A MODEL STRUCTURE FOR INCLUSION PRESPECTRA IN WHICH THE FIBRANT OBJECTS ARE SPECTRA

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1. Introduction

Among the many possible ways to construct a category equivalent to Boardman's stable category, the approach of Lewis and May [5] is very convenient for point set analysis of spectra. Their category $\mathcal{S}$ of spectra has good formal properties before passage to the stable category and it arises from an easily understood category of prespectra $\mathcal{P}$. In contrast to other categories of spectra, limit and function spectrum constructions in $\mathcal{S}$ are simple and concrete, but colimit and smash product constructions are subtle due to the mysterious behavior of the spectrification functor $L\colon \mathcal{P} \to \mathcal{S}$.

There is an old idea, mentioned to me by Mark Hovey, that it might be possible to construe spectrification as fibrant replacement with respect to a suitable Quillen closed model structure on the category $\mathcal{P}$ of prespectra. In this paper we give a partial solution to this problem. The category $\mathcal{I}$ of inclusion prespectra is intermediate between $\mathcal{P}$ and $\mathcal{S}$. We define a Quillen closed model structure on $\mathcal{I}$ for which the fibrant objects are the spectra. The homotopy category associated to this model structure is equivalent to the stable category.

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2. Categories of prespectra and spectra

Let $\mathcal{T}$ denote the category of based compactly generated weak Hausdorff spaces. If V and W are real inner product spaces we write $V \leq W$ or $W \geq V$ to denote that V is a finite dimensional subspace of W. We write $V < W$ or $W > V$ to exclude the possibility that $V = W$. When $V \leq W$ we will let $W - V$ denote the orthogonal complement of V in W. Let U be a universe—i.e. U is a countably infinite dimensional real inner product space. A prespectrum E indexed on U is a collection of based spaces $\{EV \in \mathcal{T}\}_{V<U}$ together with based maps

$$\sigma_{V,W} \colon \Sigma^{W-V} EV \to EW, \qquad V \leq W < U$$

that satisfy the evident transitivity condition. A map of prespectra $f \colon D \to E$ is a collection of based maps $\{fV \colon DV \to EV\}_{V<U}$ that strictly commute with the structure maps for D and E. Since we will not need to consider more than one universe, we leave U out of our notation and write simply $\mathcal{P}$ for this category.

A prespectrum E is said to be an inclusion prespectrum if the adjoint structure maps

$$\widetilde{\sigma}_{V,W} \colon EV \to \Omega^{W-V} EW, \qquad V \leq W < U$$

are inclusions (k-ified if necessary since we are working with compactly generated spaces). A prespectrum E is said to be a spectrum if each adjoint structure map $\widetilde{\sigma}_{V,W}$ is a homeomorphism. We write $\mathcal{I}$ and $\mathcal{S}$ respectively for the full subcategories of $\mathcal{P}$ consisting of the inclusion prespectra and spectra. Observe that we have inclusions of categories $\mathcal{S} \subset \mathcal{I} \subset \mathcal{P}$.

The technical lynch-pin of the theory of spectra is the following.

Proposition 2.1. *There exists a functor $L' \colon \mathcal{I} \to \mathcal{S}$ that is left adjoint to the inclusion $\mathcal{S} \to \mathcal{I}$. There exists a functor $L'' \colon \mathcal{P} \to \mathcal{I}$ that is left adjoint to the inclusion $\mathcal{I} \to \mathcal{P}$. Hence the composite functor $L = L' \circ L'' \colon \mathcal{P} \to \mathcal{S}$ is left adjoint to the inclusion $\mathcal{S} \to \mathcal{P}$.*

The functor L' was constructed by Peter May [7]. The functor L'' is due to Gaunce Lewis [4]. Generally speaking, the technical complexities that arise in the point set analysis of spectra are due to the mysterious behavior of the Lewis functor L''.

Proposition 2.2. *The categories $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{S}$, are bicomplete and are enriched, tensored, and cotensored over $\mathcal{T}$. In detail, let $\mathcal{A}$ denote any of these categories. The hom sets $\mathcal{A}(D,E)$ have natural basepoints and natural topologies (as spaces in $\mathcal{T}$). There exist bifunctors*

$$\wedge \colon \mathcal{A} \times \mathcal{T} \longrightarrow \mathcal{A} \qquad \text{(smash product)}$$
$$F(-,-) \colon \mathcal{T}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{A} \qquad \text{(function (pre)spectrum)}$$

that satisfy natural adjunction isomorphisms

$$\mathcal{A}(D \wedge X, E) \cong \mathcal{A}(D, F(X,E)) \cong \mathcal{T}(X, \mathcal{A}(D,E)).$$

For prespectra, the limits, colimits, smash products, and function spectra are defined spacewise in the evident way. The prespectrum level constructions of limits and function prespectra preserve inclusion prespectra and spectra. Colimits and smash products with spaces are defined for $\mathcal{I}$ (respectively $\mathcal{S}$) by applying the functor L'' (respectively L) to the constructions in $\mathcal{P}$. See [5] for more details.

In our subsequent work we will need the explicit construction of the May functor L'. Let D be any prespectrum. We define the prespectrum $L'D$ by

$$(L'D)V = \mathrm{colim}_{V \le Z < U}\ \Omega^{Z-V} DZ$$

where the colimit is taken over the maps

$$\Omega^{Z_1-V} DZ_1 \xrightarrow{\Omega^{Z_1-V}\widetilde{\sigma}_{Z_1,Z_2}} \Omega^{Z_1-V}\Omega^{Z_2-Z_1} DZ_2 \cong \Omega^{Z_2-V} DZ_2$$

for $V \le Z_1 < Z_2 < U$. Notice that if D is an inclusion prespectrum then the maps in the colimit are inclusions. The adjoint structure maps $\widetilde{\sigma}_{V,W}$ for $L'D$ are given by the compositions

$$\begin{aligned}(L'D)V = \mathrm{colim}_Z\ \Omega^{Z-V} DZ &\cong \mathrm{colim}_Z\ \Omega^{W-V}\Omega^{Z-W} DZ \\ &\longrightarrow \Omega^{W-V}\mathrm{colim}_Z\ \Omega^{Z-W} DZ = \Omega^{W-V}(L'D)W.\end{aligned}$$

Observe that if D is an inclusion prespectrum then $L'D$ is a spectrum since the functor Ω^{W-V} commutes with colimits of sequences of inclusions.

Thus we have a functor $L'\colon \mathcal{P} \to \mathcal{P}$. The evident maps $i'_D\colon D \to L'D$ define a natural transformation $i'\colon \mathrm{id}_{\mathcal{P}} \to L'$. Restricted to inclusion prespectra, the May construction defines a functor $L'\colon \mathcal{I} \to \mathcal{S}$ that is left adjoint to the inclusion $\mathcal{S} \to \mathcal{I}$. For an inclusion prespectrum D and a spectrum E the adjunction isomorphism is described by

$$\mathcal{S}(L'D, E) = \mathcal{I}(L'D, E) \xrightarrow{(i'_D)^*} \mathcal{I}(D, E).$$

3. Homotopy groups of prespectra

We recall the definition of the homotopy groups $\pi_n D$ of a prespectrum D where n is any integer (positive, zero, or negative). Let $[\ ,\ ]$ denote based homotopy classes of maps of based spaces.

Definition 3.1. For a prespectrum D and $n \ge 0$,

$$\pi_n D = \mathrm{colim}_{Z<U}\ [S^Z \wedge S^n, DZ]$$

where the colimit is taken over the evident maps. For an indexing space $V < U$ we denote

$$\pi_{-V} D = \mathrm{colim}_{V \le Z < U}\ [S^{Z-V}, DZ].$$

It turns out that, up to natural isomorphism, the functor π_{-V} depends only on the dimension of V. Thus we may unambiguosly define for $n \ge 0$

$$\pi_{-n} D = \pi_{-V} D, \qquad \dim\ V = n.$$

We have functors $\pi_n\colon \mathcal{P} \to \mathcal{A}b$ where $\mathcal{A}b$ denotes abelian groups and n is any integer.

We call a map of prespectra $f\colon D \to E$ a π_*-isomorphism if for each $n \in \mathbb{Z}$, $f_*\colon \pi_n D \to \pi_n E$ is an isomorphism. The following facts are well known and, in any case, are easily proved.

Proposition 3.2. *Let E be a spectrum. If $n \ge 0$ and $V < U$ is any indexing space, or if $n < 0$ but $n + \dim\ V \ge 0$, then there is a natural isomorphism*

$$\pi_n E \cong \pi_{n+\dim\ V} EV$$

where the right side denotes the usual space level homotopy group of the based space EV.

Corollary 3.3. *A map $f\colon D \to E$ of spectra is a π_*-isomorphism if and only if it is a level weak equivalence (meaning that for each $V < U$, $fV\colon DV \to EV$ is a weak equivalence of spaces).*

Proposition 3.4. *If D is an inclusion prespectrum then the natural map $i_D\colon D \to LD$ is a π_*-isomorphism.*

Proof. This follows from the construction of the May functor and the fact that the space level functor π_* commutes with colimits of sequences of inclusions. □

Corollary 3.5. *A map $f\colon D \to E$ of inclusion prespectra is a π_*-isomorphism if and only if $Lf\colon LD \to LE$ is a level weak equivalence of spectra.*

Proof. Apply the preceeding results to the diagram

$$\begin{array}{ccc} D & \xrightarrow{i_D} & LD \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} \\ E & \xrightarrow[i_E]{} & LE. \end{array}$$

□

4. Model structures for spaces, prespectra, and spectra

Recall from [8] that a closed model structure on a category $\mathcal{A}$ consists of three classes of morphisms of $\mathcal{A}$ which we will denote by $\mathcal{W}$, $\mathcal{F}$, and $\mathcal{C}$ (weak equivalences, fibrations, and cofibrations respectively). We also write $\mathcal{AF}$ for the class of acyclic fibrations ($\mathcal{W}\cap\mathcal{F}$) and $\mathcal{AC}$ for the class of acyclic cofibrations ($\mathcal{W}\cap\mathcal{C}$). These classes are required to satisfy axioms that are deliberately reminiscent of properties of space level homotopy equivalences, fibrations, and cofibrations. There is a "homotopy category" Ho $\mathcal{A}$ associated to a closed model structure on $\mathcal{A}$. Ho $\mathcal{A}$ is equivalent to the localization $\mathcal{A}[\mathcal{W}^{-1}]$ of $\mathcal{A}$ at the weak equivalences. Excellent expositions of the theory of model structures can be found in [2] and [3].

The standard model structure for based spaces is the following.

Proposition 4.1. *In the category $\mathcal{T}$ let $\mathcal{W}_\mathcal{T}$ denote the class of weak equivalences in the usual sense (maps that induce isomorphism in homotopy groups). Let $\mathcal{F}_\mathcal{T}$ denote the class of Serre fibrations. Let $\mathcal{C}_\mathcal{T}$ denote the class of maps that have the LLP (left lifting property) with respect to $\mathcal{W}_\mathcal{T} \cap \mathcal{F}_\mathcal{T}$. These classes comprise a Quillen closed model structure on $\mathcal{T}$.*

All objects are fibrant with respect to this model structure. The cofibrant objects are the retracts of cell complexes. Any of the various methods of CW or cellular approximation of spaces defines a cofibrant replacement functor.

The Lewis-May construction of the stable category closely parallels the above approach to based spaces. Although Lewis and May did not use the language of model categories in their work, their construction may be stated as follows.

Proposition 4.2. *In the category $\mathcal{S}$ let $\mathcal{W}_\mathcal{S}$ denote the class of maps of spectra that are level weak equivalences (equivalently π_*-isomorphisms). Let $\mathcal{F}_\mathcal{S}$ denote the class of level Serre fibrations. Let $\mathcal{C}_\mathcal{S}$ denote the class of maps that have the LLP with respect to $\mathcal{W}_\mathcal{S} \cap \mathcal{F}_\mathcal{S}$. These classes form a Quillen closed model structure on $\mathcal{S}$. The associated homotopy category is equivalent to the stable category.*

All spectra are fibrant with respect to this model structure. There is a theory of cellular spectra and CW spectra that closely imitates the space level theory. The cofibrant objects are the retracts of cellular spectra.

Finally, we consider two model structures on $\mathcal{P}$. The simpler of the two is called the *level* model structure.

Proposition 4.3. *In the category $\mathcal{P}$ let $\mathcal{W}_\ell$ denote the class of level weak equivalences. Let $\mathcal{F}_\ell$ denote the class of level Serre fibrations. Let $\mathcal{C}_\ell$ denote the class of maps that have the LLP with respect to $\mathcal{W}_\ell \cap \mathcal{F}_\ell$. These classes form a Quillen closed model structure on $\mathcal{P}$.*

The level model structure again has the property that all objects are fibrant. The level structure has the virtue of simplicity but, unfortunately, the associated homotopy category is NOT equivalent to the stable category. It does, however, restrict to the "right" model structure for spectra. The level structure is a useful stepping stone for the proofs of the *stable* model structure on $\mathcal{P}$ which consists of the following.

Proposition 4.4. *In the category $\mathcal{P}$ let $\mathcal{W}_{st}$ denote the class of π_*-isomorphisms. Let $\mathcal{C}_{st}$ denote the class of maps that have the LLP with respect to $\mathcal{W}_{st} \cap \mathcal{F}_\ell$. Let $\mathcal{F}_{st}$ denote the class of maps that have the RLP with respect to $\mathcal{W}_{st} \cap \mathcal{C}_{st}$. These classes form a Quillen closed model structure on $\mathcal{P}$. The associated homotopy category is equivalent to the stable category.*

Recall that a prespectrum D is said to be an Ω-spectrum — more logically, D should be called an Ω-prespectrum — if the adjoint structure maps $\widetilde{\sigma}_{V,W}\colon DV \to \Omega^{W-V}DW$ are weak equivalences. It can be shown that the fibrant objects with respect to the stable model structure are the Ω-spectra. More generally, we have the following fact.

Proposition 4.5. *A map of prespectra $f\colon D \to E$ is a stable fibration if and only if it is a level fibration and for each $V \le W < U$ the square*

$$\begin{array}{ccc} DV & \longrightarrow & \Omega^{W-V}DW \\ {\scriptstyle fV}\downarrow & & \downarrow{\scriptstyle \Omega^{W-V}fW} \\ EV & \longrightarrow & \Omega^{W-V}EW \end{array}$$

is a weak homotopy pullback.

The stable model structure is a topological analogue of the Bousfield-Friedlander model structure which was originally conceived for prespectra made out of simplicial sets [1]. An elegant treatment of the level and stable model structures can be achieved by slightly adapting the results of Mandell, May, Schwede, and Shipley [6]. Their paper also gives far-reaching generalizations of these model structures which apply to other categories of prespectra.

5. The spectral model structure for inclusion prespectra

We define now our proposed model structure for inclusion prespectra which we shall call the *spectral* model structure. Most of the remainder of this paper will be devoted to the proof of the following.

Proposition 5.1. *For the category $\mathcal{I}$ of inclusion prespectra let $\mathcal{W}_{spec}$ denote the class of π_*-isomorphisms. Let $\mathcal{F}_{spec}$ denote the class of maps that are level Serre fibrations and that have the property that whenever $V \leq W < U$ the square*

$$\begin{array}{ccc} DV & \longrightarrow & \Omega^{W-V} DW \\ {\scriptstyle fV}\downarrow & & \downarrow{\scriptstyle \Omega^{W-V} fW} \\ EV & \longrightarrow & \Omega^{W-V} EW \end{array}$$

is a pullback of spaces. Let $\mathcal{C}_{spec}$ denote the class of maps that have the LLP with respect to $\mathcal{W}_{spec} \cap \mathcal{F}_{spec}$. These classes constitute a Quillen closed model structure on $\mathcal{I}$. The associated homotopy category is equivalent to the stable category.

Observe that if D is any inclusion prespectrum (indeed, any prespectrum) then the final map $D \to *$ is always a level Serre fibration. Hence $D \to *$ is a spectral fibration if and only if the pullback condition is satisfied — equivalently, if and only if D is a spectrum. It follows that the fibrant objects with respect to the spectral model structure are the spectra.

Note the similarities between the spectral model structure for $\mathcal{I}$ and the stable model structure for $\mathcal{P}$. If we weaken our definition of spectral fibration and require only that the square is a weak homotopy pullback, we recover the class of stable fibrations. It can be shown that the stable and level model structures for $\mathcal{P}$ restrict to model structures on $\mathcal{I}$. We have the following comparison of model structures on $\mathcal{I}$.

$$\begin{gathered} \mathcal{W}_\ell \subset \mathcal{W}_{st} = \mathcal{W}_{spec} \\ \mathcal{F}_{spec} \subset \mathcal{F}_{st} \subset \mathcal{F}_\ell \\ \mathcal{AF}_{spec} \subset \mathcal{AF}_{st} = \mathcal{AF}_\ell \\ \mathcal{C}_\ell = \mathcal{C}_{st} \subset \mathcal{C}_{spec} \\ \mathcal{AC}_\ell \subset \mathcal{AC}_{st} \subset \mathcal{AC}_{spec}. \end{gathered}$$

6. Spectral maps of prespectra

We formalize the pullback condition in our definition of spectral fibration.

Definition 6.1. A map of prespectra $f\colon D \to E$ is spectral if whenever $V \leq W < U$ the square

$$\begin{array}{ccc} DV & \longrightarrow & \Omega^{W-V} DW \\ {\scriptstyle fV}\downarrow & & \downarrow{\scriptstyle \Omega^{W-V} fW} \\ EV & \longrightarrow & \Omega^{W-V} EW \end{array}$$

is a pullback of spaces.

Thus a spectral fibration is a spectral map of inclusion prespectra that is a level fibration. Notice that if E is a spectrum, then f is spectral if and only if D is also a spectrum. In particular, the final map $D \to *$ is spectral if and only if D is a spectrum.

The following proposition will not be explicitly needed in the sequel, but we state it anyway since it explains intuitively the relevance of spectral maps to model structures. If $\Box$ denotes a commutative square

$$\begin{array}{ccc} X & \xrightarrow{h} & D \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & E \end{array}$$

in any category (without the dashed arrow as yet), let lift $\Box$ denote the set of lifts for $\Box$ (morphisms $Y \to D$ that complete the diagram). Thus

$$\text{lift } \Box = \{\ell \colon Y \to D \mid \ell \circ f = h \text{ and } g \circ \ell = k\}.$$

If $\Box$ is a commutative square of prespectra and $V < U$ we write $\Box V$ for the square in $\mathcal{T}$ consisting of the Vth spaces.

Proposition 6.2. *If $\Box$ is a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{h} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & E \end{array}$$

of prespectra and the map g is spectral, then there exist canonical transformations

$$\phi_{V,W} \colon \text{lift } \Box W \longrightarrow \text{lift } \Box V$$

defined for $V \le W < U$ that satisfy the transitivity condition

$$\phi_{V,W} \circ \phi_{W,Z} = \phi_{V,Z}.$$

Thus $\{\text{lift } \Box V\}_{V<U}$ is an inverse system of sets with respect to the maps $\phi_{V,W}$. If ψ_V denotes the evident map lift $\Box \to$ lift $\Box V$ then $\phi_{V,W} \circ \psi_W = \psi_V$. The map

$$\{\psi_V\} \colon \text{lift } \Box \longrightarrow \lim_{V<U} \text{lift } \Box V$$

is an isomorphism.

Intuitively this proposition states that if g is spectral, then the set of lifts for $\Box$ depends only on the cofinal properties of the prespectra X and Y and the map f. This is analogous to the fact that if E is a spectrum and D is a prespectrum then

$$\mathcal{P}(D, E) \cong \lim_{V<U} \mathcal{T}(DV, EV)$$

where the limit is taken over suitable maps. Thus spectral maps of prespectra are to lifts of squares of prespectra as spectra are to maps of prespectra. We omit the proof of Proposition 6.2 since we will not need to explicitly use it.

We will, however, need the facts that the class of spectral maps is closed under retracts and pullbacks.

Proposition 6.3. *A retract of a spectral map is spectral.*

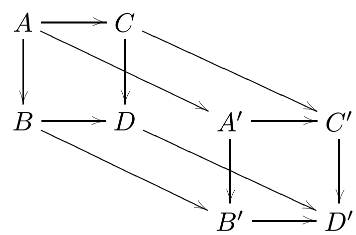
Proof. Suppose that $f \colon X \to Y$ is a retract of the spectral map $g \colon D \to E$. Consider the diagrams

$$\begin{array}{ccc} XV & \longrightarrow & \Omega^{W-V} XW \\ {\scriptstyle fV}\downarrow & & \downarrow{\scriptstyle \Omega^{W-V} fW} \\ YV & \longrightarrow & \Omega^{W-V} YW \end{array} \qquad \begin{array}{ccc} DV & \longrightarrow & \Omega^{W-V} DW \\ {\scriptstyle gV}\downarrow & & \downarrow{\scriptstyle \Omega^{W-V} gW} \\ EV & \longrightarrow & \Omega^{W-V} EW \end{array}$$

and note that the left square is a retract of the right square. It is easily proved that in any category the retract of a pullback square is a pullback square. □

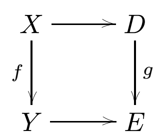
The following lemma either is well known or should be well known. Its proof is straightforward.

Lemma 6.4. *Given a commutative cubical diagram in any category*



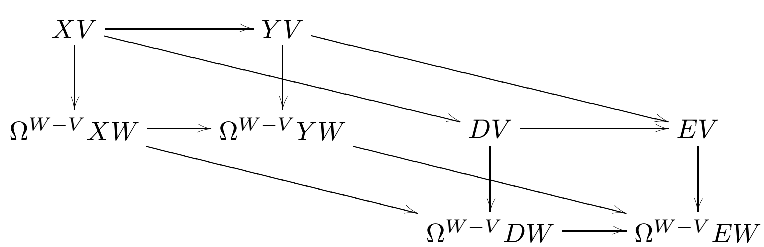
if the squares $A'B'C'D'$, $AA'CC'$, and $BB'DD'$ are pullbacks, then the square $ABCD$ is also a pullback.

Proposition 6.5. *If a square of prespectra*



is a pullback and g is spectral, then f is spectral.

Proof. Apply the preceeding lemma to the diagram



(using the fact that the functor Ω^{W-V} preserves pullbacks). □

We list without proof some further properties of spectral maps.

Proposition 6.6. *Let Λ be a small category, let $D, E\colon \Lambda \to \mathcal{P}$ be functors, and suppose that $f\colon D \to E$ is a natural transformation such that for each $\lambda \in \Lambda$ the map $f_\lambda\colon D_\lambda \to E_\lambda$ is spectral. Then the induced map on limits*

$$\{f_\lambda\}\colon \lim D_\lambda \longrightarrow \lim E_\lambda$$

is spectral. If $f\colon D \to E$ and $g\colon E \to F$ are maps of prespectra and g is spectral, then f is spectral if and only if $g \circ f$ is spectral. If $X \in \mathcal{T}$ is a space and $f\colon D \to E$ is a spectral map then the map

$$f_*\colon F(X, D) \longrightarrow F(X, E)$$

is spectral.

7. Spectral maps of inclusion prespectra

We collect here some results that we will use in our proofs of the model structure. We begin with a lemma on pullbacks of spaces.

Lemma 7.1. *Suppose that we are given sequences of spaces in* $\mathcal{T}$

$$\begin{array}{c} A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \longrightarrow \dots \\ B_1 \longrightarrow B_2 \longrightarrow \cdots \longrightarrow B_n \longrightarrow \dots \\ C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_n \longrightarrow \dots \\ D_1 \longrightarrow D_2 \longrightarrow \cdots \longrightarrow D_n \longrightarrow \dots \end{array}$$

and natural transformations $f\colon A \to B$, $g\colon A \to C$, $h\colon B \to D$, *and* $k\colon C \to D$ *such that for each* n *the square*

$$\begin{array}{ccc} A_n & \xrightarrow{g_n} & C_n \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle k_n} \\ B_n & \xrightarrow[h_n]{} & D_n \end{array}$$

is a pullback in $\mathcal{T}$*. If the maps* $B_n \to B_{n+1}$ *and* $C_n \to C_{n+1}$ *are inclusions (k-ified if necessary) then the maps* $A_n \to A_{n+1}$ *are also inclusions. If, moreover, the maps* $D_n \to D_{n+1}$ *are monomorphic then the square*

$$\begin{array}{ccc} \text{colim } A_n & \xrightarrow{\{g_n\}} & \text{colim } C_n \\ {\scriptstyle \{f_n\}}\downarrow & & \downarrow{\scriptstyle \{k_n\}} \\ \text{colim } B_n & \xrightarrow[\{h_n\}]{} & \text{colim } D_n \end{array}$$

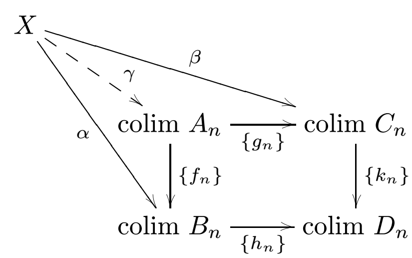
is a pullback.

Proof. If B and C are sequences of inclusions, then in the diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\{f_n, g_n\}} & B_n \times C_n \\ \downarrow & & \downarrow \\ A_{n+1} & \xrightarrow[\{f_{n+1}, g_{n+1}\}]{} & B_{n+1} \times C_{n+1} \end{array}$$

the upper and right arrows are inclusions. Therefore the left arrow is an inclusion so we see that the sequence A also consists of inclusions.

Now assume further that the maps $D_n \to D_{n+1}$ are monomorphic. Consider a test diagram

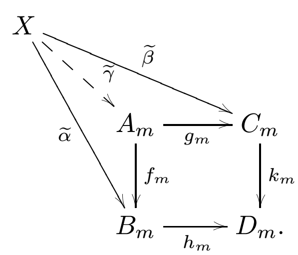


for which we must prove that the dashed arrow uniquely exists. Since we are working with compactly generated spaces, an elementary argument shows that it suffices to prove the result for the case that X is compact. We will need to use that fact that the colimit in $\mathcal{T}$ of a sequence of monomorphisms is the same as the colimit in the category of all topological spaces (for proof see [4], Appendix A). Hence the natural maps into the colimits

$$B_m \longrightarrow \text{colim } B_n$$
$$C_m \longrightarrow \text{colim } C_n$$
$$D_m \longrightarrow \text{colim } D_n$$

are monomorphic.

Now assume that X is compact. There is a m large enough that α factors through a map $\widetilde{\alpha}\colon X \to B_m$ and β factors through a map $\widetilde{\beta}\colon X \to C_m$. Since the mth square is a pullback, there is a unique map $\widetilde{\gamma}\colon X \to A_m$ that completes the diagram



Then the composite

$$X \xrightarrow{\widetilde{\gamma}} A_m \longrightarrow \text{colim } A_n$$

is the solution γ to our original diagram and it is unique. $\square$

This lemma has the following consequence.

Proposition 7.2. *The May functor $L'\colon \mathcal{I} \to \mathcal{S}$ preserves finite limits.*

Proof. We consider first pullbacks. Let us have a pullback diagram of inclusion prespectra.

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ Y & \xrightarrow[h]{} & W \end{array}$$

For $V < U$ choose a cofinal sequence $V_1 < V_2 < \dots$ of indexing spaces in U such that $V \le V_1$. Then we may apply Lemma 7.1 with

$$A_n = \Omega^{V_n - V} X V_n$$
$$B_n = \Omega^{V_n - V} Y V_n$$
$$C_n = \Omega^{V_n - V} Z V_n$$
$$D_n = \Omega^{V_n - V} W V_n$$

Recalling the definition of L' we see that colim $A_n = (L'X)V$, colim $B_n = (L'Y)V$, colim $C_n = (L'Z)V$, and colim $D_n = (L'W)V$. Hence the square

$$\begin{array}{ccc} L'X & \xrightarrow{L'g} & L'Z \\ {\scriptstyle L'f}\downarrow & & \downarrow{\scriptstyle L'k} \\ L'Y & \xrightarrow[L'h]{} & L'W \end{array}$$

is a pullback. It follows now that L' preserves finite products since the isomorphism $L'(Y \times Z) \cong L'Y \times L'Z$ is just the above case for pullbacks when $W = *$. Therefore L' preserves all finite limits. □

A significantly stronger result, due to Gaunce Lewis, states that the spectrification functor from injection prespectra to spectra preserves finite limits [5].

Henceforward, for notational convenience we will write LD rather than $L'D$ even when D is understood to be an inclusion prespectrum. In our proofs of the factorization axioms we will use the following special case of Proposition 7.2.

Corollary 7.3. *Given a pullback square of inclusion prespectra of the form*

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ D & \xrightarrow[i_D]{} & LD, \end{array}$$

the map $Lg\colon LX \to LZ$ is an isomorphism of spectra.

Proof. After spectrifying we see that Lg is a pullback of Li_D, which is the identity map on LD. A pullback of an isomorphism is an isomorphism. □

Proposition 7.4. *A map of inclusion spectra $f\colon X \to Y$ is spectral if and only if the square*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & LX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} \\ Y & \xrightarrow[i_Y]{} & LY, \end{array}$$

is a pullback.

Proof. Any map of spectra is spectral, so if the square is a pullback, it follows by Proposition 6.5 that f is spectral. Suppose now that f is spectral. We apply lemma 7.1 with

$$\begin{aligned} A_n &= XV \\ B_n &= YV \\ C_n &= \Omega^{V_n - V} XV_n \\ D_n &= \Omega^{V_n - V} YV_n \end{aligned}$$

where $V_1 < V_2 < \dots$ is a cofinal sequence of indexing spaces in U. Then colim $A_n = XV$, colim $B_n = YV$, colim $C_n = (LX)V$, and colim $D_n = (LY)V$. The conclusion follows. □

8. Spectral fibrations and cofibrations

We begin with a well known fact about spaces.

Lemma 8.1. *Given a diagram in $\mathcal{T}$*

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_n & \longrightarrow & \cdots \end{array}$$

if the maps $X_n \to X_{n+1}$ and $Y_n \to Y_{n+1}$ are inclusions and each f_n is a (acyclic) Serre fibration, then the induced map on colimits

$$\{f_n\} \colon \operatorname{colim} X_n \longrightarrow \operatorname{colim} Y_n$$

is a (acyclic) Serre fibration.

Proof. Serre fibrations are the maps that have the RLP with respect to the level zero inclusions $(D^n)_+ \to (D^n \times I)_+$ and acyclic Serre fibrations are characterized by the RLP with respect to the standard inclusions $(S^{n-1})_+ \to (D^n)_+$. Since $(S^{n-1})_+$, $(D^n)_+$, and $(D^n \times I)_+$ are compact, the result follows easily from the fact that compact spaces are small with respect to sequences of inclusions in $\mathcal{T}$. □

Proposition 8.2. *If a map of inclusion prespectra $f \colon D \to E$ is a level (acyclic) fibration then $Lf \colon LD \to LE$ is an (acyclic) fibration of spectra.*

Proof. For $V < U$ choose a cofinal sequence $V_1 \leq V_2 \leq \dots$ of indexing spaces in U such that $V \leq V_1$. Consider the diagram

$$\begin{array}{ccccc} \Omega^{V_1 - V} DV_1 & \longrightarrow & \Omega^{V_2 - V} DV_2 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \Omega^{V_1 - V} fV_1} & & \downarrow{\scriptstyle \Omega^{V_2 - V} fV_2} & & \\ \Omega^{V_1 - V} EV_1 & \longrightarrow & \Omega^{V_2 - V} EV_2 & \longrightarrow & \cdots \end{array}$$

Now we use the fact that the functors $\Omega^{V_n - V}$ preserve (acyclic) Serre fibrations. The conclusion follows from Lemma 8.1. □

Combining Proposition 7.4 and Proposition 8.2 we have the following.

Corollary 8.3. *A map of inclusion spectra $f \colon D \to E$ is a spectral (acyclic) fibration if and only if f is the pullback of an (acyclic) fibration of spectra.*

Now we consider spectral cofibrations and spectral acyclic cofibrations.

Lemma 8.4. *If $f \colon X \to Y$ is a map of inclusion prespectra (or even arbitrary prespectra) and $g \colon D \to E$ is a map of spectra, then f has the LLP with respect to g if and only if Lf has the LLP with respect to g.*

Proof. By adjunction, the diagrams

$$\begin{array}{ccc} X & \xrightarrow{h} & D \\ \downarrow{\scriptstyle f} & \nearrow{\scriptstyle \ell} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & E \end{array} \qquad\qquad \begin{array}{ccc} LX & \xrightarrow{Lh} & D \\ \downarrow{\scriptstyle Lf} & \nearrow{\scriptstyle L\ell} & \downarrow{\scriptstyle g} \\ LY & \xrightarrow[Lk]{} & E \end{array}$$

are equivalent. □

Corollary 8.5. *A map of inclusion prespectra $f\colon D \to E$ is a spectral (acyclic) cofibration if and only if $Lf\colon LD \to LE$ is a (acyclic) cofibration of spectra.*

Proof. If f is in $\mathcal{C}_{spec}$ then, by definition, f has the LLP with respect to $\mathcal{AF}_{spec}$. Certainly an acyclic fibration of spectra, regarded as a map of inclusion prespectra, is in $\mathcal{AF}_{spec}$. Thus f has the LLP with respect to acyclic fibrations of spectra and therefore Lf does also. Hence Lf is a cofibration of spectra. Conversely, if Lf is a cofibration of spectra, then Lf has the LLP with respect to acyclic fibrations of spectra and therefore f does also. Since spectral acyclic fibrations are pullbacks of acyclic fibrations of spectra, f must have the LLP with respect to $\mathcal{AF}_{spec}$. Therefore f is in $\mathcal{C}_{spec}$. Moreover, we know from Corollary 3.5 that f is in $\mathcal{W}_{spec}$ if and only if Lf is a weak equivalence of spectra. □

9. The proofs of the model structure

We are ready to prove Proposition 5.1. First note that our classes $\mathcal{W}_{spec}$, $\mathcal{F}_{spec}$, and $\mathcal{C}_{spec}$ contain all identity maps. Clearly $\mathcal{W}_{spec}$ satisfies the 2 out of 3 property.

Proposition 9.1. *The classes $\mathcal{W}_{spec}$, $\mathcal{F}_{spec}$, and $\mathcal{C}_{spec}$ are closed under retracts.*

Proof. This is obvious for $\mathcal{W}_{spec}$. Retracts of level fibrations are level fibrations and, by Proposition 6.3, retracts of spectral maps are spectral. Hence $\mathcal{F}_{spec}$ is closed under retracts. The class $\mathcal{C}_{spec}$ is necessarily closed under retracts since it is defined by the LLP with respect to the class $\mathcal{AF}_{spec}$. □

We consider now the lifting properties. One we have as a matter of definition of the class $\mathcal{C}_{spec}$. The other is given by the following.

Proposition 9.2. *The class $\mathcal{AC}_{spec} = \mathcal{W}_{spec} \cap \mathcal{C}_{spec}$ has the LLP with respect to $\mathcal{F}_{spec}$.*

Proof. Let $f \in \mathcal{AC}_{spec}$ and $g \in \mathcal{F}_{spec}$. Since g is a pullback of the fibration of spectra Lg it suffices to show that f has the LLP with respect to Lg. But by Lemma 8.4 this is equivalent to requiring that Lf has the LLP with respect to Lg. By Corollary 8.5 Lf is an acyclic cofibration of spectra and the conclusion follows. □

It remains to prove the factorization axioms. The proofs of the two cases are logically identical, so we give details for just one of the cases.

Proposition 9.3. *Any map of inclusion prespectra $f\colon D \to E$ admits a factorization*

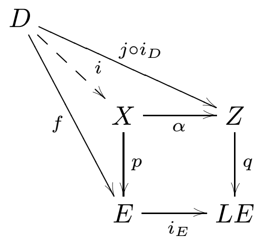
$$D \xrightarrow{i} X \xrightarrow{p} E$$

where i is in $\mathcal{AC}_{spec}$ and p is in $\mathcal{F}_{spec}$.

Proof. Consider a factorization of Lf in the category of spectra

$$LD \xrightarrow{j} Z \xrightarrow{q} LE$$

where j is an acyclic cofibration of spectra and q is a fibration of spectra. We have a diagram



where the square is a pullback and $i\colon D \to X$ is the unique dashed arrow completing the diagram. By Corollary 8.3 p is in $\mathcal{F}_{spec}$. By Corollary 7.3 $L\alpha\colon LX \to Z$ is an isomorphism of spectra. It follows that

$$j = L(j \circ i_D) = L(\alpha \circ i) = L\alpha \circ Li.$$

Therefore $Li = (L\alpha)^{-1} \circ j$. Since j is an acyclic cofibration of spectra and $(L\alpha)^{-1}$ is an isomorphism of spectra, Li is an acyclic cofibration of spectra. It follows from Corollary 8.5 that i is in $\mathcal{AC}_{spec}$. □

An identical argument establishes that:

Proposition 9.4. *Any map of inclusion prespectra $f\colon D \to E$ admits a factorization*

$$D \xrightarrow{i} X \xrightarrow{p} E$$

where i is in $\mathcal{C}_{spec}$ and p is in $\mathcal{AF}_{spec}$.

This completes the demonstration that our spectral structure is a Quillen closed model structure for $\mathcal{I}$. Moreover, it is easily argued that the associated homotopy category is equivalent to the stable category. Simply note that the inclusion prespectra that are both fibrant and cofibrant are the retracts of spectra that are cellular in the sense of [5]. The model theoretic homotopy relation on the category of these objects agrees with the usual topological notion of homotopy defined in terms of cylinders $E \wedge I_+$.

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