A GENERALIZATION OF QUILLEN'S SMALL OBJECT ARGUMENT

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ABSTRACT. We generalize the small object argument in order to allow for its application to proper classes of maps (as opposed to sets of maps in Quillen's small object argument). The necessity of such a generalization arose with the appearance of several important examples of model categories which were proven to be non-cofibrantly generated [2, 7, 8, 17]. Our current approach allows for the construction of functorial factorizations and localizations in the equivariant model category on diagrams of spaces [10] and in two different model structures on the category of pro-spaces [11, 17].

The examples above suggest a natural extension of the framework of cofibrantly generated model categories. We introduce the concept of a class-cofibrantly generated model category, which is a model category generated by classes of cofibrations and trivial cofibrations satisfying some reasonable assumptions.

1. INTRODUCTION

Quillen's definition of a model category has been slightly revised over the last decade. The changes applied to the first axiom MC1 requiring the existence of all finite limits and colimits, and to the last axiom MC5 requiring the existence of factorizations. The modern approaches to the subject [14, 16] demand the existence of all small limits and colimits in MC1. This gives some technical advantages while treating transfinite constructions, such as localizations, in model categories. The modern version of the axiom MC5 requires the factorizations to be functorial.

Functoriality of the factorizations is a very important part of the structure of Quillen's model category. Most examples of model categories have functorial factorizations, and many works on abstract homotopy theory assume that condition. For example, there are two recent constructions of homotopy limits and colimits in abstract model categories equipped with functorial factorizations [5, 14].

The most widely known model category without functorial factorizations is the category of pro-spaces or, more generally, of pro-objects (in the sense of Grothendieck) in a proper model category $\mathcal{C}$ [11, 19] and its Bousfield localization modelling the étale homotopy theory [4, 17]. The construction of functorial factorizations in these model categories was one of our main goals during the work on this paper. However, this task would not be accomplished without an observation that the well-known theorem of C.V. Meyer [21] implies immediately the existence of a *functorial* replacement of a pro-map by a levelwise pro-map. Unfortunately this construction depends on the choice of a functor which is inverse to the equivalence of categories

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constructed by Meyer. It makes our construction less explicit and perhaps not applicable for concrete computations. We provide a partial compensation for this drawback by giving in Appendix A an explicit construction of functorial fibrant replacement in the pro-categories with the strict model structure or, more precisely, by proving that Isaksen's construction of (non-functorial) factorizations becomes functorial when applied just to fibrant replacements. The purpose of the appendix is also to discuss whether the construction of factorizations in [19] can be made functorial, provided that now we have the functorial levelwise replacements. We do not arrive at a definite conclusion to this question, but we show the specific point in the proof which distinguishes between the simpler case of fibrant replacements and the case of general factorizations.

The main tool for the construction of functorial factorizations in model categories and localizations thereof is Quillen's small object argument [14, 16, 22]. However, in its original form, the argument is applicable neither to the category of diagrams with the equivariant model structure [10], nor to pro-categories, since it allows for the application in cofibrantly generated model categories only. We propose here a generalization which may be used in a wider class of model categories. The collections of generating cofibrations and generating trivial cofibrations may now form proper classes, satisfying the conditions of the following theorem:

Theorem 1.1 (The generalized small object argument). *Suppose $\mathcal{C}$ is a category containing all small colimits, and I is a class of maps in $\mathcal{C}$ satisfying the following conditions:*

1. *There exists a cardinal κ, such that each element $A \in \operatorname{dom}(I)$ is κ-small relative to I-cof;*
2. *There exists a functor $S\colon \operatorname{Map}\mathcal{C} \to \operatorname{Map}\mathcal{C}$ equipped with an augmentation $t\colon S \to \operatorname{Id}_{\operatorname{Map}\mathcal{C}}$, such that $S(f) \in I$-cof for every $f \in \operatorname{Map}\mathcal{C}$ and any morphism of maps $i \to f$ with $i \in I$ factors through the natural map $t(f)\colon S(f) \to f$.*

Then there is a functorial factorization (γ, δ) on $\mathcal{C}$ such that, for all morphisms f in $\mathcal{C}$, the map $\gamma(f)$ is in I-cof and the map $\delta(f)$ is in I-inj.

We say that a class I of maps in a category $\mathcal{C}$ *permits the generalized small object argument* if it satisfies conditions (1) and (2) of Theorem 1.1.

This theorem is the second attempt by the author to generalize the small object argument. The previous version appeared in the study of the equivariant localizations of diagrams of spaces [6]. The specific properties of the equivariant model category of D-shaped diagrams of spaces and also the non-functorial factorization technique developed by E. Dror Farjoun in [10] suggested the rather complicated technical notion of instrumentation. It is essentially a straightforward "functorialization" of Dror Farjoun's ideas. The classes of generating cofibrations and generating trivial cofibrations of diagrams satisfy the conditions of instrumentation, but we were unable to apply the same version of the argument to the category of pro-spaces. The conditions of Theorem 1.1 on the class I of maps generalize those of instrumentation, as we explain in Section 3.

Therefore, this paper shows that the two rather different homotopy theories of pro-spaces and of diagrams of spaces fit into a certain joint framework. In order to describe the similarity between the two cases let us give the following

Definition 1.2. A model category $\mathcal{C}$ is called *class-cofibrantly generated* if

(1) there exists a class I of maps in $\mathcal{C}$ (called a class of *generating cofibrations*) that permits the generalized small object argument and such that a map is a trivial fibration if and only if it has the right lifting property with respect to every element of I, and
(2) there exists a class J of maps in $\mathcal{C}$ (called a class of *generating trivial cofibrations*) that permits the generalized small object argument and such that a map is a fibration if and only if it has the right lifting property with respect to every element of J.

Obviously, the class-cofibrantly generated model categories are equipped with functorial factorizations. The categorical dual to a class-cofibrantly generated model category is called *class-fibrantly generated*.

The purpose of this paper is to show that the equivariant model structure on the diagrams of spaces is class-cofibrantly generated and the both known model structures on the category of pro-spaces are class-fibrantly generated.

The applications of Quillen's small object argument are not limited to abstract homotopy theory. A similar argument is used, for example, in the theory of categories to construct reflections in a locally presentable category with respect to a small orthogonality class [3, 1.36]. Recently another generalization of the small object argument was considered by the category theorists J. Adámek , H. Herrlich, J. Rosický and W. Tholen [1]. Their version of the argument applies to the "injective subcategory problem" in locally ranked categories – a generalization of the notion of a locally presentable category which includes topological spaces. We hope that our generalization of the small object argument will be applicable to the "orthogonal subcategory problem" and "injective subcategory problem" with respect to some reasonable classes of morphisms.

The rest of the paper is organized as follows: Section 2 is devoted to the proof of the generalized cosmall object argument. Next, we review some of our previous results about the diagrams of spaces in Section 3 and show how they fit into the newly established framework. After providing the necessary preliminaries on pro-categories in Section 4 we give our main applications of the generalized (co)small object argument in Section 5. Appendix A is devoted to an alternative, explicit construction of a functorial fibrant replacement in pro-$\mathcal{C}$. This construction is based on the construction of factorizations given by D. Isaksen [19]. We also discuss the difficulty which arises while trying to check whether the construction of general factorizations in [19] is functorial.

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2. Proof of the generalized small object argument

Proof of Theorem 1.1. Given a cardinal κ such that every domain of I is κ-small relative to I-cof, we let λ be a κ-filtered ordinal.

To any map $f: X \to Y$ we will associate a functor $Z^f: \lambda \to \mathcal{C}$ such that $Z_0^f = X$, and a natural transformation $\rho^f: Z^f \to Y$ factoring f, i.e., for each $\beta < \lambda$ the triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \nearrow_{\rho^f_\beta} & \\ & Z^f_\beta & \end{array}$$

is commutative. Each map $i^f_\beta \colon Z^f_\beta \to Z^f_{\beta+1}$ will be a pushout of a map of the form $S(f)$, i.e., $i^f_\beta \in I$-cof, since I-cof is closed under pushouts.

We will define Z^f and $\rho^f \colon Z^f \to Y$ by transfinite induction, beginning with $Z^f_0 = X$ and $\rho^f_0 = f$. If we have defined Z^f_α and ρ^f_α for all $\alpha < \beta$ for some limit ordinal β, define $Z^f_\beta = \operatorname{colim}_{\alpha<\beta} Z^f_\alpha$, and define ρ^f_β to be the map induced, naturally, by the ρ^f_α. Having defined Z^f_β and ρ^f_β, we define $Z^f_{\beta+1}$ and $\rho^f_{\beta+1}$ as follows. Consider the natural map $t(\rho^f_\beta) \colon S(\rho^f_\beta) \to \rho^f_\beta$, i.e. the following commutative square:

$$\begin{array}{ccc} A & \xrightarrow{\operatorname{dom}(t(\rho^f_\beta))} & Z^f_\beta \\ {\scriptstyle S(\rho^f_\beta)}\big\downarrow & & \big\downarrow{\scriptstyle \rho^f_\beta} \\ B & \xrightarrow[\operatorname{codom}(t(\rho^f_\beta))]{} & Y. \end{array}$$

Define $Z^f_{\beta+1}$ to be the pushout of this diagram and define $\rho^f_{\beta+1}$ to be the map naturally induced by ρ^f_β.

For each morphism $g = (g^1, g^2) \colon f_1 \to f_2$ in the category $\operatorname{Map}\mathcal{C}$, i.e., for each commutative square

$$\begin{array}{ccc} X_1 & \xrightarrow{g^1} & X_2 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{g^2} & Y_2 \end{array}$$

we define a natural transformation $\xi^g \colon Z^{f_1} \to Z^{f_2}$ by transfinite induction over small ordinals, beginning with $\xi^g_0 = g^1$. If we have defined ξ^g_α for all $\alpha < \beta$ for some limit ordinal β, define $\xi^g_\beta = \operatorname{colim}_{\alpha<\beta} \xi^g_\alpha$. Having defined ξ^g_β, we define $\xi^g_{\beta+1} \colon Z^{f_1}_{\beta+1} \to Z^{f_2}_{\beta+1}$ to be the *natural* map induced by $g_\beta = (\xi^g_\beta, g^2) \colon \rho^{f_1}_\beta \to \rho^{f_2}_\beta$, namely the *unique* map between the pushouts of the horizontal lines of the following diagram which preserves its commutativity:

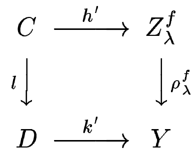
$$\begin{array}{ccccc} B_1 & \xleftarrow{S(\rho^{f_1}_\beta)} & A_1 & \xrightarrow{\operatorname{dom}(t(\rho^{f_1}_\beta))} & Z^{f_1}_\beta \\ {\scriptstyle h_2}\big\downarrow & & {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle \xi^g_\beta} \\ B_2 & \xleftarrow[S(\rho^{f_2}_\beta)]{} & A_2 & \xrightarrow[\operatorname{dom}(t(\rho^{f_2}_\beta))]{} & Z^{f_2}_\beta. \end{array}$$

In this diagram $(h_1, h_2) = S(g_\beta)$. The commutativity of the diagram follows readily, since S is a functor and t is a natural transformation.

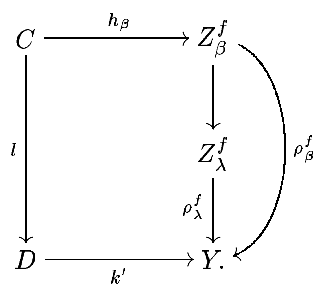
The required functorial factorization (γ, δ) is obtained when we reach the limit ordinal λ in the course of our induction. Then we define $\gamma(f) \colon X \to Z^f_\lambda$ to be the (transfinite) composition of the pushouts, and $\delta(f) = \rho^f_\lambda \colon Z^f_\lambda \to Y$. $\gamma(f) \in I$-cof since I-cof is closed under transfinite compositions.

To complete the definition of the functorial factorization (see [16, 1.1.1], [15, 1.1.1]) we need to define for each morphism $g\colon f_1 \to f_2$ a natural map $(\gamma,\delta)^g \colon Z^{f_1}_\lambda \to Z^{f_2}_\lambda$ which makes the appropriate diagram commutative. Take $(\gamma,\delta)^g = \xi^g_\lambda$.

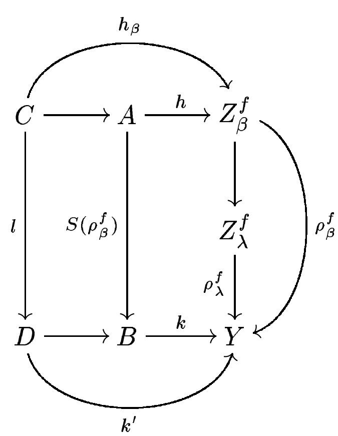
It remains to show that $\delta(f) = \rho^f_\lambda$ has the right lifting property with respect to I. To see this, suppose we have a commutative square as follows:



where l is a map of I. Due to the first condition of the theorem the object C is κ-small relative to I-cof, i.e., there is an ordinal $\beta < \lambda$ such that h' is the composite $C \xrightarrow{h_\beta} Z^f_\beta \longrightarrow Z^f_\lambda$. Hence we obtain the following commutative diagram:



The second condition of the theorem implies that there exists a factorization in the category $\operatorname{Map}\mathcal{C}$ of the map (h_β, k') through $t(\rho^f_\beta)$ which is a map of maps with domain $S(\rho^f_\beta)\colon A \to B$ and range ρ^f_β, i.e., there is a commutative diagram



where $(h,k) = t(\rho^f_\beta)$.

By construction, there is a map $B \xrightarrow{k_\beta} Z^f_{\beta+1}$ such that $k_\beta g = i_\beta h$ and $k = \rho^f_{\beta+1} k_\beta$, where i_β is the map $Z^f_\beta \to Z^f_{\beta+1}$. The composition $D \longrightarrow B \xrightarrow{k_\beta} Z^f_{\beta+1} \longrightarrow Z^f_\lambda$ is the required lift in the initial commutative square. $\square$

Remark 2.1. In all the applications we have in mind, the map $S(f)$ is a coproduct of maps from I. Hence the construction above provides us with the factorization of any map f into an I-cellular map followed by an I-injective map, as in the classical construction. But we prefer to leave the formulation of conditions on the class I of maps in the present (simpler) form, since we hope that they will be useful elsewhere and we do not see a big advantage in I-cellular maps instead of I-cofibrations.

3. Diagrams of spaces with the equivariant model structure

It was essentially shown in [6] that the category of D-shaped diagrams of spaces (by the category of spaces we mean here either the category of simplicial sets, or the category of compactly generated topological spaces with the standard simplicial model structure) is class-cofibrantly generated. In this section we show that the classes of generating cofibrations and generating trivial cofibrations permit (the new version of) the generalized small-object argument.

Let us recall first the definition of the equivariant model structure on $\mathcal{S}^D$ initially introduced in [10]. We use the word *collection* to denote a set or a proper class with respect to some fixed universe $\mathfrak{U}$. A D-diagram $\underset{\sim}{O}$ of spaces is called an *orbit* if $\operatorname{colim}_D \underset{\sim}{O} = *$. We denote by $\mathcal{O}_D$ the collection of all orbits of D (which is not necessarily a set). For any diagram $\underset{\sim}{W}$ and a map $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$, there is an induced map of simplicial sets $\operatorname{map}(\underset{\sim}{W}, f)\colon \operatorname{map}(\underset{\sim}{W}, \underset{\sim}{X}) \to \operatorname{map}(\underset{\sim}{W}, \underset{\sim}{Y})$; see [9] for details.

Definition 3.1. In the equivariant model structure on $\mathcal{S}^D$ a morphism $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ is a

- *weak equivalence* if and only if $\operatorname{map}(\underset{\sim}{O}, f)$ is a weak equivalence of spaces for any orbit $\underset{\sim}{O}$;
- *fibration* if and only if $\operatorname{map}(\underset{\sim}{O}, f)$ is a fibration of spaces for any orbit $\underset{\sim}{O}$;
- *cofibration* if and only if it has the left lifting property with respect to any trivial fibration.

The standard axioms of simplicial model categories were verified in [10] for the equivariant model structure on $\mathcal{S}^D$. Functorial factorizations were constructed in [6]. In that construction we used a different version of the generalized small-object argument, which applied only to instrumented classes of maps. The purpose of this section is to prove Proposition 3.2,which shows that Theorem 1.1 provides a more general version of the argument.

Let $I = \{\underset{\sim}{T} \otimes \partial\Delta^n \hookrightarrow \underset{\sim}{T} \otimes \Delta^n \mid \underset{\sim}{T} \in \mathcal{O}_D,\, n \geq 0\}$ and $J = \{\underset{\sim}{T} \otimes \Lambda^n_k \xrightarrow{\sim} \underset{\sim}{T} \otimes \Delta^n \mid \underset{\sim}{T} \in \mathcal{O}_D,\, n \geq k \geq 0\}$ be two classes of maps in $\mathcal{S}^D$. If the index category D is such that $\mathcal{O}_D$ is a set (this happens, for example, when D is a group), then the collections I and J are *sets* of cofibrations and the equivariant model structure on $\mathcal{S}^D$ is cofibrantly generated with I equal to the set of generating cofibrations and J equal to the set of generating trivial cofibrations. But usually I and J are proper classes of maps, and it was shown in [6] that they form classes of generating cofibrations and generating trivial cofibrations in the equivariant model structure on the D-shaped diagrams of spaces, which is class-cofibrantly generated. Our aim here is to show that the classes I and J permit the generalized small object argument. This follows from Proposition 3.2, since the classes I and J are both instrumented.

Let us recall, in an informal manner, the notion of instrumentation introduced in [6]. Instrumentation for a class I of maps in a category $\mathcal{C}$ is a formalization of the following functorial version of the classical cosolution-set condition: for any morphism f in $\mathcal{C}$ there is a *naturally* assigned set of maps $\mathcal{I}(f) = \{i \to f \mid i \in I\}$, such that for any morphism of maps $j \to f$ with $j \in I$ there exists a factorization $j \to i \to f$ with $(i \to f) \in \mathcal{I}(f)$. Additionally, every domain of a map in I is κ-small with respect to I-cell for some fixed cardinal κ.

Proposition 3.2. *Any instrumented class of maps I in a category $\mathcal{C}$ permits the generalized small object argument.*

Proof. The first condition of Theorem 1.1 is satisfied because of the same assumption for instrumented classes of maps.

Instrumentation gives rise to the augmented functor S in the following way:

$$S(f) = \coprod \mathrm{dom}(\mathcal{I}(f)) = \coprod \{i \mid (i \to f) \in \mathcal{I}(f)\}\,.$$

Naturality of $\mathcal{I}$ ensures the functoriality of S. The augmentation $t_f \colon S(f) \to f$ exists since every i is equipped with a map into f, hence their coproduct is naturally mapped into f. Certainly $S(f) \in I$-cof, and the factorization property follows from the similar property of instrumentation. □

4. Preliminaries on Pro-Categories

Definition 4.1. A small, non-empty category I is **cofiltering** if for every pair of objects i and j there exists an object k together with maps $k \to i$ and $k \to j$; and for every pair of morphisms $i \overset{f}{\underset{g}{\rightrightarrows}} j$ there exists a map $h \colon k \to i$ with $fh = gh$. A diagram is said to be **cofiltering** if its indexing category is so.

For a category $\mathcal{C}$, the category **pro-$\mathcal{C}$** has objects all cofiltering diagrams in $\mathcal{C}$, and the set of morphisms from a pro-object X indexed by a cofiltering T into a pro-object Y indexed by a cofiltering S is given by the following formula:

$$\mathrm{Hom}_{\text{pro-}\mathcal{C}}(X, Y) = \lim_s \operatorname*{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s).$$

A **pro-object** (**pro-morphism**) is an object (morphism) of pro-$\mathcal{C}$. The structure maps of diagrams which represent pro-objects are also called **bonding** maps.

This definition of morphisms in a pro-category requires, perhaps, some clarification. By definition, a pro-map $f \colon X \to Y$ is a compatible collection of maps $\{f_s \colon X \to Y_s \mid s \in S,\ f_s \in \mathrm{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s)\}$. By construction of direct limits in the category of sets, $\mathrm{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s)$ is the set of equivalence classes of the elements of $\coprod_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s)$. If for each equivalence class f_s we choose a representative $f_{t,s} \colon X_t \to Y_s$, then we obtain a representative of the pro-morphism f. This motivates the following alternative characterization of the morphisms in pro-$\mathcal{C}$ (cf. [11, 2.1.2]):

Definition 4.2. A **representative** of a morphism from a pro-object X indexed by a cofiltering I to a pro-object Y indexed by a cofiltering K is a function $\theta \colon K \to I$ (not necessarily order-preserving) and morphisms $f_k \colon X_{\theta(k)} \to Y_k$ in $\mathcal{C}$ for each $k \in K$ such that if $k \le k'$ in K, then for some $i \in I$ with $i \ge \theta(k)$ and $i \ge \theta(k')$ the following diagram commutes:

$$\begin{array}{ccccc} X_i & \xrightarrow{b_{i,\theta(k')}} & X_{\theta(k')} & \xrightarrow{f'_k} & Y_{k'} \\ & \searrow^{b_{i,\theta(k)}} & & & \downarrow b_{k',k} \\ & & X_{\theta(k)} & \xrightarrow{f_k} & Y_k . \end{array}$$

A representative $(\phi, \{g_k\})$ **rarefies** the representative $(\theta, \{f_k\})$ if for every $k \in K$, $\phi(k) \geq \theta(k)$ and there exists a bonding map $b_{\phi(k),\theta(k)} \colon X_{\phi(k)} \to X_{\theta(k)}$ with $g_k = f_k \circ b_{\phi(k),\theta(k)}$. Two representatives $(\theta, \{f_k\})$ and $(\theta', \{f'_k\})$ are called **equivalent** if there exists a representative $(\theta'', \{f''_k\})$ which rarefies both of them.

A representative $\{\theta, \{f_k\}\}$ is called **strict** [12, p. 36] if θ is a functor and the maps $\{f_k \colon X_{\theta(k)} \to Y_k \mid k \in K\}$ constitute a natural transformation $f \colon X \circ \theta \to Y$. In other words, the representative of a morphism is strict if all f_k fit into commutative squares of the following form:

$$\begin{array}{ccc} X_{\theta(k')} & \xrightarrow{f_{k'}} & Y_{k'} \\ {\scriptstyle b_{\theta(k'),\theta(k)}} \downarrow & & \downarrow {\scriptstyle b_{k',k}} \\ X_{\theta(k)} & \xrightarrow{f_k} & Y_k \end{array}$$

The proof of the following standard proposition may be found, for example, in [20, Ch. 1§1].

Proposition 4.3. *The relation between representatives defined above is an equivalence relation. The equivalence classes of representatives of morphisms from a pro-object X into a pro-object Y are in natural bijective correspondence with the elements of* $\mathrm{Hom}_{\text{pro-}\mathcal{C}}(X, Y)$.

An important technique in pro-categories is reindexing. We use two types of reindexing: for maps and for objects. Their crucial property is functoriality.

The following theorem, proven in [21], will provide us with a functorial choice of a levelwise representative of a pro-morphism:

Theorem 4.4 (C. V. Meyer)**.** *Let $\mathcal{C}$ be any category, and let*

$$F \colon \text{pro-}(\mathrm{Map}\,\mathcal{C}) \longrightarrow \mathrm{Map}(\text{pro-}\mathcal{C})$$

be the obvious functor. Then F is fully faithful and essentially surjective, i.e., the categories pro-(Map $\mathcal{C}$) *and* Map(pro-$\mathcal{C}$) *are equivalent.*

Fix once and for all the functors which induce the equivalences:

$$F \colon \text{pro-}(\mathrm{Map}\,\mathcal{C}) \leftrightarrows \mathrm{Map}(\text{pro-}\mathcal{C}) : G.$$

Beware that the pro-objects of [21] are indexed by cofiltered categories that are not necessarily small. In this paper we consider only small indexing categories. Nevertheless, Theorem 4.4 is still true, as explained in [18, 3.1, 3.5].

Corollary 4.5. *There exists a functor L:* Map(pro-$\mathcal{C}$) $\to$ Map(pro-$\mathcal{C}$) *naturally isomorphic to the identity satisfying the following property. For every $f \in$* Map(pro-$\mathcal{C}$) *the domain and the range of $L(f)$ are indexed by the same indexing category I and there exists a strict representative $\{\theta, \{f_k\}\}$ of f with $\theta = \mathrm{id}_I$. In other words, f has a levelwise representative.*

Proof. Take $L = FG$. □

We are going to use inductive arguments, therefore we need a *functorial* reindexing result that will allow us to employ induction on the number of predecessors.

Definition 4.6. A partially ordered set I is **directed** if for any $i, i' \in I$ there exists $i'' \in I$ with $i'' \geq i$ and $i'' \geq i'$; I is called **strongly** directed if $i \geq i'$ and $i \leq i'$ implies $i = i'$. A directed set I is called **cofinite** if every $i \in I$ has only a finite number of predecessors.

The following theorem [11, 2.1.6][20, p. 15, Thm. 4] supplies us with the required reindexing for pro-objects:

Theorem 4.7. *There exists a functor $M\colon \text{pro-}\mathcal{C} \to \text{pro-}\mathcal{C}$ (called the Mardešić functor), naturally equivalent to the identity, such that $M(X)$ is indexed by a cofinite strongly directed set for every X in* pro-$\mathcal{C}$.

The following proposition (see [20, p. 9, Lemma 2] for the proof) allows us to assume that whenever our pro-objects are indexed by cofinite strongly directed sets we may choose a strict representatives for each morphism.

Proposition 4.8. *Let $f\colon X \to Y$ be a pro-map. If X and Y are indexed by directed sets and the indexing category of Y is cofinite and strongly directed, then there exists a strict representative $\{\theta, \{f_k\}\}$ of f.*

5. Applications of the small object argument: Functorial factorizations in pro-$\mathcal{C}$

We discuss in this section the application of the small object argument, or more precisely of its dual, to the two different model categories on the category of pro-spaces. The strict model structure was constructed by D.A. Edwards and H.M. Hastings [11]. Weak equivalences and cofibrations are essentially levelwise in the strict model structure. The properness of $\mathcal{C}$ is the only condition required for the existence of the strict model structure on pro-$\mathcal{C}$ [19]. The localization of the strict model category on the category of pro-(simplicial sets) with respect to the class of maps rendered into strict weak equivalences by the functor P (the functor which replaces a space with its Postnikov tower) was constructed by D. Isaksen [17]. Weak equivalences in Isaksen's model structure generalize those of Artin-Mazur [4] and J.W. Grossman [13].

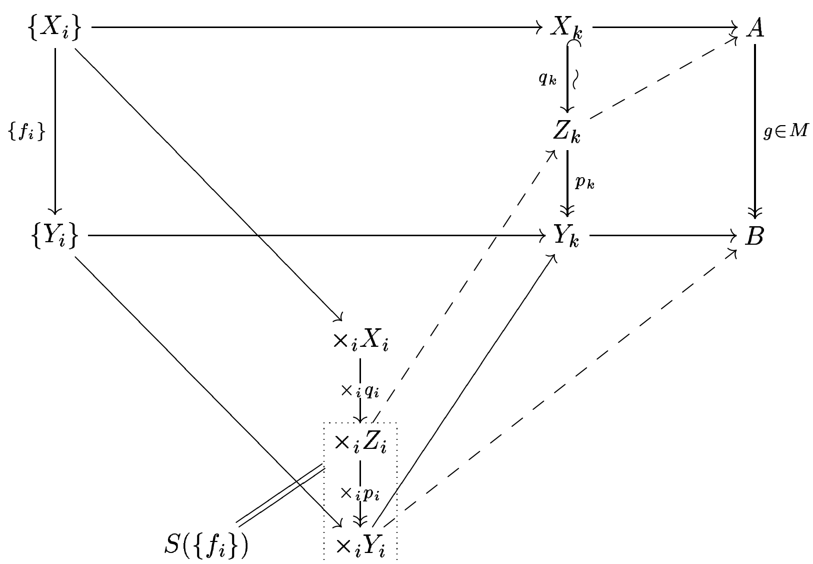
Theorem 5.1. *Let $\mathcal{C}$ be a proper model category. Then the strict model structure on* pro-$\mathcal{C}$ *may be equipped with functorial factorizations.*

Proof. We use the dual of the generalized small object argument, i.e., the generalized cosmall object argument, in order to construct the factorizations. Let M and N be the classes of fibrations and trivial fibrations between constant pro-objects respectively. Then the class of trivial cofibrations equals M-proj and the class of cofibrations between pro-objects equals N-proj [19, 5.5]. It was shown in [18] that every constant pro-object is countably cosmall.

In order to apply the generalized cosmall object argument on the class M, we need a functor $S\colon \operatorname{Map}\mathcal{C} \to \operatorname{Map}\mathcal{C}$ equipped with a coaugmentation $t\colon \mathrm{Id}_{\operatorname{Map} C} \to S$ such that for every $f \in \operatorname{Map}\mathcal{C}$, $S(f)$ is an I-cofibration and any morphism of maps $f \to i$ with $i \in I$ factors through $t(f)\colon f \to S(f)$.

We start by replacing f with a *naturally* isomorphic levelwise representative $\{f_i\}_{i\in I}$ for some cofiltering I; this is possible by Corollary 4.5. Next, for every $i \in I$, we factorize f_i into a trivial cofibration q_i followed by a fibration p_i using the functorial factorizations of the model category $\mathcal{C}$. Then define $S(f) = \times_i p_i$.

For every $i \in I$ there exists a pro-map $\varphi_i \colon \{X_i\} \to X_i$ defined by the strict representative $\{\theta(i) = i, \{f_i = id_{X_i} \colon X_i \to X_i\}\}$. The same is true for $\{Y_i\}$. These maps define the canonical maps $\{X_i\} \to \times_i X_i$ and $\{Y_i\} \to \times_i Y_i$. Composition of the first map with $\times_i q_i$ finishes the definition of the natural map $t_{\{f_i\}} \colon \{f_i\} \to S(\{f_i\})$.



In order to verify the factorization property, fix an arbitrary map $\{f_i\} \to g$ with $g \in M$ being a fibration between constant objects. It follows immediately from the definition of the morphism set between two pro-objects that any map into a constant pro-object factors through a map of the form φ_i for some $i \in I$. Applying this to the category pro-Map($\mathcal{C}$) $\cong$ Map(pro-$\mathcal{C}$), we find an index $k \in I$ such that the fixed map $\{f_i\} \to g$ factors through $X_k \to Y_k$.

In the diagram above the maps $\times_i Z_i \to Z_k$ and $\times_i Y_i \to Y_k$ are projections. The dashed map $\times_i Y_i \to B$ is the composition of the projection with the map $Y_k \to B$. Finally, the dashed map $Z_k \to A$ is a lifting in the commutative square, which exists in the model category $\mathcal{C}$.

Now we are able to apply the generalized cosmall object argument to produce for every map in pro-$\mathcal{C}$ its functorial factorization into a trivial cofibration followed by a fibration.

To obtain the second factorization we repeat the construction above for the class N of trivial fibrations between constant objects and factorize all the maps $X_i \to Y_i$ into cofibrations followed by trivial fibrations.

Hence the cosmall object argument may be applied to provide every map f with a functorial factorization into a cofibration followed by a trivial fibration in pro-$\mathcal{C}$. □

Let $\mathcal{S}$ denote the category of simplicial sets with the standard model structure. Our next goal is to construct functorial factorizations in the localized model structure of [17]. From now on the words *cofibration*, *fibration* and *weak equivalence* refer to Isaksen's model structure.

Since the procedure of (left Bousfield) localization preserves the class of cofibrations and, hence, the class of trivial fibrations, the functorial factorization into a cofibration followed by a trivial fibration was constructed in the theorem above. We keep the class N of generating trivial fibrations the same as in the strict model structure. The class of generating fibrations L is defined to be the class of all co-n-fibrations (see [17, Def. 3.2]) between constant pro-objects for all $n \in \mathbb{N}$.

Proposition 5.2. *The class of trivial cofibrations equals* L-proj.

Proof. Every element of L-proj is a strong fibration [17, Def. 6.5]; therefore every trivial cofibration has the left lifting property with respect to L by [17, Prop. 14.5]. Conversely, if a map i has the left lifting property with respect to L, then i has the left lifting property with respect to the class L' of all retracts of L-cocell complexes. By [19, Prop.5.2] L' contains all strong fibrations. But then [17, Prop. 6.6] implies that L' contains all fibrations. Therefore, i must be a trivial cofibration. □

Theorem 5.3. *Isaksen's model structure on* pro-$\mathcal{S}$ *may be equipped with functorial factorizations.*

Proof. It suffices to construct a functorial factorization of every morphism of pro-$\mathcal{S}$ into a trivial cofibration followed by a fibration. We apply the same construction as in Theorem 5.1 to the class L, except for the factorizations of the levelwise representation $\{f_i\}$.

Apply first the Mardešić functor in order to guarantee that our pro-system is indexed by a cofinite strongly directed set. Since the Mardešić functor is naturally isomorphic to the identity, we abuse notation and keep calling the indexing category I. We construct the factorizations of the maps f_i by induction on the number $n(i)$ of predecessors of i and factor f_i into an $n(i)$-cofibration q_i followed by a co-$n(i)$-fibration, which is possible by [17, Prop. 3.3]. This an induction on the set of natural numbers, since I is now cofinite.

For any element $g\colon A \to B$ of L, there is a number $n \in \mathbb{N}$ such that g is a co-n-fibration. We may always enlarge k such that $n(k) \geq n$ and hence q_k will be an n-cofibration by [17, Lemma 3.6]. Finally, the lift in the commutative square in the diagram in the proof of Theorem 5.1 exists by [17, Def. 3.2]. □

Appendix A. An explicit construction of a functorial fibrant replacement in pro-$\mathcal{C}$

The purpose of this appendix is to give an explicit construction of functorial fibrant replacements in the strict model category on pro-$\mathcal{C}$. More precisely, the purpose is to prove that the construction of fibrant replacements in [19] is functorial. We do not know whether the construction of arbitrary factorizations is functorial in [19], but in view of Remark A.2 it seems unlikely.

Proposition A.1. *If $\mathcal{C}$ is a proper model category with functorial factorizations, then there exists a functorial fibrant replacement $X \stackrel{\sim}{\hookrightarrow} \hat{X}$ in the category pro-$\mathcal{C}$ with the strict model structure.*

Proof. Given a pro-object X, we apply first the Mardešić functor in order to replace it by an isomorphic pro-object $M(X)$ indexed by a cofinite strongly directed set. Since $M(\,\cdot\,)$ is naturally isomorphic to the identity functor, we suppress the notation and quietly assume that all pro-objects are indexed by cofinite strongly directed sets.

Our objective is to find for every pro-object X indexed by a cofinite strongly directed set S, a functorial factorization of the map $f\colon X \to *$ into a trivial cofibration $i\colon X \overset{\sim}{\hookrightarrow} \hat{X}$ followed by a fibration $p\colon \hat{X} \twoheadrightarrow *$ in the strict model structure on pro-$\mathcal{C}$. We apply the factorization algorithm of [19], with a mild alteration, on the simplest level representation of f: $(id_S, \{X_s \to *\})$.

We define $\hat{X}$ by induction. The only difference between the current construction and [11, §4.3] [19] is our *vision* of the bonding maps of $\hat{X}$. In the original construction they were induced by the fibrations p_k below.

For every $s_0 \in S$ such that there are no $s \in S$ with $s \leq s_0$, define $\hat{X}_{s_0}$ by applying the functorial factorization of $\mathcal{C}$ on the map $X_{s_0} \to *$:

$$X_{s_0} \underset{i_{s_0}}{\overset{\sim}{\hookrightarrow}} \hat{X}_{s_0} \underset{p_{s_0}}{\twoheadrightarrow} \lim_{\{s<s_0\}=\emptyset} \hat{X}_s = *.$$

Suppose for induction that $\hat{X}_l$, the maps $i_l\colon X_l \to \hat{X}_l$, and the bonding maps $\hat{b}_{l,m}\colon \hat{X}_l \to \hat{X}_m$ have already been defined for $k > l > m$. We define $\hat{X}_k$ and i_k by applying the functorial factorization of $\mathcal{C}$ on the map $X_k \to \lim_{s<k} \hat{X}_s \times_* * = \lim_{s<k} \hat{X}_s$:

$$X_k \underset{i_k}{\overset{\sim}{\hookrightarrow}} \hat{X}_k \underset{p_k}{\twoheadrightarrow} \lim_{s<k} \hat{X}_s.$$

For each $l < k$ the bonding map $\hat{b}_{k,l}$ is induced by functorial factorizations in the following commutative diagram:

$$\begin{array}{ccccc}
X_k & \underset{i_k}{\overset{\sim}{\hookrightarrow}} & \hat{X}_k & \underset{p_k}{\twoheadrightarrow} & \lim_{s<k} \hat{X}_s \\
\downarrow{\scriptstyle b_{k,l}} & & \downarrow{\scriptstyle \hat{b}_{k,l}} & \swarrow{\scriptstyle a_l} & \downarrow{\scriptstyle c_l} \\
X_l & \underset{i_l}{\overset{\sim}{\hookrightarrow}} & \hat{X}_l & \underset{p_l}{\twoheadrightarrow} & \lim_{s<l} \hat{X}_s,
\end{array}$$

where the map a_l is a part of the limit cone and the map c_l is induced naturally by the previously defined bonding maps.

We have to show that $\hat{b}_{k,l} = a_l p_k$. Consider the totality of maps from X_k to X_l for all $l < k$. They induce the commutative cone $\{\hat{b}_{k,l}\colon \hat{X}_k \to \hat{X}_l \mid l < k\}$ and hence the natural map $q_k\colon \hat{X}_k \to \lim_{s<k} \hat{X}_s$. The commutativity of the cone follows from the inductive assumption that all the previously defined bonding maps were natural. Then $\hat{b}_{k,l} = a_l q_k$. But $\mathcal{C}$ is cofibrantly generated, i.e., $\hat{X}_k$ equal the colimit of the sequence $X_k = E_0 \to E_1 \to \cdots \hat{X}_k$. Every E_t is naturally mapped into $\hat{X}_l$ for each $l < k$. Therefore there exists a natural map $E_t \to \lim_{s<k} \hat{X}_s$ which equals the composition of $E_t \to \operatorname{colim}_r E_r = \hat{X}_k$ with q_k. Then the universal property of colimit implies that $p_k = q_k$.

Hence the newly defined bonding maps coincide with those introduced in [11, §4.3] [19]. The advantage of the fresh look at the bonding maps is that now it is easy to verify the functoriality of the construction.

Given a map $f\colon X \to Y$ with the range indexed by a cofinite strongly directed set K (upon application of the Mardešić functor), choose a strict representative $(\theta, \{f_k\colon X_{\theta(k)} \to Y_k\})$. The map $\hat{f}\colon \hat{X} \to \hat{Y}$ given by the representative $(\theta, \{\hat{f}_k\colon \hat{X}_{\theta(k)} \to \hat{Y}_k\})$ is defined inductively. Suppose that $\hat{f}_j$ has been already defined for all $j < k$. Then $\hat{f}_k$ is the functorially induced map in the following commutative diagram:

$$\begin{array}{ccccc} X_{\theta(k)} & \overset{\sim}{\hookrightarrow} & \hat{X}_{\theta(k)} & \twoheadrightarrow & \lim_{s<\theta(k)} \hat{X}_s \\ {\scriptstyle f_k}\downarrow & & {\scriptstyle \hat{f}_k}\downarrow & & \downarrow{\scriptstyle g_k} \\ Y_k & \overset{\sim}{\hookrightarrow} & \hat{Y}_k & \twoheadrightarrow & \lim_{j<k} \hat{Y}_j, \end{array}$$

where g_k is the natural map induced by $\hat{f}_j$ for $j < k$.

We have to verify that $\hat{f}$ is well defined, i.e., it does not depend on the choice of the representative of f. It will suffice to show that for any representative $(\theta', \{f'_k\colon X_{\theta'(k)} \to Y_k\})$ of f that rarefies the representative $(\theta, \{f_k\colon X_{\theta(k)} \to Y_k\})$, the map $\hat{f}'$ that is assigned to the representative $(\theta', \{\hat{f}'_k\})$ is equal to f. But this is obvious since $(\theta', \{\hat{f}'_k\})$ rarefies $(\theta, \{\hat{f}_k\})$:

$$\hat{f}'_k = \hat{f}_k \circ \hat{b}_{\theta'(k),\theta(k)}$$

because all three maps are induced by the *functorial* factorizations in $\mathcal{C}$.

It remains to prove that $\hat{id}_X = id_{\hat{X}}$ and if $f = gh$, then $\hat{f} = \hat{g}\hat{h}$.

To show the first equality, choose the representative $(id_S, \{id_{X_s}\colon X_s \to X_s\})$ of id_X. The construction of $\hat{id}_X$ above produces a representative $(id_S, \{\hat{id}_{X_s}\colon \hat{X}_s \to \hat{X}_s\})$ where $\hat{id}_{X_s} = id_{\hat{X}_s}$ by the functorial naturality of the factorizations of $\mathcal{C}$.

To show the second equality, choose strict representatives $(\theta, \{h_l\colon X_{\theta(l)} \to Y_l\})$ of h and $(\phi, \{g_k\colon Y_{\phi(k)} \to Z_k\})$ of g. Take $(\theta \circ \phi, \{f_k = g_k \circ h_{\phi(k)}\colon X_{\theta(\phi(k))} \to Z_k\})$ to be the representative of f. Then the functoriality of the factorizations in $\mathcal{C}$ implies that $\hat{f}_k = \hat{g}_k \circ \hat{h}_{l_k}$. Therefore $\hat{f} = \hat{g}\hat{h}$. □

Remark A.2. One can try to extend this proof for the factorization of an arbitrary map $f\colon X \to Y$ and not just $X \to *$ as we do here. The point in the current proof, which we do not know how to generalize, is the following. We would like to prove that the naturally induced bonding maps coincide with the maps that arise from the factorizations $X_s \to Z_s \to Y_s \times_{\lim_{t<s} Y_t} \lim_{t<s} Z_t$. But our argument does not work since we do not know how to present the second map in an alternative way. There is no natural map $Z_s \to Y_s$ which arises from the previously defined maps. In our case $Y_s = *$.

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