COMBINATORIAL MODELS FOR ITERATED LOOP SPACES

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1. Introduction

The purpose of this note is to record natural filtered simplicial group models for iterated loop spaces. The models are derived by classical methods for simplicial groups along with the identification for certain group theoretic kernels. The main content of the article is the study of some useful properties of these models.

One feature of these models is that some of their properties follow from standard group theoretic facts, which themselves are derived from elementary covering space theory. Another feature is that within these models one can write homotopies for certain maps in a relatively straight forward way. Specific applications will not be discussed within this note, but it should be pointed out that these models were constructed as a framework in which one could deal with certain classical problems in homotopy theory.

Before presenting details, the main group theoretic input of the constructions here will be described. Restrict to the category of pointed sets. The coproduct $A \vee B$ in this category admits a map to A by sending B to the base-point. There is an induced map on the level of free groups $Fr[A \vee B] \longrightarrow Fr[A]$ for which the kernel is the free group with basis given by the set of conjugates $B^{Fr[A]}$ in $Fr[A \vee B]$. This kernel gives a group theoretic tool for constructing simplicial group models for double loop spaces. Iteration of this construction gives methods for constructing simplicial group models for iterated loop spaces. This iteration becomes complicated, at least in terms of notation. The case of $\Omega^2\Sigma^2 X$, and $\Omega^2 X$ are both clean, and useful; it is described as a special case below.

Several well known constructions in simplicial homotopy theory will be used here. Namely Milnor's simplicial suspension functor σ, the functor F, again due to Milnor, which models the loop-suspension functor, Kan's G functor which gives a model for the loop space functor and the loop functor Ω, due to Moore, which gives a model for the loop space functor in the case where the simplicial set under consideration is a Kan complex.

The work is carried out in the category of reduced simplicial sets, whose objects are connected simplicial sets X, such that $X_i = b_i$, the base point in dimension i, for i less than or equal to the connectivity of X. This condition is imposed for simplicity, but is not a substantial restriction.

Some of the results in this article require rather elaborate notation. Thus the introduction shall only contain an extract of the contents of the paper, rather than precise statements. The combinatorial constructions given here model the functors $\Omega^n\Sigma^n$ for arbitrary connected spaces and Ω^n on n-connected spaces.

The first construction models the functor $\Omega^n\Sigma^n$. Thus for a connected reduced simplicial set X a model for $\Omega^n\Sigma^n|X|$ is given by $\Omega^{n-1}F\Sigma^{n-1}X$. This object is a

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simplicial group, which shall be denoted by $\Lambda_n X$. The objective of the following theorem is to study the structure of this simplicial group. Let G be an arbitrary group and let $W, B \subseteq G$ be two subsets. Denote by B^W the subset of G given by all conjugates of elements of B by elements of W. In particular if W contains the identity element then the set B^W contains B.

Theorem 1.1. *For each positive integer n there is a functor Λ_n from the category of pointed connected reduced simplicial sets to the category of free simplicial groups such that the geometric realization of $\Lambda_n X$ is homotopy equivalent to $\Omega^n\Sigma^n|X|$. Moreover,*

1. *For each $n \geq 1$ there is a natural transformation of functors $E : \Lambda_n \longrightarrow \Lambda_{n+1}$ whose geometric realization is the Freudenthal suspension map.*
2. *A basis for $\Lambda_n X$ is given by the graded set $B_n(X)^{Fr[A_n(X)]}$, where $B_n(X)$ is the image of the n-fold Freudenthal suspension $E^n : X \longrightarrow \Lambda_n X$ and $A_n(X)$ is its pointed complement in the canonical generating set for $F\Sigma^{n-1}X$.*
3. *The simplicial operators in $\Lambda_n X$ can be described explicitly in terms of those of X.*
4. *The groups $\Lambda_n(X)$ admit a natural increasing filtration by simplicial subgroups.*

The first interesting case of the theorem is $n = 2$. An explicit description of this case is given here. Higher values of n admit similar statements, but the notation becomes more laborious.

Proposition 1.2. *A basis for the free simplicial group $\Lambda_2 X$ in dimension d is given by all elements of the form $\mathsf{c}(w)[x]$, where $[x] = (x, 1) \in (\Sigma X)_{d+1}$ and w is a word in the free group on*

$$\bigvee_{\substack{1<i<d+1\\ q+i=d+1}} (X_q, i).$$

The simplicial operators on $\Lambda_2 X$ are the unique multiplicative extension of the functions which operate on generators as follows.

$$d_i^{\Lambda_2}(\mathsf{c}(w)[x]) = d_{i+1}^{F\Sigma}(\mathsf{c}(w)[x]) = \mathsf{c}(d_{i+1}^{F\Sigma}(w))[d_i x]$$

$$s_i^{\Lambda_2}(\mathsf{c}(w)[x]) = s_{i+1}^{F\Sigma}(\mathsf{c}(w)[x]) = \mathsf{c}(s_{i+1}^{F\Sigma}(w))[s_i x].$$

The filtration given in Theorem 1.1 in this case is described as follows. Filter the free group on the set $\bigvee_{\substack{1<i<d+1\\ q+i=d+1}} (X_q, i)$ by the reduced word length filtration. We say that a generator $\mathsf{c}(w)[x]$ is in filtration n if w is in filtration n of the reduced word length filtration. Now define $F_n\Lambda_2(X)$ to be the subgroup of $\Lambda_2(X)$ generated by all elements $\mathsf{c}(w)[x]$ of filtration at most n. It is easy to see that $F_0\Lambda_2(X)$ is the subgroup generated by the image of the Freudenthal suspension. The other filtrations are not understood at present.

The construction of a free simplicial group model for $\Omega^n X$ is analogous. Given an n-connected pointed reduced simplicial set X, let $\mathcal{L}_n(X)$ denote the free simplicial group given by $\Omega^{n-1}GX$. Here in order to present a basis for this free simplicial group, one needs to make a change of basis for the free group GX. Assuming this has been done in the appropriate manner, consider the formal $(n-1)$-fold desuspension of this generating set. This is a pointed graded set, which can be decomposed as a one-point union $A_n(X) \vee B_n(X)$.

Theorem 1.3. *Let X be a pointed n-connected reduced simplicial set and let n be a non-negative integer. There is a simplicial group $\mathcal{L}_nX$, whose geometric realization is homotopy equivalent to $\Omega^n|X|$. Moreover,*

1. *A basis for $\mathcal{L}_n(X)$ in degree n is given by the graded set $B_n(X)^{F[A_n(X)]}$.*
2. *The simplicial operators in $\mathcal{L}_n(X)$ are described explicitly in terms of those of X in the proof of this theorem.*
3. *The groups $\mathcal{L}_n(X)$ admit a natural increasing filtration by simplicial subgroups.*

Here too the first interesting case is $n = 2$. An explicit description of it is given in the following proposition. For higher values of n, the description becomes recursive and hence less explicit. The reader is referred to section 4 for details.

Proposition 1.4. *Let X be a 2-connected reduced complex. Then $(\mathcal{L}_2X)_n$ is isomorphic as a group to the quotient of the free group on symbols $\langle w, x\rangle$ with $w \in (GX)_n$ and $x \in X_{n+2}$ subject to the relations $\langle w, x\rangle = 1$ if $x = s_iy$ for some $y \in X_{n+1}$ and $i \in \{0, 1\}$. The simplicial operators are given by the formulas*

1. $s_i^{\mathcal{L}_2}\langle w, x\rangle = \langle s_i^G w, s_{i+2}x\rangle$ *for* $i \geq 0$.
2. $d_i^{\mathcal{L}_2}\langle w, x\rangle = \langle d_i^G w, d_{i+2}x\rangle$ *for* $i \geq 1$.
3. $d_0^{\mathcal{L}_2}\langle 1, x\rangle = \langle 1, d_2x\rangle\langle d_0^G d_0x, d_1x\rangle^{-1}\langle d_0^G d_0x, d_0x\rangle$ *and* $d_0^{\mathcal{L}_2}\langle w, x\rangle$ *can be computed in terms of the generators whenever w is given as an explicit word in GX.*

Viewing the last proposition as a description of pointed mapping spaces out of a 2-sphere, one is motivated to wonder about models for mapping spaces out of more general orientable surfaces. Let S_g^n denote a pointed surface of genus g and a boundary consisting of n circles wedged at the base point.

Theorem 1.5. *Let X be a 1-connected complex. Then for every $g, k \geq 0$ there is a Kan complex $M_g^k(X)$ whose geometric realization is equivalent to the pointed mapping space $Map_*(S_g^k, X)$. The simplices of $M_g^k(X)$ in dimension n are given by $(2g+k+1)$-tuples $(u_1, v_1, \cdots, u_g, v_g, a_1, \cdots a_k, w)$, where $u_i, v_j, a_r \in (GX)_n$, and $w \in (GX)_{n+1}$ are such that*

$$[u_1, v_1] \cdots [u_g, v_g] \cdot (a_1 \cdots a_k)^{-1} = d_o^G(w).$$

The simplicial operators of $M_g^k(X)$ are induced in the obvious way by those of GX. Moreover, $M_0^0(X)$ coincides with $\mathcal{L}^2(X)$.

In closing of the introduction, the reader is warned that the models constructed here are very unlikely to have any computational utility. On the other hand they do provide a context in which general homotopy theoretic phenomena can be studied.

The paper is organized as follows. In section 2, a review of some classical simplicial constructions are given. The construction Λ_n and $\mathcal{L}_n$ are discussed in sections 3 and 4 respectively. Section 5 is devoted to a comparison of the models via the Freudenthal suspension and evaluation maps. A sample calculation is given in this section. The filtrations for these models are introduced and discussed in section 6. A model for a pointed mapping space out of a surface. is given in section 7.

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2. Review of some classical simplicial constructions

As pointed out in the introduction, some classical simplicial constructions will be used in the sequel. The purpose of this preliminary section is to recall these constructions.

The reader is assumed to be familiar with the concept of a Kan complex. These are simplicial sets with an additional property that makes it possible to combinatorially define homotopy groups. The actual definition will not be recalled here as these facts are not actually used in this article. Furthermore, any simplicial set can be made into a Kan complex without changing the homotopy type of its geometric realization, but in fact simplicial groups are automatically Kan complexes. The term "a Kan complex" will thus be used to emphasize that the given simplicial set is assumed to be a Kan complex. A complex will mean an arbitrary simplicial set.

For a pointed, connected complex X the suspension ΣX is defined to be the complex, which in dimension r consists of a 1-point union

$$\bigvee_{\substack{0<i<r\\ q+i=r}} (X_q, i),$$

where (X_q, i) is a copy of X_q. Ignoring the simplicial operators momentarily, notice that the construction can be easily iterated. The following proposition summarizes the structure of the iterated suspension with the simplicial operators. Notice also that the suspension of a Kan complex is not a Kan complex in general.

Proposition 2.1. *Let X be a pointed connected reduced complex and let n be a positive integer. The the n-fold suspension $\Sigma^n X$ of X is the complex, which in dimension r is given by*

$$\bigvee_{\substack{0<i_1,i_2,\cdots,i_n<r\\ q+i_1+\cdots+i_n=r}} (X_q, i_1, i_2, \cdots, i_n),$$

where $(X_q, i_1, i_2, \cdots, i_n)$ is a copy of X_q as sets.

The simplicial operators are given inductively by the formulas

$$d_i^{\Sigma^n}(a_q, i_1, \cdots, i_n) = \begin{cases} (a_q, i_1, \cdots, i_{n-1}, i_n - 1) & \text{if } i < i_n \\ (d_{i-i_n}^{\Sigma^{n-1}}(a_q, i_1, \cdots, i_{n-1}), i_n) & \text{if } i \geq i_n \end{cases}$$

$$s_i^{\Sigma^n}(a_q, i_1, \cdots, i_n) = \begin{cases} (a_q, i_1, \cdots, i_{n-1}, i_n + 1) & \text{if } i < i_n \\ (s_{i-i_n}^{\Sigma^{n-1}}(a_q, i_1, \cdots, i_{n-1}), i_n) & \text{if } i \geq i_n \end{cases}$$

For a pointed simply-connected reduced complex X, the Moore path-complex construction PX is given by

$$(PX)_n = X_{n+1}.$$

The simplicial operators are given by $d_i^{PX} = d_{i+1}^X$, and $s_i^{PX} = s_{i+1}^X$ for $i \geq 0$.

Then there is a simplicial projection map from PX to X given by the leftover face d_0 and the Moore loop space construction ΩX is defined as the set theoretic kernel of this map, i.e. the inverse image of the base-point under the projection. If X is a Kan complex then so is ΩX and in that case $|\Omega X| \simeq \Omega|X|$ but not generally otherwise (for example, if S^n is the standard simplicial model for the n-sphere, with one non-degenerate simplex in dimension n, then $\Omega S^n = S^{n-1}$).

The Moore loop-complex construction can be iterated and the resulting complex is described in the following

Proposition 2.2. *Let n be a positive integer and let X be a pointed n-connected reduced complex. The n-fold Moore loop-complex $\Omega^n(X)$ is the complex which in dimension k consists of all simplices in X_{n+k} such that $d_i(x) = b_{n+k-1}$ for all $0 \leq i \leq n$, where b_j denotes the base-point simplex in dimension j. The simplicial operators in $\Omega^n(X)$ are given by $d_i^{\Omega^n}(x) = d_{i+n}(x)$ and $s_i^{\Omega^n}(x) = s_{i+n}(x)$.*

Let X be a pointed simply-connected reduced complex. The Kan loop-group functor G associates with X a simplicial group GX which in dimension n is freely generated by the simplices of X_{n+1} subject to the relation $s_0(x) = 1$ for any $x \in X_n$. The simplicial operators are given on generators as follows

1. $d_0^G(x) = d_0(x)^{-1}d_1(x)$
2. $d_i^G(x) = d_{i+1}(x)$ for $i \geq 1$.
3. $s_i^G(x) = s_{i+1}(x)$ for $i \geq 0$.

The complex GX is automatically a Kan complex by virtue of being a simplicial group. Its geometric realization is homotopy equivalent to $\Omega|X|$ [2].

Let X be a pointed, connected complex. The Milnor free group construction FX is the free simplicial group, which in dimension n is generated by X_n modulo the relation setting the base-point to be equal to the identity element. The face and degeneracy operators are the unique multiplicative extensions of the respective operators in X. Notice that being a simplicial group FX is automatically a Kan complex. The geometric realization of FX is homotopy equivalent to $\Omega\Sigma|X|$. The standard references for these constructions are [2, 5].

Finally recall an elementary fact about free groups. Let $Fr[-]$ denote the pointed free group functor on the category of pointed sets. Namely, $Fr[X]$ is the free group generated by X with a single relation setting the base-point to be the identity. Let A and B be arbitrary pointed sets. Then there is a natural map

$$\pi_A : Fr[A \vee B] \longrightarrow Fr[A].$$

An elementary covering space argument shows that the kernel of π_A is freely generated by the set $B^{Fr[A]}$, namely the set of all conjugates of non-trivial elements of B by arbitrary words in the free group on A [3]. This fact is also used in [1], page 139. A proof using covering spaces is given in the appendix.

For $w \in Fr[A]$ and $b \in B$ let $\mathsf{c}(w)(b) = w^{-1}bw$. Thus $\mathrm{Ker}(\pi_A)$ is freely generated by the symbols $\mathsf{c}(w)(b)$, for $b \in B$ and $w \in F[A]$.

3. The construction Λ_n

Simplicial group models for $\Omega^n\Sigma^n X$ are known in the literature. In particular there is a model for these spaces due to J. Smith [7]. Smith's construction is elegant and "homologically minimal".

The constructions in this article by contrast are large, but on the other hand have the feature that writing down explicit maps out of them becomes relatively easy.

The model for $\Omega^n\Sigma^n X$ is based on the simplicial suspension construction Σ , Milnor's F construction and the Moore loop complex construction Ω, all of which are reviewed above. Thus define $\Lambda_0(X) = X$ and for each $n \geq 1$ let

$$\Lambda_n X = \Omega^{n-1} F \Sigma^{n-1} X.$$

There is a natural simplicial map

$$E : X \longrightarrow \Lambda_1 X,$$

given by taking a simplex $x \in X$ to the generator in $\Lambda_1 X = FX$ represented by it. One may regard this map as a natural transformation from the identity functor Λ_0 to the functor Λ_1. The geometric realization of E is the usual Freudenthal suspension on the realization of X.

The first interesting case is $n = 2$, which is handled separately, as its relative simplicity provides some insight. Also, examination of the resulting model in this case will enable us to define the n-fold Freudenthal suspension map on the level of simplicial groups.

Proposition 3.1. *A basis for the free simplicial group $\Lambda_2 X$ in dimension d is given by all elements of the form $\mathsf{c}(w)[x]$, where $[x] = (x, 1) \in (\Sigma X)_{d+1}$ and w is a word in the free group on*

$$\bigvee_{\substack{1<i<d+1 \\ q+i=d+1}} (X_q, i).$$

The simplicial operators on $\Lambda_2 X$ are the unique multiplicative extension of the functions which operate on generators as follows.

$$d_i^{\Lambda_2}(\mathsf{c}(w)[x]) = d_{i+1}^{F\Sigma}(\mathsf{c}(w)[x]) = \mathsf{c}(d_{i+1}^{F\Sigma}(w))[d_i x]$$

$$s_i^{\Lambda_2}(\mathsf{c}(w)[x]) = s_{i+1}^{F\Sigma}(\mathsf{c}(w)[x]) = \mathsf{c}(s_{i+1}^{F\Sigma}(w))[s_i x].$$

Proof. By definition $(\Lambda_2 X)_d$ is equal to the kernel of the homomorphism

$$d_0^{F\Sigma} : (F\Sigma X)_{d+1} \longrightarrow (F\Sigma X)_d.$$

Notice that the group $(F\Sigma X)_{d+1}$ is generated freely by all elements of the form (a_q, i), where $0 < i < d+1$ and $i+q = d+1$. Thus the generating set can be written as

$$\bar{A}_2(X)_{d+1} \vee \bar{B}_2(X)_{d+1},$$

where $\bar{A}_2(X)_{d+1}$ is the set of all elements (a_q, i) with $i > 1$ and $\bar{B}_2(X)_{d+1}$ is the pointed complement of $\bar{A}_2(X)_{d+1}$ in $(\Sigma X)_{d+1}$. Notice that the set $\bar{A}_2(X)_{d+1}$ is exactly the image of $s_0^{F\Sigma} : (\Sigma X)_d \longrightarrow (\Sigma X)_{d+1}$. Thus the restriction of $d_0^{F\Sigma}$ to $\bar{A}_2(X)_{d+1}$ is an isomorphism of sets. Let $A_2(X)$ denote the desuspension of $\bar{A}_2(X)$ and similarly define $B_2(X)$.

The discussion above now gives that

$$(\Lambda_2 X)_d \cong Fr[B_2(X)_d^{Fr[A_2(X)_d]}]$$

as claimed. The calculation of the simplicial operators follows directly from the definitions. □

There is a natural transformation of functors

$$\Omega E_\Sigma : \Lambda_1 \longrightarrow \Lambda_2,$$

given as the unique multiplicative extension of the function given on generators by

$$\Omega E_{\Sigma X}(x) = (x, 1) = [x].$$

Notice that one can define $B_2(X)$ as the image of X under $\Omega E_{\Sigma X}$ and $A_2(X)$ as its pointed complement.

Lemma 3.2. *The geometric realization of $\Omega E_{\Sigma X}$ is the loops on the Freudenthal suspension for $\Sigma|X|$,*

$$\Omega E_{\Sigma|X|} : \Omega\Sigma|X| \longrightarrow \Omega^2\Sigma^2|X|.$$

Proof. Notice that for any simplicial set X, the composition $G\Sigma X$ is canonically isomorphic to FX. There is the simplicial Freudenthal suspension map for ΣX

$$E_{\Sigma X} : \Sigma X \longrightarrow F\Sigma X.$$

Applying Kan's G functor one obtains a multiplicative map

$$GE_{\Sigma X} : FX \longrightarrow GF\Sigma X.$$

On the other hand there is a natural map

$$\iota : \Lambda_2 X = \Omega F\Sigma X \longrightarrow GF\Sigma X$$

which is a homotopy equivalence. It follows directly from the definition that the image of $GE_{\Sigma X}$ is contained in the image of the map ι and since both maps are monomorphic, there is a unique factorization of $GE_{\Sigma X}$ through $\Lambda_2 X$, which is easily checked to coincide with $\Omega E_{\Sigma X}$. The lemma follows. □

For $n \geq 2$ define natural transformations

$$\Omega^n E_{\Sigma^n} : \Lambda_n \longrightarrow \Lambda_{n+1}$$

by composing the functors Ω^{n-1} and Σ^{n-1} with the natural transformation ΩE_Σ. More precisely,

$$\Omega^n E_{\Sigma^n} = \Omega^{n-1}(\Omega E_\Sigma)\Sigma^{n-1}.$$

It follows at once from Lemma 3.2 that for a given simplicial set X, the geometric realization of $\Omega^n E_{\Sigma^n X}$ is the (n)-fold loops on the Freudenthal suspension for $\Sigma^n X$.

These transformations encode the n-fold Freudenthal suspension

$$E^n : \Lambda_0 \longrightarrow \Lambda_n$$

in the obvious way. Namely $E^2 = \Omega E_\Sigma \circ E$ and if E^{n-1} is defined then $E^n = \Omega^{n-1}E_{\Sigma^{n-1}} \circ E^{n-1}$. Notice that $B_2(X) = E^2_X(X)$.

Lemma 3.1 generalizes to $n > 2$ as follows. Let $\bar{A}_n(X)$ denote the graded set which in dimension $d+n-1$ consists of all elements in $(\Sigma^{n-1}X)_{d+n-1}$ of the form $(a_q, i_1, \cdots, i_{n-1})$, such that $i_j > 1$ for some $1 \leq j \leq n-1$. Notice that by definition $a_q \in X_q$ and $q + i_1 + \cdots + i_{n-1} = d$. Let $\bar{B}_n(X)$ denote the pointed complement of $\bar{A}_n(X)$ in $\Sigma^{n-1}X$. Let $A_n(X)$ denote the $(n-1)$-fold desuspension of $\bar{A}_n(X)$ and define $B_n(X)$ in a similar fashion.

Proposition 3.3. *For a simplicial set X and an integer $n \geq 2$ a basis for the free simplicial group $\Lambda_n X$ is given by the graded set $B_n(X)^{Fr[\bar{A}_n(X)]}$. Furthermore, the set $B_n(X) \subseteq \Lambda_n X$ given by the image of the n-fold Freudenthal suspension $E^n_X : X \longrightarrow \Lambda_n X$.*

Proof. Consider the map of sets

$$\beta : (\Sigma^{n-1}X)_{d+n-1} \xrightarrow{d_0 \top d_1 \cdots \top d_{n-2}} \prod_{n-1} (\Sigma^{n-1}X)_{d+n-2}.$$

Then $(\Sigma^{n-1}X)_{d+n-1}$ can be written as a wedge of the inverse image under β of the base-point and its pointed complement. Examination of the definition above gives that $\bar{B}_n(X)_{d+n-1}$ is exactly the inverse image of the base-point under β in this dimension.

Notice that $(\Lambda_n X)_d$ is isomorphic to the kernel of

$$\beta : F(\Sigma^{n-1}X)_{d+n-1} \xrightarrow{d_0 \top d_1 \cdots \top d_{n-2}} \prod_{n-1} F(\Sigma^{n-1}X)_{d+n-2}.$$

By the discussion above this kernel is freely generated by $B_n(X)_d^{Fr[A_n(X)_d]}$.

For $n = 2$ the discussion above gives that for every $x \in X$ one has $\Omega E_{\Sigma X}[x] = (x, 1) \in \Lambda_2(X)$ where $[x]$ represents the class $E_X(x)$ of x in $\Lambda_1 X = FX$. In fact it follows from the definition that if $[x] \in \Lambda_{n-1}X$ denotes $E_X^{n-1}(x)$ then $\Omega^{n-1}E_{\Sigma^{n-1}X}[x] = ([x], 1) \in \Lambda_n X$. Then for each $x \in X$ one has

$$E_X^n(x) = \Omega^{n-1}E_{\Sigma^{n-1}X}(E^{n-1}(x)) = \Omega^{n-1}E_{\Sigma^{n-1}X}[x] = ([x], 1) \in \Lambda_n X$$

and it follows that $B_n(X)$ coincides with $E_X^n(X)$ as a subset of $\Lambda_n X$. □

Corollary 3.4. *For every $n \geq 2$ a basis for $\Lambda_n X$ is given by homogeneous symbols $\mathsf{c}(w)[x]$ where $x \in X$ is a non-trivial simplex, $[x] = E_X^n(x) \in \Lambda_n X$ and w is a word (possibly trivial) in the free group generated by $A_n(X)$.*

The face and degeneracy operators on $\Lambda_n X$ admit an analogous description to the case $n = 2$. The proof is also similar and is left to the reader.

Proposition 3.5. *The simplicial operators on $\Lambda_n X$ are the unique multiplicative extension of the functions which operate on generators as follows.*

$$d_i^{\Lambda_n}(\mathsf{c}(w)[x]) = d_{i+n-1}^{F\Sigma^{n-1}}(\mathsf{c}(w)[x]) = \mathsf{c}(d_{i+n-1}^{F\Sigma^{n-1}}(w))[d_i x]$$

$$s_i^{\Lambda_n}(\mathsf{c}(w)[x]) = s_{i+n-1}^{F\Sigma^{n-1}}(\mathsf{c}(w)[x]) = \mathsf{c}(s_{i+n-1}^{F\Sigma^{n-1}}(w))[s_i x].$$

The interested reader can verify now that all simplicial operators preserve the generating set except for $d_0^{\Lambda_n}$. In fact d_0 may take a generator to a word of arbitrary length. To see this consider the following example. In $\Lambda_2 X$ the element $\mathsf{c}((y,2))(x,1)$ is a generator. If $i > 0$ then

$$d_i^{\Lambda_2}(\mathsf{c}((y,2))(x,1)) = \mathsf{c}(d_{i+1}^{F\Sigma}(y,2))(d_i x, 1) = \mathsf{c}((d_{i-1}y, 2))(d_i x, 1).$$

But for $i = 0$ one gets

$$d_0^{\Lambda_2}(\mathsf{c}((y,2))(x,1)) = \mathsf{c}(d_1^{F\Sigma}(y,2))(d_0 x, 1) = \mathsf{c}((y,1))(d_0 x, 1),$$

which is a product of three generators.

The next proposition records a formula the n-fold loops on the degree q map on $\Sigma^n X$.

Proposition 3.6. *Let q be a positive integer and let $[q]$ denote the degree q map on $\Sigma|X|$. Then the n-fold loop of $[q]$ on $\Sigma^n(X)$ is given by*

$$\Omega^n[q](\mathsf{c}(w)[x]) = \mathsf{c}([q]w)[x]^q,$$

where $[q]w$ means - raise each letter in the word composing w to the q-th power.

Proof. For an integer q the degree q map on $\Sigma^n X$ is adjoint to the composition

$$X \xrightarrow{E^n} \Omega^n\Sigma^n X \xrightarrow{\langle q \rangle} \Omega^n\Sigma^n X,$$

where $\langle q \rangle$ is the q-th power map. Thus $ad[q] : X \longrightarrow \Lambda_n X$ has the form

$$ad[q](x) = [x]^q.$$

On the other hand the n-fold loop of $[q]$ can be computed in $\Lambda_n X$ as follows. On $\Lambda_1 X$ the single loops on $[q]$ is given by sending a generator $[x]$ to its q-th power $[x]^q$. Thus on $\Lambda_n X$ the n-fold loops on $[q]$ is given by sending a generator $\mathsf{c}(w)[x]$ to $\mathsf{c}([q]w)[x]^q$, as claimed. □

4. The construction $\mathcal{L}_n X$

The model below for $\Omega^n X$ is constructed in a very similar fashion to $\Lambda_n X$. Assume that the complex X is n-connected and reduced. Replace the functor $F\Sigma^{n-1}$ in the construction of $\Lambda_n X$ by Kan's loops group functor G. Thus define $\mathcal{L}_n X$ to be $\Omega^{n-1} GX$.

The generating set in this case is more complicated to describe. However, once this is done $\mathcal{L}_n X$ has some similar features to those of $\Lambda_n X$ which are described in a later section.

As before, start with an analysis of the group $\mathcal{L}_2 X = \Omega GX$. As a free group, GX is generated by the set $\bar{X} = X/s_0(\Sigma^{-1}X)$. In dimension n, $\mathcal{L}_2 X$ is given as the kernel of the homomorphism

$$d_0^G : (GX)_{n+1} \longrightarrow (GX)_n.$$

Since d_0^G is an epimorphism with right inverse s_0^G and $(GX)_n$ is free, there is a group isomorphism

$$(GX)_{n+1} \cong (\mathcal{L}_2 X)_n \rtimes (GX)_n.$$

The following definition is useful.

Definition 4.1. *An epimorphism $f : F \longrightarrow G$ of free groups is called proper if it has a right inverse s satisfying the following property: there exist sets X freely generating F and Y freely generating G, such that s maps Y into a subset of X.*

If $f : F \longrightarrow G$ is a proper epimorphism of free groups, then with the notation above, let $A(X)$ denote the subset $s(Y)$ of X. Let $B'(X)$ denote the pointed complement of $A(X)$. Let $B(X)$ denote the subset of F given by all elements $x \cdot sf(x)^{-1}$ for all $x \in B'(X)$.

Lemma 4.2. *Let $f : F \longrightarrow G$ be a proper epimorphism of free groups with kernel K. Then, with the notation above, K is freely generated by the set $B(X)^{Fr[A(X)]}$.*

Proof. Let $\alpha : B'(X) \longrightarrow B(X)$ be the map of sets given by $\alpha(x) = x \cdot sf(x)^{-1}$. Then α is an epimorphism by construction. Let $x_1, x_2 \in B'(X)$ be such that $\alpha(x_1) = \alpha(x_2)$. Then

$$\alpha(x_1) = x_1 sf(x_1)^{-1} = x_2 sf(x_2)^{-1} = \alpha(x_2).$$

Thus $x_2^{-1} x_1 = s(f(x_2)^{-1} f(x_1))$. But the left hand side of the last equation is a word in the free group on $B'(X)$, whereas the right hand side is a word in the free group on $A(X)$. Since these two sets intersect only on the base point, it follows that both sides are trivial so $x_1 = x_2$ and α is an isomorphism of sets.

Now write X as $A(X) \vee B'(X)$. There is an isomorphism of sets from X to the subset $A(X) \vee B(X)$ of F taking any element of $A(X)$ to itself and any element of $B'(X)$ to $B(X)$ via α. It is immediate that $A(X) \vee B(X)$ freely generates F because $sf(x)$ is in the free subgroup generated by $A(X)$. Thus there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{f} & G & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle Fr[1\vee\alpha]} & & \big\downarrow{\scriptstyle 1} & & \\
1 & \longrightarrow & Fr[B(X)^{Fr[A(X)]}] & \longrightarrow & Fr[A(X)\vee B(X)] & \xrightarrow{\pi_A} & Fr[A(X)] & \longrightarrow & 1
\end{array}$$

Since the right two vertical arrows are isomorphisms so is the left one and the result follows. ☐

Definition 4.3. *Define an operator κ on GX by $\kappa(w) = s_0^G d_0^G(w)$.*

Lemma 4.4. *Let X be a simplicial set and let $\bar{X}$ denote the generating set for GX as defined above. Then a point $x \in \bar{X}$ is a fixed point of κ if and only if $x = s_1 y$ for some $y \in X$.*

Proof. Since s_0^G is the unique multiplicative extension of s_1, it is clear that each fixed point x of κ in $\bar{X}$ is in the image of s_0^G. But x considered as a word in GX is of length 1 (or otherwise is the base point). On the other hand

$$\kappa(x) = s_0^G d_0^G x = s_1(d_1(x)d_0(x)^{-1}) = s_1 d_1(x) \cdot s_1 d_0(x)^{-1}$$

is of length at most 2. Hence one of the factors must be trivial and we have $x = s_1 d_i x$ for $i = 0$ or $i = 1$.

Conversely, if $x = s_1 y$ then

$$\kappa(x) = \kappa(s_0^G y) = s_0^G d_0^G s_0^G y = s_0^G y = x.$$

☐

Let X be a 2-connected reduced complex. Let $A'(X)$ denote the graded set given by the intersection of $\bar{X}$ with the fixed point set of κ on GX. Let $B''(X)$ denote the graded set given the pointed complement of $A'(X)$ in $\bar{X}$. Let $A(X)$ and $B'(X)$ denote the double desuspension of $A'(X)$ and $B''(X)$ respectively. Then

$$(GX)_{n+1} = Fr(\bar{X}_{n+2}) = F[A(X)_n \vee B'(X)_n].$$

Let $B(X)$ denote the graded set, which in dimension n is the subset of $(GX)_{n+1}$ given by all elements of the form $\alpha(x) = x\kappa(x)^{-1}$ for $x \in B'(X)_n$. Then $(GX)_{n+1}$ is freely generated by $A(X)_n \vee B(X)_n$.

The next corollary follows from Lemma 4.2.

Corollary 4.5. *The group $\mathcal{L}_2 X$ is isomorphic as a graded group to the free group $Fr[B(X)^{Fr[A(X)]}]$.*

Thus in terms of a word in the free group on the simplices of X every n-dimensional generator in the specified generating set for $\mathcal{L}_2 X$ has the form $\mathsf{c}(w)(\alpha(x))$, where w is a word in the free group on $A(X)_n$ and $x \in B'(X)_n$. A more convenient description for the group $\mathcal{L}_2 X$, analogous to the presentation of $\Lambda_2 X$ given above, will now be derived.

Notation 4.6. *For $x \in B'(X)_n$, denote $\alpha(x)$ by $[x]$. Then the generating set for $\mathcal{L}_2 X$ defined above consists of symbols of the form $\mathsf{c}(w)[x]$, where $w \in Fr[A(X)]$.*

Consider the behavior of α with respect to the simplicial operators. Notice that $d_i^{\mathcal{L}_2}$ and $s_i^{\mathcal{L}_2}$ are the multiplicative extensions of d_{i+2} and s_{i+2} respectively. Equivalently these operators can be thought of as extensions of d_{i+1}^G and s_{i+1}^G respectively. The following lemma is routine and the verification is left to the reader.

Lemma 4.7. *With the notation above, the function α satisfies the following relations*

1. *For $i > 0$, $d_i^{\mathcal{L}_2}[x] = [d_{i+2}(x)]$.*
2. *For $i \geq 0$, $s_i^{\mathcal{L}_2}[x] = [s_{i+2}(x)]$.*
3. *$d_0^{\mathcal{L}_2}[x] = d_2(x)d_1(x)^{-1}d_0(x)$.*

For any $x \in X$ define $\gamma(x) = d_2(x)d_1(x)^{-1}d_0(x)$. Then $d_0^{\mathcal{L}_2}[x] = \gamma(x)$. The operator γ will be of some significance. Its commutation properties with respect to the simplicial operators are recorded in the following lemma. Here too, the verification is easy and is left for the reader.

Lemma 4.8. *The function γ satisfies the following relations*

1. *$d_i^{\mathcal{L}_2}(\gamma(x)) = \gamma(d_{i+3}(x))$.*
2. *$s_i^{\mathcal{L}_2}(\gamma(x)) = \gamma(s_{i+3}(x))$.*
3. *$\gamma(s_0^{\mathcal{L}_2}(x)) = [x]$.*

Notice that 1 and 2 in the previous lemma imply that the set of all elements of the form $\gamma(x)$, $x \in X$ form a simplicial subset of $\mathcal{L}_2(X)$. Furthermore, 3 implies that this simplicial subset contains all elements of the form $[x] = \alpha(x)$.

The function α has a useful multiplicative property which is recorded next. For any word $w \in GX$ define $\alpha(w) = w \cdot \kappa(w)^{-1}$. By analogy to the case when w is of length 1 denote $\alpha(w)$ by $[w]$.

Lemma 4.9. *Let $w = x_1x_2 \cdots x_n \in GX$. Then*

$$[w] = [x_1] \cdot \mathsf{c}(\kappa(x_1)^{-1})([x_2]) \cdots \mathsf{c}(\kappa(x_1x_2\cdots x_{n-1})^{-1})([x_n]).$$

Proof. For w of length 1 there is nothing to prove. Let $w = x_1x_2\cdots x_n$. Then

$$\alpha(w) = x_1(x_2\cdots x_n)\kappa(x_2\cdots x_n)^{-1}\kappa(x_1)^{-1} = x_1\alpha(x_2\cdots x_n)\kappa(x_1)^{-1} = \alpha(x_1)\mathsf{c}(\kappa(x_1)^{-1})(\alpha(x_2\cdots x_n)).$$

The claim follows by induction. □

Notice that it was not assumed in the lemma that the presentation of w is reduced. Thus as an immediate corollary one obtains that for every $w \in GX$,

$$1 = \alpha(1) = \alpha(ww^{-1}) = \alpha(w)\mathsf{c}(\kappa(w)^{-1})(\alpha(w^{-1})).$$

So that $\alpha(w)^{-1} = \mathsf{c}(\kappa(w)^{-1})(\alpha(w^{-1}))$ or equivalently $\alpha(w^{-1}) = \mathsf{c}(\kappa(w))(\alpha(w)^{-1})$. In the notation used above

$$[w^{-1}] = \mathsf{c}(\kappa(w))([w]^{-1}).$$

Theorem 4.10. *Let X be a 2-connected reduced complex. Then $(\mathcal{L}_2X)_n$ is isomorphic as a group to the quotient of the free group on symbols $\mathsf{c}(w)[x]$, with $[x] \in B(X)_n$ and $w \in Fr(A(X)_n)$, by the relations $\mathsf{c}(w)[x] = 1$ if $x = 1$.*

The simplicial operators are given by the formulas

1. *$s_i^{\mathcal{L}_2}(\mathsf{c}(w)[x]) = \mathsf{c}(s_{i+1}^G(w))[s_{i+2}(x)]$ for $i \geq 0$.*
2. *$d_i^{\mathcal{L}_2}(\mathsf{c}(w)[x]) = \mathsf{c}(d_{i+1}^G(w))[d_{i+2}(x)]$ for $i \geq 1$.*
3. *$d_0^{\mathcal{L}_2}([x]) = [d_2x]\mathsf{c}(\kappa d_0x)([d_1x]^{-1}[d_0x])$ and*
   $d_0^{\mathcal{L}_2}(\mathsf{c}(w)[x]) = \mathsf{c}(\kappa d_1^G w)(\mathsf{c}([d_1^G w])(d_0^{\mathcal{L}_2}[x]))$.

Proof. The calculation of the generating set is the contents of Corollary 4.5. It remains to verify the simplicial structure of $\mathcal{L}_2X$. By the remarks above one has

$d_i^{\mathcal{L}_2}[x] = [d_{i+2}x]$ for $i > 0$ and $s_i^{\mathcal{L}_2}[x] = [s_{i+2}x]$ for $i \geq 0$. Thus

$$d_i^{\mathcal{L}_2}(\mathsf{c}(w)[x]) = \mathsf{c}(d_{i+1}^G w)[d_{i+2}x]$$

for $i > 0$ and similarly

$$s_i^{\mathcal{L}_2}(\mathsf{c}(w)[x]) = \mathsf{c}(s_{i+1}^G w)[s_{i+2}x]$$

for $i \geq 0$. This give 1 and 2 in the theorem.

For the calculation of $d_0^{\mathcal{L}_2}$ first recall that

$$d_0^{\mathcal{L}_2}[x] = \gamma(x) = d_2 x(d_1 x)^{-1} d_0 x.$$

Then observe that

$$\begin{aligned}
&[d_2 x] \cdot \mathsf{c}(\kappa d_0 x)([d_1 x]^{-1}[d_0 x]) = \\
&\quad d_2 x(\kappa d_2 x)^{-1} \cdot \mathsf{c}(\kappa d_0 x)((d_1 x(\kappa d_1 x)^{-1})^{-1} d_0 x(\kappa d_0 x)^{-1}) = \\
&\quad d_2 x(\kappa d_2 x)^{-1}(\kappa d_0 x)^{-1}(d_1 x(\kappa d_1 x)^{-1})^{-1} d_0 x = \\
&\quad d_2 x(\kappa d_2 x)^{-1}(\kappa d_0 x)^{-1}(\kappa d_1 x) d_1 x^{-1} d_0 x = \\
&\quad d_2 x(\kappa d_1^G x)^{-1}(\kappa d_0^G x) d_1 x^{-1} d_0 x = \\
&\quad d_2(x) d_1(x)^{-1} d_0(x) = \gamma(x).
\end{aligned}$$

In our notation the equation above reads

$$d_0^{\mathcal{L}_2}[x] = [d_2(x)]\mathsf{c}(\kappa d_0 x)([d_1 x]^{-1}[d_0 x]).$$

More generally, notice that

$$\begin{aligned}
&d_0^{\mathcal{L}_2}(\mathsf{c}(w)[x]) = \mathsf{c}(d_1^G w)(d_0^{\mathcal{L}_2}[x]) = \\
&\quad \mathsf{c}([d_1^G w]\kappa d_1^G(w))(d_0^{\mathcal{L}_2}[x]) = \\
&\quad \mathsf{c}(\kappa d_1^G(w))(\mathsf{c}([d_1^G w])(d_0^{\mathcal{L}_2}[x]).
\end{aligned}$$

□

Notice that lemma 4.9 gives a way of writing $[d_1^G w]$ in the theorem explicitly in terms of the generators once w is specified as a word in GX. The precise formulation is omitted.

The statement of Theorem 4.10 does not generalize well to $\mathcal{L}_m X$ for $m > 2$ as the notation becomes quite cumbersome. Before proceeding a more uniform description for the generating set is given in the next theorem.

Theorem 4.11. *Let X be a 2-connected reduced complex. Then $(\mathcal{L}_2 X)_n$ is isomorphic as a group to the quotient of the free group on symbols $\langle w, x\rangle$ with $w \in (GX)_n$ and $x \in X_{n+2}$ subject to the relations $\langle w, x\rangle = 1$ if $x = s_i y$ for some $y \in X_{n+1}$ and $i \in \{0, 1\}$. The simplicial operators are given by the formulas*

1. $s_i^{\mathcal{L}_2}\langle w, x\rangle = \langle s_i^G w, s_{i+2}x\rangle$ *for* $i \geq 0$.
2. $d_i^{\mathcal{L}_2}\langle w, x\rangle = \langle d_i^G w, d_{i+2}x\rangle$ *for* $i \geq 1$.
3. $d_0^{\mathcal{L}_2}\langle 1, x\rangle = \langle 1, d_2 x\rangle\langle d_0^G d_0 x, d_1 x\rangle^{-1}\langle d_0^G d_0 x, d_0 x\rangle$ *and*
   $d_0^{\mathcal{L}_2}\langle w, x\rangle = \mathsf{c}(w)(d_0^{\mathcal{L}_2}\langle 1, x\rangle)$

Proof. Let $U_2 X$ denote the graded free group generated in dimension n by the symbols described in the theorem subject to the given relations. Then there is a map

$$\phi : U_2 X \longrightarrow \mathcal{L}_2 X$$

sending $\langle w, x\rangle$ to $\mathsf{c}(s_0^G w)[x]$.

To check that ϕ is well defined notice that if $x = s_0 y$ then $\kappa(x) = 1$ and so $[x] = x = s_1 y = 1$ in GX. Hence generators of the form $\langle w, x\rangle$ with $x = s_0 y$ are in the kernel of ϕ. On the other hand if $x = s_1 y$ for some y then $[x] = 1$ in GX since in this case $\kappa(x) = x$. Consequently ϕ is well defined. Furthermore, since $A(X)$ is isomorphic as a set to GX via the degeneracy operator s_0^G, it follows that ϕ is an epimorphism. To prove that ϕ is an isomorphism it remains to show that its kernel vanishes. Thus it suffices to show that if $w \in (GX)_n$ and $x \in X_{n+2}$ then $\mathsf{c}(s_0^G w)[x] = 1$ if and only if $x = s_i y$ for some y and $i \in \{0, 1\}$.

Indeed $\mathsf{c}(s_0^G w)[x] = 1$ if and only if $[x] = 1$ as an element of GX. But $[x] = x\kappa(x)^{-1}$. Thus $[x] = 1$ if and only if $x = \kappa(x)$. By Lemma 4.4 this holds for $x \in \bar{X}$ if and only if $x = s_1 y$ for some y, whence the result.

The simplicial operators can be calculated directly from Theorem 4.10 using the identification above. Notice that $d_0^{\mathcal{L}_2}\langle w, x\rangle$ is not given in terms of the generating set. However, one can work out the explicit expression using Lemma 4.9 and the observation that $w = [w]\kappa(w)$. $\square$

The next theorem generalizes Theorem 4.11 to $n \geq 2$.

Theorem 4.12. *Let X be an m-connected reduced complex for $m \geq 2$. Then $(\mathcal{L}_m X)_n$ is isomorphic as a group to the quotient of the free group on symbols*

$$\langle w_{m-1}, w_{m-2}, \cdots, w_1, x\rangle, \quad x \in X_{n+m} \quad \text{and} \quad w_i \in (\mathcal{L}_i X)_{n+m-i-1},$$

subject to the relations $\langle w_{m-1}, w_{m-2}, \cdots, w_1, x\rangle = 1$ if either

1. *for some $s < m$ the element $\langle w_{s-1}, \cdots w_1, x\rangle$ represents the identity in $\mathcal{L}_s X$ or*
2. *there exists a generator $\langle v_{m-2}, \cdots, v_1, y\rangle$ in $\mathcal{L}_{m-1}X$ such that*

$$s_0^{\mathcal{L}_{m-1}}\langle v_{m-2}, \cdots, v_1, y\rangle = \langle w_{m-2}, \cdots, w_1, x\rangle.$$

The simplicial operators are defined recursively as follows.

1. $s_i^{\mathcal{L}_m}\langle w_{m-1}, w_{m-2}, \cdots, w_1, x\rangle = \langle s_i^{\mathcal{L}_{m-1}} w_{m-1}, s_{i+1}^{\mathcal{L}_{m-2}} w_{m-2} \cdots s_{i+m-2}^{\mathcal{L}_1} w_1, s_{i+m} x\rangle$ *for each $i \geq 0$.*
2. $d_i^{\mathcal{L}_m}\langle w_{m-1}, w_{m-2}, \cdots, w_1, x\rangle = \langle d_i^{\mathcal{L}_{m-1}} w_{m-1}, d_{i+1}^{\mathcal{L}_{m-2}} w_{m-2} \cdots d_{i+m-2}^{\mathcal{L}_1} w_1, d_{i+m} x\rangle$ *for each $i \geq 1$.*
3. *The face operator $d_0^{\mathcal{L}_m}$ can be expanded recursively using the formula* $d_0^{\mathcal{L}_m}\langle w_{m-1}, w_{m-2}, \cdots, w_1, x\rangle = \mathsf{c}(w_{m-1})(d_1^{\mathcal{L}_{m-1}}(u)\cdot d_0^{\mathcal{L}_{m-1}}(u)^{-1})$, *where u denotes the generator $\langle w_{m-2}, \cdots, w_1, x\rangle$ of $\mathcal{L}_{m-1}X$.*

Proof. The statement of the theorem for $m = 2$ is the contents of Theorem 4.11. For $u \in \mathcal{L}_r X$ define $\kappa_r(u) = s_0^{\mathcal{L}_r} d_0^{\mathcal{L}_r}(u)$. For $m > 2$ assume by induction that the theorem holds for $m - 1$.

For each $1 \leq s \leq m-1$ let $G_s X$ denote the subset of $\mathcal{L}_s X$ given by the symbols

$$\langle w_{s-1}, w_{s-2}, \cdots, w_1, x\rangle, \quad x \in X_{n+s} \quad \text{and} \quad w_i \in (\mathcal{L}_i X)_{n+s-i-1}.$$

Then by induction hypothesis $G_s X$ is closed under $s_0^{\mathcal{L}_s}$. Let $R_s X$ denote the subset of $G_s X$ given by

$$\left\{ \begin{array}{l|c} \langle w_{s-1}, w_{s-2}, \cdots, w_1, x\rangle & \exists\langle v_{s-2}, \cdots, v_1, y\rangle \in G_{s-1}X, \\ & s_0^{\mathcal{L}_{s-1}}\langle v_{s-2}, \cdots, v_1, y\rangle = \langle w_{s-2}, \cdots, w_1, x\rangle \text{ or} \\ & \langle w_{j-1}, \cdots, w_1, x\rangle = 1 \text{ in } \mathcal{L}_j X \text{ for some } j < s-1 \end{array} \right\}.$$

Let $\bar{G}_s X$ denote the quotient set $G_s X/R_s X$. Then by induction hypothesis $\mathcal{L}_s X$ is freely generated in the pointed sense by $\bar{G}_s X$, for each $s \le m-1$.

Consider the projection

$$d_0^{\mathcal{L}_{m-1}} : P\mathcal{L}_{m-1}X \longrightarrow \mathcal{L}_{m-1}X,$$

whose kernel is by definition $\mathcal{L}_m X$. The formula for $s_0^{\mathcal{L}_{m-1}}$ gives that $d_0^{\mathcal{L}_{m-1}}$ is a proper epimorphism of free groups with right inverse given by $s_0^{\mathcal{L}_{m-1}}$.

Write

$$\bar{G}_{m-1}X = A_m(X) \vee B'_m(X),$$

where $A_m(X) = s_0^{\mathcal{L}_{m-1}}(\bar{G}_{m-1}X)$ and $B'_m(X) = \bar{G}_{m-1}X/s_0^{\mathcal{L}_{m-1}}(\bar{G}_{m-1}X)$. Let

$$B_m(X) = \{[u] := u \cdot \kappa_{m-1}(u)^{-1} \,|\, u \in B'_m(X)\} \subset \mathcal{L}_m X.$$

Then by Lemma 4.2 there is an isomorphism

$$\mathcal{L}_m X \cong Fr[B_m(X)^{Fr[A_m(X)]}].$$

Notice that the group $Fr[A_m(X)]$ is isomorphic with the appropriate dimension shift to $\mathcal{L}_{m-1}X$.

It is easy to observe, as in the proof of Lemma 4.2, that the correspondence $u \mapsto [u] := u \cdot \kappa_{m-1}(u)^{-1}$ is $1-1$. Furthermore, if $u \in B'_m(X)$, then $[u] = 1$ in $\mathcal{L}_m X$ if and only if $u = 1$ in $\mathcal{L}_{m-1}X$. To see this notice that if $[u] = 1$ then $u = \kappa_{m-1}(u)$. Hence in this case u is in the image of $s_0^{\mathcal{L}_{m-1}}$, which means that it is trivial as an element of $B'_m(X)$. Finally notice that

$$B'_m(X) = \bar{G}_{m-1}X/s_0^{\mathcal{L}_{m-1}}(\bar{G}_{m-1}X) = G_{m-1}X/s_0^{\mathcal{L}_{m-1}}(\bar{G}_{m-1}X) \cup R_{m-1}.$$

Putting all this together, the presentation of $\mathcal{L}_m X$ follows.

Notice that the symbol $\langle w_{m-1}, \cdots, w_1, x\rangle$ denotes the element

$$\mathsf{c}(s_0^{\mathcal{L}_{m-1}} w_{m-1})([\langle w_{m-2}, \cdots, w_1, x\rangle])$$

in $P\mathcal{L}_{m-1}X$, where $[u] := u\kappa_{m-1}(u)^{-1}$. Thus for every $i \ge 0$ one has

$$s_i^{\mathcal{L}_m}\langle w_{m-1}, \cdots, w_1, x\rangle = s_{i+1}^{\mathcal{L}_{m-1}}(\mathsf{c}(s_0^{\mathcal{L}_{m-1}} w_{m-1})[\langle w_{m-2}, \cdots, w_1, x\rangle]) = \mathsf{c}(s_0^{\mathcal{L}_{m-1}} s_i^{\mathcal{L}_{m-1}} w_{m-1})[s_{i+1}^{\mathcal{L}_{m-1}}\langle w_{m-2}, \cdots, w_1, x\rangle],$$

and the formula for the degeneracy operators follows by induction. The formula for the face operators $d_i^{\mathcal{L}_m}$ for $i \ge 1$ follows in a similar fashion.

Finally let $u = \langle w_{m-2}, \cdots, w_1, x\rangle$. Then we have

$$d_0^{\mathcal{L}_m}\langle w_{m-1}, w_{m-2}, \cdots w_1, x\rangle = d_1^{\mathcal{L}_{m-1}}(\mathsf{c}(s_0^{\mathcal{L}_{m-1}} w_{m-1})(u \cdot \kappa_{m-1}(u)^{-1}) = \mathsf{c}(w_{m-1})(d_1^{\mathcal{L}_{m-1}}(u) \cdot d_0^{\mathcal{L}_{m-1}}(u)^{-1}).$$

This gives the recursive formula for $d_0^{\mathcal{L}_m}$ and thus completes the proof. □

Example 4.13. *The calculation of $d_0^{\mathcal{L}_3}$ on a typical generator of the form $\langle w_2, w_1, x\rangle$ is given next. By the theorem one has*

$$d_0^{\mathcal{L}_3}\langle w_2, w_1, x\rangle = \mathsf{c}(w_2)(d_1^{\mathcal{L}_2}\langle w_1, x\rangle \cdot d_0^{\mathcal{L}_2}\langle w_1, x\rangle^{-1}) = \mathsf{c}(w_2)((\langle d_1 w_1, d_3 x\rangle \cdot (\mathsf{c}(w_1)(d_0^{\mathcal{L}_2}\langle 1, x\rangle^{-1}))).$$

Expanding this further can be done by using the identities $w_i = [w_i]\kappa_i(w_i)$, where in each case $[w_i]$ means $w_i\kappa_i(w_i)^{-1}$. In each case one needs to know an explicit *decomposition of w_i as a word in its respective group. The simplest instance of this occurs for $w_i = 1$ for $i = 1, 2$. In this case one has*

$$d_0^{\mathcal{L}_3}\langle 1,1,x\rangle = \langle 1,1,d_3x\rangle(\langle 1,1,d_2x\rangle\langle 1,d_0^G d_0x,d_1x\rangle^{-1}\langle 1,d_0^G d_0x,d_0x\rangle)^{-1}.$$

5. Comparison of the models

Recall that $\Lambda_n X = \Omega^{n-1}F\Sigma^{n-1}X$, whereas $\mathcal{L}_n X = \Omega^{n-1}GX$. The following theorem summarizes some obvious relationships between the two constructions.

Proposition 5.1. *Let Y be connected then*

1. *If $X = \Sigma^n Y$ then $\mathcal{L}_n X = \Lambda_n Y$.*
2. *If $Y = \mathcal{L}_n Z$ then Y is a retract of $\Lambda_n Y$ via explicit maps*

$$\mathcal{L}_n Z \xrightarrow{E^n} \Lambda_n\mathcal{L}_n Z \xrightarrow{\mathcal{L}_n ev^n} \mathcal{L}_n Z.$$

Proof. The first statement follows at once from the definitions and the observation that the functors $G\Sigma$ and F are naturally isomorphic.

For the second statement recall that the k-fold Freudenthal suspension map was described earlier. There is a natural way to write the simplicial analogue of the map $\Omega^n ev^n$. The construction becomes a triviality once it is observed that there is a natural map $ev : \Sigma\Omega W \longrightarrow W$ for every simplicial set W (categorically this is just the fact that Σ is left adjoint to Ω in the category of simplicial sets). Once this map is constructed, apply its $(n-2)$-fold suspension to $W = \Omega^{n-2}GZ$ to get a map

$$\Sigma^{n-2}ev : \Sigma^{n-1}\Omega^{n-1}GZ \longrightarrow \Sigma^{n-2}\Omega^{n-2}GZ.$$

Inductively one gets a map

$$ev^{n-1} : \Sigma^{n-1}\Omega^{n-1}GZ \longrightarrow GZ.$$

The target being a simplicial group this map extends to a simplicial homomorphism

$$\Omega ad(ev^{n-1}) : F\Sigma^{n-1}\Omega^{n-1}GZ \longrightarrow GZ.$$

This simplicial homomorphism can now be looped $n-1$ times to give the desired simplicial homomorphism

$$\Omega^n ev^n : \Lambda_n\mathcal{L}_n Z \longrightarrow \mathcal{L}_n Z.$$

Thus define $ev : \Sigma\Omega Y \longrightarrow Y$ as follows. Let $(y,j) \in (\Sigma\Omega Y)_n$ be any element. Hence y is a simplex of dimension $n-j+1$ in Y with the additional property that $d_0(y) = *$. Define

$$ev(y,j) = s_0^{j-1}(y).$$

It is a routine verification that this map is simplicial, and natural in Y.

□

For an explicit example, consider $\Omega^2 ev^2 : \Lambda_2\mathcal{L}_2 X \longrightarrow \mathcal{L}_2 X$. Thus start with $ev : \Sigma\mathcal{L}_2 X \longrightarrow GX$. A typical element has the form (w,j) and is sent by definition to $(s_0^G)^{j-1}w$ under ev. In particular an element of the form $[w] = (w,1)$ is sent to w. A generator for $\Lambda_2\mathcal{L}_2 X$ has the form $\mathsf{c}((u_1,i_1)\cdots(u_k,i_k))[w]$. Thus $\Omega^2 ev^2$ is the unique multiplicative extension of the map sending an element of the form above to

$$\mathsf{c}((s_0^G)^{i_1-1}(u_1)\cdots(s_0^G)^{i_k-1}(u_k))(w).$$

Notice that for any simplicial set X, the construction of the Moore loops ΩX makes no use of the simplicial operator s_0. Thus ΩX has an extra operator which we may denote by s_{-1}. Notice however that the use of the word "operator" here is misleading. Namely, if $x \in X$ is a non-trivial simplex such that $d_0 x$ is the base point so that x is in fact a simplex in ΩX, then $s_{-1}x$ is not in ΩX, since $d_0^{\Omega} s_{-1} x = d_1 s_0 x = x$. To fix this one may think about s_{-1} as a function from ΩX to X, which is degree preserving. Notice that an i-fold iteration of s_{-1} may be thought of as a function of degree $i-1$ from ΩX to X.

For every $i \geq 0$ and $n \geq 2$ define an operator of degree -1

$$\phi_i : \mathcal{L}_n X \longrightarrow \mathcal{L}_{n-1} X$$

by

$$\phi_i = (s_{-1}^{\mathcal{L}_n})^i (d_0^{\mathcal{L}_n})^i,$$

where $(s_{-1}^{\mathcal{L}_n})^i$ stands for $(s_0^{\mathcal{L}_{n-1}})^i$ considered as an operator from $\mathcal{L}_n$ to $\mathcal{L}_{n-1}$. Notice that if X is n-connected and reduced and $w \in (\mathcal{L}_n X)_k$ then $\phi_k(w) = 1$. Notice also that the operators ϕ_i defined above for $\mathcal{L}_n X$ are inherited by $\Lambda_n X$ by 1 of Proposition 5.1. The following lemma summarizes the commutation features of ϕ_i with the simplicial operators.

Lemma 5.2. *Let for $i \geq 1$ and $n \geq 2$, let $\phi_i : \mathcal{L}_n X \longrightarrow \mathcal{L}_{n-1} X$ denote the operator defined above. Then*

1. $d_j^{\mathcal{L}_{n-1}} \phi_i = \phi_{i-1} d_j^{\mathcal{L}_n}$ *if* $j \leq i$
2. $d_j^{\mathcal{L}_{n-1}} \phi_i = \phi_i d_{j-1}^{\mathcal{L}_n}$ *if* $j > i$
3. $s_j^{\mathcal{L}_{n-1}} \phi_i = \phi_{i+1} s_j^{\mathcal{L}_n}$ *if* $j \leq i$
4. $s_j^{\mathcal{L}_{n-1}} \phi_i = \phi_i s_{j-1}^{\mathcal{L}_n}$ *if* $j > i$.

Proof. We verify 1 and 3. The other two identities follow by analogy and are left to the reader. Notice that

$$d_j^{\mathcal{L}_{n-1}} \phi_i = d_j^{\mathcal{L}_{n-1}} (s_0^{\mathcal{L}_{n-1}})^i (d_1^{\mathcal{L}_{n-1}})^i.$$

One then uses the simplicial identities to verify that for $j < i$

$$d_j s_0^i d_1^i = s_0^{i-1} d_1^{i-1} d_{j+1}.$$

Back in $\mathcal{L}_n X$ the right hand side is $\phi_i d_j^{\mathcal{L}_n}$. This gives 1. To verify 3 notice that the left hand side is

$$s_j s_0^i d_1^i = s_0^{i+1} d_1^i.$$

On the other hand the right hand side is

$$s_0^{i+1} d_1^{i+1} s_{j+1} = s_0^{i+1} d_1^{i-k} s_{j-k} d_1^{k+1} = s_0^{i+1} d_1^{i-j+1} s_1 d_1^j = s_0^{i+1} d_1^{i-j} d_1^j = s_0^{i+1} d_1^i$$

implying 3. □

The operators ϕ_i become useful when one writes explicit homotopies on these models. The idea of how these are used is demonstrated below in writing an explicit null-homotopy for the commutator map on an iterated loop space.

The standard 1-simplex $\Delta[1]$ is the simplicial set which in dimension n has $n+2$ simplices denoted $\langle 0^i, 1^{n+1-i} \rangle$ for $0 \leq i \leq n+1$. Faces and degeneracies are given by the formulas

$$d_j \langle 0^i, 1^{n+1-i} \rangle = \begin{cases} \langle 0^{i-1}, 1^{n+1-i} \rangle & j < i \\ \langle 0^i, 1^{n-i} \rangle & j \geq i \end{cases}$$

$$s_j\langle 0^i, 1^{n+1-i}\rangle = \begin{cases} \langle 0^{i+1}, 1^{n+1-i}\rangle & j < i \\ \langle 0^i, 1^{n+2-i}\rangle & j \geq i \end{cases}$$

If $h : \Delta[1] \times X \longrightarrow Y$ is a homotopy (i.e. a simplicial map), denote by h_i the function defined for $x_n \in X_n$ by

$$h_i(x_n) = h(\langle 0^i, 1^{n+1-i}\rangle, x_n).$$

Notice that h_i is only defined on x_n with $n \geq i - 1$.

Let X be an n-connected reduced complex, $n \geq 2$, and let G denote $\mathcal{L}_n X$. Define

$$\tau : \Delta[1] \times G \times G \longrightarrow G$$

by

$$\tau_i(u, v) = [u, \phi_i(v)],$$

for every pair of words $u, v \in G$.

Lemma 5.3. *The map τ defined above is a simplicial null-homotopy for the commutator map.*

Proof. For $u, v \in G_k$ one evidently has

$$\tau_{k+1}(u, v) = [u, 1] = 1 \text{ and } \tau_0(u, v) = [u, v].$$

The verification that τ is a simplicial map is a standard exercise, using the notation given above for the 1-simplex, Lemma 5.2 and the simplicial identities, is left to the reader. □

6. Filtrations for Λ_n and $\mathcal{L}_n$

There is a filtration of $\Lambda_n X$ by simplicial subgroups obtained as follows. Recall that $\Lambda_n X$ is freely generated by elements of the form of the form $\mathsf{c}(w)[x]$. The word w is an element in a free group $Fr[A_n(X)]$ and thus has the usual reduced word length filtration. Give $\mathsf{c}(w)[x]$ filtration k if w has filtration k in the word length filtration. Let $F_j\Lambda_2 X$ be the subgroup of $\Lambda_n X$ generated by all $\mathsf{c}(w)[x]$ of filtration at most j.

Lemma 6.1. *The simplicial groups $F_j\Lambda_n X$ endow $\Lambda_n X$ with the structure of a filtered simplicial group with $F_0\Lambda_n X$ isomorphic to FX.*

Proof. First notice that the simplicial operators preserve the generating set of the group $F_j\Lambda_n X$. This ensures that each $F_j\Lambda_n X$ is indeed a simplicial subgroup of $\Lambda_n X$. Filtration zero, $F_0\Lambda_n X$, is exactly the subgroup generated by the image of the Freudenthal suspension $E^n : X \longrightarrow \Lambda_n X$. Thus $F_0\Lambda_n X = FX$ as claimed. □

The filtration on $\mathcal{L}^n X$ arises in a similar way, but describing it requires some preparation. First, let X be a simplicial set and let Y be a subset, which is not necessarily simplicial. Define the simplicial envelope of Y in X, $Env_X(Y)$ to be the simplicial subset of X given by the intersection of all simplicial subsets containing Y. Thus $Env_X(Y)$ is the minimal simplicial subset of X containing Y. Obviously $Env_X(Y)$ is obtained from Y by adding in all iterations of faces and degeneracies of simplices in Y.

If G is a simplicial group and X a subset of G, then the simplicial closure $Env_G(X)$ is not necessarily a simplicial subgroup. However the following holds.

Lemma 6.2. *Let G be a simplicial group and let X be a subset such that $Env_G(X)$ is obtained from X by adjoining to X all elements of G of the form $d_0^i(x)$, $x \in X$ for some $i \geq 0$. Let $\langle X \rangle$ denote the subgroup of G generated by X. Then $Env_G(\langle X \rangle)$ is a simplicial subgroup of G. Moreover*

$$Env_G(\langle X \rangle) = \langle Env_G(X) \rangle.$$

Proof. Since $Env_G(X)$ contains X, the subgroup $\langle Env_G(X) \rangle$ contains $\langle X \rangle$. By definition $Env_G(\langle X \rangle)$ is the intersection of all simplicial subsets of G containing $\langle X \rangle$. Hence, $\langle Env_G(X) \rangle \supseteq Env_G(\langle X \rangle)$. Next notice that $Env_G(X)$ is obtained from X by adjoining all elements of the form $d_0(y)$, $y \in X$. This follows at once from the identity $d_0^k = d_0 d_1 \cdots d_{k-1}$ and the assumption that X is closed under all simplicial operators except possibly d_0. Thus every element of $\langle Env_G(X) \rangle$ can be written in the form

$$w = x_1^{\epsilon_1} d_0(y_1)^{\delta_1} \cdots x_r^{\epsilon_r} d_0(y_r)^{\delta_r},$$

where $x_i, y_j \in X$ and $\epsilon_i, \delta_j \in \{-1, 0, 1\}$. But

$$x_1^{\epsilon_1} d_0(y_1)^{\delta_1} \cdots x_r^{\epsilon_r} d_0(y_r)^{\delta_r} = d_0(s_0(x_1)^{\epsilon_1} d_0(y_1)^{\delta_1} \cdots s_0(x_r)^{\epsilon_r} d_0(y_r)^{\delta_r})$$

and the right hand side is the image under d_0 of an element in $\langle X \rangle$. This gives and inclusion the other way and the proof is complete. □

We are now ready to define the filtration for $\mathcal{L}_k X$. The group $\mathcal{L}_k X$ is freely generated by symbols $\langle w_{k-1}, w_{k-2}, \cdots, w_1, x \rangle$ with $w_j \in \mathcal{L}_j X$. Let $T_r \mathcal{L}_k X$ denote the subset of the generating set consisting of all generators of the form above such that w_{k-1} is a word of reduced length at most r. Notice that $T_r \mathcal{L}_k X$ is closed under all simplicial operators with the exception of $d_0^{\mathcal{L}_k}$. Thus the simplicial subset $Env_{\mathcal{L}_k X}(T_r \mathcal{L}_k X)$ is obtained from $T_r \mathcal{L}_k X$ by adjoining all the images of its elements under $d_0^{\mathcal{L}_k}$. Consequently $Env_{\mathcal{L}_k X}(\langle T_r \mathcal{L}_k X \rangle)$ is a simplicial subgroup of $\mathcal{L}_k X$ and is in fact equal to the simplicial subgroup generated by $Env_{\mathcal{L}_k X}(T_r \mathcal{L}_k X)$. Define

$$F_p \mathcal{L}_k X = Env_{\mathcal{L}_k X}(\langle T_p \mathcal{L}_k X \rangle).$$

7. A simplicial model for $Map_*(S_g^{n,k}, X)$

Let g be a non-negative integer and let S_g denote the closed Riemann surface of genus g. If k is another non-negative integer, let S_g^k denote a Riemann surface of genus g and k distinct boundary components. In particular S_g^0 is identified with S_g. Let X be an arbitrary space. In various applications one is interested in the space of maps from S_g^k to X. In this section simplicial models for $Map_*(S_g^k, X)$ will be constructed, where here X denote an arbitrary simplicial set.

The intuition for this construction comes from the, rather simple, homotopy theory of these spaces. First notice that up to homotopy, a Riemann surface with $k > 0$ boundary components, is equivalent to a singular surface whose boundary consists of a wedge of k circles. Furthermore, as a cell complex the surface S_g^k can be constructed by means of the cofibration sequence

$$S_z^1 \xrightarrow{\alpha_g^k} \{\bigvee_{i=1}^{g} (S_{x_i}^1 \vee S_{y_i}^1)\} \vee \bigvee_{j=1}^{k} S_{b_j}^1 \longrightarrow S_g^k,$$

where subscripts on the circles denote the generators of the respective fundamental groups and the map α_g^k is specified up to homotopy by its effect on fundamental groups as

$$\alpha_g^k = [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] \cdot b_j^{-1} \cdots b_1^{-1}.$$

For an arbitrary pointed space X, the mapping space functor $Map_*(-, X)$ turns the cofibration above into a fibration and $Map_*(S_g^k, X)$ is defined as the fibre space in this fibration, where the projection map is induced by α_g^k.

For non-negative integers g and k consider the system

$$(GX)^{2g+k} \xrightarrow{a_g^k} GX \xleftarrow{d_0} PGX.$$

The map a_g^k is given as follows

$$a_g^k(u_1, v_1, \cdots, u_g, v_g, z_1, \cdots, z_k) = [u_1, v_1][u_2, v_2] \cdots [u_g, v_g](z_1 \cdots z_k)^{-1}.$$

Let $M_g^k(X)$ denote the pull-back of this system.

Theorem 7.1. *If X is a simply-connected reduced complex then $M_g^k(X)$ is a Kan complex and its geometric realization is homotopy equivalent to $Map_*(S_g^k, |X|)$. The simplicial operators in $M_g^k(X)$ are induced from those of GX and PGX.*

Proof. First notice that $M_g^k(X)$ is a Kan complex as a pull-back space in a diagram of Kan complexes. That the maps a_g^k model the geometric maps induced by α_g^k is obvious by construction. Notice also that since the projection $d_0 : PGX \longrightarrow GX$ is a Kan fibration, the homotopy type of $M_g^k(X)$ is a homotopy invariant of X. This completes the proof. □

Theorem 1.5 follows at once from Theorem 7.1. In particular for $g = k = 0$ the bottom left hand side corner of the pull-back square above is trivial and so $M_0^0(X)$ is simply the kernel of d_0^G, namely the model $\mathcal{L}_2(X)$.

Notice that in general $M_g^k(X)$ has a natural action of the simplicial group $\mathcal{L}_2X$, with homotopy orbit space equivalent to $(GX)^{2g+k}$.

8. Appendix

This appendix is dedicated to the proof of the following lemma, which is well known but we are not aware of an appropriate reference.

Lemma 8.1. *Let A and B be free groups generated by sets X and Y respectively. Then the kernel of the natural projection $\pi : A*B \longrightarrow A$ is the free group generated by the set Y^A, namely by all conjugates of elements in Y by words in A.*

Proof. Let A, and B be free groups. Let $\tilde{C}$ denote a tree on which A acts properly discontinuously with the orbit space $\tilde{C}/A = C$. Similarly, let $\tilde{D}$ denote a tree on which B acts properly discontinuously with the orbit space $\tilde{D}/B = D$. Without loss of generality we may label the edges of C and D by X and Y respectively. Then $\tilde{C}$ has one vertex for any element in A and one edge between two elements if their difference is a member of the generating set X. Similarly for $\tilde{D}$.

Fix a base-point in D, which is the image of any vertex in $\tilde{D}$, and attach one copy of D to each vertex in $\tilde{C}$ by identifying each vertex of $\tilde{C}$ with this fixed base-point in D. Call the resulting adjunction space E.

Then the action of A on $\tilde{C}$ descends to a properly discontinuous action of A on E with quotient $C \vee D$. Thus there is a split short exact sequence of groups

$$1 \to \pi_1(E) \to \pi_1(C \vee D) \to A \to 1.$$

Notice that the fundamental group $\pi_1(E)$ is a free group which has basis given by the ordered pairs (α, β) where α runs over all vertices of $\tilde{C}$ and β runs over all edges of D (indexed by Y).

A choice of basis is gotten as follows: Choose a path from the vertex α to the base-point. The basis element (α, β) is represented the class on the loop $\alpha^{-1} \cdot \beta \cdot \alpha$ in $\pi_1(E)$ by standard covering space theory. The image of $\pi_1(E) \to \pi_1(C \vee D)$ is thus given precisely by the subgroup freely generated by Y^A. The results follows. □

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