DERIVED CATEGORIES AND PROJECTIVE CLASSES

J. DANIEL CHRISTENSEN

ABSTRACT. An important example of a model category is the category of unbounded chain complexes of R-modules, which has as its homotopy category the derived category of the ring R. This example shows that traditional homological algebra is encompassed by Quillen's homotopical algebra. The goal of this paper is to show that more general forms of homological algebra also fit into Quillen's framework. Specifically, any set of objects in a complete and cocomplete abelian category $\mathcal{A}$ generates a projective class on $\mathcal{A}$, which is exactly the information needed to do homological algebra in $\mathcal{A}$. The main result is that if the generating objects are "small" in an appropriate sense, then the category of chain complexes of objects of $\mathcal{A}$ has a model category structure which reflects the homological algebra of the projective class. The motivation for this work is the construction of the "pure derived category" of a ring R. Finally, we explain how the category of simplicial objects in a possibly non-abelian category can be equipped with a model category structure reflecting a given projective class.

CONTENTS

INTRODUCTION

An important example of a model category is the category Ch of unbounded chain complexes of R-modules, which has as its homotopy category the derived category $\mathcal{D}$ of the ring R. The formation of a projective resolution is an example of cofibrant replacement, and traditional derived functors are examples of derived functors in the model category sense. This example shows that traditional homological algebra is encompassed by Quillen's homotopical algebra, and indeed this unification was one of the main points of Quillen's work [14].

The goal of this paper is to illustrate that more general forms of homological algebra also fit into Quillen's framework. In any abelian category $\mathcal{A}$ there is a natural notion of "projective object" and "exact sequence." However, it is sometimes useful to impose different definitions of these terms.

If this is done in a way that satisfies some natural axioms, what is obtained is a "projective class," which is exactly the information needed to do homological algebra in $\mathcal{A}$. Our main result is that given any projective class satisfying a set-theoretic hypothesis, the category of chain complexes of objects of $\mathcal{A}$ has a model category structure which reflects the homological algebra of the projective class. The motivation for this work is the construction of the "pure derived category" of a ring R. Pure homological algebra has applications to phantom maps in the stable homotopy category and in the (usual) derived category of a ring, and these connections will be described.

When $\mathcal{A}$ has enough projectives, the projective objects and exact sequences form a projective class. Therefore the results of this paper apply to traditional homological algebra as well. Even in this special case, it is not a trivial fact that the category of *unbounded* chain complexes can be given a model category structure, and indeed Quillen restricted himself to the bounded below case. I know of three other written proofs that the category of unbounded chain complexes is a model category [7, 8, 10], which do the case of R-modules, but this was probably known to others as well.

An important corollary of the fact that a derived category $\mathcal{D}$ is the homotopy category of a model category is that $\mathcal{D}(X, Y)$ is a set (as opposed to a proper class) for any two chain complexes X and Y. This is not the case in general, and much work on derived categories ignores this possibility. It is not just a pedantic point; if one uses the morphisms in the derived category to index constructions in other categories or to define cohomology groups, one needs to know that the indexing class is actually a set. Recently, the unbounded case has been handled under various assumptions on $\mathcal{A}$. (See Weibel [16] Remark 10.4.5, which credits Gabber, and Exercise 10.4.5, which credits Lewis, May and Steinberger [13]. See also Kriz and May [12, Part III].) The assumptions that appear in the present paper are different from those that have appeared before and the proof is somewhat easier (because of our use of the theory of cofibrantly generated model categories), so this paper may be of some interest even in this special case.

Another consequence of the fact that Ch is a model category is the existence of resolutions coming from cofibrant and fibrant approximations, and the related derived functors. Some of these are discussed in [1] and [15]. We do not discuss these topics here, but just mention that the model category point of view can provide a framework for some of this material.

We also briefly discuss the category of non-negatively graded chain complexes. In this case we describe a model category structure that works for an arbitrary projective class, without any set-theoretic hypotheses. More generally, we show that under appropriate hypotheses a projective class on a possibly non-abelian category $\mathcal{A}$ determines a model category structure on the category of simplicial objects in $\mathcal{A}$. As an example, we deduce that the category of equivariant simplicial sets has various model category structures.

We now briefly outline the paper. In Section 1 we give the axioms for a projective class and mention a few examples. In Section 3 we describe the model category structure on the category of chain complexes which takes into account a given projective class. Then we state the set-theoretic assumption needed and prove our main result, using the recognition lemma for cofibrantly generated categories, which is recalled in Section 2. In Section 4 we give two examples, the traditional derived category of R-modules and the pure derived category, and we describe why the pure derived category is interesting. In the final section we discuss the bounded below case, which works for any projective class, and describe a result for simplicial objects in a possibly non-abelian category.

1. Projective classes

In this section we explain the notion of a projective class, which is the information necessary in order to do homological algebra. Intuitively, a projective class is a choice of which sort of "elements" we wish to think about.

The elements of a set X correspond bijectively to the maps from a singleton to X, and the elements of an abelian group A correspond bijectively to the maps from $\mathbb{Z}$ to A. Motivated by this, we call a map $P \to A$ in any category a **P-element of A**. If we don't wish to mention P, we call such a map a **generalized element of A**. A map $A \to B$ in any category is determined by what it does on generalized elements. If $\mathcal{P}$ is a collection of objects, then a **$\mathcal{P}$-element** means a P-element for some P in $\mathcal{P}$.

Let $\mathcal{A}$ be a **pointed category**, *i.e.,* a category in which initial and terminal objects exist and agree. In a pointed category, any initial (equivalently, terminal) object is called a **zero object**. Let P be an object of $\mathcal{A}$. A sequence

$$A \longrightarrow B \longrightarrow C$$

is said to be **P-exact** if the composite $A \to C$ is the zero map (the unique map which factors through a zero object) and

$$\mathcal{A}(P, A) \longrightarrow \mathcal{A}(P, B) \longrightarrow \mathcal{A}(P, C)$$

is an exact sequence of pointed sets (the base points being the zero maps). The latter can be rephrased as the condition that $A \to B \to C$ induces an exact sequence of P-elements. A **$\mathcal{P}$-exact sequence** is one which is P-exact for all P in $\mathcal{P}$. A map $A \to B$ is **P-epic** (resp. **$\mathcal{P}$-epic**) if it induces a surjection of P-elements (resp. $\mathcal{P}$-elements).

A **projective class** on $\mathcal{A}$ is a collection $\mathcal{P}$ of objects of $\mathcal{A}$ and a collection $\mathcal{E}$ of sequences $A \to B \to C$ in $\mathcal{A}$ such that

(*i*) $\mathcal{E}$ is precisely the collection of all $\mathcal{P}$-exact sequences;

(*ii*) $\mathcal{P}$ is precisely the collection of all objects P such that each sequence in $\mathcal{E}$ is P-exact;

(*iii*) any map $A \to B$ can be extended to a sequence $P \to A \to B$ in $\mathcal{E}$ with P in $\mathcal{P}$.

When a collection $\mathcal{P}$ is part of a projective class $(\mathcal{P}, \mathcal{E})$, the projective class is unique, and so we say that $\mathcal{P}$ determines a projective class or even that $\mathcal{P}$ is a projective class. An object of $\mathcal{P}$ is called a **$\mathcal{P}$-projective**. Condition (iii) says that there are "enough $\mathcal{P}$-projectives."

Example 1.1. Let $\mathcal{A}$ be the category of pointed sets, let $\mathcal{P}$ be the collection of all pointed sets and let $\mathcal{E}$ be the collection of all exact sequences of pointed sets. Then $\mathcal{E}$ is precisely the collection of $\mathcal{P}$-exact sequences, and $\mathcal{P}$ is a projective class.

Example 1.2. For an associative ring R, let $\mathcal{A}$ be the category of left R-modules, let $\mathcal{P}$ be the collection of all summands of free R-modules and let $\mathcal{E}$ be the collection of all exact sequences of R-modules. Then $\mathcal{E}$ is precisely the collection of $\mathcal{P}$-exact sequences, and $\mathcal{P}$ is a projective class.

The above two examples are "categorical" projective classes in the sense that the $\mathcal{P}$-epimorphisms are just the epimorphisms and the $\mathcal{P}$-projectives are the categorical projectives, *i.e.,* those objects P such that maps from P lift through epimorphisms.

Here are two examples of non-categorical projective classes.

Example 1.3. Let $\mathcal{A}$ be the category of left R-modules, as in Example 1.2. Let $\mathcal{P}$ consist of all summands of sums of finitely presented modules and define $\mathcal{E}$ to consist of all $\mathcal{P}$-exact sequences. Then $\mathcal{P}$ is a projective class. A short exact sequence is $\mathcal{P}$-exact iff it remains exact after tensoring with every right module.

Example 1.4. Let $\mathcal{S}$ be the homotopy category of spectra and let $\mathcal{P}$ consist of all retracts of wedges of finite spectra. Then $\mathcal{P}$ determines a projective class, and a map is a $\mathcal{P}$-epimorphism iff its cofibre is a phantom map.

Examples 1.2 and 1.3 will be discussed further in Section 4. Example 1.4 is studied in [2], along with similar examples.

We note that all of the examples mentioned above are **determined by a set** in the sense of the following lemma, whose proof is easy.

Lemma 1.5. *Suppose $\mathcal{F}$ is any* set *of objects in a pointed category with coproducts. Let $\mathcal{E}$ be the collection of $\mathcal{F}$-exact sequences and let $\mathcal{P}$ be the collection of all objects P such that every sequence in $\mathcal{E}$ is P-exact. Then $\mathcal{P}$ is the collection of retracts of coproducts of objects of $\mathcal{F}$ and $(\mathcal{P}, \mathcal{E})$ is a projective class.* $\square$

A projective class is precisely the information needed to form projective resolutions and define derived functors. For further information, we refer the reader to [6] for the general theory and to [2] in the special case of a triangulated category.

2. Cofibrantly generated model categories

In this section we briefly recall the basics of cofibrantly generated model categories. This material will be used in the next section to prove our main result. For more details, see the books by Dwyer, Hirschhorn and Kan [5] and Hirschhorn [9]. We assume knowledge of the basics of model categories, for which [4] is an excellent reference.

We will always assume our model categories to be complete and cocomplete.

In the following, a cardinal number is thought of as the first ordinal with that cardinality.

Definition 2.1. *Given an ordinal κ, a κ-sequence in a category $\mathcal{M}$ is a diagram*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\lambda \longrightarrow X_{\lambda+1} \longrightarrow \cdots$$

indexed by the ordinals less than κ, such that for each limit ordinal γ less than κ the natural map $\mathrm{colim}_{\lambda < \gamma} X_\lambda \to X_\gamma$ *is an isomorphism.*

*A cardinal γ is said to be **regular** if any sum of fewer than γ cardinals each less than γ is less than γ. The first few regular cardinals are 0, 1, 2 and $\aleph_0$.*

*An object P is said to be **small relative to a subcategory** $\mathcal{B}$ if there exists a cardinal κ such that for any regular cardinal $\gamma \geq \kappa$ and any γ-sequence X in $\mathcal{M}$ taking values in $\mathcal{B}$, the natural map*

$\operatorname{colim}_{\lambda<\gamma} \mathcal{M}(P, X_\lambda) \to \mathcal{M}(P, \operatorname{colim}_{\lambda<\gamma} X_\lambda)$ *is an isomorphism, where the last colimit is taken in* $\mathcal{M}$.

Definition 2.2. *Let* I *be a class of maps in a cocomplete category. A map is said to be **I-injective** if it has the right lifting property with respect to each map in* I, *and we write* I-inj *for the category containing these maps. A map is said to be an **I-cofibration** if it has the left lifting property with respect to each map in* I-inj, *and we write* I-cof *for the category containing these maps. A map is said to be **I-cellular** if it is a transfinite composite of pushouts of coproducts of maps in* I, *and we write* I-cell *for the category containing these maps.*

Note that I-cell is a subcategory of I-cof.

Definition 2.3. *A **cofibrantly generated model category** is a model category* $\mathcal{M}$ *for which there exist sets* I *and* J *of morphisms with domains which are small relative to* I-cof *and* J-cof, *respectively, such that* I-cof *is the category of cofibrations and* J-cof *is the category of acyclic cofibrations. It follows that* I-inj *is the category of acyclic fibrations and that* J-inj *is the category of fibrations.*

For example, take $\mathcal{M}$ to be the category of spaces and take $I = \{S^n \to B^{n+1}\}$ and $J = \{B^n \times 0 \to B^n \times [0,1]\}$.

Proposition 2.4. *(Recognition Lemma.) Let* $\mathcal{M}$ *be a category which is complete and cocomplete, let* W *be a class of maps which is closed under retracts and satisfies the two-out-of-three axiom, and let* I *and* J *be sets of maps with domains which are small relative to* I-cell *and* J-cell, *respectively, such that*

(*i*) J-cell $\subseteq I$-cof $\cap W$ *and* I-inj $\subseteq J$-inj $\cap W$, *and*
(*ii*) J-cof $\supseteq I$-cell $\cap W$ *or* I-inj $\supseteq J$-inj $\cap W$.

Then $\mathcal{M}$ *is cofibrantly generated by* I *and* J, *and* W *is the subcategory of weak equivalences. Moreover, a map is in* I-cof *if and only if it is a retract of a map in* I-cell, *and similarly for* J-cof *and* J-cell. $\qquad \square$

The proof, which uses the small object argument, can be found in [5] and [9].

3. DERIVED CATEGORIES

Let $\mathcal{A}$ be a complete and cocomplete abelian category equipped with a projective class $\mathcal{P}$. Our goal is to construct a model category structure on the category of chain complexes of objects of $\mathcal{A}$ in a way that takes into account the projective class $\mathcal{P}$. At a certain point we will need a set-theoretic assumption, but we proceed as far as possible without it.

We write Ch for the category of unbounded chain complexes of objects of $\mathcal{A}$ and degree zero chain maps. To fix notation, assume that the differentials lower degree. For an object X of Ch, define $Z_n X := \ker(d : X_n \to X_{n-1})$ and $B_n X := \operatorname{im}(d : X_{n+1} \to X_n)$, and write $H_n X$ for the quotient. Given an object P of $\mathcal{A}$ and a chain complex X, we write $\mathcal{A}(P, X)$ for the chain complex which has the abelian group $\mathcal{A}(P, X_n)$ in degree n. This is the chain complex of P-elements of X. (See Section 1 for a discussion of P-elements.) The cycles, boundaries and homology classes in $\mathcal{A}(P, X)$ are called **P-cycles**, **P-boundaries** and **P-homology classes** in X. Note that a P-cycle is the same as a P-element of $Z_n X$, but that not every P-element of $B_n X$ is necessarily a P-boundary (unless P is a categorical projective).

Using this terminology, we can say which maps in our category should be weak equivalences, fibrations and cofibrations.

Definition 3.1. *Consider a chain map $f : X \to Y$. The map f is said to be a **quasi-isomorphism** if it induces an isomorphism $H_n X \to H_n Y$ for each n. The map f is said to be a **weak equivalence** if for each $\mathcal{P}$-projective P the chain map $\mathcal{A}(P, X) \to \mathcal{A}(P, Y)$ is a quasi-isomorphism, i.e., if f induces an isomorphism of P-homology groups. The map f is said to be a **fibration** if for each $\mathcal{P}$-projective P the chain map $\mathcal{A}(P, X) \to \mathcal{A}(P, Y)$ is degreewise epic, i.e., if f induces a surjection of P-elements. And f is said to be a **cofibration** if each $f_n : X_n \to Y_n$ is split-monic and the complex of cokernels is "cofibrant": a complex is **cofibrant** if for some ordinal κ it is a retract of a complex which is a colimit of a κ-sequence*

$$0 = C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$$

*such that each map is degreewise split-monic and the successive quotients are complexes of $\mathcal{P}$-projective objects with zero differential. As usual, a map is said to be an **acyclic fibration** (resp. **acyclic cofibration**) if it is both a fibration (resp. cofibration) and a weak equivalence.*

It will be useful to have the following notation. We write Σ for the usual automorphism of Ch, which sends a complex X to the complex ΣX with $(\Sigma X)_n = X_{n-1}$ and $d_{\Sigma X} = -d_X$. The functor Σ is defined on morphisms by $(\Sigma f)_n = f_{n-1}$. Given an object A in $\mathcal{A}$, we consider A as a chain complex concentrated in degree 0. Thus we write $\Sigma^n A$ for the complex with A concentrated in degree n. The maps from $\Sigma^n A$ to a complex X biject with the A-elements of $Z_n X$. Similarly, we write $D^n A$ for the complex which has A in degrees $n-1$ and n connected by the identity map, and which is zero in other degrees, and we find that the maps from $D^n A$ to a complex X biject with the A-elements of X_n. There is a natural map $\Sigma^{n-1} A \to D^n A$.

The following lemma will be used to prove our main result.

Lemma 3.2. *Consider an object $A \in \mathcal{A}$ and a map $f : X \to Y$ of chain complexes.*

(i) *The map f has the RLP with respect to each of the maps $0 \to D^n A$, $n \in \mathbb{Z}$, if and only if the induced map of A-elements is a surjection.*

(ii) *The map f has the RLP with respect to each of the maps $\Sigma^{n-1} A \to D^n A$, $n \in \mathbb{Z}$, if and only if the induced map of A-elements is a surjection and a quasi-isomorphism.*

Proof. We begin with (*i*): The map f has the RLP with respect to the map $0 \to D^n A$ if and only if each map $D^n A \to Y$ factors through f, *i.e.,* if and only if each A-element of Y_n is in the image of f_n.

Now (*ii*): The map f has the RLP with respect to the map $\Sigma^{n-1} A \to D^n A$ if and only if for each A-element y of Y_n whose boundary is the image of an A-cycle x of X_{n-1}, there is an A-element x' of X_n which hits y under f and x under the differential. (In other words, if and only if the natural map $X_n \to Z_{n-1} X \times_{Z_{n-1} Y} Y_n$ induces a surjection of A-elements.)

So suppose that f has the RLP with respect to each map $\Sigma^{n-1} A \to D^n A$. As a preliminary result, we prove that f induces a surjection of A-cycles. Suppose we are given an A-cycle y of Y. Its boundary is zero and is thus in the image of the A-cycle 0 of X. Therefore y is the image of an A-cycle x'.

It follows immediately that f induces a surjection in A-homology.

Now we prove that f induces a surjection of A-elements. Suppose we are given an A-element y of Y. By the above argument, its boundary, which is an A-cycle, is the image of an A-cycle x of X. Thus, by the characterization of maps f having the RLP, we see that there is an A-element x' which hits y.

Finally, we prove that f induces an injection in A-homology. Suppose x is an A-cycle of X whose image under f is the boundary of an A-element y. Then, using the characterization of the RLP, we see that there is an A-element x' whose boundary is x. This shows that f induces an injection in A-homology.

We have proved that if f has the RLP with respect to the maps $\Sigma^{n-1} A \to D^n A$, then f induces an isomorphism in A-homology and a surjection of A-elements. The proof of the converse is similar. $\square$

Corollary 3.3. *A map $f : X \to Y$ is a fibration (resp. acyclic fibration) if and only if it has the RLP with respect to the map $0 \to D^n P$ (resp. $\Sigma^{n-1} P \to D^n P$) for each $\mathcal{P}$-projective P and each $n \in \mathbb{Z}$.* $\square$

We want to claim that the above definitions lead to a model category structure on the category Ch. In order to prove this, we need to assume that there is a *set* $\mathcal{S}$ of $\mathcal{P}$-projectives such that a map $f : A \to B$ is $\mathcal{P}$-epic if and only if f induces a surjection of P-elements for each P in $\mathcal{S}$. This implies that for any B there is a $\mathcal{P}$-epimorphism $P \to B$ with P a coproduct of objects from $\mathcal{S}$, and that every $\mathcal{P}$-projective object is a retract of such a coproduct. We also need to assume that each P in $\mathcal{S}$ is small relative to the subcategory $K := \{(1,0) : A \to A \oplus Q \mid A \in \mathcal{A} \text{ and } Q \in \mathcal{P}\}$. (See the previous section for the definition of "small.") When these conditions hold we say that $\mathcal{P}$ is **determined by a set of small objects**.

Theorem 3.4. *Assume that $\mathcal{P}$ is determined by a set $\mathcal{S}$ of small objects. Then the category* Ch *is a cofibrantly generated model category with the following generating sets:*

$$I := \{\Sigma^{n-1} P \to D^n P \mid P \in \mathcal{S},\, n \in \mathbb{Z}\},$$

$$J := \{0 \to D^n P \mid P \in \mathcal{S},\, n \in \mathbb{Z}\}.$$

Moreover, the weak equivalences, fibrations, cofibrations and cofibrant objects are as described in Definition 3.1, and every object is fibrant.

Proof. In this proof we use the terms weak equivalence, (acyclic) fibration, (acyclic) cofibration, and cofibrant as defined in Definition 3.1.

We check the hypotheses of the Recognition Lemma from the previous section. Since $\mathcal{A}$ is complete and cocomplete, so is Ch; limits and colimits are taken degreewise. The class of weak equivalences is easily seen to be closed under retracts and to satisfy the two-out-of-three condition. The zero chain complex is certainly small relative to J-cell. We assumed that each P in $\mathcal{S}$ is small relative to $K \subseteq \mathcal{A}$, and it follows that each $\Sigma^k P$ is small relative to I-cell $\subseteq$ Ch. In more detail: A map is in I-cell if and only if it is a degreewise split monomorphism whose cokernel is a transfinite colimit of degreewise split monomorphisms whose cokernels are complexes of $\mathcal{P}$-projectives with zero differential. In particular, every map in I-cell is a degreewise split monomorphism whose cokernel is a complex of $\mathcal{P}$-projectives. Let P be an object of $\mathcal{S}$. For large enough regular cardinals

κ, P is small with respect to κ-sequences of maps in K. Suppose that for such a κ we have a κ-sequence $X^0 \to X^1 \to \cdots$ of maps in I-cell with colimit X. A chain map from $\Sigma^k P$ to X is just a map $f : P \to X_k$ such that $df = 0$. Given such an f, it factors through some $g : P \to X_k^\alpha$, since P is small relative to K, and the sequence $X_k^0 \to X_k^1 \to \cdots$ is in K. Moreover, since $df = 0$, it follows that $dg = 0$, at least after passing to X_k^β for some $\beta \geq \alpha$, again using smallness. Thus the map $\operatorname{colim} \operatorname{Ch}(\Sigma^k P, X^\lambda) \to \operatorname{Ch}(\Sigma^k P, X)$ is surjective. An easier argument shows that it is monic, and therefore $\Sigma^k P$ is small relative to I-cell.

Since the projectives in $\mathcal{S}$ may be used to test whether a map is a fibration or weak equivalence, Corollary 3.3 tells us that I-inj is the collection of acyclic fibrations and that J-inj is the collection of fibrations. Thus we have an equality I-inj $= J$-inj $\cap W$, giving us two of the inclusions required by the Recognition Lemma.

We now prove that J-cell $\subseteq I$-cof $\cap W$. Since I-inj $\subseteq J$-inj, it is clear that J-cof $\subseteq I$-cof, so in particular J-cell $\subseteq I$-cof. We must prove that J-cell $\subseteq W$, *i.e.,* that each map which is a transfinite composite of pushouts of coproducts of maps in J is a weak equivalence. A map in J is of the form $0 \to D^n P$ for some $P \in \mathcal{S}$. Thus a coproduct of maps in J is of the form $0 \to C$ for some contractible complex C. A pushout of such a coproduct is of the form $X \to X \oplus C$, again with C a contractible complex. And a transfinite composite of such maps is of the same form as well. In particular, a transfinite composite of pushouts of coproducts of maps from J is a weak equivalence.

We can now apply the Recognition Lemma and conclude that Ch is a model category with weak equivalences as described in Definition 3.1, cofibrations given by the maps in I-cof and fibrations given by the maps in J-inj. We saw above that the maps in J-inj are precisely those we called fibrations in Definition 3.1, and it is clear from this that every object is fibrant.

It remains to check that the maps in I-cof are as described in Definition 3.1. We continue use the words "cofibration" and "cofibrant" as defined in Definition 3.1. Suppose that i is in I-cof. By 2.4, i is a retract of a map j which is a transfinite composite of pushouts of coproducts of maps in I. As above, one can show that j is degreewise split-monic and that the complex C of cokernels is cofibrant. It follows that i is degreewise split-monic and that the complex of cokernels is a retract of C and is therefore cofibrant. That is, i is a cofibration. And now it is clear that if the map $0 \to X$ is in I-cof, then X is cofibrant.

Using that I-cof is closed under transfinite compositions and retracts, one can show that every cofibration is in I-cof and in particular that if X is cofibrant then the map $0 \to X$ is in I-cof. $\quad\square$

For the rest of this section, assume that the projective class $\mathcal{P}$ is determined by a set of small objects, so that Ch is a model category. The homotopy category of Ch, formed by inverting the weak equivalences, is called the **derived category** of $\mathcal{A}$ (with respect to $\mathcal{P}$). It is denoted $\mathcal{D}$, and a fundamental result in model category theory asserts that $\mathcal{D}(X, Y)$ is a set for each X and Y. In Exercise 10.4.5 of [16], Weibel outlines an argument which proves that $\mathcal{D}(X, Y)$ is a set when there are enough (categorical) projectives, $\mathcal{P}$ is the categorical projective class, and $\mathcal{A}$ satisfies AB5. A connection between Weibel's hypotheses and ours is that if $\mathcal{A}$ has enough projectives which are small with respect to all filtered diagrams in $\mathcal{A}$, then AB5 holds. Of course, the smallness condition needed for our theorem is weaker than this in that smallness is only required for diagrams indexed by large cardinals and taking values in the subcategory K.

We now show that the notion of homotopy determined from the model category structure (see [4] or [14]) corresponds to the usual notion of chain homotopy.

Definition 3.5. *If M is an object in a model category $\mathcal{C}$, a **good cylinder object** for M is an object $M \times I$ and a factorization $M \coprod M \xrightarrow{i} M \times I \xrightarrow{p} M$ of the codiagonal map, with i a cofibration and p a weak equivalence. (Despite the notation, $M \times I$ is not in general a product of M with an object I.) A **left homotopy** between maps $f, g : M \to N$ is a map $H : M \times I \to N$ such that the composite Hi is equal to $f \coprod g : M \coprod M \to N$, for some good cylinder object $M \times I$.*

The notion of **good path object** N^I for N is dual to that of good cylinder object and leads to the notion of **right homotopy**. The following standard result can be found in [4, Section 4], for example.

Lemma 3.6. *For M cofibrant and N fibrant, two maps $f, g : M \to N$ are left homotopic if and only if they are right homotopic, and both of these relations are equivalence relations and respect composition. Moreover, if $M \times I$ is a fixed good cylinder object for M, then f and g are left homotopic if and only if they are left homotopic using $M \times I$; similarly for a fixed good path object.* $\square$

Because of the lemma, for M cofibrant and N fibrant we have a well-defined relation of **homotopy** on maps $M \to N$. One can show that the homotopy category of $\mathcal{C}$, which is by definition the category of fractions formed by inverting the weak equivalences, is equivalent to the category consisting of objects which are both fibrant and cofibrant with morphisms being homotopy classes of morphisms.

Now we return to the study of the model category Ch.

Lemma 3.7. *Let M and N be objects of Ch with M cofibrant. Two maps $M \to N$ are homotopic if and only if they are chain homotopic.*

Proof. We construct a factorization $N \to N^I \to N \times N$ of the diagonal map $\Delta : N \to N \times N$ in the following way. Let N^I be the chain complex which has $N_n \oplus N_{n+1} \oplus N_n$ in degree n. We describe the differential by saying that it sends a generalized element $(n, \bar{n}, n')$ in $(N^I)_n$ to $(dn, n - n' - d\bar{n}, dn')$. Let $\alpha : N \to N^I$ be the map which sends n to $(n, 0, n)$ and let $\beta : N^I \to N \times N$ be the map which sends $(n, \bar{n}, n')$ to (n, n'). One can check easily that N^I is a chain complex and that α and β are chain maps whose composite is Δ. The map α is a chain homotopy equivalence with chain homotopy inverse sending $(n, \bar{n}, n')$ to n; this implies that it induces a chain homotopy equivalence of generalized elements and is thus a weak equivalence. The map β is degreewise split-epi; this implies that it induces a split epimorphism of generalized elements and is thus a fibration. Therefore N^I is a good path object for N.

It is easy to see that a chain homotopy between two maps $M \to N$ is the same as a right homotopy using the path object N^I. By Lemma 3.6, two maps are homotopic if and only if they are right homotopic using N^I. Thus the model category notion of homotopy is the same as the notion of chain homotopy when the source is cofibrant. $\square$

There is a dual proof which proceeds by constructing a specific good cylinder object $M \times I$ for M such that a left homotopy using $M \times I$ is the same as a chain homotopy.

Corollary 3.8. *Let A and B be objects of $\mathcal{A}$ considered as chain complexes concentrated in degree 0. Then $\mathcal{D}(A, \Sigma^n B) \cong \operatorname{Ext}_{\mathcal{P}}^n(A, B)$.*

Here we denote by $\operatorname{Ext}_{\mathcal{P}}^n(-, B)$ the right derived functors of $\mathcal{A}(-, B)$ with respect to the projective class $\mathcal{P}$. The group $\operatorname{Ext}_{\mathcal{P}}^n(A, B)$ can be calculated by forming a $\mathcal{P}$-projective $\mathcal{P}$-exact resolution of A, applying $\mathcal{A}(-, B)$, and taking cohomology. One can show that $\operatorname{Ext}_{\mathcal{P}}^n(A, B)$ classifies equivalence classes of $\mathcal{P}$-exact sequences $0 \to B \to C_1 \to \cdots \to C_n \to A \to 0$.

Proof. The group $\mathcal{D}(A, \Sigma^n B)$ may be calculated by choosing a cofibrant replacement A' for A and computing the homotopy classes of maps from A' to $\Sigma^n B$. (Recall that all objects are fibrant, so there is no need to take a fibrant replacement for $\Sigma^n B$.) A $\mathcal{P}$-projective $\mathcal{P}$-exact resolution P of A serves as a cofibrant replacement for A, and by Lemma 3.7 the homotopy relation on $\operatorname{Ch}(P, \Sigma^n B)$ is chain homotopy, so it follows that $\mathcal{D}(A, \Sigma^n B)$ is isomorphic to $\operatorname{Ext}_{\mathcal{P}}^n(A, B)$. $\qquad\square$

More generally, a similar argument shows that the derived functors of a functor F can be expressed as the cohomology of the derived functor of F in the model category sense. To make the story complete, we next show that the shift functor Σ corresponds to the notion of suspension that the category $\mathcal{D}$ obtains as the homotopy category of a pointed model category.

Definition 3.9. *Let $\mathcal{C}$ be a pointed model category. If M is cofibrant, we define the **suspension** ΣM of M to be the cofibre of the map $M \coprod M \to M \times I$ for some good cylinder object $M \times I$. (The cofibre of a map $X \to Y$ is the pushout $* \coprod_X Y$, where $*$ is the zero object.) ΣM is cofibrant and well-defined up to homotopy equivalence.*

The loop object ΩN of a fibrant object N is defined dually. These operations induce adjoint functors on the homotopy category. A straightforward argument based on the path object described above (and a dual cylinder object) proves the following lemma.

Lemma 3.10. *In the model category Ch, the functor Σ defined in Definition 3.9 can be taken to be the usual suspension, so that $(\Sigma X)_n = X_{n-1}$ and $d_{\Sigma X} = -d_X$. Similarly, ΩX can be taken to be the complex $\Sigma^{-1} X$. That is, $(\Omega X)_n = X_{n+1}$ and $d_{\Omega X} = -d_X$.* $\qquad\square$

In particular, Σ and Ω are inverse functors. Hovey [10] has shown that this implies that cofibre sequences and fibre sequences agree (up to the usual sign) and that Σ and the cofibre sequences give rise to a triangulation of the homotopy category. (See [14, Section I.3] for the definition of cofibre and fibre sequences in any pointed model category.) Applying Hovey's result to our situation we can be more explicit. Given a map $f : L \to M$ of chain complexes, the **standard triangle** on f is the sequence $L \to M \to N \to \Sigma L$, where $N_n = M_n \oplus L_{n-1}$, $d_N(m, l) = (d_M m + f l, -d_L l)$ on generalized elements, and the maps are the inclusion and projection.

Corollary 3.11. *The category $\mathcal{D}$ is triangulated. A sequence $L' \to M' \to N' \to \Sigma L'$ is a triangle if and only if it is isomorphic in $\mathcal{D}$ to a standard triangle.*

Proof. That $\mathcal{D}$ is triangulated follows from Hovey's result, since we have shown that Σ is a self-equivalence. The identification of the triangles uses the definition of fibre sequences from [14] and the construction of the path object from the proof of Lemma 3.7. There is also a dual proof using cofibre sequences and cylinder objects. $\qquad\square$

The results of this section probably apply with $\mathcal{A}$ replaced by a complete and cocomplete additive category.

4. The pure and categorical derived categories

In this section we let R be an associative ring with unit and we take for $\mathcal{A}$ the category of left R-modules. We are concerned with two projective classes on the category $\mathcal{A}$. The first is the categorical projective class $\mathcal{C}$ whose projectives are summands of free modules, whose exact sequences are the usual exact sequences, and whose epimorphisms are the surjections. The second is the pure projective class $\mathcal{P}$ whose projectives are summands of sums of finitely presented modules. A short exact sequence is $\mathcal{P}$-exact iff it remains exact after tensoring with any right module. A map is a $\mathcal{P}$-epimorphism iff it appears in a $\mathcal{P}$-exact short exact sequence. We say pure projective instead of $\mathcal{P}$-projective, and similarly for pure exact and pure epimorphism. We assume that the reader has some familiarity with these projective classes. A brief summary with further references may be found in [2, Section 9]. As usual, we write $\mathrm{Ext}^*(-, B)$ (resp. $\mathrm{PExt}^*(-, B)$) for the derived functors of $\mathcal{A}(-, B)$ with respect to the categorical (resp. pure) projective class.

Both of these projective classes are determined by sets of small objects: $\mathcal{C}$ is determined by $\{R\}$ and $\mathcal{P}$ is determined by any set of finitely presented modules containing a representative from each isomorphism class. Thus we get two model category structures on Ch and two derived categories, the categorical derived category $\mathcal{D}_{\mathcal{C}}$ and the pure derived category $\mathcal{D}_{\mathcal{P}}$. We refer to the pure weak equivalences in Ch as pure quasi-isomorphisms, and as usual call the categorical weak equivalences simply quasi-isomorphisms. Similarly, we talk of pure fibrations and fibrations, pure cofibrations and cofibrations, etc.

Pure homological algebra is of interest in stable homotopy theory because of the following result.

Theorem 4.1. [3] *Phantom maps from a spectrum X to an Eilenberg-Mac Lane spectrum HG are given by* $\mathrm{PExt}^1_Z(H_{-1}X, G)$, *that is, by maps of degree one from $H_{-1}X$ to G in the pure derived category of abelian groups.*

The two derived categories are connected in various ways. For example, since every categorical projective is pure projective, it follows that every pure quasi-isomorphism is a quasi-isomorphism. This implies that there is a unique functor $R : \mathcal{D}_{\mathcal{P}} \to \mathcal{D}_{\mathcal{C}}$ commuting with the functors from Ch. This functor has a left adjoint L. We can see this in the following way: the identity functor on Ch is adjoint to itself. It is easy to see that every pure fibration is a fibration and that every pure acyclic fibration is an acyclic fibration. Therefore, by [4, Theorem 9.7], there is an induced pair of adjoint functors between the pure derived category and the categorical derived category, and the right adjoint is the functor R mentioned above. This right adjoint is the identity on objects, since everything is fibrant, and induces the natural map $\mathrm{PExt}^*(A, B) \to \mathrm{Ext}^*(A, B)$ for R-modules A and B. The left adjoint sends a complex X in $\mathcal{D}_{\mathcal{C}}$ to a (categorical) cofibrant replacement $\tilde{X}$ for X. Similar adjoint functors exist whenever one has two projective classes on a category, one containing the other.

In addition to the connection between phantom maps and pure homological algebra, the author is interested in the pure derived category as a tool for connecting the global pure dimension of a ring

R to the behaviour of phantom maps in $\mathcal{D}_{\mathcal{C}}$ and $\mathcal{D}_{\mathcal{P}}$ under composition. This work will hopefully appear in a future paper.

5. SIMPLICIAL OBJECTS AND THE BOUNDED BELOW DERIVED CATEGORY

In this section, let $\mathcal{A}$ be a complete and cocomplete pointed category with a projective class $\mathcal{P}$. We do not assume that $\mathcal{A}$ is abelian. Write $s\mathcal{A}$ for the category of simplicial objects in $\mathcal{A}$. Given a simplicial object X and an object P, write $\mathcal{A}(P, X)$ for the simplicial set which has $\mathcal{A}(P, X_n)$ in degree n. Consider the following conditions:

(*) For each X in $s\mathcal{A}$ and each P in $\mathcal{P}$, $\mathcal{A}(P, X)$ is a fibrant simplicial set.

(**) $\mathcal{P}$ is determined by a set $\mathcal{S}$ of small objects. Here we require that each P in $\mathcal{S}$ be small with respect to all monomorphisms in $\mathcal{A}$, not just the split monomorphisms with $\mathcal{P}$-projective cokernels.

Theorem 5.1. *Let $\mathcal{A}$ be a complete and cocomplete pointed category with a projective class $\mathcal{P}$. If $\mathcal{A}$ and $\mathcal{P}$ satisfy* (*) *or* (**), *then $s\mathcal{A}$ is a simplicial model category. The weak equivalences* (*resp. fibrations*) *are the maps f such that $\mathcal{A}(P, f)$ is a weak equivalence* (*resp. fibration*) *of simplicial sets for each P in $\mathcal{S}$.* (*Here we write $\mathcal{S}$ for the set of small objects which determines $\mathcal{P}$ when* (**) *holds, and we take $\mathcal{S}$ to be equal to $\mathcal{P}$ when* (*) *holds.*) *In general the cofibrations are determined by the lifting property. When* (**) *holds, $s\mathcal{A}$ is cofibrantly generated by the sets*

$$I := \{ P \otimes \dot{\Delta}[n] \to P \otimes \Delta[n] \mid P \in \mathcal{S}, \, n \geq 0 \},$$

$$J := \{ P \otimes V[n, k] \to P \otimes \Delta[n] \mid P \in \mathcal{S}, \, 0 < n \geq k \geq 0 \}.$$

In the above, $\Delta[n]$ denotes the standard n-simplex simplicial set; $\dot{\Delta}[n]$ denotes its boundary; and $V[n, k]$ denotes the subcomplex with the n-cell and its kth face removed. For an object P of $\mathcal{A}$ and a set K, $P \otimes K$ denotes the coproduct of copies of P indexed by K. When K is a simplicial set, $P \otimes K$ denotes the simplicial object in $\mathcal{A}$ which has $P \otimes K_n$ in degree n.

If $\mathcal{A}$ is abelian, then (*) holds. Moreover, in this case there is an equivalence of categories given by the normalization functor $s\mathcal{A} \to \mathrm{Ch}^+$, where Ch^+ denotes the category of non-negatively graded chain complexes of objects of $\mathcal{A}$. Thus we can deduce:

Corollary 5.2. *Let $\mathcal{A}$ be a complete and cocomplete abelian category with a projective class $\mathcal{P}$. Then Ch^+ is a model category. A map f is a weak equivalence iff $\mathcal{A}(P, f)$ is a quasi-isomorphism for each P in $\mathcal{P}$. A map f is a fibration iff $\mathcal{A}(P, f)$ is surjective in positive degrees* (*but not necessarily in degree 0*) *for each P in $\mathcal{P}$. A map is a cofibration iff it is degreewise split-monic with degreewise $\mathcal{P}$-projective cokernel. Every complex is fibrant, and a complex is cofibrant iff it is a complex of $\mathcal{P}$-projectives.* $\qquad\square$

Note that no conditions on the projective class are required, and that the description of the cofibrations is simpler than in the unbounded case.

As another example of the theorem, let G be a group and consider G-simplicial sets, or equivalently, simplicial objects in the category $\mathcal{A}$ of (say, left) G-sets. Let $\mathcal{F}$ be a family of subgroups of G, and consider the set $\mathcal{S} = \{ G/H \mid H \in \mathcal{F} \}$ of homogeneous spaces. These are small, and determine a projective class $\mathcal{P}$. Thus, using case (**) of the above theorem, we can deduce:

Corollary 5.3. *Let G be a group and let $\mathcal{F}$ be a family of subgroups of G. The category of G-equivariant simplicial sets has a model category structure in which the weak equivalences are precisely the maps which induce a weak equivalence on H-fixed points for each H in $\mathcal{F}$.* $\qquad\square$

We omit the proof of Theorem 5.1, and simply note that it follows the argument in [14, Section II.4] fairly closely. It is a bit simpler, in that Quillen spends part of the time (specifically Proposition 2) proving that effective epimorphisms give rise to a projective class, although he doesn't use this terminology. It is also a bit more complicated, in the (**) case, in that a transfinite version of Kan's Ex^∞ functor [11] is required, since we make a weaker smallness assumption. Quillen's argument in this case can be interpreted as a verification of the hypotheses of the recognition lemma for cofibrantly generated model categories (Proposition 2.4).

References

[1] Luchezar L. Avramov, Hans-Bjørn Foxby, and Stephen Halperin. Unpublished manuscript, 1997.

[2] J. Daniel Christensen. Ideals in triangulated categories: Phantoms, ghosts and skeleta. *Adv. in Math.*, 136:284–339, 1998.

[3] J. Daniel Christensen and Neil P. Strickland. Phantom maps and homology theories. *Topology*, 37:339–364, 1998.

[4] W. G. Dwyer and J. Spalinski. Homotopy theories and model categories. In I. M. James, editor, *Handbook of Algebraic Topology*, pages 73–126. North-Holland, 1995.

[5] William G. Dwyer, Philip S. Hirschhorn, and Daniel M. Kan. Model categories and more general abstract homotopy theory. In preparation, available at http://www-math.mit.edu/∼psh, 1997.

[6] Samuel Eilenberg and J. C. Moore. Foundations of relative homological algebra. *Mem. Amer. Math. Soc.*, 55, 1965.

[7] Jesper Grodal. Personal communication.

[8] Vladimir Hinich. Homological algebra of homotopy algebras. *Comm. Algebra*, 25:3291–3323, 1997.

[9] Philip S. Hirschhorn. Localization of model categories. Preprint, available at http://www-math.mit.edu/∼psh, 1997.

[10] Mark Hovey. Model categories. Preprint, available at http://hopf.math.purdue.edu, 1997.

[11] D. Kan. Fill in this entry.

[12] Igor Kriz and J. P. May. Operads, algebras, modules and motives. *Astérisque*, 233, 1995.

[13] L. Gaunce Lewis, J. Peter May, and M. Steinberger. *Equivariant Stable Homotopy Theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer–Verlag, New York, 1986. With contributions by Jim E. McClure.

[14] Daniel G. Quillen. *Homotopical Algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer-Verlag, 1967.

[15] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[16] C. A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1994.

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, 3400 N. CHARLES ST., BALTIMORE, MD 21218-2686

E-mail address: jdc@math.jhu.edu