

A HOMOTOPY THEORETIC REALIZATION OF STRING TOPOLOGY

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ABSTRACT. Let M be a closed, oriented manifold of dimension d . Let LM be the space of smooth loops in M . In [2] Chas and Sullivan defined a product on the homology $H_*(LM)$ of degree $-d$. They then investigated other structure that this product induces, including a Lie algebra structure on $H_*(LM)$, and an induced product on the S^1 equivariant homology, $H_*^{S^1}(LM)$. These algebraic structures, as well as others, came under the general heading of the “String topology” of M . In this paper we will describe a realization of the Chas - Sullivan loop product in terms of a ring spectrum structure on the Thom spectrum of a certain virtual bundle over the loop space. We show that this ring spectrum structure extends to an operad action of the the “cactus operad”, originally defined by Voronov, which is equivalent to operad of framed disks in \mathbb{R}^2 . We then describe a cosimplicial model of this spectrum and, by applying the singular cochain functor to this cosimplicial spectrum we show that this ring structure can be interpreted as the cup product in the Hochschild cohomology, $HH^*(C^*(M); C^*(M))$.

INTRODUCTION

Let M^d be a closed, oriented d - dimensional manifold, and let $LM = C^\infty(S^1, M)$ be the space of smooth loops in M . In [2] Chas and Sullivan described an intersection product on the homology, $H_*(LM)$, having total degree $-d$,

$$\circ : H_q(LM) \otimes H_r(LM) \rightarrow H_{q+r-d}(LM).$$

In this paper we show that this product is realized by a geometric structure, not on the loop space itself, but on the Thom spectrum of a certain bundle over LM . We describe this structure both homotopy theoretically and simplicially, and in so doing describe the relationship of the Chas - Sullivan product to the cup product in Hochschild cohomology. We now make these statements more precise.

Consider the standard parameterization of the circle by the unit interval, $exp : [0, 1] \rightarrow S^1$ defined by $exp(t) = e^{2\pi it}$. With respect to this parameterization we can regard a loop $\gamma \in LM$ as a map $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = \gamma(1)$. Consider the evaluation map

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$$\begin{aligned} ev : LM &\rightarrow M \\ \gamma &\rightarrow \gamma(1). \end{aligned}$$

Now let $\iota : M \rightarrow \mathbb{R}^{N+d}$ be a fixed embedding of M into codimension N Euclidean space. Let $\nu^N \rightarrow M$ be the N - dimensional normal bundle. Let $Th(\nu^N)$ be the Thom space of this bundle. Recall the famous result of Atiyah [1] that $Th(\nu^N)$ is Spanier - Whitehead dual to M_+ . Said more precisely, let M^{-TM} be the spectrum given by desuspending this Thom space,

$$M^{-TM} = \Sigma^{-(N+d)}Th(\nu^N).$$

Then if M_+ denotes M with a disjoint basepoint, there are maps of spectra

$$S^0 \rightarrow M_+ \wedge M^{-TM} \quad \text{and} \quad M_+ \wedge M^{-TM} \rightarrow S^0$$

that establish M^{-TM} as the S - dual of M_+ . Said another way, these maps induce an equivalence with the function spectrum $M^{-TM} \simeq Map(M_+, S^0)$. In particular in homology we have isomorphisms

$$\begin{aligned} H^q(M_+) &\cong H_{-q}(M^{-TM}) \\ H^{-q}(M^{-TM}) &\cong H_q(M_+) \end{aligned}$$

for all $q \in \mathbb{Z}$. These duality isomorphism are induced by the compositions

$$H^{-q}(M^{-TM}) \xrightarrow[\cong]{\tau} H^{-q+d}(M) \xrightarrow[\cong]{\rho} H_q(M)$$

where τ is the Thom isomorphism, and ρ is Poincare duality.

Notice by duality, the diagonal map $\Delta : M \rightarrow M \times M$ induces a map of spectra

$$\Delta^* : M^{-TM} \wedge M^{-TM} \rightarrow M^{-TM}$$

that makes M^{-TM} into a ring spectrum with unit. The unit $S^0 \rightarrow M^{-TM}$ is the map dual to the projection $M_+ \rightarrow S^0$.

Now let $Th(ev^*(\nu^N))$ be the Thom space of the pull back bundle $ev^*(\nu^N) \rightarrow LM$. Define the spectrum

$$LM^{-TM} = \Sigma^{-(N+d)}Th(ev^*(\nu^N))$$

The goal of this paper is to define and study a product structure on the spectrum LM^{-TM} which among other properties makes the evaluation map $ev : LM^{-TM} \rightarrow M^{-TM}$ a map of

ring spectra. Here, by abuse of notation, ev is referring the map of Thom spectra induced by the evaluation map $ev : LM \rightarrow M$. We will prove the following theorem.

Theorem 1. *The spectrum LM^{-TM} is a homotopy commutative ring spectrum with unit, whose multiplication*

$$\mu : LM^{-TM} \wedge LM^{-TM} \longrightarrow LM^{-TM}$$

satisfies the following properties.

1. *The evaluation map $ev : LM^{-TM} \rightarrow M^{-TM}$ is a map of ring spectra.*
2. *There is a map of ring spectra $\rho : LM^{-TM} \rightarrow \Sigma^\infty(\Omega M_+)$ where the target is the suspension spectrum of the based loop space with a disjoint basepoint. Its ring structure is induced by the usual product on the based loop space. In homology the map ρ_* is given by the composition*

$$\rho_* : H_q(LM^{-TM}) \xrightarrow[\cong]{\tau} H_{q+d}LM \xrightarrow{\iota} H_q(\Omega M)$$

where like above, τ is the Thom isomorphism, and the map ι takes a $(q+d)$ -cycle in LM and intersects it with the based loop space viewed as a codimension d -submanifold.

3. *The ring structure is compatible with the Chas - Sullivan homology product in the sense that the following diagram commutes:*

$$\begin{array}{ccccc} H_q(LM^{-TM}) \otimes H_r(LM^{-TM}) & \longrightarrow & H_{q+r}(LM^{-TM} \wedge LM^{-TM}) & \xrightarrow{\rho_*} & H_{q+r}(LM^{-TM}) \\ \tau \downarrow \cong & & & & \cong \downarrow \tau \\ H_{q+d}(LM) \otimes H_{r+d}(LM) & \xrightarrow{\circ} & & & H_{q+r+d}(LM) \end{array}$$

Remark. In [2] Chas and Sullivan define a regrading of the homology of the loop space

$$\mathbb{H}_q = H_{q+d}(LM)$$

with respect to which the product \circ is of total degree zero. We observe that the Thom isomorphism defines an isomorphism

$$\mathbb{H}_* \cong H_*(LM^{-TM})$$

which respects gradings, and by the above theorem is an isomorphism of rings, where the ring structure on the right hand side comes from the ring spectrum structure of LM^{-TM} .

Next we show that the ring structure $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$ extends to an operad structure of the ‘‘cactus operad’’ \mathcal{C} defined originally by Voronov. This operad is

homotopy equivalent to the operad of framed little disks in \mathbb{R}^2 . We will recall the definition of the cactus operad \mathcal{C} in section 2, where we will prove the following theorem.

Theorem 2. *There are maps of spectra*

$$\zeta_k : (\mathcal{C}_k)_+ \wedge_{\Sigma_k} (LM^{-TM})^{(k)} \longrightarrow LM^{-TM}$$

giving LM^{-TM} the structure of a \mathcal{C} - operad ring spectrum, compatible with the ring structure $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$.

Our next result has to do with the simplicial structure of LM^{-TM} , and the resulting simplicial description of the product.

Let S_*^1 be the simplicial set decomposition of the circle which has one zero simplex and one nondegenerate one simplex. In this decomposition there are $n + 1$ n - simplices, all of which are degenerate for $n > 1$. We write this as $S_n^1 = \{n + 1\}$. Now given any space X , there is a resulting cosimplicial model for the free loop space, $LX = \text{Map}(S^1, X)$, \mathbb{X}_* . The n - simplices of \mathbb{X}_* are given by maps

$$\mathbb{X}_n = \text{Map}(S_n^1, X) = \text{Map}(\{n + 1\}, X) = X^{n+1}.$$

Of course the coface and codegeneracy maps of $(LX)_*$ are dual to the face and degeneracy maps of S_*^1 .

Our next result states that there is a similar cosimplicial model for LM^{-TM} .

Theorem 3. *For M a closed, oriented manifold, the spectrum LM^{-TM} has the structure of a cosimplicial spectrum which we write as $\mathbb{T}\mathbb{X}_*$. The k simplices of $\mathbb{T}\mathbb{X}_*$ are given by*

$$\mathbb{T}\mathbb{X}_k = (X^k)_+ \wedge M^{-TM}.$$

This cosimplicial structure has the following properties.

1. *The ring structure of LM^{-TM} is realized on the (co)simplicial level by pairings*

$$\mu_k : \left((X^k)_+ \wedge M^{-TM} \right) \wedge \left((X^r)_+ \wedge M^{-TM} \right) \rightarrow (X^{k+r})_+ \wedge M^{-TM}$$

defined by

$$\mu_k(x_1, \dots, x_k; u) \wedge (y_1, \dots, y_r; v) = (x_1, \dots, x_k, y_1, \dots, y_r; \Delta^*(u \wedge v))$$

where Δ^ is the ring structure defined on M^{-TM} described earlier (dual to the diagonal map $\Delta : M \rightarrow M \times M$).*

2. Applying the singular chain functor $C_*(-)$ to the cosimplicial space \mathbb{TX}_* , we get a natural chain homotopy equivalence between the chains of LM^{-TM} and the Hochschild cochain complex

$$f_* : C_*(LM^{-TM}) \cong C_*(\mathbb{TX}_*) \cong CH^*(C^*(M); C^*(M)).$$

Here the notation $CH^*(A; A)$ refers to the Hochschild cochain complex of the form

$$A \rightarrow \text{Hom}(A; A) \rightarrow \cdots \rightarrow \text{Hom}(A^{\otimes n}; A) \rightarrow \text{Hom}(A^{\otimes n+1}; A) \rightarrow \cdots$$

Furthermore, the pairing on the chains $C_*(LM^{-TM})$ induced by the ring spectrum structure corresponds via the chain homotopy equivalence f_* to the cup product pairing in $CH^*(C^*(M); C^*(M))$. This gives ring isomorphisms in homology,

$$\mathbb{H}_* \cong H_*(LM^{-TM}) \xrightarrow[\cong]{f_*} HH^*(C^*(M); C^*(M)).$$

Remark. The fact that the Chas - Sullivan product is realized as the cup product in Hochschild cohomology was also observed by T. Tradler, and will appear in his CUNY Ph.D thesis.

The paper will be organized as follows. In section 1 we will show how to realize the Chas - Sullivan product using the Pontrjagin - Thom construction for embedded, finite codimension manifolds. We will use this to prove theorem 1. In section 2 we define the cactus operad \mathcal{C} and prove theorem 2. In section 3 we will recall the cosimplicial study of the loop space done by the second author in [4], apply the Thom spectrum construction to it and use it to prove theorem 3.

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1. THE RING STRUCTURE ON LM^{-TM} : THE PROOF OF THEOREM 1

In this section we will describe the ring spectrum structure of the Thom spectrum LM^{-TM} defined in the introduction, discuss some its properties, and prove theorem 1. We begin by restating it.

Theorem 4. *The spectrum LM^{-TM} is a homotopy commutative ring spectrum with unit, whose multiplication*

$$\mu : LM^{-TM} \wedge LM^{-TM} \longrightarrow LM^{-TM}$$

satisfies the following properties.

1. *The evaluation map $ev : LM^{-TM} \rightarrow M^{-TM}$ is a map of ring spectra.*
2. *There is a map of ring spectra $\rho : LM^{-TM} \rightarrow \Sigma^\infty(\Omega M_+)$ where the target is the suspension spectrum of the based loop space with a disjoint basepoint. Its ring structure is induced by the usual product on the based loop space. In homology the map ρ_* is given by the composition*

$$\rho_* : H_q(LM^{-TM}) \xrightarrow[\cong]{u_*} H_{q+d}LM \xrightarrow{\iota} H_q(\Omega M)$$

where u_ is the Thom isomorphism, and the map ι takes a $(q+d)$ - cycle in LM and intersects it with the based loop space viewed as a codimension d - submanifold.*

3. *The ring structure is compatible with the Chas - Sullivan homology product in the sense that the following diagram commutes:*

$$\begin{array}{ccc} H_q(LM^{-TM}) \otimes H_r(LM^{-TM}) & \longrightarrow & H_{q+r}(LM^{-TM} \wedge LM^{-TM}) \xrightarrow{\mu_*} H_{q+r}(LM^{-TM}) \\ u_* \downarrow \cong & & \cong \downarrow u_* \\ H_{q+d}(LM) \otimes H_{r+d}(LM) & \xrightarrow{\circ} & H_{q+r+d}(LM) \end{array}$$

Proof. The multiplicative structure $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$ will be defined using the Pontrjagin - Thom construction. We therefore begin by recalling some properties of this construction.

Let

$$e : P^k \hookrightarrow N^{n+k}$$

be an embedding of closed, oriented manifolds. Let ν_e be a tubular neighborhood of $e(P^k)$, which we identify with the total space of the normal bundle of the embedding. Let

$$\tau : P^k \rightarrow \nu_e \cup \infty$$

be the Pontrjagin - Thom collapse map to the one point compactification, defined by

$$\tau(x) = \begin{cases} x & \text{if } x \in \nu_e \\ \infty & \text{if } x \notin \nu_e. \end{cases}$$

If we identify the compactification with the Thom space of the normal bundle, $\nu_e \cup \infty \cong P^{\nu_e}$, then in homology, after applying the Thom isomorphism $u_* : H_q(P^{\nu_e}) \xrightarrow{\cong} H_{q-n}(P^k)$, we get the “push-forward”, or “umkehr” map,

$$e_! : H_q(N^k) \xrightarrow{\tau_*} H_q(P^{\nu_e}) \xrightarrow[\cong]{u_*} H_{q-n}(P^k).$$

Recall that in the case of the diagonal embedding of a d - dimensional closed oriented manifold,

$$\Delta : M^d \rightarrow M^d \times M^d$$

that the normal bundle is isomorphic to the tangent bundle,

$$\nu_\Delta \cong TM$$

so that the Pontrjagin - Thom map is a map $\tau : M \times M \rightarrow M^{TM}$. Furthermore, the push - forward map in homology,

$$\Delta_! : H_*(M^d \times M^d) \xrightarrow{\tau_*} H_*(M^{TM}) \xrightarrow[\cong]{u_*} H_{*-d}(M^d).$$

is simply the intersection product.

Now the Pontrjagin - Thom construction also applies when one has a vector bundle over the ambient manifold of an embedding. That is, if one has an embedding $e : P^k \hookrightarrow N^{n+k}$ as above, and if one has a vector bundle (or virtual bundle) $\zeta \rightarrow N$, then one obtains a Pontrjagin - Thom map

$$\tau : \zeta \cup \infty \rightarrow \nu(\iota^*(\zeta)) \cup \infty$$

where $\nu(\iota^*(\zeta))$ is the tubular neighborhood of the induced embedding of total spaces $\iota^*(\zeta) \hookrightarrow \zeta$. Now $\zeta \cup \infty$ is the Thom space N^ζ , and $\nu(\iota^*(\zeta)) \cup \infty$ is the Thom space $P^{\iota^*(\zeta) \oplus \nu_e}$. So the Pontrjagin map is a map

$$\tau : N^\zeta \rightarrow P^{\iota^*(\zeta) \oplus \nu_e}.$$

Moreover this construction works when ζ is a virtual bundle over N as well. In this case when $\zeta = -E$, where $E \rightarrow N$ is a k - dimensional vector bundle over N , then the Thom spectrum $N^\zeta = N^{-E}$ is defined as follows. Suppose the bundle E is embedded in a $k + M$

dimensional trivial bundle, $E \hookrightarrow N \times \mathbb{R}^{k+M}$. Let E^\perp be the M - dimensional orthogonal complement bundle to this embedding $E^\perp \rightarrow N$. Then

$$N^{-E} = \Sigma^{-(N+k)} N^{E^\perp}.$$

Notice that the Thom isomorphism is of the form $u_* : H^q(N) \cong H^{q-k}(N^{-E})$.

In particular, applying the Pontrjagin - Thom construction to the diagonal embedding $\Delta : M \hookrightarrow M \times M$, using the virtual bundle $-TM \times -TM$ over $M \times M$, we get a map of Thom spectra,

$$\tau : (M \times M)^{-TM \times -TM} \rightarrow M^{TM \oplus \Delta^*(-TM \times -TM)}$$

or,

$$\tau : M^{-TM} \wedge M^{-TM} \rightarrow M^{-TM}.$$

In homology, this map still realizes the intersection pairing on $H_*(M)$, after applying the Thom isomorphism. The map τ defines a ring spectrum structure on M^{-TM} that is well known to be the Spanier - Whitehead dual of the diagonal map $\Delta : M \rightarrow M \times M$.

To construct the ring spectrum pairing $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$, we basically “pull back” the structure τ over the loop space.

To make this precise, let $ev \times ev : LM \times LM \rightarrow M \times M$ be the product of the evaluation maps, and define $LM \times_M LM$ to be the fiber product, or pull back:

$$(1.1) \quad \begin{array}{ccc} LM \times_M LM & \xrightarrow{\tilde{\Delta}} & LM \times LM \\ ev \downarrow & & \downarrow ev \times ev \\ M & \xrightarrow[\Delta]{} & M \times M. \end{array}$$

Notice that $LM \times_M LM$ is a codimension d submanifold of the infinite dimensional manifold $LM \times LM$, and can be thought of as

$$LM \times_M LM = \{(\alpha, \beta) \in LM \times LM \text{ such that } \alpha(0) = \beta(0)\}.$$

Notice that there is also a natural map from $LM \times_M LM$ to the loop space LM defined by first applying α and then β . That is,

$$(1.2) \quad \begin{array}{ccc} \gamma : LM \times_M LM & \rightarrow & LM \\ & & (\alpha, \beta) \rightarrow \alpha * \beta \end{array}$$

where

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice that when restricted to the product of the based loop spaces, $\Omega M \times \Omega M \subset LM \times_M LM$, then γ is just the H - space product on the based loop space, $\Omega M \times \Omega M \rightarrow \Omega M$.

Notice that by its definition (1.1) the embedding $\tilde{\Delta} : LM \times_M LM \hookrightarrow LM \times LM$ has a tubular neighborhood $\nu(\tilde{\Delta})$ defined to be the inverse image of the tubular neighborhood of the diagonal $\Delta : M \hookrightarrow M \times M$:

$$\nu(\tilde{\Delta}) = ev^{-1}(\nu(\Delta)).$$

Therefore this neighborhood is homeomorphic to the total space of the d - dimensional vector bundle given by pulling back the normal bundle of the embedding Δ , which is the tangent bundle of M :

$$\nu(\tilde{\Delta}) \cong ev^*(\nu_\Delta) = ev^*(TM).$$

Thus there is a Pontrjagin - Thom construction

$$\tau : LM \times LM \rightarrow LM \times_M LM^{ev^*(TM)}.$$

As described earlier, we ease the notation by referring to this Thom spectrum as $(LM \times_M LM)^{TM}$. By the naturality of the Pontrjagin - Thom construction, we have a commutative diagram of spectra,

$$(1.3) \quad \begin{array}{ccc} LM \times LM & \xrightarrow{\tau} & (LM \times_M LM)^{TM} \\ ev \downarrow & & \downarrow ev \\ M \times M & \xrightarrow{\tau} & M^{TM} \end{array}$$

Since, as observed before, $\tau_* : H_*(M \times M) \rightarrow H_*(M^{TM}) \cong H_{*-d}(M)$ is the intersection product, then in homology,

$$H_*(LM \times LM) \xrightarrow{\tau_*} H_*((LM \times_M LM)^{TM}) \xrightarrow{u_*} H_{*-d}(LM \times_M LM)$$

can be viewed (as is done in Chas - Sullivan [2]) as taking a cycle in $LM \times LM$, and “intersecting” it with the codimension d submanifold $LM \times_M LM$.

Now observe that the map $\gamma : LM \times_M LM \rightarrow LM$ defined above (1.2) preserves the evaluation map. That is, the following diagram commutes:

$$(1.4) \quad \begin{array}{ccc} LM \times_M LM & \xrightarrow{\gamma} & LM \\ ev \downarrow & & \downarrow ev \\ M & \xrightarrow{\quad} & M \\ & = & \end{array}$$

Thus γ induces a map of bundles $\gamma : ev^*(TM) \rightarrow ev^*(TM)$, and therefore a map of Thom spectra,

$$\gamma : (LM \times_M LM)^{TM} \rightarrow LM^{TM}.$$

Now consider the composition

$$(1.5) \quad \tilde{\mu} : LM \times LM \xrightarrow{\tau} (LM \times_M LM)^{TM} \xrightarrow{\gamma} LM^{TM}$$

In homology, the homomorphism

$$(1.6) \quad H_*(LM \times LM) \xrightarrow{\tilde{\mu}_*} H_*(LM^{TM}) \xrightarrow[\cong]{u_*} H_{*-d}(LM)$$

takes a cycle in $LM \times LM$, intersects in with the codimension d - submanifold $LM \times_M LM$, and maps it via γ to LM . This is the definition of the Chas - Sullivan product $H_*(LM)$.

Now as we did before with the diagonal embedding, we can perform the Pontrjagin - Thom construction when we pull back the virtual bundle $-TM \times -TM$ over $LM \times LM$. That is, we get a map of Thom spectra

$$\tau : (LM \times LM)^{(ev \times ev)^*(-TM \times -TM)} \longrightarrow (LM \times_M LM)^{ev^*(TM) \oplus ev^*(\Delta^*(-TM \times -TM))}.$$

But since $ev^*(\Delta^*(-TM \times -TM)) = ev^*(-2TM)$, we have

$$\tau : LM^{-TM} \wedge LM^{-TM} \longrightarrow (LM \times_M LM)^{TM \oplus -2TM} = (LM \times_M LM)^{-TM}.$$

Now by the commutativity of (1.4), γ induces a map of Thom spectra,

$$\gamma : (LM \times_M LM)^{-TM} \rightarrow LM^{-TM}$$

and so we can define the ring structure on the Thom spectrum LM^{-TM} to be the composition

$$(1.7) \quad \mu : LM^{-TM} \wedge LM^{-TM} \xrightarrow{\tau} (LM \times_M LM)^{-TM} \xrightarrow{\gamma} LM^{-TM}.$$

A few properties of this map μ are now immediately verifiable.

First, μ is associative. This follows from the naturality of the Pontrjagin - Thom construction, and the fact that the map γ is associative. (Strictly speaking, formula (1.2) is A_∞ - associative as is the usual formula for the product on the based loop space, ΩM .

However the standard trick of replacing ΩM with ‘‘Moore loops’’ changes the A_∞ structure to a strictly associative structure. The same technique applies to the map γ . Otherwise, the spectrum LM^{-TM} will have the structure of an A_∞ ring spectrum.)

Also, notice that LM^{-TM} has a unit, $\iota : S^0 \rightarrow LM^{-TM}$, defined by the composition

$$\iota : S^0 \xrightarrow{j} M^{-TM} \xrightarrow{\sigma} LM^{-TM}$$

where j is the unit of the ring spectrum structure of M^{-TM} , and σ is the map of Thom spectra induced by the section of the evaluation map $ev : LM \rightarrow M$ defined by viewing points in M as constant loops.

Thus $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$ defines an associative ring spectrum.

Notice furthermore that in homology, after applying the Thom isomorphism, μ_* induces the same homomorphism as $\tilde{\mu}_*$, and so by (1.6) the following diagram commutes:

$$(1.8) \quad \begin{array}{ccc} H_{q-2d}(LM^{-TM} \wedge LM^{-TM}) & \xrightarrow{\mu_*} & H_{q-2d}(LM^{-TM}) \\ u_* \downarrow \cong & & \cong \downarrow u_* \\ H_q(LM \times LM) & \xrightarrow{\circ} & H_{q-d}(LM) \end{array}$$

where $\circ : H_q(LM \times LM) \rightarrow H_{q-d}(LM)$ is the Chas - Sullivan product. This proves part (3) of theorem 1.

Now by the naturality of the Pontrjagin - Thom construction, the following diagram of Thom spectra commutes (compare (1.3))

$$(1.9) \quad \begin{array}{ccc} LM^{-TM} \wedge LM^{-TM} & \xrightarrow{\mu} & LM^{-TM} \\ ev \times ev \downarrow & & \downarrow ev \\ M^{-TM} \wedge M^{-TM} & \xrightarrow{\tau} & M^{-TM} \end{array}$$

Thus the evaluation map $ev : LM^{-TM} \rightarrow M^{-TM}$ is a map of ring spectra, which proves part 1 of theorem 1.

We now verify part 2 of theorem 1. Let $x_0 \in M$ be a base point, and consider the following pullback diagram:

$$(1.10) \quad \begin{array}{ccc} \Omega M & \xrightarrow{j} & LM \\ p \downarrow & & \downarrow ev \\ x_0 & \xrightarrow{i} & M \end{array}$$

Thus the embedding $j : \Omega M \hookrightarrow LM$ is an embedding of a codimension d submanifold, and has tubular neighborhood, $\nu(j)$ equal to the inverse image of the tubular neighborhood of the inclusion of the basepoint $i : x_0 \hookrightarrow M$. This tubular neighborhood is simply a disk D^d , and so

$$\nu(j) \cong \Omega M \times D^d \cong \epsilon^d$$

where ϵ^d reflects the d dimensional trivial bundle over ΩM . Thus the Thom - Pontrjagin construction makes sense, and is a map

$$\tau : LM \rightarrow \Omega M^{\epsilon^d} = \Sigma^d(\Omega M_+)$$

where the last space is the d - fold suspension of ΩM with a disjoint basepoint. In homology, the homomorphism

$$\tau_* : H_q(LM) \rightarrow H_q(\Sigma^d(\Omega M_+)) = H_{q-d}(\Omega M)$$

denotes the map that is obtained by intersecting a q -cycle in LM with the codimension d submanifold ΩM .

By performing the Pontrjagin - Thom construction after pulling back the virtual bundle $-TM$ over LM , we get a map of Thom spectra

$$\tau : LM^{-TM} \rightarrow (\Omega M)^{\epsilon^d \oplus j^* ev^*(-TM)}.$$

But by the commutativity of diagram (1.10), $j^* ev^*(-TM) = p^* i^*(-TM)$, which is the trivial, virtual $-d$ dimensional bundle which we denote ϵ^{-d} . So the Pontrjagin - Thom map is therefore a map of spectra

$$(1.11) \quad \rho : LM^{-TM} \xrightarrow{\tau} (\Omega M)^{\epsilon^d \oplus j^* ev^*(-TM)} = (\Omega M)^{\epsilon^d \oplus \epsilon^{-d}} = \Sigma^\infty(\Omega M_+)$$

where here, like before, $\Sigma^\infty(\Omega M_+)$ denotes the suspension spectrum of the based loop space of M with a disjoint basepoint. To complete the proof of theorem 1 we need to prove that $\rho : LM^{-TM} \rightarrow \Sigma^\infty(\Omega M_+)$ is a map of ring spectra. Toward this end, consider the following diagram of pull back squares:

$$(1.12) \quad \begin{array}{ccccc} \Omega M \times \Omega M & \xrightarrow{\iota} & LM \times_M LM & \xrightarrow{\tilde{\Delta}} & LM \times LM \\ \downarrow & & \downarrow ev & & \downarrow ev \times ev \\ x_0 & \xrightarrow{i} & M & \xrightarrow{\Delta} & M \times M. \end{array}$$

This gives Pontrjagin - Thom maps

$$(1.13) \quad LM^{-TM} \wedge LM^{-TM} \xrightarrow{\tau} (LM \times_M LM)^{-TM} \xrightarrow{\tau} \Sigma^\infty(\Omega M_+) \wedge \Sigma^\infty(\Omega M_+).$$

Notice that by the naturality of the Pontrjagin - Thom construction, the above composition is equal to

$$\rho \wedge \rho : LM^{-TM} \wedge LM^{-TM} \longrightarrow \Sigma^\infty(\Omega M_+) \wedge \Sigma^\infty(\Omega M_+).$$

Now notice that by the formula for the map $\gamma : LM \times_M LM \rightarrow LM$, the following diagram commutes:

$$\begin{array}{ccc} \Omega M \times \Omega M & \xrightarrow{\iota} & LM \times_M LM \\ m \downarrow & & \downarrow \gamma \\ \Omega M & \xrightarrow{\quad \subset \quad} & LM \end{array}$$

where m is the usual multiplication on the based loop space. Pulling back the virtual bundle $-TM$ over LM , and applying the Pontrjagin - Thom construction, we then get a commutative diagram of spectra,

$$(1.14) \quad \begin{array}{ccc} LM^{-TM} \wedge LM^{-TM} & \xrightarrow{\tau} & (LM \times_M LM)^{-TM} & \xrightarrow{\tau} & \Sigma^\infty(\Omega M_+) \wedge \Sigma^\infty(\Omega M_+) \\ & & \gamma \downarrow & & \downarrow m \\ & & LM^{-TM} & \xrightarrow{\rho} & \Sigma^\infty(\Omega M_+) \end{array}$$

Now as observed above (1.13), the top horizontal composition $\tau \circ \tau$ is equal to $\rho \wedge \rho : LM^{-TM} \wedge LM^{-TM} \rightarrow \Sigma^\infty(\Omega M_+) \wedge \Sigma^\infty(\Omega M_+)$. Also, $\gamma \circ \tau$ is, by definition, the ring structure $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$. Thus the following diagram of spectra commutes:

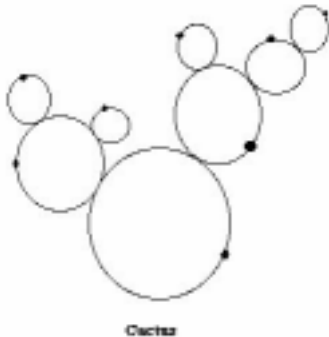
$$\begin{array}{ccc} LM^{-TM} \wedge LM^{-TM} & \xrightarrow{\rho \wedge \rho} & \Sigma^\infty(\Omega M_+) \wedge \Sigma^\infty(\Omega M_+) \\ \mu \downarrow & & \downarrow m \\ LM^{-TM} & \xrightarrow{\rho} & \Sigma^\infty(\Omega M_+). \end{array}$$

Thus ρ is a map of ring spectra, which completes the proof of theorem 1. \square

2. THE OPERAD STRUCTURE

In this section we describe an operad structure on the spectrum LM^{-TM} . This is the “cactus operad” \mathcal{C} . This operad was introduced by Voronov. It has the homotopy of the operad of framed little disks in \mathbb{R}^2 . According to Getzler’s result [3] this is precisely what is needed to induce the Batalin - Vilkovisky algebra structure in homology. Therefore this structure can be viewed as a homotopy theoretic realization of the BV - algebra structure Chas and Sullivan show to exist on the chains of the loop space.

We begin by recalling the definition of cactus operad \mathcal{C} . A point in the space \mathcal{C}_k is a collection of k oriented circles c_1, \dots, c_k , with radii r_i so that $\sum_{i=1}^k r_i = 1$. Each circle has a marked point $x_i \in c_i$. Moreover the circles can intersect each other at a finite number of points (vertices) to create a “cactus - type configuration”. Strictly speaking this means that the dual graph of this configuration is a tree. That is, the “cactus” (i.e the union of the circles) must be connected and have no “extra loops”. (This is the tree condition on the dual graph.) The edges coming into any vertex are also equipped with a cyclic ordering.



Notice that a cactus (i.e a point in \mathcal{C}_k) comes equipped with a well defined map from the unit circle to the boundary of the cactus. That is, the map begins at the marked point $x_1 \in c_1$, then traverses the circle c_1 in the direction of its orientation in a length preserving manner. When a point of intersection with another circle is reached, the loop then traverses that circle, in the direction of its orientation. This path is continued until it eventually arrives back at the original basepoint $x_1 \in c_1$. Given a cactus $c = \langle c_1, \dots, c_k \rangle \in \mathcal{C}_k$ we let $\delta_c : S^1 \rightarrow c_1 \cup \dots \cup c_k$ be this loop that traverses the boundary of the cactus.

Notice that \mathcal{C}_k has a free action of the symmetric group Σ_k defined by permuting the ordering of the circles. The operad action

$$\xi : \mathcal{C}_k \times (\mathcal{C}_{j_1} \times \dots \times \mathcal{C}_{j_k}) \rightarrow \mathcal{C}_j$$

where $j = j_1 + \cdots + j_k$ is defined as follows. Let $c \times (c^1, \dots, c^k) \in \mathcal{C}_k \times (\mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k})$. Scale the cactus c^1 down so that its total radius is the radius r_1 of the first component c_1 of $c \in \mathcal{C}_k$. Similarly, scale each of the cacti c^i down so that its total radius is the radius r_i of the i^{th} component c_i of c . By using the loops δ_{c^i} (scaled down appropriately) we identify the component circles c_i with the boundary of the cacti c^i . This has the effect of replacing the k component circles c_1, \dots, c_k making up the cactus c , by the k cacti, c^1, \dots, c^k . This produces a cactus with $j_1 + \cdots + j_k = j$ component circles.

Our goal in this section is to prove theorem 2 of the introduction. We now restate that theorem.

Theorem 5. *There are maps of spectra*

$$\zeta_k : (\mathcal{C}_k)_+ \wedge_{\Sigma_k} (LM^{-TM})^{(k)} \longrightarrow LM^{-TM}$$

giving LM^{-TM} the structure of a \mathcal{C} - operad ring spectrum, compatible with the ring structure $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$.

Proof. Given a cactus $c = \langle c_1, \dots, c_k \rangle \in \mathcal{C}_k$, define $L_c M$ to be the mapping space

$$L_c M = \text{Map}(c, M).$$

This space consists of maps from the union $c_1 \cup \cdots \cup c_k \rightarrow M$. The map from the circle $\delta_c : S^1 \rightarrow c_1 \cup \cdots \cup c_k$ defines a map from $L_c M$ to the loop space,

$$(2.1) \quad \begin{aligned} \gamma_c : L_c M &\rightarrow LM \\ f &\rightarrow f \circ \delta_c \end{aligned}$$

Now $L_c M$ can also be viewed as the pullback of an evaluation mapping of the product $(LM)^k$ defined as follows. For each component of the cactus c_i , let m_i be the number of points on the circle c_i that intersect other components of the cactus. Let $m_c = m_1 + \cdots + m_k$. We define an evaluation map

$$ev_c : (LM)^k \longrightarrow (M)^{m_c}$$

as follows. On the circle c_i , let y_1, \dots, y_{m_i} be the points that intersect other circles in the cactus c . Assume that these points are ordered according to the orientation of the circle c_i beginning at the marked point $x_i \in c_i$. Let $s_i : S^1 \rightarrow c_i$ be the identification of the unit circle with c_i obtained by scaling down the unit circle so as to have radius r_i , and rotating

it so the basepoint $1 \in S^1$ is mapped to the marked point $x_i \in c_i$. Let u_1, \dots, u_{m_i} be the points on the unit circle corresponding to $y_1, \dots, y_{m_i} \in c_i$ under the map s_i . Define

$$(2.2) \quad \begin{aligned} ev_{c_i} : LM &\rightarrow (M)^{m_i} \\ \sigma &\rightarrow (\sigma(u_1), \dots, \sigma(u_{m_i})) \end{aligned}$$

Now define

$$(2.3) \quad ev_c = ev_{c_1} \times \dots \times ev_{c_{m_i}} : LM \rightarrow (M)^{m_1} \times \dots \times (M)^{m_i} = (M)^{m_c}$$

Now let $w_1, \dots, w_{n_c} \in c_1 \cup \dots \cup c_k$ denote all the points in the cactus that lie in more than one component. For each such point w_i , let μ_i be the number of components of the cactus on which w_i lies. We think of μ_i as the ‘‘multiplicity’’ of the intersection point w_i . Notice that we have the relation

$$(2.4) \quad \sum_{i=1}^{n_c} \mu_i = m_c.$$

The ‘‘tree’’ condition on the dual of the cactus also imposes the following relation:

$$(2.5) \quad m_c - n_c = k - 1$$

Now consider the diagonal mapping

$$\Delta_c : (M)^{n_c} \longrightarrow (M)^{m_c}$$

defined by the composition

$$\Delta_c : (M)^{n_c} \xrightarrow{\Delta_{\mu_1} \times \dots \times \Delta_{\mu_{n_c}}} (M)^{\mu_1} \times \dots \times (M)^{\mu_{n_c}} \xrightarrow{=} (M)^{m_c}$$

where $\Delta_{\mu_i} : M \rightarrow (M)^{\mu_i}$ is the μ_i -fold diagonal. Observe that the following is a cartesian pull - back square:

$$(2.6) \quad \begin{array}{ccc} L_c M & \xrightarrow{\tilde{\Delta}_c} & (LM)^k \\ ev_{int} \downarrow & & \downarrow ev_c \\ (M)^{n_c} & \xrightarrow{\Delta_c} & (M)^{m_c} \end{array}$$

where $ev_{int} : L_c M \rightarrow (M)^{n_c}$ evaluates a map $f : c \rightarrow M$ at the n_c intersection points, $w_1, \dots, w_{n_c} \in c_1 \cup \dots \cup c_k$.

The normal bundle $\nu(\Delta_c)$ of the diagonal embedding

$$\Delta_c : (M)^{n_c} \xrightarrow{\Delta_{\mu_1} \times \dots \times \Delta_{\mu_{n_c}}} (M)^{\mu_1} \times \dots \times (M)^{\mu_{n_c}} \xrightarrow{=} (M)^{m_c}$$

is equal to

$$(\mu_1 - 1)TM \times \cdots \times (\mu_{n_c} - 1)TM \longrightarrow (M)^{n_c},$$

where $(q)TM$ denotes the q -fold direct sum of TM with itself as a bundle over M . Notice that since $\sum_{i=1}^{n_c} \mu_i = m_c$ and $m_c - n_c = k - 1$ (2.5), then if $\Delta : M \hookrightarrow (M)^{n_c}$ is the full diagonal map then the pull back of this normal bundle

$$(2.7) \quad \Delta^*(\nu(\Delta_c)) \cong (k - 1)TM \longrightarrow M.$$

Now by pulling back the tubular neighborhood (and the resulting normal bundle) of Δ_c over L_cM , we have a resulting Pontrjagin - Thom to the Thom space,

$$\tau : (LM)^k \longrightarrow (L_cM)^{ev_{int}^*(\nu(\Delta_c))}$$

Notice that by (2.7), we have a bundle isomorphism

$$ev_{int}^*(\nu(\Delta_c)) \cong ev^*((k - 1)TM)$$

where $ev : L_cM \rightarrow M$ is evaluation at $x_1 \in c = c_1 \cup \cdots \cup c_k$. In particular the map $\gamma_c : L_cM \rightarrow LM$ (2.1) is covered by a map of bundles,

$$(2.8) \quad \begin{array}{ccc} ev_{int}^*(\nu(\Delta_c)) & \xrightarrow{\gamma_c} & (k - 1)TM \\ \downarrow & & \downarrow \\ L_cM & \xrightarrow{\gamma_c} & LM. \end{array}$$

Now, like what we did in section 1, let $(-TM)^k \rightarrow (LM)^k$ denote the pull back of the k -fold exterior product of the virtual bundle $-TM \rightarrow TM$ via the k -fold product of the evaluation map at the basepoint, $(ev)^k : (LM)^k \rightarrow (M)^k$. Then performing the Pontrjagin - Thom construction on this bundle, we get a map of Thom spectra,

$$(2.9) \quad \tau : (LM^{-TM})^{(k)} \longrightarrow (L_cM)^{ev_{int}^*(\nu(\Delta_c)) \oplus \tilde{\Delta}_c^*((-TM)^k)}$$

Now by (2.8) we have a map of virtual bundles

$$\begin{array}{ccc} ev_{int}^*(\nu(\Delta_c)) \oplus \tilde{\Delta}_c^*((-TM)^k) & \xrightarrow{\gamma_c} & (k - 1)TM \oplus (-k)TM \xrightarrow{=} -TM \\ \downarrow & & \downarrow \\ L_cM & \xrightarrow{\gamma_c} & LM. \end{array}$$

and therefore a map of Thom spectra

$$\gamma_c : (L_cM)^{ev_{int}^*(\nu(\Delta_c)) \oplus \tilde{\Delta}_c^*((-TM)^k)} \rightarrow LM^{-TM}$$

We then define the map of spectra $\zeta(c) : (LM^{-TM})^{(k)} \rightarrow LM^{-TM}$ to be the composition

$$(2.10) \quad \zeta(c) = \gamma_c \circ \tau : (LM^{-TM})^{(k)} \longrightarrow (L_c M)^{ev_{int}^*(\nu(\Delta_c)) \oplus \tilde{\Delta}_c^*((-TM)^k)} \rightarrow LM^{-TM}.$$

This then defines the basic structure map

$$(2.11) \quad \begin{aligned} \zeta_k : (\mathcal{C}_k)_+ \wedge (LM^{-TM})^{(k)} &\longrightarrow LM^{-TM} \\ (c ; u_1 \wedge \cdots \wedge u_k) &\longrightarrow \zeta(c)(u_1 \wedge \cdots \wedge u_k). \end{aligned}$$

By checking the definition one sees that these maps descend to the orbit of the Σ_k - action,

$$(2.12) \quad \zeta_k : (\mathcal{C}_k)_+ \wedge_{\Sigma_k} (LM^{-TM})^{(k)} \longrightarrow LM^{-TM}.$$

Furthermore, checking the definitions, and in particular using the naturality of the Pontrjagin - Thom constructions, one sees that the maps ζ_k fit together compatibly to define the \mathcal{C} - operad structure of the spectrum LM^{-TM} , as claimed.

We end with two observations. First, let $c^0 \in \mathcal{C}_2$ be the 2- component cactus $c^0 = \langle c_1, c_2 \rangle$, where c_1 and c_2 are both circles of radii 1/2, both oriented counter - clockwise, intersecting at one point. The point of intersection is defined to be the marked point of each circle, $x_1 = x_2 \in c_1 \cup c_2$. Then observe that the induced pairing, $\zeta(c^0)$ is equal to the ring spectrum multiplication μ defined in section 1,

$$\zeta(c^0) = \mu : LM^{-TM} \wedge LM^{-TM} \longrightarrow LM^{-TM}.$$

Thus the operad structure is compatible with the ring spectrum structure.

Second, let $\sigma(c^0) \in \mathcal{C}_2$ denote the same 2 - component cactus as c^0 , permuted by the action of the nontrivial element of the symmetric group, $\sigma \in \Sigma_2$. So $c^1 = \langle c_2, c_1 \rangle$. Then notice that the following diagram of spectra commutes:

$$\begin{array}{ccc} LM^{-TM} \wedge LM^{-TM} & \xrightarrow{\zeta(\sigma(c^0))} & LM^{-TM} \\ T \downarrow & & \downarrow = \\ LM^{-TM} \wedge LM^{-TM} & \xrightarrow{\mu} & LM^{-TM} \end{array}$$

where in this diagram T switches the two factors.

Since the operad space \mathcal{C}_2 is connected, a path between c^0 and $\sigma(c^0)$ yields the homotopy commutativity of the ring spectrum structure μ of LM^{-TM} .

□

3. A COSIMPLICIAL DESCRIPTION OF LM AND LM^{-TM} AND A PROOF OF THEOREM 3

In this section we describe a cosimplicial model for the spectrum LM^{-TM} . We then describe the ring spectrum structure simplicially. This cosimplicial model will then give a natural way of relating the singular chains $C_*(LM^{-TM})$ to the Hochschild cochain complex $CH^*(C^*(M), C^*(M))$, and in particular relate the simplicial model for the ring structure of LM^{-TM} to the cup product structure in this cochain complex. This will allow us to prove theorem 3.

We begin by reviewing the cosimplicial model of the loop space LX for any space X , coming from a simplicial decomposition of the circle S^1 . We refer the reader to [4] for details.

Let S_*^1 be the simplicial set decomposition of the circle which has one zero simplex and one nondegenerate one simplex. In this decomposition there are $n + 1$ n -simplices, all of which are degenerate for $n > 1$. We write this as $S_n^1 = \{n + 1\}$. Now given any space X , there is a resulting cosimplicial model for the free loop space, LX , which we call \mathbb{X}_* . The n -simplices of \mathbb{X}_* are given by maps

$$\mathbb{X}_n = \text{Map}(S_n^1, X) = \text{Map}(\{n + 1\}, X) = X^{n+1}.$$

Of course the coface and codegeneracy maps of \mathbb{X}_* are dual to the face and degeneracy maps of S_*^1 . They are given by the formulas

$$(3.1) \quad \begin{aligned} \delta_i(x_0, \dots, x_{n-1}) &= (x_0, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_{n-1}), \quad 0 \leq i \leq n-1 \\ \delta_n(x_0, \dots, x_{n-1}) &= (x_0, x_1, \dots, x_{n-1}, x_0) \\ \sigma_i(x_0, \dots, x_{n+1}) &= (x_0, \dots, x_i, x_{i+2}, \dots, x_{n+1}), \quad 0 \leq i \leq n \end{aligned}$$

Since the geometric realization of S_*^1 is homeomorphic to the circle,

$$S^1 \cong |S_*^1|,$$

the “total complex” or geometric corealization of \mathbb{X}_* is homeomorphic to the loop space,

$$LX \cong \text{Tot}(\mathbb{X}_*).$$

This was studied in detail by the second author in [4], and in particular the following interpretation of this result was given. For each k , let Δ^k be the standard k -simplex:

$$\Delta^k = \{(x_1, \dots, x_k) : 0 \leq x_1 \leq x_2 \leq \dots \leq x_k \leq 1\}.$$

Consider the maps

$$(3.2) \quad \begin{aligned} f_k &: \Delta^k \times LX \longrightarrow X^{k+1} \\ (x_1, \dots, x_k) \times \gamma &\rightarrow (\gamma(0), \gamma(x_1), \dots, \gamma(x_k)). \end{aligned}$$

Let $\bar{f}_k : LX \rightarrow \text{Map}(\Delta^k, X^{k+1})$ be the adjoint of f_k . Then the following was proven in [4].

Theorem 6. *Let X be any space, and let $f : LX \rightarrow \prod_{k \geq 0} \text{Map}(\Delta^k, X^{k+1})$ be the product of the maps \bar{f}_k . Then f is a homeomorphism onto its image. Furthermore, the image consists of sequences of maps $\{\phi_k\}$ which commute with the coface and codegeneracy operators. We call this space of sequences of maps $\text{Map}_{\Delta^*}(\Delta^*, X^{*+1})$ and this is the total space of the cosimplicial space $\text{Tot}(\mathbb{X}_*)$.*

By applying singular cochains to the maps f_k , one obtains maps

$$f_k^* : C^*(X)^{\otimes k+1} \rightarrow C^{*-k}(LX).$$

The following was also observed in [4].

Theorem 7. *For any space X , the homomorphisms $f_k^* : C^*(X)^{\otimes k+1} \rightarrow C^{*-k}(LX)$ fit together to define a chain map from the Hochschild complex of the cochains of X to the cochains of the free loop space,*

$$f^* : CH_*(C^*(X)) \rightarrow C^*(L(X))$$

which is a chain homotopy equivalence when X is simply connected. Hence it induces an isomorphism in homology

$$f^* : HH_*(C^*(X)) \xrightarrow{\cong} H^*(L(X)).$$

Remark. Let us clarify some notation. Given an algebra (or differential graded algebra) A , the the Hochschild complex of A , $CH_*(A)$ is a complex of the form

$$\dots \xrightarrow{b} A^{\otimes n+2} \xrightarrow{b} A^{\otimes n+1} \xrightarrow{b} \dots \xrightarrow{b} A \otimes A \xrightarrow{b} A.$$

The homology of this algebra is denoted $HH_*(A)$. More generally if M is a bimodule over A , we denote by $CH_*(A; M)$ the Hochschild complex of the form

$$\dots \xrightarrow{b} A^{\otimes n+1} \otimes M \xrightarrow{b} A^{\otimes n} \otimes M \xrightarrow{b} \dots \xrightarrow{b} A \otimes M \xrightarrow{b} M.$$

The homology of this complex is denoted $HH_*(A; M)$. So in particular if $M = A$ we see that $HH_*(A; A) = HH_*(A)$. Dually, we denote by $CH^*(A; M)$ the Hochschild cochain complex of the form

$$M \xrightarrow{b^*} Hom(A; M) \xrightarrow{b^*} \dots \xrightarrow{b^*} Hom(A^{\otimes n}; M) \xrightarrow{b^*} Hom(A^{\otimes n+1}; M) \xrightarrow{b^*} \dots$$

Its cohomology is denoted $HH^*(A; M)$. By dualizing theorem 6 we obtain the following.

Corollary 8. *For any simply connected space X , there is a chain homotopy equivalence from the singular chains of the loop space to the Hochschild cochain complex*

$$f_* : C_*(LX) \rightarrow CH^*(C^*(X); C_*(X))$$

and so an isomorphism in homology,

$$f_* : H_*(LX) \xrightarrow{\cong} HH^*(C^*(X); C_*(X)).$$

Notice that the cochain complex $CH^*(C^*(X); C_*(X))$ does not in general have a natural product structure. This is because the coefficients, $C_*(X)$, is not in general a ring. Notice however that the Hochschild complex $CH^*(C^*(X), C^*(X))$ does in fact have a cup product coming from the algebra structure of $C^*(X)$. Of course when X is a closed, oriented manifold of dimension d , Poincare duality gives a chain homotopy equivalence, $C_*(X) \cong C^{d-*}(X)$, and so the cochain complex $CH^*(C^*(X); C_*(X))$ inherits an algebra structure. Therefore by the above corollary, $H_*(LX)$ inherits an algebra structure in this case. We will see that this indeed realizes the Chas - Sullivan product. We will show this by showing that when M is a closed, oriented d - manifold, the Thom spectrum LM^{-TM} inherits a cosimplicial structure from \mathbb{X}_* for which the analogue of theorem 7 will yield a natural chain homotopy equivalence $C_*(LM^{-TM}) \cong CH^*(C^*(M), C^*(M))$.

To begin, notice that by the definitions 3.2, the following diagrams commute:

$$\begin{array}{ccc} \Delta^k \times LM & \xrightarrow{f_k} & M^{k+1} \\ e \downarrow & & \downarrow p_1 \\ M & \xrightarrow{\quad} & M \\ & = & \end{array}$$

where the left hand vertical map is the evaluation, $e((t_1, \dots, t_k); \gamma) = \gamma(0)$, and the right hand vertical map is the projection onto the first coordinate. Pulling back the virtual bundle $-T(M)$ defines a map of virtual bundles

$$(f_k)_* : e^*(-TM) \longrightarrow p_1^*(-TM),$$

and therefore maps of Thom spectra, (which by abuse of notation we still call f_k)

$$(3.3) \quad f_k : (\Delta_k)_+ \wedge LM^{-TM} \longrightarrow M^{-TM} \wedge (M^k)_+.$$

By taking adjoints, we get a map of spectra,

$$f : LM^{-TM} \xrightarrow{\prod_k f_k} \prod_k \text{Map}((\Delta_k)_+; M^{-TM} \wedge (M^k)_+)$$

where on the right hand side the mapping spaces are maps of unital spectra. This map is just the induced map of Thom spectra of the map $f : LX \longrightarrow \prod_{k \geq 0} \text{Map}(\Delta^k, X^{k+1})$ described in theorem 6. The following result is induced by theorem 6 by passing to Thom spectra.

Let $\mathbb{T}\mathbb{X}_*$ be the cosimplicial spectrum defined to be the cosimplicial Thom spectrum of the cosimplicial virtual bundle $-TM$. That is, the virtual bundle over the k simplices $\mathbb{X}_k = M^{k+1}$ is $p_1^*(-TM)$. Said more explicitly, $\mathbb{T}\mathbb{X}_*$ is the cosimplicial spectrum whose k -simplices are the spectrum

$$\mathbb{T}\mathbb{X}_k = M^{-TM} \wedge (M^k)_+.$$

To describe the coface and codegeneracy maps, consider the maps

$$\mu_L : M^{-TM} \rightarrow M_+ \wedge M^{-TM} \quad \text{and} \quad \mu_R : M^{-TM} \rightarrow M^{-TM} \wedge M_+$$

of Thom spectra induced by the diagonal map $\Delta : M \rightarrow M \times M$. μ_L and μ_R are the maps of Thom spectra induced by the maps of virtual bundles $\Delta_* : -TM \rightarrow p_L^*(-TM)$ and $\Delta_* : -TM \rightarrow p_R^*(-TM)$, where p_L and p_R are the projection maps $M \times M \rightarrow M$ onto the left and right coordinates respectively. We then have the following formulas for the coface and codegeneracy maps:

$$(3.4) \quad \begin{aligned} \delta_0(u; x_1, \dots, x_{k-1}) &= (v_R; y_R, x_1, \dots, x_{k-1}) \\ \delta_i(u; x_1, \dots, x_{k-1}) &= (u; x_1, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_{k-1}), \quad 1 \leq i \leq k-1 \\ \delta_k(u; x_1, \dots, x_{k-1}) &= (v_L; x_1, \dots, x_{k-1}, y_L), \end{aligned}$$

where $\mu_R(u) = (v_R, y_R)$, $\mu_L(u) = (y_L, v_L)$ and

$$\sigma_i(u; x_1, \dots, x_{k+1}) = (u; x_1, \dots, x_i, x_{i+2}, \dots, x_{k+1}), \quad 0 \leq i \leq k$$

The following result is simply the application of the Thom spectrum functor for the virtual bundle $-TM$ to theorem 6.

Theorem 9. *Let M be any closed, d - dimensional manifold, and let*

$$f : LM^{-TM} \xrightarrow{\prod_k f_k} \prod_k \text{Map}((\Delta_k)_+; M^{-TM} \wedge (M^k)_+)$$

be the product of the maps of spectra f_k , as defined above (3.3). Then f is a homeomorphism onto its image. Furthermore, the image consists of sequences of maps $\{\phi_k\}$ which commute with the coface and codegeneracy operators. We call this space of sequences of maps $\text{Map}_{\Delta^}(\Delta^*, M^{-TM} \wedge (M^k)_+)$ and this is the total space of the cosimplicial spectrum $\text{Tot}(\mathbb{T}\mathbb{X}_*)$.*

Now for M orientable, recall Atiyah's S - duality between M_+ and M^{-TM} described in the introduction [1]. This defines a chain homotopy equivalence between the cochains $C^*(M^{-TM})$ (defined to be the appropriate desuspension of the cochains of the Thom space of the normal bundle of a fixed embedding $M \hookrightarrow \mathbb{R}^N$) and the chains of the manifold

$$C^*(M^{-TM}) \cong C_{-*}(M_+).$$

The maps $f_k : (\Delta_k)_+ \wedge LM^{-TM} \longrightarrow M^{-TM} \wedge (M^k)_+$ then define maps of cochains,

$$f_k^* : C_{-*}(M) \otimes C^*(M)^{\otimes k} \cong C^*(M^{-TM} \wedge (M^k)_+) \longrightarrow C^{*-k}(LM^{-TM}).$$

Taking the dual we get a map of chain complexes

$$\begin{aligned} (f_k)_* : C_{*-k}(LM^{-TM}) &\longrightarrow \text{Hom}(C^*(M)^{\otimes k} \otimes C_{-*}(M); \mathbb{Z}) \\ &\cong \text{Hom}(C^*(M)^{\otimes k}; C^*(M)) \\ &= CH^k(C^*(M); C^*(M)) \end{aligned}$$

The following is then a consequence of corollary 8, by passing to Thom spectra.

Corollary 10. *For any oriented, closed manifold M , the chain maps $(f_k)_*$ fit together to define a chain homotopy equivalence from the chains of Thom spectrum LM^{-TM} to the Hochschild cochain complex*

$$f_* : C_*(LM^{-TM}) \rightarrow CH^*(C^*(M); C^*(M))$$

and so an isomorphism in homology,

$$f_* : H_*(LM^{-TM}) \xrightarrow{\cong} HH^*(C^*(M); C^*(M)).$$

As mentioned in the introduction, the Hochschild cochain complex $CH^*(C^*(M); C^*(M))$ has a cup product structure. Namely, for any algebra A , if

$$\phi \in CH^k(A; A) = Hom(A^{\otimes k}; A) \quad \text{and} \quad \psi \in CH^r(A; A) = Hom(A^{\otimes r}; A),$$

then

$$\phi \cup \psi \in CH^{k+r}(A; A) = Hom(A^{\otimes k+r}; A)$$

is defined by

$$\phi \cup \psi(a_1 \otimes \cdots \otimes a_k \otimes a_{k+1} \otimes \cdots \otimes a_{k+r}) = \phi(a_1 \otimes \cdots \otimes a_k) \psi(a_{k+1} \otimes \cdots \otimes a_{k+r}).$$

For $A = C^*(M)$ (where the algebra structure is the cup product in $C^*(M)$), by taking adjoints, we can think of this as a pairing

$$(3.5) \quad \begin{aligned} \cup : (C_*(M)^{\otimes k} \otimes C^*(M)) \otimes (C_*(M)^{\otimes r} \otimes C^*(M)) &\longrightarrow C_*(M)^{\otimes k+r} \otimes C^*(M) \\ (\alpha_1 \otimes \cdots \otimes \alpha_k \otimes \theta) \otimes (\beta_1 \otimes \cdots \otimes \beta_r \otimes \rho) &\longrightarrow \alpha_1 \otimes \cdots \otimes \alpha_k \otimes \beta_1 \otimes \cdots \otimes \beta_r \otimes \theta \cup \rho. \end{aligned}$$

Now recall that by S -duality, there is a ring spectrum structure

$$\Delta^* : M^{-TM} \wedge M^{-TM} \longrightarrow M^{-TM}$$

dual to the diagonal map $\Delta : M \rightarrow M \times M$. Passing to chains, $\Delta^* : C_*(M^{-TM}) \otimes C_*(M^{-TM}) \rightarrow C_*(M^{-TM})$, is, with respect to the duality identification $C_*(M^{-TM}) \cong C^*(M)$, therefore the cup product on cochains

$$\Delta^* = \cup : C^*(M) \otimes C^*(M) \rightarrow C^*(M).$$

Thus formula (3.5) for the cup product in Hochschild cochains is therefore realized by the map

$$(3.6) \quad \begin{aligned} \tilde{\mu}_{k,r} : [M^{-TM} \wedge (M^k)_+] \wedge [M^{-TM} \wedge (M^r)_+] &\longrightarrow M^{-TM} \wedge (M^{k+r})_+ \\ (u; x_1, \dots, x_k) \wedge (v; y_1, \dots, y_r) &\longrightarrow (\Delta^*(u, v); x_1, \dots, x_k, y_1, \dots, y_r) \end{aligned}$$

The maps $\tilde{\mu}_{k,r}$ define maps of the simplices

$$\tilde{\mu}_{k,r} : \mathbb{T}\mathbb{X}_k \wedge \mathbb{T}\mathbb{X}_r \rightarrow \mathbb{T}\mathbb{X}_{k+r}$$

and it is straight forward to check that these maps preserve the coface and codegeneracy operators, and so define a map of the geometric corealization (“total spectra”)

$$\tilde{\mu} : Tot(\mathbb{T}\mathbb{X}_*) \wedge Tot(\mathbb{T}\mathbb{X}_*) \longrightarrow Tot(\mathbb{T}\mathbb{X}_*).$$

This proves the following.

Theorem 11. *Using the homeomorphism $f : LM^{-TM} \cong Tot(\mathbb{T}\mathbb{X}_*)$ of theorem 9, LM^{-TM} inherits the structure of a ring spectrum,*

$$\tilde{\mu} : LM^{-TM} \wedge LM^{-TM} \longrightarrow LM^{-TM}$$

which is compatible with the cup product in Hochschild cohomology. That is, with respect to the chain homotopy equivalence

$$f_* : C_*(LM^{-TM}) \rightarrow CH^*(C^*(M); C^*(M))$$

of corollary 10, the following diagram of chain complexes commutes:

$$\begin{array}{ccc} C_*(LM^{-TM}) \otimes C_*(LM^{-TM}) & \xrightarrow{\tilde{\mu}} & C_*(LM^{-TM}) \\ f_* \otimes f_* \downarrow \cong & & \cong \downarrow f_* \\ CH^*(C^*(M); C^*(M)) \otimes CH^*(C^*(M); C^*(M)) & \xrightarrow{\cup} & CH^*(C^*(M); C^*(M)). \end{array}$$

In view of theorems 9, 10, and 11, theorem 3 will therefore be proven once we prove the following.

Theorem 12. *Let $\mu : LM^{-TM} \wedge LM^{-TM} \rightarrow LM^{-TM}$ be the ring spectrum structure defined in section 1. Then the structures μ and $\tilde{\mu} : Tot(\mathbb{T}\mathbb{X}_*) \wedge Tot(\mathbb{T}\mathbb{X}_*) \rightarrow Tot(\mathbb{T}\mathbb{X}_*)$ are compatible in the sense that the following diagram homotopy commutes:*

$$\begin{array}{ccc} LM^{-TM} \wedge LM^{-TM} & \xrightarrow{\mu} & LM^{-TM} \\ f \wedge f \downarrow \simeq & & \simeq \downarrow f \\ Tot(\mathbb{T}\mathbb{X}_*) \wedge Tot(\mathbb{T}\mathbb{X}_*) & \xrightarrow{\tilde{\mu}} & Tot(\mathbb{T}\mathbb{X}_*). \end{array}$$

Proof. Let $w_{k,r} : \Delta^{k+r} \rightarrow \Delta^k \times \Delta^r$ be the Alexander - Whitney diagonal map. That is, for $(x_1, \dots, x_{k+r}) \in \Delta^{k+r}$, then $w_{k,r}(x_1, \dots, x_{k+r}) = (x_1, \dots, x_k) \times (x_{k+1}, \dots, x_{k+r}) \in \Delta^k \times \Delta^r$. By the definition of the cosimplicial structure of $\mathbb{T}\mathbb{X}_*$, to prove the theorem it suffices to prove that the following diagrams of spectra commute:

(a)

$$\begin{array}{ccc}
\Delta_+^{k+r} \wedge LM^{-TM} \wedge LM^{-TM} & \xrightarrow{1 \wedge \mu} & \Delta_+^{k+r} \wedge LM^{-TM} \\
\downarrow w_{k,r} \wedge 1 & & \\
(\Delta^k \times \Delta^r)_+ \wedge LM^{-TM} \wedge LM^{-TM} & & \downarrow f_{k+r} \\
\downarrow f_k \wedge f_r & & \\
M^{-TM} \wedge (M^k)_+ \wedge M^{-TM} \wedge (M^r)_+ & \xrightarrow{\tilde{\mu}} & M^{-TM} \wedge (M^{k+r})_+
\end{array}$$

We verify the commutativity of these diagrams in several steps. First observe that the maps

$$f_k \times f_r : \Delta^k \times \Delta^r \times LM \times LM \rightarrow M^{k+1} \times M^{r+1}$$

restrict to $\Delta^k \times \Delta^r \times LM \times_M LM$ to define a map $f_{k,r}$ whose image is in $M \times M^k \times M^r$ making the following diagram commute:

$$\begin{array}{ccc}
\Delta^k \times \Delta^r \times LM \times_M LM & \xrightarrow{\hookrightarrow} & \Delta^k \times \Delta^r \times LM \times LM \\
f_{k,r} \downarrow & & \downarrow f_k \times f_r \\
M \times M^k \times M^r & \xrightarrow{\Delta} & M^{k+1} \times M^{r+1}
\end{array}$$

where the bottom horizontal map is the diagonal map:

$$\Delta(m \times (x_1, \dots, x_k) \times (y_1, \dots, y_r)) = (m, x_1, \dots, x_k) \times (m, y_1, \dots, y_r).$$

By the naturality of the Pontrjagin - Thom construction, we therefore have a commutative diagram of spectra

(b)

$$\begin{array}{ccc}
(\Delta^k \times \Delta^r)_+ \wedge LM^{-TM} \wedge LM^{-TM} & \xrightarrow{1 \wedge \tau} & (\Delta^k \times \Delta^r)_+ \wedge (LM \times_M LM)^{-TM} \\
\downarrow f_k \wedge f_r & & \downarrow f_{k,r} \\
M^{-TM} \wedge (M^k)_+ \wedge M^{-TM} \wedge (M^r)_+ & \xrightarrow{\tau} & M^{-TM} \wedge (M^k)_+ \wedge (M^r)_+.
\end{array}$$

Notice further that by the definition of the maps f_n , $f_{k,r}$ and the loop composition $\gamma : LM \times_M LM \rightarrow LM$ defined in the last section, the following diagram commutes:

$$\begin{array}{ccc}
\Delta^{k+r} \times (LM \times_M LM) & \xrightarrow{1 \times \gamma} & \Delta^{k+r} \times LM \\
w_{k,r} \times 1 \downarrow & & \\
\Delta^k \times \Delta^r \times (LM \times_M LM) & & \downarrow f_{k+r} \\
f_{k,r} \downarrow & & \\
M \times (M^k) \times (M^r) & \xrightarrow{=} & M^{k+r+1}.
\end{array}$$

Passing to Thom spectra this yields the following commutative diagram:

(c)

$$\begin{array}{ccc}
(\Delta^{k+r})_+ \wedge (LM \times_M LM)^{-TM} & \xrightarrow{1 \wedge \gamma} & (\Delta^{k+r})_+ \wedge LM^{-TM} \\
w_{k,r} \wedge 1 \downarrow & & \\
(\Delta^k \times \Delta^r)_+ \wedge (LM \times_M LM)^{-TM} & & \downarrow f_{k+r} \\
f_{k,r} \downarrow & & \\
M^{-TM} \wedge (M^k)_+ \wedge (M^r)_+ & \xrightarrow{=} & M^{-TM} \wedge M_+^{k+r}.
\end{array}$$

Now consider the following diagram of spectra:

$$\begin{array}{ccccc}
(\Delta^{k+r})_+ \wedge LM^{-TM} \wedge LM^{-TM} & \xrightarrow{1 \wedge \tau} & (\Delta^{k+r})_+ \wedge (LM \times_M LM)^{-TM} & \xrightarrow{1 \wedge \gamma} & (\Delta^{k+r})_+ \wedge LM^{-TM} \\
w_{k,r} \wedge 1 \downarrow & & w_{k,r} \wedge 1 \downarrow & & \\
(\Delta^k \times \Delta^r)_+ \wedge LM^{-TM} \wedge LM^{-TM} & \xrightarrow[1 \wedge \tau]{} & (\Delta^k \times \Delta^r)_+ \wedge (LM \times_M LM)^{-TM} & & \downarrow f_{k+r} \\
f_k \wedge f_r \downarrow & & \downarrow f_{k,r} & & \\
M^{-TM} \wedge (M^k)_+ \wedge M^{-TM} \wedge (M^r)_+ & \xrightarrow[\tau]{} & M^{-TM} \wedge (M^{k+r})_+ & \xrightarrow{=} & M^{-TM} \wedge (M^{k+r})_+.
\end{array}$$

Now the top left square in this diagram clearly commutes. The bottom left diagram is diagram (b) above, and so it commutes. The right hand rectangle is diagram (c) above, so it commutes. Therefore the outside of this diagram commutes. Now the top horizontal composition is, by definition the map

$$1 \wedge \mu : (\Delta^{k+r})_+ \wedge LM^{-TM} \wedge LM^{-TM} \longrightarrow (\Delta^{k+r})_+ \wedge LM^{-TM}.$$

The bottom horizontal map is seen to be

$$\tilde{\mu} : M^{-TM} \wedge (M^k)_+ \wedge M^{-TM} \wedge (M^r)_+ \longrightarrow M^{-TM} \wedge (M^{k+r})_+$$

by recalling that the ring multiplication $\Delta^* : M^{-TM} \wedge M^{-TM} \rightarrow M^{-TM}$ is the Pontrjagin - Thom map $\tau : M^{-TM} \wedge M^{-TM} \rightarrow M^{-TM}$ applied to the diagonal embedding $\Delta : M \hookrightarrow M \times M$.

With these identifications, the outside of this diagram is then diagram (a) above. As observed earlier, the commutativity of diagram (a) proves this theorem, and this completes the proof of theorem 2. \square

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