

EQUIVARIANT CELLULAR HOMOLOGY AND ITS APPLICATIONS

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ABSTRACT. In this work we develop a cellular equivariant homology functor and apply it to prove an equivariant Euler-Poincaré formula and an equivariant Lefschetz theorem.

1. INTRODUCTION

Let D be an arbitrary small topologically enriched category. In this paper we develop a D -CW-homology functor which allows for easy computation of the *ordinary* D -equivariant homology defined by E. Dror Farjoun in [1]. Our approach is a generalization of the G -CW-(co)homology functor constructed by S.J. Willson in [13] for the case of G being compact Lie group.

Then we apply the D -CW-homology functor to obtain:

(i) Equivariant Euler-Poincaré formula:

$$(1) \quad \chi^D(\underline{X}) = \sum_{n=0}^{\infty} (-1)^n \widetilde{rk}_{\text{HS}}(H_n^D(\underline{X}; \mathcal{I}))$$

This formula establishes a connection between the equivariant homology and an equivariant Euler characteristic; $\widetilde{rk}_{\text{HS}}(\cdot)$ is a slight modification of Hattori-Stallings rank (originally defined in [8],[11]).

(ii) Equivariant Lefschetz theorem: Let \underline{X} be a triangulated D -space, $f : \underline{X} \rightarrow \underline{X}$ an equivariant map. If the equivariant Lefschetz number

$$(2) \quad \Lambda_D(f) = \sum_{n=0}^{\infty} (-1)^n \widetilde{tr}_{\text{HS}}(H_n^D(f; \mathcal{I}))$$

is not equal to zero, then there are f -invariant orbits in \underline{X} , moreover the orbit types of the invariant orbits may be recovered from $\Lambda_D(f)$.

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2. PRELIMINARIES

2.1. D -spaces. Let \mathcal{Top} denote the category of the compactly generated Hausdorff topological spaces. Fix an arbitrary small category D enriched over \mathcal{Top} . We work in the category \mathcal{Top}^D of functors from D to \mathcal{Top} . The objects of this category

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are called topological diagrams or just D -spaces. The arrows in \mathcal{Top}^D are natural transformations of functors or *equivariant maps*.

2.2. D -homotopy. An equivariant homotopy between two D -maps $f, g : \underline{X} \rightarrow \underline{Y}$, where $\underline{X}, \underline{Y}$ are D -diagrams, is a D -map $H : \underline{X} \times I \rightarrow \underline{Y}$, where I denotes the constant D -space $I(d) = [0, 1]$. A *homotopy equivalence* $f : \underline{X} \rightarrow \underline{Y}$ is a map with a (two sided) D -homotopy inverse.

2.3. D -orbits. We recall now the central concept of the D -homotopy theory (introduced in [1],[3]) – that of D -orbit. A D -orbit is a D -space $T : D \rightarrow \mathcal{Top}$, such that $\text{colim}_D T = \{*\}$. A **free** D -orbit generated in $d \in \text{ob}(D)$ is $\mathcal{Top}^D \ni F^d = \text{hom}_D(d, \cdot)$, i.e. $F^d(d') = \text{hom}_D(d, d')$ and $F^d(d' \rightarrow d'')$ is given by the composition. Clearly F^d is a D -orbit. A D -space \underline{X} is called free iff for any $s \in \text{colim}_D \underline{X}$ the full orbit T_s lying over s is free.

2.4. D -CW-complexes. A D -cell is a D -space of the form $T \times e^n$, where T is a D -orbit and e^n is the standard n -cell. An *attaching map* of this D -cell to some D -space \underline{X} is a map $\phi : T \times \partial e^n \rightarrow \underline{X}$.

A (relative) D -CW-complex $(\underline{X}, \underline{X}_{-1})$ is a D -space \underline{X} together with a filtration $\underline{X}_{-1} \subset \underline{X}_0 \subset \dots \subset \underline{X}_n \subset \underline{X}_{n+1} \subset \dots \subset \underline{X} = \text{colim}_n \underline{X}_n$, such that \underline{X}_{n+1} is obtained from \underline{X}_n by attaching a set of n -dimensional D -cells. Namely one has a push-out diagram of D -spaces:

$$\begin{array}{ccc} \coprod_i (T_i \times \partial e^n) & \xrightarrow{\phi} & \underline{X}_{n-1} \\ \downarrow & & \downarrow \\ \coprod_i (T_i \times e^n) & \xrightarrow{\Phi} & \underline{X}_n \end{array}$$

If $\underline{X}_{-1} = \emptyset$ we call the D -CW-complex *absolute*.

Let \underline{X} be a D -CW-complex. A D -subspace $\underline{Y} \subset \underline{X}$ is called the *cellular subspace* if \underline{Y} has a D -CW-structure such that each cell of \underline{Y} is also a cell of \underline{X} .

2.5. The category of orbits. The *category of orbits* \mathcal{O} is a full topological subcategory of \mathcal{Top}^D generated by all D -orbits.

Usually \mathcal{O} is not a small category. For example for $D = J = (\bullet \rightarrow \bullet)$, then $\mathcal{O} \cong \mathcal{Top}$. A model category has been constructed for the D -spaces of arbitrary orbit type in [2]. We will be interested in the diagrams which are homotopy equivalent to the finite D -CW-complexes, i.e. only finite number of orbit types appear in such diagram. We collect those orbits into the a full subcategory \mathcal{O}' of \mathcal{O} with a finite amount of objects.

2.6. Orbit point $(\cdot)^\mathcal{O}$ and realization $|\cdot|_D$ functors. Suppose \mathcal{O} is a *small* category of D -orbits. An orbit point functor $(\cdot)^\mathcal{O} : \mathcal{Top}^D \rightarrow \mathcal{Top}^{\mathcal{O}^{\text{op}}}$ is a generalization to the diagram case of Bredon's fixed point functor. For any D -space \underline{X} (usually of type \mathcal{O}) $(\underline{X})^\mathcal{O}$ is a \mathcal{O}^{op} diagram such that $(\underline{X})^\mathcal{O}(T) = \text{hom}_D(T, \underline{X})$ for all $T \in \text{ob}(\mathcal{O})$ and the arrows of the diagram are induced by composition with the maps between orbits.

If $f : \underline{X} \rightarrow \underline{Y}$ is an equivariant map between two D -spaces, then there exist an \mathcal{O}^{op} -equivariant map $f^\mathcal{O} : \underline{X}^\mathcal{O} \rightarrow \underline{Y}^\mathcal{O}$, which is obtained from f by composition:

$$\underline{X}^\mathcal{O}(T) = \text{hom}_D(T, \underline{X}) \ni g \xrightarrow{f^\mathcal{O}} f \circ g \in \text{hom}_D(T, \underline{Y}) = \underline{Y}^\mathcal{O}(T)$$

The fundamental property of $(\cdot)^\mathcal{O}$ functor is that for any D -space \underline{X} the \mathcal{O}^{op} -space $(\underline{X})^\mathcal{O}$ is \mathcal{O}^{op} -free [1, 3.7].

Example 2.1. Consider a free D -space \underline{X} . And let the orbit category \mathcal{O} consists of all the free orbits. Then \mathcal{O} is isomorphic to D^{op} as a category and $(\underline{X})^\mathcal{O} \cong \underline{X}$ (Yoneda's lemma).

Another easy case occurs then \underline{X} is a D -CW-space. We shall discuss it in the next section.

There exist a left adjoint to $(\cdot)^\mathcal{O}$. It is called *realization functor* $|\cdot|_D$, since it takes an \mathcal{O}^{op} -space and produce a D -space with the prescribed orbit point data (up to local weak equivalence). Realization functor in the group case has been constructed by A.D.Elmendorf in [7] and has been generalized to the arbitrary diagram case by W.Dwyer and D.Kan in [6]. Compare also [1].

2.7. Equivariant Euler characteristic. Let \underline{X} be a finite D -CW-complex, then \underline{X} is of type \mathcal{O} for some category of orbits with finite amount of objects. We define the Equivariant Euler characteristic to be the Universal Additive Invariant [10, I.5] $(U(D), \chi^D)$. Or, equivalently, we say that $\chi^D(\underline{X}) \in U(D)$ is equal to the alternating sum of the orbit types over the dimensions of the cells of \underline{X} in the free abelian group $U(D) = \bigoplus_{T \in Iso(h\mathcal{O})} \mathbb{Z}$ generated by the homotopy types of orbits.

3. EQUIVARIANT CELLULAR HOMOLOGY

3.1. \mathcal{O}^{op} -CW structure on the orbit point space of a D -CW-complex. Let \mathcal{C} denote a subcategory of $\mathcal{Top}^{\mathcal{O}^{\text{op}}}$ which is obtained as the image of \mathcal{Top}^D under the functor $(\cdot)^\mathcal{O}$. Recall that there is the inclusion of the categories $\iota : D \hookrightarrow \mathcal{O}^{\text{op}}$, where $\iota(d) = F^d$, for each $d \in ob(D)$. Hence there is a functor $\text{Res} : \mathcal{Top}^{\mathcal{O}^{\text{op}}} \rightarrow \mathcal{Top}^D$. By abuse of notation we denote by Res also $\text{Res}|_{\mathcal{C}}$.

Lemma 3.1. *The functor $(\cdot)^\mathcal{O}$ is fully faithful.*

Proof. The faithfulness is clear. We have to show only that for any map $f_0 : \underline{X}^\mathcal{O} \rightarrow \underline{Y}^\mathcal{O}$ there exists a map $f : \underline{X} \rightarrow \underline{Y}$ s.t. $f_0 = f^\mathcal{O}$. Take $f = \text{Res}(f_0)$, then if \underline{X} and \underline{Y} were orbits the result follows from the bijective correspondence induced by the Yoneda's lemma: $\text{hom}_D(\underline{X}, \underline{Y}) = \text{hom}_{\mathcal{O}^{\text{op}}}(\underline{X}^\mathcal{O}, \underline{Y}^\mathcal{O})$. The general claim will follow from the comparison of f_0 and $f^\mathcal{O}$ orbitwise, i.e. by their action on each full orbit. Fortunately the functor $(\cdot)^\mathcal{O}$, being right adjoint, commutes with taking full orbit (pullback). \square

Lemma 3.2. *The pair of functors $\text{Res}(\cdot) : \mathcal{C} \leftrightarrow \mathcal{Top}^D : (\cdot)^\mathcal{O}$ induce the equivalence of the categories \mathcal{C} and \mathcal{Top}^D .*

Proof. We need to construct the natural isomorphisms of the functors $\text{id}_{\mathcal{Top}^D} \cong \text{Res}((\cdot)^\mathcal{O})$ and $\text{id}_{\mathcal{C}} \cong (\text{Res}(\cdot))^\mathcal{O}$.

Let $\underline{X} \in \mathcal{Top}^D$, then $\text{Res}(\underline{X}^\mathcal{O}) \cong \underline{X}$ because the generalized lemma of Yoneda [9] induces the objectwise homeomorphisms and the equivariance is preserved by the naturality of the Yoneda's isomorphism. But an equivariant map which is the objectwise homeomorphism is an isomorphism of D -spaces, hence the first isomorphism of functors.

Let $\underline{X}^\mathcal{O} \in \mathcal{C}$, then $\text{Res}(\underline{X}^\mathcal{O}) \cong \underline{X}$ by the first homeomorphism, then $(\text{Res}(\underline{X}^\mathcal{O}))^\mathcal{O} \cong \underline{X}^\mathcal{O}$. Hence the second isomorphism. \square

Proposition 3.3. *Let \underline{X} be a (pointed) D -CW-space of orbit type \mathcal{O} , where \mathcal{O} is a small category of orbits. Consider the \mathcal{O}^{op} -space $\underline{X}^{\mathcal{O}}$ to be the orbit point space of \underline{X} .*

Then $\underline{X}^{\mathcal{O}}$ has \mathcal{O}^{op} -CW structure which corresponds to the D -CW structure of \underline{X} in the following sense: let $\underline{pt}_D \subseteq \underline{X}_0 \subseteq \underline{X}_1 \subseteq \cdots \subseteq \underline{X}_n \subseteq \cdots \subseteq \underline{X} = \text{colim}_n \underline{X}_n$ is a D -CW filtration of \underline{X} , such that each \underline{X}_n is a push-out:

$$\begin{array}{ccc} \coprod_i T_i \times S^{n-1} & \xrightarrow{\phi} & \underline{X}_{n-1} \\ \downarrow & & \downarrow i_n \\ \coprod_i T_i \times D^n & \xrightarrow{\Phi} & \underline{X}_n \end{array}$$

then there exist a \mathcal{O}^{op} -CW-filtration: $\underline{pt}_{\mathcal{O}^{\text{op}}} \subseteq \underline{X}_0^{\mathcal{O}} \subseteq \underline{X}_1^{\mathcal{O}} \subseteq \cdots \subseteq \underline{X}_n^{\mathcal{O}} \subseteq \cdots \subseteq \underline{X}^{\mathcal{O}} = \text{colim}_n \underline{X}_n^{\mathcal{O}}$, such that $\underline{X}_n^{\mathcal{O}} = (\underline{X}_n)^{\mathcal{O}}$, and

$$\begin{array}{ccc} \coprod_i F^{T_i} \times S^{n-1} & \xrightarrow{\phi^{\mathcal{O}}} & \underline{X}_{n-1}^{\mathcal{O}} \\ \downarrow & & \downarrow i_n^{\mathcal{O}} \\ \coprod_i F^{T_i} \times D^n & \xrightarrow{\Phi^{\mathcal{O}}} & \underline{X}_n^{\mathcal{O}} \end{array}$$

is a push-out square.

Proof. We proceed by the induction on the skeleton of \underline{X} .

$$\begin{aligned} \underline{X}_0^{\mathcal{O}}(T) &= (\coprod_i T_i)^{\mathcal{O}}(T) = \coprod_i ((T_i)^{\mathcal{O}}(T)) = \coprod_i \text{hom}_D(T, T_i) = \\ &= \coprod_i \text{hom}_{\mathcal{O}^{\text{op}}}(F^T, F^{T_i}) = \coprod_i F^{T_i}(T). \end{aligned}$$

Hence the base of the induction.

Suppose we know the claim for \underline{X}_n . Then it follows for \underline{X}_{n+1} since $(\cdot)^{\mathcal{O}}$ is both left and right adjoint, so commutes both with push-outs and products. \square

3.2. D -CW-homology functor. The construction of the (co)homology functor in [1, 4.16] depends on the specific D -CW-decomposition of $\underline{X}^{\mathcal{O}}$. We apply this construction to the cellular structure of $\underline{X}^{\mathcal{O}}$, which was constructed in 3.3 and obtain the required D -CW-homology functor.

3.3. Isotropy ring \mathcal{I} . In [13] a universal coefficient system for the G -equivariant homology have been developed (where G is a compact Lie group). Let us generalize this approach to the coefficient systems for the classical D -homology theory. Suppose \mathcal{O}' is a small, full subcategory of the orbit category \mathcal{O} . Let \underline{X} be a D -space of orbit type \mathcal{O}' . Then a coefficient system for the ordinary (co)homology is a homotopy (co)functor $M : \mathcal{O}' \rightarrow (R - \text{mod})$.

Definition 3.4. Let R be a commutative ring. An *isotropy ring* $\mathcal{I} = I_D^{\mathcal{R}, \mathcal{O}'}$ is generated by $\text{mor}(h\mathcal{O}')$ as a free $R - \text{mod}$. Define the multiplication on the generators by

$$fg = \begin{cases} f \circ g, & \text{if } \text{codom}(g) = \text{dom}(f) \\ 0, & \text{otherwise} \end{cases}$$

and extend the definition to the rest of the elements of \mathcal{I} by linearity.

Proposition 3.5. *The category \mathcal{M} of the left \mathcal{I} -modules which satisfy:*

$$(3) \quad \forall M \in \text{ob}(\mathcal{M}), M = \bigoplus_{T \in \text{ob}(h\mathcal{O}')} 1_T M$$

(where $\{1_T M\}_{T \in \text{ob}(h\mathcal{O}')}$ are left R -modules) and the category of $R(h\mathcal{O}')$ -mod of functors from $h\mathcal{O}'$ to the category of left R -modules are equivalent.

Proof. Let us define a pair of functors which induce the required equivalence:

$$\zeta : \mathcal{M} \rightleftarrows R(h\mathcal{O}')$$

Let $M \in \text{ob}(\mathcal{M})$, $T \in \text{ob}(h\mathcal{O}')$, then define

$$\zeta M(T) = 1_T M.$$

If $\text{mor}(h\mathcal{O}') \ni f : T_1 \rightarrow T_2$, then define

$$\zeta M(f)(1_{T_1} m) = f 1_{T_1} m = (1_{T_2} f) 1_{T_1} m \in 1_{T_2} M.$$

Obviously the morphisms of the left \mathcal{I} modules correspond to the natural transformations of the functors.

Given a $R(h\mathcal{O}')$ -module N , then

$$\xi N = \bigoplus_{T \in \text{ob}(h\mathcal{O}')} N(T), \text{ as a left } R\text{-module.}$$

Define the left \mathcal{I} -module structure on ξN by $f(\dots, n, \dots) = (\dots, fn, \dots)$, where

$$N(\text{codom}(f)) \ni fn = \begin{cases} f(n), & \text{if } n \in N(\text{dom}(f)) \\ 0, & \text{otherwise} \end{cases}$$

Now it is clear that the defined functors provide the equivalence of the categories. \square

Remark 3.6. The ring \mathcal{I} considered as a left \mathcal{I} -module is an object of \mathcal{M} , because $\mathcal{I} \cong \bigoplus_{T \in \text{ob}(h\mathcal{O}')} 1_T \mathcal{I}$ (as left R -modules) by the construction. But it also carries an obvious structure of the right \mathcal{I} module, so the $\zeta \mathcal{I}(T)$.

Remark 3.7. If $\text{ob}(h\mathcal{O}')$ is finite then the ring \mathcal{I} has a two-sided identity element $1 = \sum_{T \in \text{ob}(h\mathcal{O}')} 1_T$ together with its decomposition into the sum of the orthogonal idempotents and the condition (3) is redundant.

Definition 3.8. The augmentation $\varphi : \mathcal{I} \rightarrow \bigoplus_{T \in \text{Iso}(\text{ob}(h\mathcal{O}'))} R$ is defined for any

$$\mathcal{I} \ni g = \sum_{T \in \text{ob}(h\mathcal{O}')} \sum_{f \in \text{mor}(T, T)} r_f f + \sum_{h \in \text{mor}(T_1, T_2), T_1 \neq T_2} s_h h$$

(only a finite number of $r_f, s_h \in R$ is non equal to zero) to be

$$\varphi(g) = (\dots, \sum_{f \in \text{mor}(T, T)} r_f, \dots) \in \bigoplus_{T \in \text{Iso}(\text{ob}(h\mathcal{O}'))} R$$

Remark 3.9. The idempotents in \mathcal{I} which correspond to the D -homotopy equivalent orbits are identified under φ . Apparently, φ is an epimorphism of rings. Consider the abelization functor $Ab : (\text{Rings}) \rightarrow \text{Ab}$ which corresponds to a ring its additive group divided by the commutator subgroup. Then $Ab(\varphi) : Ab(\mathcal{I}) \rightarrow \bigoplus_{T \in \text{Iso}(\text{ob}(h\mathcal{O}'))} R$. The last map will be used to obtain a generalization of the Euler-Poincaré formula.

4. APPLICATIONS

Let \underline{X} be a finite D -CW-complex of type \mathcal{O}' for some orbit category \mathcal{O}' with $\text{obj}(\mathcal{O}')$ a finite set.

4.1. Equivariant Euler-Poincaré formula. We remind that the equivariant Euler characteristic lies in the abelian group $U(D) \cong \bigoplus_{\text{Iso}(\text{obj}(h\mathcal{O}'))} \mathbb{Z}$, so in order to apply Hattori–Stallings machinery we need to choose a coefficient system for the equivariant homology such that the resulting chain complex and homology groups will be endowed with the module structure over some ring S which allows an epimorphism $\varepsilon : Ab(S) \rightarrow U(D)$.

Our choice of the coefficient system for the equivariant homology will be the isotropy ring $\mathcal{I} = I_{\mathbb{D}}^{\mathbb{Z}, \mathcal{O}'}$ taken over itself as a left module.

Lemma 4.1. *Let \underline{X} be a finite D -CW-complex. Suppose \underline{X} has n_q q -dimensional cells and $t_1 + \dots + t_s = n_q$, t_i is the number of q -dimensional cells of the same homotopy type $T_i \in \text{Iso}(\text{obj}(h\mathcal{O}'))$. Then $\mathcal{C}_q(\underline{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I} \cong \zeta\mathcal{I}(T_1)^{t_1} \oplus \dots \oplus \zeta\mathcal{I}(T_s)^{t_s}$ as a left \mathbb{Z} -module.*

Proof. Let $t_i = r_{i1} + \dots + r_{ik}$, where r_{ij} is the number of q -dimensional cells of type $T_{ij} \in \text{obj}(\mathcal{O})$ of homotopy type T_i . By the construction of the equivariant homology $\mathcal{C}_q(\underline{X}) = \bigoplus_{i=1}^s (\bigoplus_{j=1}^k \mathbb{Z}(\text{hom}_{\mathcal{O}'}(?, T_{ij})^{r_{ij}}))$. The dual Yoneda isomorphism [9, p.74] implies:

$$\mathcal{C}_q(\underline{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I} \cong \bigoplus_{i=1}^s (\bigoplus_{j=1}^k \zeta\mathcal{I}(T_{ij})^{r_{ij}}) \cong \bigoplus_{i=1}^s \bigoplus_{j=1}^k (1_{T_{ij}} \mathcal{I})^{r_{ij}},$$

If T_{ij_1} is isomorphic to T_{ij_2} in $h\mathcal{O}'$ then there is an obvious isomorphism of the left \mathbb{Z} -modules and right \mathcal{I} -modules $1_{T_{ij_1}} \mathcal{I} \cong 1_{T_{ij_2}} \mathcal{I}$. Let us choose a representative T_i of each isomorphism class of objects in $h\mathcal{O}'$, then

$$\mathcal{C}_q(\underline{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I} \cong \bigoplus_{i=1}^s (1_{T_i} \mathcal{I})^{(\sum_{j=1}^k r_{ij})} \cong \bigoplus_{i=1}^s (1_{T_i} \mathcal{I})^{t_i} \cong \bigoplus_{i=1}^s (\zeta\mathcal{I}(T_i))^{t_i}$$

□

Because of 3.6 the equivariant chain complex $\{\mathcal{C}_q(\underline{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I}\}_{q=0}^{\dim \underline{X}}$ is a complex of *projective* right \mathcal{I} -modules and the equivariant homology is endowed with the right \mathcal{I} -module structure.

Notation: $\chi_{\text{HS}}(\cdot)$ means Euler characteristic of a \mathcal{I} differential complex with respect to $rk_{\text{HS}}(\cdot)$.

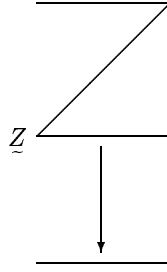
Proposition 4.2. *Let $K_* = \mathcal{C}_*(\underline{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I}$ be a right \mathcal{I} -complex, then $\chi^D(\underline{X}) = Ab(\varphi)(\chi_{\text{HS}}(K_*))$ whenever left side is defined.*

Proof. It is easy to see that $rk_{\text{HS}}(1_T \mathcal{I}) = 1_T \in Ab(\mathcal{I})$. Lemma 4.1 together with 3.9 completes the proof. □

Now we combine 4.2 with the additivity properties of the Hattori–Stallings rank and obtain the following

Theorem 4.3. $\chi^D(\underline{X}) = Ab(\varphi)(\sum_{n=0}^{\infty} (-1)^n rk_{\text{HS}} H_n^D(\underline{X}; \zeta\mathcal{I}))$, whenever the left side is defined.

Example 4.4. Consider the J -diagram:



\underline{Z} has two 0-cells of type $T_2 = [\downarrow]$ and one 1-cell of type $T_3 = [\downarrow]$, hence $\chi^J(\underline{Z}) = 2[\downarrow] - [\downarrow]$.

The category \mathcal{O} of orbits contains two objects: T_2, T_3 . The cellular chain complex tensored with the coefficients $\mathcal{I} = \mathcal{I}_J^{\mathbb{Z}, \{T_2, T_3\}}$ becomes:

$$\cdots \rightarrow 0 \rightarrow 1_{T_3} \mathcal{I} \xrightarrow{\partial_1} (1_{T_2} \mathcal{I})^2$$

and $\partial_1 = 0$ from the orbit type considerations.

$U(J) = \mathbb{Z} \oplus \mathbb{Z}$ in that case. And $H_0^J(\underline{Z}, \mathcal{I}) = (1_{T_2} \mathcal{I})^2$, $H_1^J(\underline{Z}, \mathcal{I}) = 1_{T_3} \mathcal{I}$ are right \mathcal{I} -modules. Hence, $\chi^J(\underline{Z}) = (2, 0) - (0, 1) = (2, -1)$.

Let us, for comparison, calculate the J -equivariant homology of \underline{Z} with $\mathbb{Z}^{\mathcal{O}'}$ coefficients: $H_i^J(\underline{Z}, \mathbb{Z}^{\mathcal{O}'}) = H_i(\text{colim}_J \underline{Z}, \mathbb{Z})$ (see [1, 5.2]). Then $\text{colim}_J \underline{Z} = I = [0, 1]$ and

$$H_i^J(\underline{Z}, \mathbb{Z}^{\mathcal{O}'}) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

We can see that $\mathbb{Z}^{\mathcal{O}'}$ coefficients are inappropriate to the Euler-Poincaré formula.

4.2. Equivariant Lefschetz theorem. Using cellular equivariant homology functor we are able now to prove a version of the equivariant Lefschetz theorem.

Some result of the Lefschetz type in the equivariant setting may be obtained already by applying the ordinary Lefschetz theorem: consider an equivariant map $f : \underline{X} \rightarrow \underline{X}$, where \underline{X} is a diagram over small category D , then if the Lefschetz number $\Lambda(\text{colim}_D \underline{X}) \neq 0$ there are f -invariant D -orbits in \underline{X} . However the advantage of using the equivariant homology and equivariant Lefschetz number $\Lambda_D(\underline{X}) \in U(D)$ is that we obtain the specific information about orbit type of the invariant orbit.

First we give a technical

Definition 4.5. A D -CW-complex \underline{X} will be called the triangulated D -space if the natural CW-structure of $\text{colim} \underline{X}$ also triangulates $\text{colim} \underline{X}$.

The following lemma will be used in the proof of the equivariant Lefschetz theorem.

Lemma 4.6. *Let \underline{X} be a triangulated diagram, then for any refinement Y of the triangulation of $\text{colim} \underline{X}$, there exists a D -CW-complex \underline{X}' , such that \underline{X}' is D -homeomorphic to \underline{X} and $\text{colim} \underline{X}' = Y$ (as the triangulated spaces). \underline{X}' will be called the refinement of \underline{X} .*

Proof. Consider a new simplex Δ in the triangulation of Y . It lies in some old simplex of $\text{colim } \underline{X} : \Delta \in \Delta'$. Then consider the pull-back:

$$\lim \left(\begin{array}{ccc} & \underline{X} & \\ & \downarrow & \\ \Delta & \hookrightarrow & \text{colim}_D \underline{X} \end{array} \right) = T \times \Delta,$$

where T is the orbit which lies over Δ' .

We've obtained the cell of the new D -CW-complex \underline{X}' . Continuing in the same way for the rest of the simplices of Y completes the construction of \underline{X}' . Hence D -CW-complex \underline{X}' has the same underlying topological diagram as \underline{X} , therefore they are D -homeomorphic. \square

Definition 4.7. Let $f : \underline{X} \rightarrow \underline{X}$ be a map of the finite triangulated D -space \underline{X} of orbit type \mathcal{O}' , where \mathcal{O}' is an orbit category with the finite number n of objects. Let $\mathcal{I} = \mathcal{I}_D^{\mathbb{Z}, \mathcal{O}'}$. Then the equivariant Lefschetz number of f :

$$U(D) \ni (\lambda_1, \dots, \lambda_n) = \Lambda_D(f) = \text{Ab}(\varphi) \left(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(H_k(f; \mathcal{I})) \right)$$

Theorem 4.8. Let \underline{X} be a finite triangulated diagram over D . $f : \underline{X} \rightarrow \underline{X}$ be a D -map. $\Lambda_D(f) = (\lambda_1, \dots, \lambda_n) \in U(D)$ - Lefschetz number of f . Then if there is no f -invariant orbit of type T_m , $\lambda_m = 0$.

Proof. A simplex in $\text{colim } \underline{X}$ will be called of type T if the overlaying orbit is of type T in \underline{X} . Then the condition that there are no invariant orbits of type T_m is equivalent to the condition that there are no fixed points in the simplices of type T_m .

Since \underline{X} is a finite triangulated diagram, $\text{colim } \underline{X}$ is a finite triangulated space, hence it is a compact metric space. If there are no fixed points of type T_m , then there exists a refinement Y of the triangulation such that if Δ is a simplex of type T_m in Y , $\Delta \cap (\text{colim } f)(\Delta) = \emptyset$.

Consider the refinement \underline{X}' of \underline{X} , which exists by lemma 4.6. Since $\underline{X}' \cong \underline{X}$, $H_*^D(\underline{X}'; \mathcal{I}) = H_*^D(\underline{X}; \mathcal{I})$, $\Lambda(f') = \Lambda(f)$, where $f' : \underline{X}' \rightarrow \underline{X}'$ is equal to f , $\lambda'_m = \lambda_m$. Therefore, it is enough to show that $\lambda'_m = 0$.

Now,

$$\Lambda_D(f') = \text{Ab}(\varphi) \left(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(H_k(f'; \mathcal{I})) \right) = \text{Ab}(\varphi) \left(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(\mathcal{C}_k(f'; \mathcal{I})) \right),$$

where $\mathcal{C}_k(f'; \mathcal{I})$ is the map induced by f on the chains $\mathcal{C}_k(\underline{X}; \mathcal{I}) = \mathcal{C}_k(\underline{X}) \otimes_{\mathcal{O}' \mathcal{I}} \mathcal{I} = (1_{T_1} \mathcal{I})^{t_1} \oplus \dots \oplus (1_{T_n} \mathcal{I})^{t_n}$ as \mathcal{I} -module. Because of the property: $\Delta \cap (\text{colim } f)(\Delta) = \emptyset$ for *any* simplex Δ of type T_m , the induced map on $\mathcal{C}_k(\underline{X}; \mathcal{I})$ will take the generator 1_{T_m} corresponding to Δ outside the submodule $1_{T_m} \mathcal{I}$, that it generates. Then the m -th entry of $\text{Ab}(\varphi) \left(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(\mathcal{C}_k(f'; \mathcal{I})) \right)$ will be zero. This is true for all k , hence $\lambda_m = 0$. \square

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