = *$. The weak equivalences which we would like to consider arise naturally from the relation of equivariant homotopy. By the generalized Bredon theorem [5] a map $f : \underset{\sim}{X} \to \underset{\sim}{Y}$ is an equivariant homotopy equivalence between diagrams which are both cofibrant and fibrant iff $map(O, f) : map(O, \underset{\sim}{X}) \to map(O, \underset{\sim}{Y})$ is a weak equivalence of spaces for any orbit O. A model category, *generated by the collection of orbits*, on diagrams of spaces was constructed in [4] with a map f being a weak equivalence (reps. fibration) iff $map(O, f)$ is a weak equivalence (resp. fibration) for any orbit O. In the sequel we consider only this model category on diagrams of spaces. The simplest example of a non-cofibrantly generated model category is given by the following

Theorem 1.1. *If $J = (\bullet \to \bullet)$ is the category with two objects and only one non-identity morphism, then the functor category $\mathcal{M} = \mathcal{S}^J$ of maps of spaces with the model structure as above is not cofibrantly generated.*

However, not every small category gives rise to a non-cofibrantly generated model category of diagrams. For example, if we take G to be a group, then the above model structure on $\mathcal{S}^G$ is cofibrantly generated. We conclude the paper by using this example to produce many other examples of the same nature.

Date: December 12, 2001.

1991 *Mathematics Subject Classification.* Primary 55U35; Secondary 55P91, 18G55.

Key words and phrases. model category, equivariant homotopy, non-cofibrantly generated.

The author is a fellow of the Marie Curie Training Site hosted by the Centre de Recerca Matemàtica (Barcelona), grant nr. HPMT-CT-2000-00075 of the European Commission.

Acknowledgments. I would like to thank C. Casacuberta and E. Dror Farjoun for helpful conversations about the subject matter of this paper.

2. Preliminaries

By an *orbit over a point* in the colimit of a diagram $\underset{\sim}{X}$ we mean the pull back of the canonical map $f\colon \underset{\sim}{X} \to \operatorname{colim} \underset{\sim}{X}$ over $g\colon * \to \operatorname{colim} \underset{\sim}{X}$. Let D be any small category enriched over $\mathcal{S}$. We denote by $\mathcal{O}$ the collection of all orbits of D. By collection we mean a set or a proper class with respect to some fixed universe $\mathfrak{U}$. The operator $\operatorname{codom}(\cdot)$ applied to a collection of maps returns the collection of ranges. Given a set I of maps in $\mathcal{M} = \mathcal{S}^D$, we denote by I-cell the collection of relative I-cellular complexes and by abs-I-cell the collection of (absolute) I-cellular complexes. See [7, 2.1.9] for precise definitions.

Definition 2.1. Let $\mathcal{X} = \{\underset{\sim}{X}_\alpha\}_{\alpha \in A}$ be a collection of D-shaped diagrams of spaces. The *collection of orbits* of $\mathcal{X}$, denoted by $\Omega(\mathcal{X}) \subset \mathcal{O}$, consists of all orbits $O \in \mathcal{O}$ such that there exists $\alpha \in A$ and a point $x \in \operatorname{colim} \underset{\sim}{X}_\alpha$ with O being the orbit over x.

Lemma 2.2. *Let I be a set of cofibrations in the model category $\mathcal{M}$ of D-shaped diagrams of spaces. Then $\Omega($abs-I-cell$) \subset \Omega(\operatorname{codom}(I))$.*

Proof. Let $\underset{\sim}{X} \in \mathcal{M}$ be any I-cellular complex. We proceed by transfinite induction on the I-cellular filtration of $\underset{\sim}{X}$. $\underset{\sim}{X}_{-1} = \emptyset$, hence $\underset{\sim}{X}_0 \in \operatorname{codom}(I)$ and in particular $\Omega(\underset{\sim}{X}_0) \subset \Omega(\operatorname{codom}(I))$.

Suppose $\underset{\sim}{X}_\beta$ satisfies $\Omega(\underset{\sim}{X}_\beta) \subset \Omega(\operatorname{codom}(I))$. We need to show that $\underset{\sim}{X}_{\beta+1}$, which is obtained from $\underset{\sim}{X}_\beta$ by attaching a map $I \ni f : \underset{\sim}{A} \hookrightarrow \underset{\sim}{B}$, satisfies $\Omega(\underset{\sim}{X}_{\beta+1}) \subset \Omega(\operatorname{codom}(I))$.

$$\begin{array}{ccc} \underset{\sim}{A} & \xrightarrow{\varphi} & \underset{\sim}{X}_\beta \\ {\scriptstyle f}\downarrow & \text{push-out} & \downarrow{\scriptstyle f'} \\ \underset{\sim}{B} & \longrightarrow & \underset{\sim}{X}_{\beta+1} \end{array}$$

Let O_s be an orbit over a point $s \in \operatorname{colim} \underset{\sim}{X}_{\beta+1} = \operatorname{colim} \underset{\sim}{X}_\beta \amalg_{\operatorname{colim} \underset{\sim}{A}} \operatorname{colim} \underset{\sim}{B}$. Considering two cases, $s \in \operatorname{colim} \underset{\sim}{X}_\beta \subset \operatorname{colim} \underset{\sim}{X}_{\beta+1}$ and $s \notin \operatorname{colim} \underset{\sim}{X}_\beta$, we find out that in the first case O_s equals the corresponding orbit of $\underset{\sim}{X}_\beta$ and in the second case O_s is some orbit of $\underset{\sim}{B}$. This follows immediately from the fact that the diagrams

$$\begin{array}{ccc} \underset{\sim}{X}_\beta & \xrightarrow{f'} & \underset{\sim}{X}_{\beta+1} \\ \downarrow & & \downarrow \\ \operatorname{colim} \underset{\sim}{X}_\beta & \longrightarrow & \operatorname{colim} \underset{\sim}{X}_{\beta+1} \end{array} \qquad \begin{array}{ccc} \underset{\sim}{B}/\underset{\sim}{A} & \xrightarrow{\cong} & \underset{\sim}{X}_{\beta+1}/\underset{\sim}{X}_\beta \\ \downarrow & & \downarrow \\ \operatorname{colim}(\underset{\sim}{B}/\underset{\sim}{A}) & \xrightarrow{\cong} & \operatorname{colim}(\underset{\sim}{X}_{\beta+1}/\underset{\sim}{X}_\beta) \end{array}$$

are pull-backs. The first square is a pull-back by [4, 2.1] and the second by the observation that horizontal maps are isomorphisms. Hence $\Omega(\underset{\sim}{X}_{\beta+1}) \subset \Omega(\operatorname{codom}(I))$.

Obviously, if β is a limit ordinal, then

$$\Omega(\underset{\sim}{X}_\beta) = \bigcup_{\lambda<\beta} \Omega(\underset{\sim}{X}_\lambda) \subset \Omega(\operatorname{codom}(I)).$$

Hence $\Omega(\text{abs-}I\text{-cell}) \subset \Omega(\operatorname{codom}(I))$. □

3. Proof of Theorem 1.1

Let us prove first a slightly more general result.

Proposition 3.1. *Let D be a small category enriched over $\mathcal{S}$ which admits a proper class of orbits $\mathcal{O}_D$. Then the model category $\mathcal{M}$ on the D-shaped diagrams of spaces generated by the orbits is not cofibrantly generated.*

Proof. We argue by contradiction. Suppose the model category $\mathcal{M}$ generated by the proper class of orbits $\mathcal{O}_D$ is cofibrantly generated. Let I be the set of generating cofibrations, then any cofibration is a retract of an I-cellular map. (This follows from Quillen's small object argument; see [7, 2.1.15].) In particular, any orbit of $\mathcal{O}_D$ is a retract of an I-cellular complex. Hence any orbit is a retract of some orbit of an I-cellular space. But by 2.2 the whole collection of orbits of I-cellular complexes form a set, hence the contradiction. □

In particular, the model category $\mathcal{S}^J$ is generated by the proper class of orbits $\mathcal{O}_J = \{X \to *\}$, where X runs through all the objects of $\mathcal{S}$. Therefore, by Proposition 3.1, $\mathcal{S}^J$ is not cofibrantly generated, hence the main result 1.1.

4. More examples

Let us conclude by giving more examples of non-cofibrantly generated model categories. Proposition 3.1 implies that $\mathcal{M} = \mathcal{S}^D$ is not cofibrantly generated iff $\mathcal{O}_D$ is a proper class. We have already indicated in the introduction that if we take $D = G$ to be a group, then $\mathcal{O}_G = \{G/H | H < G\}$ is a set, hence $\mathcal{S}^G$ is cofibrantly generated. The same holds for groupoids. However, the following proposition provides us with a large family of examples.

Proposition 4.1. *Let D be a small category which admits a fully faithful functor $i : K \to D$, where K is a category with two objects k_1, k_2, at least one arrow $f : k_1 \to k_2$ and no arrows in the opposite direction. Then $\mathcal{O}_D$ consists of a proper class of orbits.*

Proof. First define for each space $X \in \mathcal{S}$ an orbit over K, i.e. a functor $T_X : K \to \mathcal{S}$, by $T_X(k_1) = X$, $T_X(k_2) = *$, $T_X(g) = id_X$ for any $g \in mor(k_1, k_1)$ and T_X on the elements of $mor(k_1, k_2)$, $mor(k_2, k_2)$ has a unique definition, since $*$ is the final object of $\mathcal{S}$. Obviously T_X is an orbit over K.

Next we define for each T_X a D-orbit O_X by extending the definition of T_X to the whole D. More precisely, $O_X = \mathrm{Lan}_i T_X$. We need to check that O_X is an orbit. It follows from the fact that colimit is itself a left Kan extension along a functor into the trivial category. But any two left Kan extensions commute since they may be represented as coends, and for the coends there is a "Fubini" theorem. See [8, X] for the details.

The functor i is taken to be fully faithful, hence $O_X(i(k_1)) = X$; therefore we obtain a proper class of D-orbits of the form O_X. □

Question 4.2. Let D be a monoid which is not a group. Is $\mathcal{S}^D$ cofibrantly generated?

References

[1] J. Adámek, H. Herrlich, J. Rosický and W. Tholen, *Weak factorization systems and topological functors*, preprint, 2000.

[2] D. Christensen and M. Hovey, *Quillen model structures for relative homological algebra*, preprint, 2001.
[3] E. Dror Farjoun, *Homotopy and homology of diagrams of spaces*, Proceedings of the Conference on Algebraic Topology, Seattle 1985, Lecture Notes in Math. 1286, 93–134, Springer-Verlag, 1987.
[4] E. Dror Farjoun, *Homotopy theories for diagrams of spaces*, Proc. Amer. Math. Soc. 101, 181–189, 1987.
[5] E. Dror Farjoun and A. Zabrodsky, *Homotopy equivalence between diagrams of spaces*, J. Pure Appl. Alg. 41, 169–182, 1986.
[6] D. Isaksen, *A model structure on the category of pro-simplicial sets*, Trans. Amer. Math. Soc. 353, no. 7, 2805–2841, 2001.
[7] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs vol. 63, Amer. Math. Soc., 1999.
[8] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Math. 5, Springer-Verlag, 1971.

Centre de Recerca Matemàtica, Apartat 50, E-08193 Bellaterra, Spain
E-mail address: cboris@crm.es