On The homotopy theory of p-completed classifying spaces

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0. Introduction

Let G be a discrete group and let BG denote its classifying space. Recall that a group is said to be perfect if it is equal to it's own commutator subgroup. If G is an arbitrary group, then write BG^+ for the Quillen "plus" construction applied to BG with respect to the unique maximal normal perfect subgroup ΠG of G [**32**]. The space BG^+ can be obtained by attaching 2 and 3 cells to BG and has the defining properties:

1. There is a natural map $BG \longrightarrow BG^+$, which induces a homology isomorphism (with any simple coefficients), and
2. $\pi_1(BG^+) \cong G/\Pi G$.

Since Quillen's first defined the higher algebraic K-groups of a ring R, using the "plus" construction, and computed the K-theory of finite fields, the homotopy type of BG^+, for G finite and perfect has been a subject of interest as the transition between the homotopy theory of BG and BG^+ is dramatic. Two important classical examples illustrate this transition.

1. $G = \Sigma_\infty$ is the colimit of the n-th symmetric group and BG^+ is the space $\mathbf{Q}_0S^0$. [**2, 11, 31**]
2. $G = SL(q)$ is The colimit of the special linear groups $SL(n, q)$ over the field of q elements $\mathbb{F}_q$ and BG^+ gives, after localization at a suitable prime, the space "image of J" [**32**].

In addition, Kan and Thurston showed that any path-connected CW-complex has the homotopy type of BG^+ for a suitable group G (which is an extremely "large" infinite group in their construction) [**15**]. However, if G is assumed to be finite, then there are strong restrictions placed on the homotopy type of BG^+. One feature is that the homotopy groups are entirely torsion and are non-trivial in arbitrarily large degrees [**22**]. In addition, in the cases for which G is finite, it will be seen below that BG^+ behaves through the eyes of homotopy groups as if it were a finite complex.

Furthermore, the homotopy groups of BG^+ for G finite frequently have direct summands given by the homotopy groups of various classical finite complexes such as spheres and mod-p^r Moore spaces [**8, 22, 23, 18**]. In contrast, for finite groups G the space BG^+ does not admit an essential map to any simply-connected finite

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complex [**27**]. Thus one must consider different constructions in order to get a better hold of these relationships.

One general principle in homotopy theory is to use the classical fact that the homotopy groups of any (reasonable) space X are isomorphic to the homotopy groups of the loop space of X with the degrees of all groups shifted down by one dimension. Thus if one knew additional features of the loop space, then one could extract information about X. In case $X = BG^+$, the loop space or possibly iterated loop spaces are frequently homotopy equivalent to a product of more familiar spaces. In addition, the homology of a loop space supports additional and informative structure; namely the homology with field coefficients is a Hopf algebra.

This view is illustrated by the next two examples. Start with $G = SL(2,5)$, the special linear group over the field of 5 elements. Here the loop space of BG^+ is homotopy equivalent to the homotopy theoretic fibre of a degree 120 map from the 3-sphere to itself, as recalled in section 1 here. Thus the homotopy theory associated to $SL(2,5)$ is given in terms of standard spaces and maps.

The space ΩBG^+ here does not split as a product in the above example . However, after looping twice more, it splits into two factors given by the localization at 2, 3, and 5 of $\Omega_0^3 S^3$ and $\Omega_0^4 S^3$. Here by Ω_0^k we mean the component of the constant map of the respective k-fold loop space. Thus the homotopy groups of BG^+ are given by the homotopy groups of the 3-sphere at the primes 2,3, and 5.

A second example, originally discussed in [**23**], is given by the groups $D(p)$, defined for a certain family of primes p. The loop space of BG^+ is homotopy equivalent to a product, where one factor is given by the loop space on a mod-p Moore space. Thus the the mod-p homology of ΩBG^+ contains a tensor algebra on two generators.

This is an example of a different nature and it is not clear what the other factors are. It will be illustrated in section 1 below. A similar example, where a complete splitting of ΩBG^+ in terms of atomic (indecomposable) spaces will also be discussed in a separate section. The atomic factors are all related to Moore spaces and homotopy theoretic fibres of self maps of spheres.

We warn the reader at this point that there is an abuse of terminology here. Namely, the last two families of examples mentioned above, involve groups that are not perfect but rather p-perfect. Thus the "plus" construction does not mean the usual Quillen "plus" construction but rather the p-local analogue discussed below.

These examples illustrate the fact that the homotopy theory of the loop spaces may be given by more familiar and sometimes "handleable" spaces. Thus one useful principle in our analysis is to consider the structure of ΩBG^+ and its iterated loop spaces, rather than to deal directly with BG^+ itself. .

Connections with classical homotopy theory go further as will be illustrated next. Using the standard fibration obtained from a faithful representation of G in $SU(n)$, it will be seen below that the loop space of BG^+ is homotopy equivalent to the homotopy theoretic fibre of a map from $SU(n)$ to a Poincaré complex Z, where the map is a rational equivalence. Thus the homotopy theory of BG^+ is very closely related to that of both $SU(n)$ and Z, (It is not clear whether Z has the homotopy type of a closed manifold in general).

The features of Z impinge on both the homotopy theory of classical groups and, an as yet unsolved problem known as the Moore finite exponent conjecture (this latter conjecture is that if the rational homotopy groups of a simply-connected finite complex are a totally finite dimensional vector space, then the p-torsion in the homotopy of this space has a universally bounded order). For example, the Moore conjecture for Z is satisfied if and only if BG^+ has an exponent for its homotopy groups. One wonders whether the group theory available will then impinge directly on the homotopy theory of either the finite complex Z above or $SU(n)$.

Let us illustrate this last question more precisely. It is not known whether the homotopy groups of the Lie group $SU(3)$ have any elements of order 8 (although it is easy to see that there are infinitely many elements of order 4 and no elements of order 32).

Consider the case of $G = M_{11}$ or $SL(3,3)$. A theorem due to J. Martino and S. Priddy implies that the 2-completed classifying spaces of these groups Coincide. Using work in [**22**], it is easy to see that if BG^+ has an element of order 32 in its homotopy, then there must exist elements of order 8 in the homotopy groups of $SU(3)$. On the other hand it is believed that in fact 4 is an exponent for the homotopy of $SU(3)$. This conjecture would be supported if 16 turns out to be a bound for the order of the 2-primary component of $\pi_*(BG^+)$. Thus the homotopy theory associated to the "plus" construction has a close relation to the homotopy theory of classical groups together with basic questions about Lie groups which seem, at the moment, to be somewhat tricky.

In dealing with this subject a p-local, preferably functorial, analogue of the "plus" construction is useful. Such a construction does exist, namely the partial R-completion functor of Bousfield and Kan [**4**], which is functorial and defined with respect to any ring R (commutative with unit). Thus in 1 and 2 above one has to replace integral homology theory by homology with coefficients in R and take for the analogue of ΠG, the maximal normal R-perfect subgroup of G. In particular, using the terminology of [**4**], if a space X is R-good then the partial R-completion and the usual R-completion on X yield homotopy equivalent spaces. A special case of non-simply-connected spaces which are nevertheless, R-good are those spaces with R-perfect fundamental group, i.e. such that the first mod-R homology vanishes. Also, if X is a space with finite homotopy (or finite integral homology) groups then the R-completion of X with respect to $R = \mathbb{F}_p$ and $\mathbb{Z}_{(p)}$ give homotopy equivalent spaces. Notice that this is precisely the case when one deals with classifying spaces of finite groups. Thus from this point on we discuss the homotopy type of $BG^\wedge_p$, the p-completion of BG, assuming that G is finite and p-perfect. Notice that if G happens to be perfect then it is in particular p-perfect and if it is also finite then $BG^+ \simeq \prod_{p|ord(G)} BG^\wedge_p$. Thus no information is lost in the transition to the p-local analogue.

The interest in these questions arose in the summer 1990, during a bar-room conversation between the first author and Dave Benson, the last posed the question of what can be said about the homotopy theory of BG^+ when G is a finite perfect group. We both would like to thank Dave for many conversations about finite groups and homotopy theory.

This paper represents a survey of some work directed toward answering Benson's question. In particular, questions about $BG^\wedge_p$ for finite p-perfect groups G, why they are interesting, and how they fit with other problems are considered here. Some general theorems in the subject are known and will be discussed. However, there are many questions that are still to be solved. Numerous examples are given which both illustrate the theory and justify related conjectures. Some new results are included in section 6.

1. Motivating examples

This section gives two basic examples as motivation for much of what is done below. The first is of a very simple nature, whereas the second is a bit more tricky and in a sense belongs to a different class of examples.

The first example arises at once from the Poincaré homology 3-sphere given by the orbit space $SO(3)/A_5$. Namely, let $SO(3)$ denote the special orthogonal group of rank 3 and let A_5 be the alternating group on 5 letters. There are fibrations

$$SO(3)/A_5 \longrightarrow BA_5 \longrightarrow BSO(3)$$

and

$$(SO(3)/A_5)^+ \longrightarrow BA_5^+ \longrightarrow BSO(3).$$

The space $(SO(3)/A_5)^+$ is now simply-connected and has the homology of the 3-sphere. Thus it is homotopy equivalent to the 3-sphere. On the other hand, the 3-sphere double covers $SO(3)$. A direct check gives the following

PROPOSITION 1.1. *The 2-connected cover of BA_5^+ is a classifying space for the fibre of the degree 120 map from the 3-sphere to itself. Thus BA_5^+, after looping sufficiently often, is homotopy equivalent to a product of loop spaces on spheres.*

The degree 120 map on the 3-sphere is null-homotopic after passage to triple loop spaces by [**33, 7**]. The splitting follows as one has a multiplicative fibration with a cross-section.

Similar, though sometimes more peculiar behavior propagates for other finite groups. However the example of A_5 has the feature that its cohomology and, apparently as a result, the associated homotopy type are very simple. Examples of this form inspired the spherical resolvability conjecture due to the first author [**8**], which we mention again below. All of those example share the common property that the cohomology of the group under consideration admits a filtration with a symmetric associated graded. This last observation was only made after the spherical resolvability conjecture was disproved by the second author and a closer inspection of the positive examples verified the observation above.

This leads us directly to our second example. A family of p-perfect groups, for particular primes p, for which the associated homotopy type is much larger than in the previous example.

Consider primes $p \geq 13$, such that $4|p-1$. Then the cyclic group $\mathbb{Z}/8\mathbb{Z}$ operates on an elementary abelian p-group V of rank 2. Let $D(p)$ denote the group given by the semi-direct product. There is an 8-dimensional unitary faithful representation of $D(p)$ and the p-completion of the resulting orbit space $(U(8)/D(p))^\wedge_p$ has a 16-dimensional mod-p Moore space $P^{16}(p)$ as a retract. This together with the fibration

$$\Omega(U(8)/D(p))^\wedge_p \longrightarrow \Omega BD(p)^\wedge_p \longrightarrow U(8)^\wedge_p$$

is used in [**23**] to show the following.

PROPOSITION 1.2. *The loop space $\Omega P^{16}(p)$ is a non-multiplicative retract of $\Omega BD(p)^\wedge_p$. Thus $\Omega BD(p)^\wedge_p$ splits as a product where one factor is given by $\Omega P^{16}(p)$.*

Although the nature of the other factor is not known, this is enough to conclude that each group $D(p)$ forms an example of a totally different nature than our first example. Indeed by [**29**], $\Omega P^{16}(p)$ is homotopy equivalent to $S^{15}\{p\}\times\Omega(\vee_\alpha P^{n_\alpha}(p))$. Hence the homotopy type resulting from this example is much "larger" than the previous one. Notice that Proposition 1.2 implies that the homotopy of $BD(p)^\wedge_p$ has infinitely many elements of order precisely p^2.

These two examples turn out to be generic in the subject. Whereas the first is nicely behaved and can be shown to satisfy nearly any reasonable conjecture one can make about it, the second is much harder to analyze in general. The unexpected fact is that within the family of all finite p-perfect groups there is a dichotomy of this sort. Namely, every group G belongs either to the first family or to the second. We shall elaborate more on this in the following sections.

2. Loop space homology for $BG^\wedge_p$

In this section we discuss some results about the loop space homology of spaces of the form $BG^\wedge_p$ for G finite and p-perfect. A large part of the section however is devoted to discussion of observations and problems which appear intriguing.

2.1. Torsion in loop space homology. The first type of general behavior for spaces of the form $\Omega BG^\wedge_p$ we discuss is that their homology groups have exponents, a fact that is analogous to the homology of the group having an exponent bounded above by the order of the group [**22**].

This type of behavior, where an infinite dimensional space X and its loop space have exponents for their homology groups appears to be relatively rare in nature. The only other class of examples, of which we are aware is those spaces which are suspensions and which have an exponent for their homology groups. It is also known that if X is a finite torsion complex then its loop space homology has an exponent [**20**]. The family of p-completed classifying spaces of non trivial finite groups has an empty intersection with both classes mentioned above.

An explicit construction, which approximates the singular chain complex for $BG^\wedge_p$ was given in [**22**]. The model is built as an algebraic "plus"-construction for the classical bar construction on the group G. More precisely, given a chain complex C, one divides out by the 1-skeleton and then adds algebraic cells in dimension 2 to kill the extra homology created by the first step. This construction in fact does not depend on the original complex having trivial first homology. The result is a new complex $P_1(C)$ together with a map $C \xrightarrow{\iota} P_1(C)$ inducing a homology isomorphism in dimensions larger than 1. In case the first homology was trivial to begin with, the map ι induces a homology isomorphism in all dimensions. This algebraic "plus" construction works in the category of connected differential graded coalgebras over a principal ideal domain. In the particular case, where C is the reduced bar construction on G and the ground ring is $\mathbb{Z}_{(p)}$ or $\mathbb{Z}^\wedge_p$, the resulting differential graded coalgebra gives a model for the cellular chains on $BG^\wedge_p$. This construction is of course a direct analogue of the geometric "plus" construction due to Quillen.

The construction $P_1(-)$ on arbitrary connected differential graded coalgebras has a geometric analogue. If one looks closely at how Quillen describes the "plus" construction and tries to perform it on a general connected CW complex X, dropping the perfectness assumption, one obtains a space, which we denote $P_1(X)$, and a map $X \longrightarrow P_1(X)$, which induces an isomorphism on homology groups in dimensions larger than one. It is the cellular chains on $P_1(X)$ that the algebraic construction $P_1(-)$ models. We summarize all this in the following theorem. For any CW complex Y let $C(Y)$ denote the cellular chains on Y over a fixed principal ideal domain R.

THEOREM 2.1. *Let X be a connected CW complex. Assume that $C(X)$ has a strictly associative coproduct, making it a differential graded R-coalgebra. Then*

1. *the differential graded R-coalgebras $P_1(C(X))$ and $C(P_1(X))$ are quasi isomorphic.*
2. *if $X = BG$ for G finite and p-perfect and $R = \mathbb{Z}_{(p)}$ or $\mathbb{Z}_p^\wedge$ then $C(X)$ is the mod-R bar construction $B[G]$ on G and $P_1(B[G])$ is quasi isomorphic to $C(BG_p^\wedge)$.*

The idea is naive and the construction $P_1(-)$ suffers from lack of functoriality both on the algebraic and the geometric level. However it provides a tool in doing certain calculations. The construction is given explicitly in [**22**] and [**20**].

Let G be a finite p-perfect group. In order to study the loop space homology of $BG_p^\wedge$, one applies the loop space algebra functor Ω to the differential graded coalgebra $P_1B[G]$ to get a model for the chains on $\Omega BG_p^\wedge$. The first proof of the following theorem was obtained by using this differential graded algebra to construct an explicit null-homotopy for the homomorphism given by multiplication by the order of G.

THEOREM 2.2. *Let G be a finite p-perfect group. Then the highest power of p dividing the order of G is an exponent for $\tilde{H}_*(\Omega BG_p^\wedge; \mathbb{Z}_p^\wedge)$.*

Going through the details of these constructions and calculations fall beyond the scope of this article. However a different and easier proof of this theorem was suggested to the authors by Bill Dwyer shortly after the result was first announced. His proof has never appeared in print and is thus given here for the first time.

Consider a finite p-perfect group G. The natural map $BG \longrightarrow BG_p^\wedge$ induces an isomorphism on homology with coefficients in $\mathbb{Z}_p^\wedge$. Thus its homotopy fibre, which we denote by A^pG is a mod-p acyclic space. Now consider the principal fibration

$$\Omega BG_p^\wedge \longrightarrow A^pG \xrightarrow{h} BG, \tag{1}$$

obtained by pulling back the path-loop fibration over $BG_p^\wedge$ along the completion map. Let EG denote the universal bundle over BG. Taking the pull-back once more, this time of the universal covering $EG \longrightarrow BG$ along the map h, we obtain a covering

$$L^pG \longrightarrow A^pG,$$

where L^pG is homotopy equivalent to $\Omega BG_p^\wedge$. Since the fibration (1) is principal and its fibre connected, the action of $\pi_1(BG) = G$ on the homology groups of L^pG is trivial. Consequently the restriction

$$res : H_*(L^pG; \mathbb{Z}_p^\wedge) \longrightarrow H_*(A^pG; \mathbb{Z}_p^\wedge)$$

followed by the homology transfer is given by the following formula

$$tr \circ res(x) = \sum_{g \in G} g_* x = \sum_{g \in G} x = |G|x$$

We have thus factored multiplication by the order of the group through the homology of $\mathbb{Z}_p^\wedge$-acyclic space. The theorem follows immediately.

Dwyer's observation is the core of another general result on $\Omega BG^\wedge_p$, namely that it has a stable homotopy exponent. This will be discussed below.

It is a classical fact that the exponent for the homology of a finite group G is bounded above by the order of G. By the theorem above, the same result is correct for the p-local loop space homology of $BG^\wedge_p$. However, the actual bounds which occur for certain choices of examples vary. Consider for instance the special case when the Sylow 2-subgroup of G is the semi-dihedral group of order 2^n and G is 2-perfect. The homology of G has exponent 2^{n-1}. However the situation changes in loop space homology. Namely the order of torsion in the loop space homology of $BG^\wedge_2$ grows by a factor of two and is exactly 2^n. Thus looping increases the exponent although the universal bounds are the same [**22**]. More examples could be given here, but we omit them as a detailed calculation might appear a bit too lengthy..

Our model for the chains on $BG^\wedge_p$ remains useful in at least two ways. The first way is that it provides an explicit chain complex for computing the loop space homology of $BG^\wedge_p$ in much the same way as the classical bar construction provides an explicit chain complex for computing the homology of G. Furthermore, like the bar complex, this chain complex depends entirely on the structure of G.

A second use of the model is that it can be used to generalize Theorem 2.2. Indeed, consider a finite, not necessarily perfect group G. Then the algebraic model for $P_1(BG)$ given by $P_1(B[G])$ is exploited in [**20**] to prove the following

Theorem 2.3. *Let G be an arbitrary finite group. If p^r is the highest power of p, dividing the order of G, then p^{3r} annihilates the reduced p-local homology of $P_1(BG)$.*

The method of proof is identical to the original proof of Theorem 2.2, except for one extra observation. Namely, consider the endomorphism ϕ of the graded module underlying the cobar construction $\Omega P_1(B[G])$, given by multiplication by the order of G. In [**20**] an obstruction is constructed, which measures the failure of ϕ to be null-homotopic. However, the third iteration of this obstruction is shown to vanish, thus proving the theorem.

2.2. Spherically resolvable spaces. A different aspect of the homological behavior for $\Omega BG^\wedge_p$ is discussed next. As mentioned above, a conjecture in the subject, due to the first author, was that $\Omega BG^\wedge_p$ is spherically resolvable of finite length for G finite and p-perfect [**8**]. The conjecture was disproved by the second author in [**23**], using a homological method which we next describe.

For a space X, the property of being spherically resolvable of finite length is roughly defined by the requirement that X is the total space of a fibration, which can be obtained by iteratively fibering the base and fibre in terms of spheres and their loop spaces. Finite length here means, of course, that this decomposition is required to be finite. Although this may seem somewhat artificial, we confine ourselves in the sequel to spaces, which are spherically resolvable of finite length with the resolving spaces being of the form $\Omega^n S^{n+k}$, rather than allowing arbitrary iterated loop spaces.

If X is spherically resolvable of finite length, then it is easy to see that its mod-p homology cannot grow "too fast". More precisely, if one considers the (well known) mod-p homology of any potential resolving space $\Omega^n S^{n+k}$, one sees that the rate of growth of this graded vector space is constant if $n = 1$ and hyper-polynomial if

$n > 1$. By hyper-polynomial growth we mean that the the rank grows faster than polynomially but not as fast as exponentially. Thus the same restriction on the growth applies to X, by iteratively using the Serre spectral sequence to bound the growth.

Let G be a finite p-perfect group and consider a faithful representation of G in $SU(n)$ for some n. Then we obtain a fibration

$$(SU(n)/G)^{\wedge}_p \longrightarrow BG^{\wedge}_p \longrightarrow BSU(n)^{\wedge}_p.$$

Since the space $(SU(n)/G)^{\wedge}_p$ is the p-completion of a finite complex, a result of Felix, Halperin and Thomas [**13**] gives that its mod-p loop space homology grows either polynomially or sub-exponentially (i.e. as fast as $\lambda^{d^{1/n}}$, where $\lambda > 1$ and $n \geq 1$). For those spaces X whose loop space homology grows polynomially, it is also proven in [**13**] that $H_*(\Omega X; \mathbb{F}_p)$ is a finitely generated nilpotent Hopf algebra. Now back to our context, notice that since $SU(n)$ is both finite and spherically resolvable, the fibration above yields the following theorem, originally proven in [**18**]

Theorem 2.4. *For a finite p-perfect group G*

1. $\Omega BG^{\wedge}_p$ *is spherically resolvable if and only if* $\Omega(SU(n)/G)^{\wedge}_p$ *is.*
2. $H_*(\Omega BG^{\wedge}_p; \mathbb{F}_p)$ *grows either polynomially or sub-exponentially.*
3. *If* $H_*(\Omega BG^{\wedge}_p; \mathbb{F}_p)$ *grows polynomially then it is a finitely generated nilpotent Hopf algebra.*

Unfortunately, $\Omega BG^{\wedge}_p$ is not spherically resolvable of finite length in general. In [**23, 18**] the second author produced examples of groups G, such that the loop space homology of $BG^{\wedge}_p$ grows exponentially. Thus the conjecture is not satisfied for these examples.

One family of counter examples was already given in the preceding section. Indeed the groups $D(p)$, defined above do not satisfy the spherical resolvability conjecture, since, as we pointed out, the loop space on a Moore space splits off as a retract of $\Omega BD(p)^{\wedge}_p$ and the first has exponentially growing mod-p homology. An even simpler family of counter examples is given in [**18**]. For every prime p, there is an action of the cyclic group of order 3 on the abelian group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let $E(p)$ denote the respective semi direct product. Then for $p \neq 3$, the group $E(p)$ is p-perfect and has a rather simple mod-p cohomology ring. However for $p \geq 5$, the mod-p homology of the loop space contains a tensor algebra on more than one generator and thus grows exponentially. A complete splitting of $\Omega BE(p)^{\wedge}_p$ was obtained by the second author and J. Wu [**24**] and is given in a separate section below. From this splitting it becomes apparent that $\Omega BE(p)^{\wedge}_p$ is a very "large" space in spite of the naive appearance of $H^*(BE(p); \mathbb{F}_p)$.

These examples suggest that the spherical resolvability conjecture fails because there are not enough symmetries in the group to make the cohomology "nice". This last sentence is very vague. However, we do not yet have enough data to replace it with a more precise phrase.

One could consider the following questions.

1. What is the subset of the set of all finite p-perfect groups, consisting of groups G such that the homology of $\Omega BG^{\wedge}_p$ satisfies polynomial growth?
2. Is the "spherical resolvability" conjecture satisfied for this class of groups?

It would be very interesting if one could find a group theoretic characterization which would provide an answer to either one of these questions.

2.3. Periodic homology. A particularly nice family of groups in this context is given by those groups G, where the loop space homology of $BG^{\wedge}_p$ is periodic, namely it is a (possibly non-commutative) ring of Krull dimension 1. These of course satisfy polynomial growth and thus are finitely generated and nilpotent as Hopf algebras. A classical theorem of R. Swan gives the groups which have periodic cohomology. At odd primes these are precisely the groups whose Sylow-p subgroup is cyclic and at the prime 2 the Sylow subgroup is either cyclic or generalized quaternion. Of course the cohomology rings of those groups all have Krull dimension 1 and those which in addition are p-perfect, at the respective prime p, form the first useful examples in analyzing the structure of $\Omega BG^{\wedge}_p$.

Indeed if G is a finite p-perfect group with periodic mod-p cohomology then $\Omega BG^{\wedge}_p$ also has periodic mod-p homology [**8**]. The converse is not true as there are examples of groups G where the loop space homology is periodic, but the mod-p cohomology of G has Krull dimension 2. One such family of examples is given in [**14**]. It is not clear what sort of implications for G are obtained by the assumption that $\Omega BG^{\wedge}_p$ has periodic homology.

2.4. Finite generation and the existence of elements of infinite height. The next question is whether the mod-p loop space homology of $BG^{\wedge}_p$ is always a finitely generated algebra. Of course, by the remarks above, it is finitely generated if it satisfies polynomial growth. Thus the question remains what happens at the other extreme, namely when the loop space homology grows semi-exponentially. In the few examples that have been worked out the answer appears to be yes but there is no general theorem to that effect. One related example is the loop space of a mod-p Moore space where the mod-p homology is a tensor algebra on two generators and is thus finitely generated, but grows quickly. Other examples are the groups $D(p)$ and $E(p)$ described above. As we shall observe later the mod-p homology of $\Omega BE(p)^{\wedge}_p$ is finitely generated as an algebra although its growth is exponential. The analogous result for $D(p)$ is not known.

Notice that if finite generation holds in $H_*(\Omega BG^{\wedge}_p;\mathbb{F}_p)$, then the fact that this homology is infinite dimensional implies that there must exist at least one element of infinite height. In all of the examples, which have been studied successfully, this has always been the case, even if the finite generation question has not been settled yet (the groups $D(p)$ for instance). The existence of an element of infinite height seems likely to be true in general but there is no theorem to that effect when the loop space homology fails to grow polynomially.

Another related fact is the following. In section 3 we prove that the homotopy of $BG^{\wedge}_p$ is non-trivial in arbitrarily high degrees by using a Moore-Postnikov type argument. It is thus appropriate to point out that this result is actually implied by the existence of an element of infinite height in $H_*(\Omega BG^{\wedge}_p;\mathbb{F}_p)$. Indeed suppose that the $\pi_i(BG^{\wedge}_p) = 0$ for all sufficiently large i . Then $BG^{\wedge}_p$ is equivalent to a finite Moore-Postnikov section. After looping, the homology ring mod-p has no elements of infinite height as in Moore and Smith [**30**].

2.5. Loop space homology as a Lie algebra. As the loop space homology is an associative algebra, it can be thought of as a Lie algebra in the usual way, defining $[x,y] = xy - (-1)^{|x||y|}yx$. Being in fact a Hopf algebra, the sub module of primitives becomes a sub Lie algebra with respect to this structure

The structure of the underlying Lie algebra of primitives is part of what forces various spaces to split as products and forces the loop space to not be "resolvable" by finitely many spheres in some examples. For instance in studying the groups $D(p)$ and $E(p)$, the respective loop spaces have the property that the underlying mod-p homology Lie algebra of primitives contains a free Lie algebra on two generators.

On the other hand, the example arising from A_5 has an abelian Lie algebra associated to it, as is also the case in mod-2 homology for all finite simple groups of 2-rank two with the possible exception of $U(3,4)$, the unitary group of rank 3 over $\mathbb{F}_4$ [**14**].

As the dichotomy theorem 2.4 states, the Lie algebra associated to a group G either grows polynomially, in which case it is nilpotent, or it grows sub exponentially, in which case one may conjecture that the Lie algebra of primitives contains a free Lie algebra on at least two generators.

Consider the polynomially growing case first. Then $H_*(\Omega BG^\wedge_p;\mathbb{F}_p)$ is a nilpotent Lie algebra. However, we have no examples, where the homology is known to be non-abelian and is nilpotent. In other words all the examples of this case familiar to us are abelian. An interesting question would be what is true in general here.

One could try to construct a family of finite p-perfect groups $G(n)$ such that the homology Lie algebra of $\Omega BG(n)^\wedge_p$ is nilpotent of rank exactly n. Indeed, as we see no reason to believe that the actual loop space has any reasonable homotopy commutativity properties, we suggest that any nilpotency rank is possible in our context. It remains to be seen whether or not this is the case.

Another interesting aspect of this question is how does the group theory associated to G reflects itself in the rank of nilpotency of the associated loop space homology.

Next look at the sub exponential case. It is not known whether or not this case can be nilpotent. Our evidence suggest that the answer to that is no, but there is no general theorem. Notice that these questions are all closely related to each other. Namely, if an algebra is finitely generated and nilpotent then it obviously satisfies polynomial growth. Our observation above is that the converse also holds for $H_*(\Omega BG^\wedge_p;\mathbb{F}_p)$. On the other hand an algebra which is growing sub exponentially and is nilpotent cannot be finitely generated. Thus one might conjecture that in the sub exponential case the algebras under consideration are non-nilpotent.

One might like to find a direct relation between the cohomology of G and the loop space homology of $BG^\wedge_p$. An intuitive guess is the following. Suppose that the mod-p cohomology of G is Cohen-Macaulay, namely that there exists a polynomial subalgebra P of $H^*(BG;\mathbb{F}_p)$ over which it is a finitely generated free module. Then one can consider the algebra quotient $\Omega = H^*(BG;\mathbb{F}_p)//P$. It is a finite dimensional algebra and one might guess that if this algebra contains distinct indecomposable elements whose cup product is zero, then the loop space homology Lie algebra contains a free Lie algebra on at least two generators.

2.6. The loop space homology functor. The functor on the category of finite groups given by loop space homology seems to be interesting enough to deserve its own name. Rather then defining this functor on the category of finite p-perfect groups, we define it on the category of all finite groups and specify a ring of coefficients in the definition as follows.

Let $\mathcal{G}$ denote the category of all groups and let $\mathcal{FG}$ denote the full sub category consisting of all finite groups. Let R be a commutative ring with a unit R. Let $\mathcal{M}_R$ denote the category of R-modules. Let $\mathcal{A}lg_R$ denote the category of connected graded R-algebras. If R is a field, let $\mathcal{H}_R$ denote the category of connected Hopf algebras over R. For a fixed prime p let $\mathcal{H}_{\mathcal{A}_p}$ denote the category of connected Hopf algebras over the mod-p Steenrod algebra. Finally let $\mathcal{L}_p$ denote the category of graded restricted Lie algebras over $\mathbb{F}_p$.

For a group $G \in \mathcal{G}$ define

$$\mathcal{L}SH_*(G; R) := H_*(\Omega C^R BG\langle 1\rangle; R),$$

where $C^R(-)$ denotes the partial R-completion functor of Bousfield and Kan, mentioned in the introduction.

If G is finite and $R = \mathbb{F}_p$ then $C^R BG \simeq BG^\wedge_p$ and $BG^\wedge_p\langle 1\rangle \simeq B\Pi^p G^\wedge_p$. Thus in that case $\mathcal{L}SH_*(G; \mathbb{F}_p) = H_*(\Omega B\Pi^p G^\wedge_p; \mathbb{F}_p)$. We summarize this in the following theorem.

THEOREM 2.5. *Let R be a commutative ring with a unit. There is a bifunctor $\mathcal{L}SH_*(-;-)$ defined on the product category $\mathcal{G} \times \mathcal{M}_R$ which takes values in $\mathcal{M}_R$. Furthermore,*

1. *The functor $\mathcal{L}SH_*(-; R)$ takes values in $\mathcal{A}lg_R$.*
2. *If $R = \mathbb{F}_p$ then the values of $\mathcal{L}SH_*(-; R)$ can be thought of as objects of either $\mathcal{H}_{\mathcal{A}_p}$ or $\mathcal{L}_p$.*

Furthermore, with respect to both categories in (2) the functor $\mathcal{L}SH_*(-; \mathbb{F}_p)$ takes products into coproducts and preserves inductive colimits.

PROOF. The only points which requires proof are that the functor $\mathcal{L}SH_*(-; \mathbb{F}_p)$ takes products into coproducts and preserves inductive colimits. First observe that $\Omega B(G \times H)^\wedge_p \simeq \Omega BG^\wedge_p \times \Omega BH^\wedge_p$. Thus

$$\mathcal{L}SH_*(G \times H; \mathbb{F}_p) \cong \mathcal{L}SH_*(G; \mathbb{F}_p) \otimes \mathcal{L}SH_*(H; \mathbb{F}_p).$$

Since the tensor product is the coproduct in both $\mathcal{H}_{\mathcal{A}_p}$ and $\mathcal{L}_p$, the first statement follows.

Next, notice that $\mathcal{L}SH_*(-; \mathbb{F}_p)$ is a composition of functors which all preserve inductive colimits. The second statement follows. $\square$

An immediate corollary of the Kan-Thurston theorem is that for $R = \mathbb{Z}$ the homology of every connected loop space is of the form $\mathcal{L}SH_*(G; \mathbb{Z})$ for a suitable group G. One may wonder about abstract properties of this functor. In particular it would be interesting if one could come up with a purely algebraic description of it. The major obstacle at this point to doing so is the fact that the algebraic "plus" construction is not functorial. However we believe that this problem can be solved. Notice that a purely algebraic construction for a differential graded algebra whose homology gives $\mathcal{L}SH_*(G; R)$ would be desirable as it would imply that all the algebraic invariants discussed above, the Hopf algebra, Lie algebra, growth features, nilpotence questions etc., are somehow encoded completely in the structure of the groups under consideration.

3. The stable and unstable homotopy of $\Omega BG_p^\wedge$

Spaces of the form $\Omega BG_p^\wedge$ appear to have some intriguing homotopy properties. In this section some known general facts and examples of the behavior of the homotopy of $\Omega BG_p^\wedge$ are given. Relations to the existence of exponents in homotopy are discussed.

3.1. The stable homotopy theory of $\Omega BG_p^\wedge$. Here one can prove the following.

THEOREM 3.1. *Let G be a finite p-perfect group. Then the highest power of p dividing the order of G is an exponent for the stable homotopy of $\Omega BG_p^\wedge$.*

As mentioned above, Dwyer's alternative proof of Theorem 2.2, inspired the proof of 3.1. The reader will recognize the similarity at once.

Let G be a finite p-perfect group. Consider the principal bundle discussed in the preceding section

$$L^pG \xrightarrow{\pi} A^pG,$$

where A^pG is mod-p acyclic and L^pG is homotopy equivalent to $\Omega BG_p^\wedge$. Consider the stable adjoint of the Kahn-Priddy stable transfer map

$$\bar{tr} : A^pG \longrightarrow \mathbf{Q}(L^pG).$$

This map is null-homotopic, since A^pG is mod-p acyclic and $Q(L^pG)$ is a loop space on a p-local space.

Thus consider the composite $\bar{tr} \circ \pi : L^pG \longrightarrow \mathbf{Q}(L^pG)$. On one hand it is obviously null-homotopic since $\bar{tr}$ is. On the other hand it is not hard to show that, due to the fact that G operates trivially up to homotopy on L^pG, this composition gives precisely the adjoint of the degree $|G|$ map on $\Sigma^\infty \Omega BG_p^\wedge$. Thus $|G|$ annihilates the stable homotopy of $\Omega BG_p^\wedge$.

In contrast to the above theorem, the following is proven in [**22**]

THEOREM 3.2. *Let G be a finite group. Then the p-completed classifying space $BG_p^\wedge$ does not have a stable homotopy exponent.*

3.2. Unstable homotopy theory. We now turn to the unstable homotopy of $BG_p^\wedge$. A first question one could ask is how much of it is there? Throughout this article we consistently stress the point that in many respects $\Omega BG_p^\wedge$ behaves like it is trying to be a finite complex (but for technical reasons cannot be a finite complex). Indeed one can show that the homotopy of $BG_p^\wedge$ has a common feature with that of a finite complex.

THEOREM 3.3. *Let G be a finite group containing a non-trivial p-perfect subgroup of order divisible by p. Then $BG_p^\wedge$ has infinitely many non-vanishing k-invariants; in particular infinitely many non-trivial homotopy groups.*

The proof for this is rather simple. Namely, let G be a finite p-perfect group and consider a faithful representation of G in $SU(n)$ for a suitable n. Then from the fibration

$$SU(n)/G \longrightarrow BG \longrightarrow BSU(n),$$

it follows that $\Omega BG_p^\wedge$ is the fibre of the p-completion of the projection from $SU(n)$ to $SU(n)/G$. Thus by [**16**] $\Omega BG_p^\wedge$ has a quasi-bounded cohomology module;

that is the cyclic module over the Steenrod algebra generated by every element $x \in H^*(\Omega BG_p^\wedge;\mathbb{F}_p)$ is a finite dimensional vector space. By Miller's theorem, the Sullivan conjecture, spaces with this property have the feature that they do not accept any essential map from BG where G is a finite group.

This property of $\Omega BG_p^\wedge$ is exploited as follows. Recall that $\Omega BG_p^\wedge$ is up to homotopy a finite cover of an acyclic space A^pG. One observes that A^pG cannot be a product of a finite Postnikov section and a generalized Eilenberg MacLane space. Then one assumes that A^pG is a finite Postnikov section and observes that in this case $\Omega BG_p^\wedge$ has to be the target of an essential map from $K(\pi,n)$, where π is the top homotopy group of A^pG. Notice that π is a finite group. Thus by looping $n-1$ times a contradiction to Miller's theorem is obtained and the proof is complete.

A second view of this last theorem is to ask whether the mod-p homology ring of $\Omega BG_p^\wedge$ has an element of infinite height. If that were the case, then this theorem would follow at once from [**30**].

The question now becomes, what can be said about the homotopy of $\Omega BG_p^\wedge$. An easy observation gives an upper bound on the connectivity of $BG_p^\wedge$.

Proposition 3.4. *Let G be a finite p-perfect group and let n be the minimal dimension of a faithful unitary representation of G. Then $BG_p^\wedge$ is at most $(2n-1)$-connected.*

Let a faithful representation of G in $U(n)$ be given. Then, as usual, we obtain a principal fibration

$$U(n)_p^\wedge \longrightarrow (U(n)/G)_p^\wedge \longrightarrow BG_p^\wedge.$$

Since the fibration is principal and since the top dimensional indecomposable in the fibre line of the associated Serre spectral sequence appears in dimension $2n-1$, it follows at once that if $BG_p^\wedge$ is $2n$-connected then the spectral sequence collapses. But this implies a contradiction, since $BG_p^\wedge$ is infinite dimensional, whereas $(U(n)/G)_p^\wedge$ is finite dimensional.

Notice that one obtains better estimates on the first non-vanishing group by using the Steenrod algebra. We omit the precise statement.

In all the examples where one has a reasonable hold on the homotopy type there is an exponent for the homotopy of $BG_p^\wedge$. There are basically two families of examples. The first is the family of groups which satisfy the spherical resolvability conjecture. Since spheres are known to have homotopy exponents, so does any space which is spherically resolvable of finite length. A list of such examples appears in [**22**] and includes finite groups of classical Lie type and many more examples.

The second family of groups consists of those which fail to satisfy the spherical resolvability conjecture (and, as observed above, there are such groups). In examples there is not much known. But for at least one family of finite groups G it is known that $\Omega BG_p^\wedge$ splits as a product of familiar pieces, all of which have been shown by Cohen, Moore and Neisendorfer to have homotopy exponents. Thus it seems to be reasonable to conjecture that the existence of exponents is a general feature of $BG_p^\wedge$ for G finite. We shall collect all known examples in an appendix at the end of the paper.

4. The Postnikov tower of $BG_p^\wedge$

A major viewpoint in our subject is that the functor $\Omega(B(-)_p^\wedge)$ applied to a p-perfect group G gives a loop space whose structure is encoded in a large part in the group structure of G. Thus, since our groups G are always assumed to be finite, we could ask what finiteness properties does $\Omega BG^\wedge_p$ have? Several such properties were already mentioned or conjectured to be true in the discussion above. Here we provide another finiteness condition on $\Omega BG^\wedge_p$, namely we discuss the fact that for many groups G, any space X with the same mod-p cohomology as $BG^\wedge_p$ and the same Moore-Postnikov tower through a range has the property that ΩX is homotopy equivalent to $\Omega BG^\wedge_p$.

Recall that a commutative algebra with a unit A is said to be Cohen-Macaulay if its Krull dimension is equal to its depth. Another way of saying the same thing is that there exists a finitely generated polynomial subalgebra P in A such that A is a finitely generated free module over P. Many finite groups have the property that their mod-p cohomology is Cohen-Macaulay. By abuse of terminology we say in this case that the group itself is Cohen Macaulay at p. For instance all abelian groups have this property. Every group G such that the cohomology of its Sylow p-subgroup is Cohen-Macaulay has the same property itself and some groups whose Sylow p-subgroups are not Cohen-Macaulay are still Cohen-Macaulay. We should point out though that being Cohen-Macaulay is a rather strong condition on G and if a finite group is chosen at random, chances are its cohomology won't have this property (for instance most symmetric groups are not Cohen-Macaulay at most primes dividing their orders).

From now on use the phrase G is a CM_p group to say that $H^*(BG;\mathbb{F}_p)$ is Cohen-Macaulay. For finite CM_p groups G the following well-known lemma holds. A reference for a slightly more general version is [**9**]. For a finite group G, the p-rank, $rk_p(G)$ is the maximal rank of an elementary abelian subgroup of G.

LEMMA 4.1. *Let G be a finite group of p-rank r. Then there is a faithful unitary representation ρ of G with the following property:*

1. *there are precisely r non-zero Chern classes $c_{i_1}, \cdots c_{i_r}$ for ρ and*
2. *$H^*(BG;\mathbb{F}_p)$ is a finitely generated module over the polynomial subring generated by $c_{i_1}, \cdots c_{i_r}$.*

In particular if G is CM_p and a representation as in the lemma is given then $H^*(BG;\mathbb{F}_p)$ is a finitely generated free module over the polynomial subring generated by the non-zero Chern classes. Representation with the properties granted by the lemma will be called an admissible representations for G. The existence of admissible representations for any CM_p group is an essential ingredient in the proof of the theorem below. Details are in [**21**], where the theorem is proven assuming the existence of admissible representations rather than having it granted for every CM_p group.

For any space X and a positive integer d, let $X[d]$ denote the d-th stage of the Postnikov tower for X. We say that X and Y have the same d-th Postnikov stage if there is a map $l_Y : Y \longrightarrow X[d]$, inducing an isomorphism on $\pi_i(-)$ for $0 \le i \le d$.

DEFINITION 4.2. *Two spaces X and Y are said to be comparable of degree d if*

1. *X and Y have the same d-th Postnikov stage $X[d]$.*
2. *$H^*(X) \cong H^*(Y)$ as algebras.*
3. *There is an isomorphism $\phi : H^*(X) \longrightarrow H^*(Y)$ realizing 2, such that $\phi \circ l^*_X = l^*_Y$ in mod-p cohomology, up to dimension d.*

THEOREM 4.3. *Let G be a finite p-perfect CM_p group. Assume that G has an admissible representation in $SU(n)$. Suppose that X and $BG_p^\wedge$ are comparable of degree n^2-1. Then $\Omega BG_p^\wedge \simeq \Omega X$.*

A sketch of the proof follows. First, let $X[n]$ be the n-th stage of the Postnikov tower for a space X. Then for any space C such that $\dim C \leq n+1$, the canonical map $l : X \longrightarrow X[n]$ induces an epimorphism of sets

$$l_* : [C; X] \longrightarrow [C; X[n]].$$

Furthermore, if $\dim C \leq n$, then l_* is an isomorphism.

Next, as a consequence of the big collapse theorem of L. Smith [**34**], we get that if G is a finite p-perfect group which admits an admissible representation ρ in $SU(n)$ for some n, then

$$H^*(SU(n)/G) \cong H^*(BG)/(P) \otimes E$$

as an $H^*(BG)$-module, where P is the polynomial subalgebra of $H^*(BG)$, generated by the non-zero Chern classes of ρ, E is a trivial $H^*(BG)$-module and is isomorphic to an exterior algebra on generators corresponding to the regular sequence generating $Ker(\rho^*)$. Moreover, if $p \neq 2$ then this isomorphism is an isomorphism of algebras.

Using the first observation, one obtains the following commutative diagram of fibrations.

$$\begin{tikzcd}
 & \Omega F_X \arrow[r] \arrow[d] & \Omega X \arrow[r] \arrow[d] & \Omega B[n^2-1] \arrow[d] \\
\Omega F_G \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & BG_p^\wedge\langle n^2-1\rangle \arrow[d] \\
\Omega BG_p^\wedge \arrow[r] \arrow[d] & SU(n)_p^\wedge \arrow[r, "j"] \arrow[d] & (SU(n)/G)_p^\wedge \arrow[r] \arrow[d, "\pi"'] & BG_p^\wedge \arrow[d] \\
\Omega B[n^2-1] \arrow[r] & X\langle n^2-1\rangle \arrow[r] & X \arrow[r] & B[n^2-1]
\end{tikzcd}$$

Since F_X is the (n^2-1)-connected cover of X and $SU(n)$ is (n^2-1)-dimensional, it follows that $\pi \circ j$ is null homotopic. Hence j lifts to a map $l : SU(n)_p^\wedge \longrightarrow S$. This in turn induces a map $f : \Omega BG_p^\wedge \longrightarrow \Omega X$, which is obviously a homotopy equivalence if and only if l is a homotopy equivalence.

One now proceeds by showing that any lift l of j has the property that the induced map on $\mathbb{Z}_{(p)}$ homology is an isomorphism in dimension i if $i \leq n^2-2$ and a split monomorphism if $i = n^2-1$. One then argues, by using the Eilenberg-Moore spectral sequence, that S is at most (n^2-1)-dimensional. This is done by observing that the spectral sequence for the fibration

$$SU(n)_p^\wedge \longrightarrow (SU(n)/G)_p^\wedge \longrightarrow BG_p^\wedge$$

collapses at the E_2 term and that, by our algebraic assumptions, the E_2 term for the fibration

$$S \longrightarrow (SU(n)/G)_p^\wedge \longrightarrow BG_p^\wedge$$

is identical to the one above. Finally one observes that S is rationally equivalent to $SU(n)$ and hence is precisely (n^2-1)-dimensional. This implies that any lift l induces an isomorphism on homology and the proof is complete.

5. Homotopy decompositions of $\Omega BG_p^\wedge$ for certain examples

We say that a space Y is a costable factor of another space X if for some $n \geq 0$ Y is a factor of $\Omega^n X$. This notion is dual to that of a stable summand. It is well known that for a finite group G the classifying space $BG_p^\wedge$ stably splits into a finite number of irreducible components. Thus we present the dual question, namely what are the costable building blocks of $BG_p^\wedge$. The answer suggested by numerous examples appears to be that these are related to iterated loop spaces of finite complexes. Here, once more, there is no general theorem to that effect, at the moment.

In this section we present an example of a group G, which has the property that $\Omega BG_p^\wedge$ splits as a product, where the factors are loop spaces on Moore spaces and fibres of degree maps on spheres. This decomposition is a theorem due to Jie Wu and the second author. We sketch the proof as a more detailed account will appear elsewhere.

The groups under consideration are the groups $E(p)$ defined in section 2 above for $p \geq 5$.

There is a faithful representation of $E(p)$ in $SU(3)$. The explicit map is given in [**18**]. One important fact about this representation is the following

Proposition 5.1. *For each prime $p \geq 5$ there is an isomorphism of algebras*

$$H^*(SU(3)/E(p); \mathbb{F}_p) \cong P[a_2, b_3, b_3', c_5, c_5', d_6]/R,$$

where degrees are given by subscripts and the set of relations R consists of the equalities $ad = cb = c'b'$ together with setting every other product being equal to zero. The only non trivial Steenrod operations in $H^*(SU(3)/E(p))$ are $\beta_r a = b$ and $\beta_1 c = d$.

Let X denote $(SU(3)/E(p))_p^\wedge$. The observation made in [**18**] is the following

Proposition 5.2. *There is a homotopy equivalence*

$$X^{(6)} \simeq P^3(p) \vee P^6(p) \vee S^3 \vee S^5,$$

where $X^{(6)}$ is the 6-skeleton of X.

The proof involves a mildly tricky argument, using an improvement on a theorem of I. James, due to J. Harper. Proposition 5.2 gives us enough information to proceed.

For primes $p \geq 3$ there is a mod-p homotopy equivalence $SU(3) \simeq S^3 \times S^5$. Thus there is a map $\pi : X \longrightarrow SU(3)$, which is degree 1 on the cells represented by b' and c'. Let F denote the homotopy fibre of π. Notice that the composition of π with the projection to each one of the spheres admits a right homotopy inverse. This observation is used to prove the following

Proposition 5.3. *There are homotopy equivalences*

1. $\Omega X \simeq \Omega F \times \Omega SU(3)$ *and*
2. $\Omega BE(p^r)_p^\wedge \simeq \Omega F \times S^3\{p\} \times S^5\{p\}$.

Thus to complete the decomposition one needs to understand the homotopy type of F. For any two pointed spaces X and Y we denote by $X \rtimes Y$ the half smash product $X \times Y/ * \times Y$. Indeed it is an easy observation that there is a homotopy equivalence

$$F \simeq (P^3(p) \vee P^6(p)) \rtimes \Omega SU(3).$$

Finally the formula

$$\Omega(X \rtimes Y) \simeq \Omega X \times \Omega\Sigma(Y \wedge \Omega X)$$

gives that

$$\Omega F \simeq \Omega(P \rtimes \Omega S) \simeq \Omega P \times \Omega\Sigma(\Omega S \wedge \Omega P) \simeq \Omega P \times \Omega(\Sigma\Omega S \wedge \Omega P),$$

where P denotes the wedge $P^3(p) \vee P^6(p)$ and S denotes $SU(3)^\wedge_p$. But since S is a product of two spheres, $\Sigma\Omega S$ is an infinite wedge of spheres and by [**28**], $\Sigma\Omega S \wedge \Omega P$ is an infinite wedge of mod-p Moore spaces.

The known results of [**28**] about the homotopy of Moore spaces and spaces of the form $S^{2n+1}\{p^r\}$ at odd primes now imply the following.

Corollary 5.4. *The groups $\pi_* BE(p)^\wedge_p$ have an exponent p^2 and the p^2 power map on the double loop space $\Omega^2_0 BE(p)^\wedge_p$ is null-homotopic.*

6. Relation with certain finite complexes

Here we present some relationships between spaces of the form $BG^\wedge_p$ for G finite and p-perfect and finite complexes. Part of this appears here for the first time.

An unexpected fact that sheds new light on some of what is said above is that for any finite p-perfect group G, the loop space $\Omega BG^\wedge_p$ is a retract of the loop space of a finite complex. The statement of the main theorem we discuss here, in its full generality, requires the use of p-compact groups and maps which we call homotopy representations into such gadgets. However to simplify the discussion, we will restrict ourselves to using compact Lie groups instead. Conceptually there is no difference. The reader interested in full details is referred to [**19**].

Let G be a finite p-perfect group. Let L be a compact connected Lie group, such that there exists a faithful representation of G in L. Then one obtains a fibration

$$L/G \longrightarrow BG \longrightarrow BL,$$

which p-completion preserves, since BL is simply-connected. Notice that the assumption that ρ is faithful implies that L/G is a compact manifold, in particular a finite complex. Consider the following diagram, where the rows are fibrations

$$\begin{tikzcd}
\Omega BG^\wedge_p \arrow[r] \arrow[d] & L^\wedge_p \arrow[r, "g"] \arrow[d, "\phi"'] & (L/G)^\wedge_p \arrow[r, "\alpha"] \arrow[d] & BG^\wedge_p \arrow[d, "="'] \\
\Omega BG^\wedge_p \arrow[r, "\delta"] & F_\rho \arrow[r] & S \arrow[r, "\rho"] & BG^\wedge_p.
\end{tikzcd}$$

In the diagram, S is the cofibre of g, the map ρ extends α and F_ρ is the homotopy fibre of ρ. An old result of Ganea [**12**] gives a formula for $F\rho$. Indeed there is a homotopy equivalence

$$F_\rho \simeq \Sigma(L^\wedge_p \wedge \Omega BG^\wedge_p).$$

In particular the map ϕ in the diagram is null-homotopic and hence by commutativity so is δ. It follows that $\Omega\rho$ has a right homotopy inverse and so

$$\Omega S \simeq \Omega BG^{\wedge}_p \times \Omega\Sigma(L \wedge \Omega BG^{\wedge}_p).$$

Notice that S is

1. a p-torsion space, since g is a rational equivalence and
2. has finite mod-p homology.

Thus S has the homotopy type of the p-completion of a the finite complex given by the cofibre of the projection $L \longrightarrow L/G$. This implies that, given a CW structure on BG with B_kG denoting the k-skeleton, there is an integer n such that the skeletal inclusion

$$B_nG^{\wedge}_p \overset{i}{\hookrightarrow} BG^{\wedge}_p$$

has the property that Ωi has a right homotopy inverse. The minimal such n is called the p-essential dimension of G. It is obviously an invariant of the group and will be denoted by $ed_p(G)$. We have thus proven

Theorem 6.1. *Every finite p-perfect group has a finite p-essential dimension.*

The terminology "essential dimension" here is motivated by the fact that if $f : BG \longrightarrow X$ is any map with X p-complete and such that $conn(X) > ed_p(G)$ then Ωf is null-homotopic.

Here is a simple example of an application of this concept. Let G be any proper subgroup of H, where H itself is the normalizer of a $\mathbb{Z}/p\mathbb{Z}$ in the symmetric group on p letters. If G contains $\mathbb{Z}/p\mathbb{Z}$ and is p-perfect, then the natural map $BG^{\wedge}_p \xrightarrow{\iota} BH^{\wedge}_p$ gives an injection in mod-p cohomology. However, $\Omega\iota$ is null-homotopic. Notice that an immediate consequence of this is that ι induces the zero map on homotopy group. One should really consider this observation in contrast to the fact that if one suspends ι at least once it admits a right homotopy inverse [**6**].

Theorem 6.1 explains some of the results above in a new way. In particular it gives a new proof of the fact that $H_*(\Omega BG^{\wedge}_p; \mathbb{Z}^{\wedge}_p)$ has an exponent for G finite and p-perfect. Indeed it is shown in [**20**] that every finite torsion complex has this property. A bound is not easy to determine by this method though.

The following is another easy application of Theorem 6.1. Its proof appeared originally in [**19**]. Here we present a slightly stronger statement.

Theorem 6.2. *Let G be a finite p-perfect group. Let $E : BG^{\wedge}_p \longrightarrow \Omega\Sigma BG^{\wedge}_p$ denote the Freudenthal suspension map. Then ΩE is an element of finite order in the group $[\Omega BG^{\wedge}_p; \Omega^2\Sigma BG^{\wedge}_p]$. Furthermore, let S be as i-connected p-torsion space together with a map $f : S \longrightarrow BG^{\wedge}_p$ such that Ωf has a right homotopy inverse. Then a bound on the order of ΩE is given by the order of the identity map in the group $[\Sigma S; \Sigma S]$.*

Proof. First notice that a space as in the statement of the theorem always exists by the proof of Theorem 6.1. Next observe that the adjoint to the composite

$$S \xrightarrow{f} BG^{\wedge}_p \xrightarrow{E} \Omega\Sigma BG^{\wedge}_p \xrightarrow{\langle p^r \rangle} \Omega\Sigma BG^{\wedge}_p$$

is the suspension of f followed by the degree p^r map $[p^r]$ on $\Sigma BG^{\wedge}_p$. Furthermore, $[p^r] \circ \Sigma f \simeq \Sigma f \circ [p^r]$. Hence if p^r is the order of the identity element in the group $[\Sigma S; \Sigma S]$ then $\langle p^r \rangle \circ E \circ f$ is null homotopic. But upon looping Ωf has a right homotopy inverse and it follows that $\langle p^r\rangle \circ \Omega E$ is null-homotopic, which proves our statement. $\square$

COROLLARY 6.3. *Let G be a finite p-perfect group. Let F be any representable cohomology theory. Then the natural map*

$$F^n(BG^\wedge_p) \xrightarrow{\Omega} F^{n-1}(\Omega BG^\wedge_p),$$

given by the looping functor has an image of finite exponent. Furthermore, a bound is universally given by the order of $\Omega E \in [\Omega BG^\wedge_p; \Omega^2\Sigma BG^\wedge_p]$.

PROOF. Let F_n denote the n-th space of an Ω-spectrum representing F. Then any element of $x \in F^n(BG^\wedge_p)$ is represented by a map $x : BG^\wedge_p \longrightarrow F_n$. Since F_n is a loop space, x factors through $\Omega\Sigma BG^\wedge_p$ as $\Omega(\mathrm{ad}x) \circ E$, where E is the Freudenthal suspension. Let $\bar{x}$ denote $\Omega(\mathrm{ad}x)$. Then under the group homomorphism

$$[BG^\wedge_p; \Omega\Sigma BG^\wedge_p] \xrightarrow{\bar{x}_*} F^n(BG^\wedge_p)$$

the Freudenthal suspension E is taken to x. By naturality of the looping functor,

$$\Omega\bar{x}_* : [\Omega BG^\wedge_p; \Omega^2\Sigma BG^\wedge_p] \longrightarrow F^{n-1}(\Omega BG^\wedge_p)$$

takes ΩE to $\Omega\bar{x}$. But ΩE is an element of finite order and the result follows. $\square$

Another application of Theorem 6.1 comes next. Let G be a finite p-perfect group and choose any faithful representation ρ of G in $U(n)$. In the examples which have been understood, the map $\Omega B\rho^\wedge_p : \Omega BG^\wedge_p \longrightarrow U(n)^\wedge_p$ has been nontrivial, which might lead one to believe that this should always be the case. However, the following theorem will imply the existence of many faithful representations ρ of a finite p-perfect group G, where the induced map $\Omega B\rho^\wedge_p$ is null-homotopic.

THEOREM 6.4. *Let G be a finite p-perfect group and let ρ be a representation of G in $L(n)$, where $L(n)$ is either $O(n)$, $U(n)$ or $Sp(n)$. For a positive integer k let $k\rho$ denote the k-fold Whitney sum of ρ. Then there exists an integer r such that*

$$\Omega B(p^r\rho) : \Omega BG^\wedge_p \longrightarrow L(p^r n)^\wedge_p$$

is null-homotopic.

PROOF. Let L denote the colimit of the groups $L(n)$. Let $\iota_n : L(n) \longrightarrow L$ denote the inclusion. Recall that the k-fold Whitney sum induces the k-th power map on the infinite loop space BL in the sense that the diagram

$$\begin{tikzcd} BL(n) \arrow[r] \arrow[d, "\iota_n"'] & BL(n)^{\times k} \arrow[r] & BL(kn) \arrow[d, "\iota_{kn}"'] \\ BL \arrow[rr, "\langle k\rangle"] & & BL \end{tikzcd}$$

commutes. Let w_k denote the composite on the top row of the diagram.

Let S be a 1-connected p-torsion space together with a map $f : S \longrightarrow BG^\wedge_p$, such that Ωf has a right homotopy inverse. Without loss of generality we may assume that the representation ρ is faithful and and thus that S is of dimension at most one more than the dimension of $L(n)$. To justify this, notice that the proof of Theorem 6.1 implies the existence of a complex S of this dimension satisfying our requirement, provided the representation ρ is faithful. If ρ isn't faithful, then it factors through a faithful representation of a quotient group H of G, and thus it suffices to prove the claim for H.

Consider the p-completion of the diagram above. If p^s is the order of identity in $[\Sigma S;\Sigma S]$ then we have

$$\iota_{p^s n}\circ B(p^s\rho)^\wedge_p\circ f=\iota_{p^s n}\circ w_{p^r}\circ B\rho^\wedge_p\circ f\simeq\langle p^s\rangle\circ\iota_n\circ B\rho^\wedge_p\circ f$$

and the right hand side is null-homotopic. Thus the map $B(p^s\rho)^\wedge_p\circ f$ lifts to the fibre $(L/L(p^sn))^\wedge_p$ of $\iota_{p^s n}$, which is (p^sn-1), $(2p^sn-1)$ or $(4p^sn+1)$-connected, according to L being O, U or Sp. On the other hand $\dim(S)\leq\dim(L(n))+1$. Thus if r is chosen so that $\dim(L(n))+1\leq\mathrm{conn}(L/L(p^rn))$, then $B(p^r\rho)^\wedge_p\circ f$ is null-homotopic. But Ωf has a right homotopy inverse, implying that for such r, $\Omega B(p^r\rho)^\wedge_p$ vanishes. □

Corollary 6.5. *Let G be any finite group. Let $L(n)$ be as in Theorem 6.4. Then there exists a positive integer r such that for any representation ρ of G in $L(n)$, the induced map $\Omega B(p^r\rho)^\wedge_p$ is null-homotopic.*

Proof. We may assume that G contains a non-trivial p-perfect subgroup or otherwise the claim is redundant. Thus $\Omega BG^\wedge_p$ is a finite disjoint union of copies of $\Omega B\Pi^pG^\wedge_p$. The result follows at once from Theorem 6.4 and the fact that any finite group has a finite number of distinct quotient groups, each of which having at most a finite number of non-equivalent representations in $L(n)$. □

Corollary 6.6. *Let G be a finite p-perfect group and let a faithful representation ρ of G in $L(n)$ be given, where $L(n)$ is as in Theorem 6.4. Then there exists a positive integer r such that*

$$\Omega(L(p^rn)/G)^\wedge_p\simeq\Omega L(p^rn)^\wedge_p\times\Omega BG^\wedge_p,$$

where G acts on $L(p^rn)$ via the faithful representation $p^r\rho$.

Proof. This follows at once from the fact that for a suitable r, the map

$$\Omega BG^\wedge_p\xrightarrow{\Omega B(p^r\rho)^\wedge_p}L(p^rn)^\wedge_p$$

is null-homotopic. □

The discussion above might motivate one to wonder about the order of the identity map in the group $[\Sigma S;\Sigma S]$, where S is the space produced in the proof of Theorem 6.1. We are not able at this point to answer that question, but a first step would be getting an upper bound on the exponent of the integral homology of S.

Theorem 6.7. *Let L be a connected compact Lie group with torsion free p-adic cohomology and let G be a finite p-perfect group. Let $\rho : G\longrightarrow L$ be a faithful representation of G in L. Let S denote the cofibre of the natural map $L^\wedge_p\longrightarrow(L/G)^\wedge_p$. Then the reduced p-adic homology of S is annihilated by the maximal power of p dividing the order of G.*

Proof. consider the (non-completed) projection $L\longrightarrow L/G$. Let S' denote its cofibre. By the proof of Theorem 6.1, $S\simeq(S')^\wedge_p$. Thus it suffices to bound the torsion of the homology of S'. But S' is the cofibre of a fibre inclusion and hence Ganea's theorem, mentioned in the proof of Theorem 6.1, applies to compute its universal cover. Indeed the map $L/G\longrightarrow BG$ factors through S' by the universal property of cofibrations. By Ganea's theorem the fibre of the map resulting map $S'\longrightarrow BG$ is given by

$$\Sigma(L\wedge\Omega BG)\simeq\Sigma(L\wedge G)\simeq\bigvee_{|G|-1}\Sigma L.$$

Thus there is a G covering spaces over S', with a homologically torsion free total space. The usual transfer argument now implies the result. □

7. Problems and conjectures

Since many of the problems concerning the homotopy theoretic features of $BG^{\wedge}_p$ remain unsolved, a list of them together with examples is given below.

7.1. Homotopy exponents. The first question we present about the homotopy type of $BG^{\wedge}_p$ for G finite and p-perfect is whether or not there exist an exponent for the unstable homotopy $\pi_*(BG^{\wedge}_p)$. Evidence suggest that the answer should be yes and furthermore, that the order of the Sylow p-subgroup of G should be an upper bound for this exponent. We mention the examples discussed in [**22**] of groups G, such that $\Omega BG^{\wedge}_p$ is spherically resolvable of finite length. For most of those examples it is possible to show, using the resolution that the order of the Sylow p-subgroup is indeed an upper bound for the exponent of $\pi_*(BG^{\wedge}_p)$.

It can also be shown that if G is a 2-perfect group containing the dihedral group D_{2^n} as a Sylow 2-subgroup, then 2^n does not bound the torsion in $\pi_*(BG^{\wedge}_2)$ but 2^{n+1} does. This is possibly due to the fact that for such a group G the universal cover of $BG^{\wedge}_2$ is the 2-completed classifying space of the 2-universal central extension of G, which contains the generalized quaternion group $Q_{2^{n+1}}$ as a Sylow 2-subgroup.

Recall the example of $\Omega BE(p)^{\wedge}_p$, discussed above, which splits in terms of loop spaces on Moore spaces and fibres of degree p self maps on spheres. Thus an exponent in unstable homotopy is obtained for those examples as well by the known results of Moore, Neisendorfer and the first author. Furthermore, notice that if the group theory could be used to inform on the homotopy theory, then there would be an alternative proof that a Moore space has an exponent.

Also Theorem 6.2 gives that the image of

$$E^n_* : \pi_*(BG^{\wedge}_p) \longrightarrow \pi_*(\Omega^n\Sigma^n BG^{\wedge}_p),$$

where E^n is the n-fold Freudenthal suspension map has an exponent.

The Moore finite exponent conjecture is that if X is a 1-connected finite complex then the torsion in $\pi_*(X)$ has an exponent if and only if X is elliptic, namely if its rational homotopy is a finite dimensional graded vector space. If X is p-complete then the same conjecture can be stated replacing the rationals by the p-adic rationals $\mathbb{Q}_p$. Above we exhibited fibrations of the form

$$L^{\wedge}_p \longrightarrow (L/G)^{\wedge}_p \longrightarrow BG^{\wedge}_p$$

where L is a compact Lie group. Obviously both $L^{\wedge}_p$ and $(L/G)^{\wedge}_p$ have finite dimensional $\mathbb{Q}_p$ homotopy. Thus if the Moore conjecture were true it would imply the existence of unstable homotopy exponents for $BG^{\wedge}_p$. On the other hand a general exponents theorem for $BG^{\wedge}_p$ will automatically produce a large family of positive examples to the Moore conjecture, namely the spaces $(L/G)^{\wedge}_p$.

With this in mind we are now ready to state our first conjecture. Recall that a group G is said to be p-superperfect if $BG^{\wedge}_p$ is 2-connected.

CONJECTURE 7.1. *Let G be a finite p-superperfect group. Then the order of the Sylow p-subgroup of G is an exponent for $\pi_*(BG^{\wedge}_p)$.*

In all the examples, which we could work out, the p^r-th power map on $\Omega^2 BG_p^\wedge$, for G finite and p-superperfect, is null-homotopic. Here p^r is, as above the order of the Sylow p-subgroup of G. Thus we state an even stronger conjecture

CONJECTURE 7.2. *Let G be a finite p-superperfect group. Then the order of the Sylow p-subgroup of G is an H-space exponent for $\Omega^2 BG_p^\wedge$.*

7.2. Stable and unstable splitting. Next we turn to the problem of expressing the homotopy type of $BG_p^\wedge$, possibly after looping frequently enough, in terms of more familiar spaces. General information concerning the stable structure of $BG_p^\wedge$ is given in works of Martino and Priddy [**25**], Benson and Feshbach [**3**] and others.

We have seen that $\Omega BG_p^\wedge$ is a retract of the loop space on a finite complex, for any finite p-perfect group G. We have also demonstrated an example of a case where $\Omega BG_p^\wedge$ splits completely in terms of loop spaces on Moore spaces and fibres on degree p maps on spheres. Thus one gets the (possibly superficial) impression that $\Omega BG_p^\wedge$ is doing its best to be a finite complex but fails to do so for technical reasons. Let G be a finite p-perfect group. Consider first the stable type of $\Omega BG_p^\wedge$. Some of our evidence suggests the following

CONJECTURE 7.3. *Let G be a finite p-perfect group. Then $\Omega BG_p^\wedge$ stably splits as a wedge of finite complexes. Furthermore, an iterated (finite) suspension of $\Omega BG_p^\wedge$ splits.*

Except for the few examples we know, one can consider Theorem 2.2 above. It gives a property, the existence of exponents in loop space homology, that $\Omega BG_p^\wedge$ has in common with finite complexes and homological properties are preserved by suspensions. Also the existence of stable homotopy exponents supports our last conjecture as it is true for any finite torsion complex that its stable homotopy has an exponent.

From an unstable point of view, it is not clear at all that $\Omega BG_p^\wedge$ or an iterated loop space splits in terms of familiar spaces. In the examples we know it does, but we do not believe this is a general phenomenon.

7.3. Homological problems. Next we recapitulate on some problems concerning loop space homology. Consider $H_*(\Omega BG_p^\wedge; \mathbb{F}_p)$. In all the known examples this is a finitely generated algebra. It is sometimes commutative but most of the time not so (as the homology ring contains a tensor algebra on two generators). The strongest conjecture one can make here is that this is always the case.

CONJECTURE 7.4. *The homology algebra $H_*(\Omega BG_p^\wedge; \mathbb{F}_p)$ is finitely generated for every finite p-perfect group G.*

However, we would not claim that there is enough evidence to make this conjecture plausible. Several weaker statements could be easier to verify. First notice that $H_*(\Omega BG_p^\wedge; \mathbb{F}_p)$ is always infinite dimensional. Thus finite generation would imply that there is always an element of infinite height in $H_*(\Omega BG_p^\wedge; \mathbb{F}_p)$.

CONJECTURE 7.5. *For every finite p-perfect group G there exists an element of infinite height in $H_*(\Omega BG_p^\wedge; \mathbb{F}_p)$.*

Another conjecture already mentioned above concerns the nilpotency degree of the loop space homology Lie algebra.

Conjecture 7.6. *For every positive integer $1 \leq n \leq \infty$ and every prime p, there is a finite p-perfect group G with the property that $\mathcal{L}SH_*(G;\mathbb{F}_p)$ is a nilpotent Lie algebra of rank precisely n.*

This conjecture is known for n at the extremes namely if $n = 1$ or ∞.

7.4. The p-essential dimension. Recall that the essential dimension of a group G is defined to be the least positive integer n such that for any CW model for BG, the natural map

$$B_nG_p^\wedge \longrightarrow BG_p^\wedge$$

admits a right homotopy inverse after looping. This mysterious invariant of groups might have interesting applications. Indeed some were already discussed above. However the upper bound suggested by the proof of Theorem 6.1 is not satisfactory. Indeed this bound is one more than the minimal dimension of a faithful representation of G, or more generally than that of a faithful homotopy representation. For details on the use of homotopy representations in this context the reader is referred to [**10**]. Although the use of those improves on the result we get by using Lie groups, it should be pointed out, by means of examples that one can do better. One example is the groups $E(p)$ discussed above, where the existing approximation is $ed_p(E(p)) \leq 9$, whereas in fact the decomposition of the previous section gives that $ed_p(E(p)) \leq 6$. Another example, this time out of our direct context is of the group $SL(3,\mathbb{Z})$ at the prime 2. There, the upper bound on the 2-essential dimension given by [**19**] is 6 whereas one can show that the actual 2-essential dimension is 4 [**17**]. The question is thus what determines the p-essential dimension of a p-perfect group G. The following conjecture has been verified in all known examples. For a graded algebra A, let QA denote the module of indecomposable elements. If A is finitely generated let t_A denote the top dimension in which QA is non-zero.

Conjecture 7.7. *Let G be a finite p-perfect group and let A denote its mod-p cohomology algebra. Then*

$$ed_p(G) \leq t_A.$$

It is worth pointing out though that for the group $SL(3,\mathbb{Z})$ the bound given by this conjecture is not best possible. However it is precisely this example which motivates the following

Conjecture 7.8. *Let H be a central extension of a finite p-perfect group by a finite abelian p-group. Then $ed_p(G) \leq ed_p(H)$. Thus for a finite p-perfect group let $\tilde{G}$ denote the p-universal central extension. Let A denote the mod-p cohomology of $\tilde{G}$. Then with the notation above*

$$ed_p(G) \leq t_A.$$

8. Appendix

A list of examples where the homotopy type of $\Omega BG_p^\wedge$ has been worked out appears below. Details are omitted. The interested reader is referred to the appropriate references.

8.1. Periodic p-Sylow subgroup. If G has periodic mod-p cohomology and is p-perfect then $\Omega BG_p^\wedge$ is homotopy equivalent to $S^{2n-1}\{p^r\}$ for appropriate n and r [**8**].

8.2. Groups of Lie type. Let $G(\mathbb{C})$ be a complex reductive connected Lie group. Let $G(\mathbb{F}_q)$ denote the corresponding finite group of Lie type over the field of q elements. Let p be a prime not dividing q and suppose that the integral cohomology of $G(\mathbb{C})$ has no p-torsion. Then $\Omega BG(\mathbb{F}_q)^{\wedge}_p$ is spherically resolvable of finite length [**22**]. This applies in particular to the groups $SL(n,\mathbb{F}_q)$ and $Sp(n,\mathbb{F}_q)$ at all primes p-different from the characteristic. In those examples the resolution is given by spaces of the form $S^{2n-1}\{p^r\}$ which allows one to conclude that the order of the Sylow p-subgroup is an upper bound for the order of torsion in $\pi_*(BG^{\wedge}_p)$.

8.3. Clark-Ewing groups. These are groups of the form $T \rtimes W$, where T is a finite product of cyclic groups of order p^r (r fixed) and W is a p-adic pseudo-reflection group. The resulting groups have the property that their mod-p cohomology is a symmetric algebra. It is shown in [**22**] that most of those are resolvable of finite length, where the resolving spaces are of the form $S^{2n-1}\{p^r\}$. Thus, as for groups of Lie type, there is a homotopy exponent for those examples as well.

8.4. Finite simple groups of 2-rank two. Let G be a finite simple group of 2-rank two. Then one of the following holds

1. G has dihedral Sylow 2-subgroups and G is isomorphic to either A_7 or $PSL(2,\mathbb{F}_q)$ for q odd,
2. G has semi-dihedral Sylow 2-subgroup and so G is isomorphic to either $PSL(3,\mathbb{F}_q)$ for $q \equiv 3(\text{mod } 4), U(3,\mathbb{F}_q)$ for $q \equiv 1(\text{mod } 4)$, or M_{11},
3. G has wreathed Sylow 2-subgroup and G is isomorphic to either $PSL(3,\mathbb{F}_q)$ for $q \equiv 1(\text{mod } 4)$, or $U(3,\mathbb{F}_q)$ for $q \equiv 3(\text{mod } 4)$; or
4. Γ is isomorphic to $U(3,\mathbb{F}_4)$.

In case 1 it is shown in [**22**] that the 1-connected cover of $\Omega BG^{\wedge}_2$ is homotopy equivalent to $S^3\{2^r\}$ for the appropriate r. If G has as semi-dihedral Sylow 2-subgroup then a combination of results from [**22**] and [**14**] gives that $\Omega BG^{\wedge}_2$ is spherically resolvable. In fact $\Omega BG^{\wedge}_2$ fibres over $S^5\{2\}$ with fibre $S^3\{2^r\}$ for a suitable r. It is worth pointing out though that work of Martino and Priddy shows that the 2-completed classifying spaces of $SL(3,\mathbb{F}_3)$ and M_{11} are homotopy equivalent even before looping [**26**]. The existence of spherical resolutions of length 4 in case 3 follows from [**22**]. The type of resolutions is similar to case 2. The only homotopy type in this family that has not been decided is case 4, namely $G = U(3,\mathbb{F}_4)$. Here one might guess that it behaves like one of the "exponentially growing" examples discussed above.

8.5. The sporadic simple group J_1. Using an unpublished theorem of J. Harper which characterizes the 2-local homotopy type of the Lie group G_2 cohomologically, it is shown in [**1**, pp. 281-282] that the homotopy fibre of a map $(BJ_1)^{\wedge}_2$ to $(BG_2)^{\wedge}_2$ is $(G_2)^{\wedge}_2$. Thus $\Omega(BJ_1)^{\wedge}_2$ is the homotopy fibre of a self map of $(G_2)^{\wedge}_2$.

8.6. The groups $D(p)$ and $E(p)$. The groups $D(p)$ and $E(p)$ are defined above. The homotopy type of $\Omega BD(p)^{\wedge}_p$ hasn't been worked out completely. It is known though that $\Omega P^{16}(p)$ splits off as a retract of the first. A complete decomposition for $\Omega E(p)^{\wedge}_p$ was given in section 5.

8.7. Amalgamated products of finite groups. Amalgamated products of finite p-perfect groups are also p-perfect. They are generally infinite but if the amalgam is finite then the resulting groups have finite virtual cohomological dimension [**5**]. Thus by [**19**] for such groups G, the loop space $\Omega BG^{\wedge}_p$ behaves very much as if G were finite. In particular they have finite p-essential dimension. The p-local homology of such groups is entirely torsion and in fact has an exponent bounded above by the highest exponent for the factors. Thus having a finite p-essential dimension implies, in conjunction with [**20**], that $BG_p^\wedge$ has an integral loop space homology exponent.

There are interesting examples in the literature of finite groups which have the cohomology of an amalgamated product of some of their subgroups. One example of this sort is given by the finite simple group M_{12} [**1**].

8.8. The group $SL(3,\mathbb{Z})$. The special linear group of rank 3 over the integers is in fact mildly out of our context being infinite. However, it can be described as a generalized amalgamated product of finite subgroups and thus exhibits a somewhat similar behavior as one expects from a finite group. For instance its integral cohomology is entirely torsion (2 and 3 torsion). Thus the space $\Omega BSL(3,\mathbb{Z})_p^\wedge$ is only interesting at 2 and 3. At the prime 3 the homotopy type has been worked out in [**8**] by the first author. More recently the second author worked out the homotopy type at $p = 2$ [**17**]. In both cases the answer is given in terms of Moore spaces and fibres of degree p^r maps on spheres.

References

1. A. Adem and R. J. Milgram, *Cohomology of Finite Groups*, Grundlehren der Math. Wiss., vol. 309, Springer Verlag, 1995.
2. S. Araki and T. Kudo, *On $H_*(\Omega^2 S^n;\mathbb{Z}/2\mathbb{Z})$*, Proc. Japan Acad. **32** (1956), 333–335.
3. D. J. Benson and M. Feshbach, *Stable splittings of classifying spaces of finite groups*, Topology **31** (1992), no. 1, 157–176.
4. A. Bousfield and D. Kan, *Homotopy limits completions and localizations*, vol. 304, Springer Verlag, 1972.
5. K. Brown, *Cohomology of Groups*, Springer Verlag, 1982.
6. F. R. Cohen, *Splitting certain suspensions via self maps*, Ill. Jour. Math. **20** (1976), no. 2, 336–347.
7. ———, *Two-primary analogues of Selick's theorem and the Kahn-Priddy theroem for the 3-sphere*, Topology 23(1984) **23** (1984), 401–421.
8. ———, *Some remarks on the homotopy theory associated to perfect groups*, Lecture Notes in Mathematics, vol. 1509, Springer-Verlag, 1990.
9. P. Donovan, *Spectral duality for block cohomology*, Jour. of Algebra **88** (1984), no. 2, 330–345.
10. W. Dwyer and C. Wilkerson, *Homotopy fixed point methods for Lie groups and finite loop spaces*, Ann. Math. **139** (1994), no. 2, 395–442.
11. E. Dyer and R. K. Lashof, *Homology of iterated loop spaces*, Amer. Jour. Math. **84** (1962), 35–88.
12. T. Ganea, *A Generalization of the homology and homotopy suspension*, Comment. Math. Helv. **39** (1965), 295–322.
13. Y. Felix; S. Halperin and J.-C. Thomas, *Elliptic spaces II*, Enseign. Math. (2) **39** (1993), 25–32.
14. F. R. Cohen; J. R. Harper and R. Levi, *On the homotopy theory associated to certain finite groups of 2-rank two*, Homotopy theory and its applications, Contemporary Mathematics, vol. 188, 1995.
15. D. Kan and W. Thurston, *Every connected space has the homology of a $K(\pi,1)$*, Topology **15** (1976), 253–258.
16. J. Lannes and L. Schwartz, *A propo de conjecture de Serre et Sullivan*, Invent. Math. **83** (1986), 593–603.
17. R. Levi, *A Note on the homotopy type of $BSL(3,\mathbb{Z})_2^\wedge$*, to appear.
18. ———, *On homological rate of growth and the homotopy type of $\Omega BG_p^\wedge$*, To appear in Math. Zeit.
19. ———, *On p-completed classifying spaces of discrete groups and finite complexes* , to appear in Journal of the London Math. Soc.

20. ———, *Torsion in loop space homology of rationally contractible spaces*, to appear in Jour. Pure and App. Alg.
21. ———, *A comparison criterion for certain loop spaces*, The Čech centennial, Contemporary Mathematics, vol. 181, 1995, pp. 335–367.
22. ———, *On finite groups and homotopy theory*, Mem. AMS **567** (1995).
23. ———, *A counter example to a conjecture of Cohen*, Algebraic Topology: New trends in localization and periodicity, Birkhäuser, 1996, pp. 261–269.
24. R. Levi and J. Wu, *On* $\Omega BSL(2, \mathbb{F}_9)^\wedge_3$ *and realated spaces*, to appear.
25. J. Martino and S. Priddy, *The complete stable splitting for the classifying space of a finite group*, Topology **31** (1992), no. 1, 143–156.
26. J. R Martino and S. B. Priddy, *Unstable homotopy classification of* $BG^\wedge_p$, Math. Proc. Cambridge Philos. Soc **119** (1996), no. 1, 119–137.
27. H. Miller, *The Sullivan Conjecture on Maps from Classifying Spaces*, Ann. of Math. **120** (1984), no. 1, 39–87.
28. F. R. Cohen; J. C. Moore and J. Neisendorfer, *Exponents in homotopy theory*, Ann. of Math. Stud. **113** (1979), 3–34.
29. ———, *Torsion in homotopy groups*, Ann. of Math. **109** (1979), no. 1, 121–168.
30. J. C. Moore and L. Smith, *Hopf algebras and multiplicative fibrations II*, Amer. Jour. Math. **90** (1968), 1113–1150.
31. M. Nakaoka, *Homology of the infinite symmetric group*, Ann. of Math. **73** (1961), no. 2, 229–257.
32. D. Quillen, *On the cohomology and* K*-theory of the general linear groups over a finite field*, Ann. of Math. **96** (1972), no. 2, 552–586.
33. P. Selick, *Odd primary torsion in* $\pi_k S^3$), Topology **17** (1978), no. 4, 407–412.
34. L. Smith, *Homological algebra and the Eilenberg-Moore spectral sequence*, Trans. Amer. Math. Soc. **129** (1967), 58–93.

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