# TRACTABLE FORMULAS FOR $v_1$-PERIODIC HOMOTOPY GROUPS OF $SU(n)$ WHEN $n \le p^2 - p + 1$

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## 1. Statement of results

Let $p$ be a fixed odd prime. The ($p$-local) $v_1$-periodic homotopy groups, $v_1^{-1}\pi_*(X)$, of a space $X$ were defined in [8]. Roughly speaking, they tell the portion of the $p$-local homotopy groups of $X$ detected by $K$-theory. For a spherically resolved space $X$, each group $v_1^{-1}\pi_i(X)$ is a direct summand of some actual homotopy group $\pi_L(X)$. We will focus on the case where $X$ is one of the special unitary groups $SU(n)$, which are spherically resolved by the fibrations

$$SU(n-1) \to SU(n) \to S^{2n-1}. \tag{1.1}$$

Let $\nu(m) = \nu_p(m)$ denote the exponent of $p$ in $m$. Let

$$e_p(k,n) = \min\{\nu_p(j!S(k,j)) \ : \ n \le j \le k\}, \tag{1.2}$$

where $S(k,j)$ is the Stirling number of the second kind, which satisfies

$$j!S(k,j) = \sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i}i^k. \tag{1.3}$$

The following result was proved in [7].

**Theorem 1.4.** *If $k \ge n$, then $v_1^{-1}\pi_{2k}(SU(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$, while $v_1^{-1}\pi_{2k-1}(SU(n))$ has order $p^{e_p(k,n)}$, but is not necessarily cyclic.*

Periodicity in $v_1^{-1}\pi_*(SU(n))$ would allow one to determine $v_1^{-1}\pi_{2k}(SU(n))$ for smaller or negative values of $k$ from this, if one wished.

Theorem 1.4 at first glance appears to be all that one might want to know about $v_1^{-1}\pi_*(SU(n))$. However, it has two drawbacks. One is that it only gives the orders

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of the odd groups, and not their actual structure, and the other has to do with the intractability of the formula (1.2).

The numbers $e_p(k,n)$ which occur in Theorem 1.4 are in some sense given explicitly in (1.2), and some of them can be computed by a computer, but it seems to be very difficult to obtain useful general formulas for them from (1.2). Indeed, despite efforts in [11], [9], [6], and [7], the only general results obtained from (1.2) seem to be a (sharp) lower bound for $e_p(k,n)$ when $n \le p$, and the inequality $\max\{e_p(k,n) : k \in \mathbf{Z}\} \ge n-1$. These were proved in [6], using Fermat's Little Theorem. Since $SU(n)$ localized at $p$ splits as a product of spheres when $n \le p$, the first result can also be obtained easily from results for spheres. The second result is more useful, since it implies (1.7).

The first main result of this paper is a tractable formula for $v_1^{-1}\pi_{2k}(SU(n))$, provided $n \le p^2 - p + 1$.

**Theorem 1.5.** *Suppose $p$ is odd, $k = N + (p-1)m$ with $1 \le N < p$, and*

$$N + 1 + (p-1)i \le n < N + 1 + (p-1)(i+1)$$

*with $0 \le i \le p-1$. Define $\hat{m}$ by $0 \le \hat{m} < p$ and $m \equiv \hat{m} \bmod p$. Then $v_1^{-1}\pi_{2k}(SU(n)) \approx \mathbf{Z}/p^e$, where $e$ is equal to*

$$\begin{cases} i+1 & \text{if } i < N \text{ and } i < \hat{m} \\ i & \text{if } N \le i \text{ and } (i < \hat{m} \text{ or } \hat{m} = 0) \\ \min(N + (p-1)\hat{m},\ i + \nu(m - \hat{m}) + 1) & \text{if } i < N \text{ and } \hat{m} \le i \\ \min\Big(N + (p-1)\hat{m} + 1, & \\ \qquad i + \nu(m - \hat{m} + (-1)^{\hat{m}}\hat{m}\binom{N}{\hat{m}}p^{\hat{m}(p-1)})\Big) & \text{if } N \le i \text{ and } 1 \le \hat{m} \le i \end{cases}$$

The smallest $n$ for which this theorem fails to give complete information is $n = N + 1 + (p-1)i$ with $N = 1$ and $i = p$.

The proof of this result makes no use of formula (1.2); it is an independent calculation of $v_1^{-1}\pi_*(SU(n))$. These separate calculations of the same homotopy group give a topological proof of a result in number theory, a tractable evaluation of

$$\min\{\nu_p(j!S(k,j)) \ : \ n \le j \le k\}$$

when $n \le p^2 - p + 1$, as given in Theorem 1.5. For example, the following corollary is easily obtained from Theorem 1.5 and (1.2).

**Corollary 1.6.** *Suppose $p$ is an odd prime, $0 \le \hat{m} \le i < N < p$, and $i + \nu_p(\ell) + 2 \le N + (p-1)\hat{m}$. Then*

$$\nu_p(S(N + (p-1)\hat{m} + (p-1)p\ell, N + (p-1)i)) = \nu_p(p\ell).$$

We view this merely as an amusing offshoot of our work.

An application of Theorem 1.5 to homotopy theory is a slightly improved lower bound for the $p$-exponent of $SU(n)$. Recall that the $p$-exponent of a space $X$, denoted $\exp_p(X)$, is defined to be the largest $e$ such that some homotopy group of $X$ contains an element of order $p^e$. As noted at the outset, if $X$ is spherically resolved, its $p$-exponent will be at least as large as that of the order of any element of $v_1^{-1}\pi_*(X)$. It was shown in [7] that

$$\exp_p(SU(n)) \ge n - 1. \tag{1.7}$$

This can be improved by 1 for certain values of $n$, as given in the following corollary.

**Corollary 1.8.** *If $p$ is odd, then $\exp_p(SU(n)) \ge n$ if, for some $i < p$,*

$$i(p-1) + 2 \le n \le ip + 1.$$

*Proof.* We use Theorem 1.5 to determine $\max\{e_p(k,n) : k \in \mathbf{Z}\}$. In the notation of Theorem 1.5, this maximum will occur when $n = N + 1 + (p-1)i$ and $\hat{m} = i$. This maximum will equal $n - 1$ if $i < N$, and $n$ if $i \ge N$. The expression in the corollary is obtained as $N + 1 + (p-1)i$ for $1 \le N \le i$. ■

The other main result is an explicit determination of the groups $v_{2k-1}(SU(n))$ when $n \le p^2 - p + 1$. Recall that Theorem 1.4 only gave their order.

**Theorem 1.9.** *Use the notation of Theorem 1.5, and let $t = \min\{N, \nu(m) + 1\}$. Then*

$$v_1^{-1}\pi_{2k-1}(SU(n)) \approx \begin{cases} \mathbf{Z}/p^e & \text{if } i \le N \\ \mathbf{Z}/p \oplus \mathbf{Z}/p^{e-1} & \text{if } i > N \text{ and } \hat{m} \ne 0 \\ \mathbf{Z}/p^t \oplus \mathbf{Z}/p^{i-t} & \text{if } i > N \text{ and } \hat{m} = 0 \text{ and } t \le i - N \\ \mathbf{Z}/p^N \oplus \mathbf{Z}/p^{i-N} & \text{if } i > N \text{ and } \hat{m} = 0 \text{ and } t \ge i - N \end{cases}$$

A description of the conditions under which these groups are cyclic can be given without resorting to all the special notation of Theorem 1.5. The following result is easily obtained from Theorem 1.9.

**Corollary 1.10.** *Let $\overline{k}$ be defined by $1 \le \overline{k} \le p-1$ and $\overline{k} \equiv k$ mod $p-1$. Then $v_1^{-1}\pi_{2k-1}(SU(n))$ is cyclic if $n < (\overline{k}+1)p$, and is the direct sum of two cyclic summands if*

$$(\overline{k}+1)p \le n \le p^2 - p + 1.$$

The proofs of both of our main theorems will involve a delicate analysis of the unstable Novikov spectral sequence (UNSS) based on $BP$. We let $E_2^{s,t}(X)$ denote the $E_2$-term of this spectral sequence. The first part of Theorem 1.4 was an immediate consequence of the following result, the first part of which was proved in [3], and the second part in [7].

**Theorem 1.11.** (1) *If $k \ge n$, then $E_2^{1,2k+1}(SU(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$.*
(2) *If $k \ge n$, then $v_1^{-1}\pi_{2k}(SU(n)) \approx E_2^{1,2k+1}(SU(n))$.*

Theorem 1.5 is proved by computing $E_2^{1,2k+1}(SU(n))$ by increasing induction on $n$. By contrast, Theorem 1.11(1) was proved by computing $E_2^{1,2k+1}(SU(n))$ by downward induction on $n$, starting with $E_2^{1,2k+1}(SU(k)) \approx \mathbf{Z}/k!$. The methods of calculation of the UNSS used in proving Theorems 1.5 and 1.9 extend those of [5]; we think that the methods introduced here of calculating this spectral sequence for multicell complexes should be useful for other computations.

In Section 2, we present the requisite background on the UNSS. In Section 3, we outline the proof of Theorem 1.5, with details relegated to Section 4. In Section 5, we prove Theorem 1.9. This paper overlaps substantially with the second author's thesis, [12].

## 2. Background in the UNSS

Let $BP$ be the Brown-Peterson spectrum corresponding to the prime $p$. Then

$$BP_* = \pi_*(BP) \approx Z_{(p)}[v_1, v_2, \dots],$$

where $v_i$ are the Hazewinkel generators of $BP_*$. Let $\Gamma = BP_*(BP) \approx BP_*[t_1, t_2, \dots]$, where $t_i$ are Quillen's generators. We have $|v_i| = |t_i| = 2(p^i - 1)$. Let $c : BP_*(BP) \to BP_*(BP)$ be the conjugation, and define $h_i = c(t_i)$. Then $\Gamma = BP_*[h_1, h_2, \dots]$. Let $\eta = \eta_R : BP_* \to BP_*(BP)$ be the right unit. We write $h_i v_j$ interchangeably with $\eta(v_j)h_i$; this is the right action of $BP_*$ on $\Gamma$.

Let $M$ be a $\Gamma$-comodule with coaction map $\psi_M : M \to \Gamma \otimes_{BP_*} M$. The stable cobar complex $C(T(M))$ is defined as follows:

$$C^s(T(M)) = \Gamma \otimes_{BP_*} \Gamma \otimes \cdots \otimes_{BP_*} \Gamma \otimes_{BP_*} M,$$

with $s$ copies of $\Gamma$. We use the cobar notation, and write $\gamma_0 [\gamma_1| \ldots |\gamma_s] m$ for $\gamma_0 \otimes \gamma_1 \otimes \ldots \otimes \gamma_s \otimes m$. The differential $d$ is given by

$$\begin{aligned} d(\gamma_0[\gamma_1|\cdots|\gamma_s]m) &= \sum \gamma_0' \left[\gamma_0''|\cdots|\gamma_s\right] m \\ &+\sum_{j=1}^{s}(-1)^j \gamma_0 \left[\gamma_1|\cdots|\gamma_j'|\gamma_j''|\cdots|\gamma_s\right] m + (-1)^{s+1} \sum \gamma_0 \left[\gamma_1|\cdots|\gamma_s|\gamma'\right] m'' \end{aligned} \tag{2.1}$$

where $\psi(\gamma_i) = \sum \gamma_j' \otimes \gamma_j''$ for $0 \le j \le s$ and $\psi_M(m) = \sum \gamma' \otimes m''$.

The unstable cobar complex $\{C^*(U(M)), d\}$ is a subcomplex of $\{C^*(T(M)), d\}$, consisting of terms satisfying an unstable condition, introduced in the following definition.

**Definition 2.2.** [4, p. 243] *If $M$ is a nonnegatively graded free left $A$-module, then $U(M)$ is defined to be the $BP_*$-span of*

$$\{h^I \otimes m \ : \ 2(i_1 + i_2 + i_3 + \cdots) < |m|\} \subset \Gamma \otimes_{BP_*} M,$$

*where $I = (i_1, i_2, \ldots)$ is a sequence containing only finitely many nonzero $i_j$'s, and $h^I = h_1^{i_1} h_2^{i_2} \cdots$.*

Define $U^s(M) = U(U^{s-1}(M))$, and $C^s(U(M)) = U^s(M)$. If $M$ is a $\Gamma$-comodule, then $U(M) \subset T(M) = \Gamma \otimes M$ as a sub-$\Gamma$-comodule, and so the differential $d$ of the stable cobar complex $\{C^s(T(M)), d\}$ induces a differential on $\{C^s(U(M))\}$ which defines a complex $\{C^s(U(M)), d\}$. We will usually replace it by the chain-equivalent reduced complex obtained by replacing $U(M)$ by $\ker(U(M) \xrightarrow{\epsilon} M)$. This has the effect of only looking at terms which have positive grading in each position. The homology groups of this unstable cobar complex are denoted by $\mathrm{Ext}_{\mathcal{U}}^{s,t}(M)$.

It was proved in [4] that, if $X$ is a simply-connected $CW$-space, there is a spectral sequence $\{E_r^{s,t}(X), d_r\}$ which converges to the homotopy groups of $X$ localized at $p$, and if the integral cohomology $H^*(X)$ is a free algebra, then

$$E_2^{s,t}(X) = \mathrm{Ext}_{\mathcal{U}}^{s,t}(P(BP_*X)),$$

where $P(BP_*X)$ denotes the sub-$\Gamma$-comodule of $BP_*X$ consisting of the primitives under the coproduct. This is the UNSS for the space $X$.

The following basic formulas were used or proved in [5].

**Lemma 2.3.** *(1) $v_1 = ph_1 + \eta(v_1)$ and $\psi(h_1) = h_1 \otimes 1 + 1 \otimes h_1$;*

*(2) $v_2 = ph_2 + (1 - p^{p-1})h_1^p v_1 + \eta(v_2) - (p+1)v_1^p h_1 + \sum_{i=2}^p a_i v_1^{p+1-i} p^i h_1^i$, where $a_i \in Z$;*

*(3) $d(v_1) = \eta(v_1^n) - v_1^n$ and*

$$d(v^a h^b v^c) = (\eta(v^a) - v^a) \otimes h^b v^c - v^a \bar{\psi}(h^b) v^c - v^a h^b \otimes (\eta(v^c) - v^c)$$

*where $\bar{\psi}(h^b) = \psi(h^b) - h^b \otimes 1 - 1 \otimes h^b$.*

The first part of this lemma will be used very frequently in the context of replacing $ph_1$ by $v_1 - \eta(v_1)$.

Let $\sum^{F^*} h_i = c(\sum^F c(h_i)) = c(\sum^F t_i)$, where $x +_F y$ is the formal group sum defined by $x +_F y = \exp(\log x + \log y)$ with $\log x = \sum_{i \geq 0} m_i x^{i+1}$, $\exp x = \sum_{i \geq 0} b_i x^{i+1}$, and $\exp(\log x) = \log(\exp x) = x$. Here $\{m_i\}$ and $\{b_i\}$ are two different polynomial generators sets for $H_*BP$ with $|m_i| = |b_i| = 2(p^i - 1)$. Then the following lemma of Bendersky ([3]) is useful.

**Lemma 2.4.** *The primitives $P(BP_*(SU(n))$ form a free $BP_*$-module generated by elements $\{x_3, x_5, x_7, ..., x_{2n-1}\}$ with coaction given by*

$$\psi(x_{2k+1}) = \sum_j (\sum_i^{F^*} h_i)^j_{k-j} \otimes x_{2j+1}.$$

The subscript $k - j$ refers to the component in grading $q(k - j)$. Here we have introduced the notation $q = 2(p - 1)$, which will be used frequently throughout the paper.

We will need the following explicit computation.

**Proposition 2.5.** *Mod terms of degree greater than $3q$,*

$$\sum_i^{F^*} h_i = 1 + h_1 - h_1 v_1 + h_1 v_1^2 - \tfrac{p-1}{2} h_1^2 v_1.$$

*The only terms $h_1^j$ which appear in $\sum^{F^*} h_i$ are $1 + h_1$.*

*Proof.* From

$$\begin{aligned} x &= \exp(\log x) \\ &= b_0 \log x + b_1(\log x)^2 + b_2(\log x)^3 + \cdots \\ &= b_0 \sum_{i\ge 0} m_i x^{p^i} + b_1(\sum_{i\ge 0} m_i x^{p^i})^2 + b_2(\sum_{i\ge 0} m_i x^{p^i})^3 + \cdots, \end{aligned}$$

we deduce that the first nonzero $b_i$'s are $b_0 = 1$, $b_{p-1} = -m_1$, $b_{2p-2} = pm_1^2$, and $b_{3p-3} = \binom{p}{2} - p(2p-1)$. Then, mod terms of degree greater than $3q$,

$$\begin{aligned} \sum_i^{F^*} h_i &= c(\sum_i^F t_i) \\ &= c\Big(\sum_{i\ge 0} b_i(\sum_{j,k\ge 0} m_j t_k^{p^j})^{i+1}\Big) \\ &= c\Big(1 + m_1 + t_1 - m_1(1 + m_1 + t_1)^p + pm_1^2(1 + m_1 + t_1)^{2p-1} \\ &\quad +(\tbinom{p}{2} - p(2p-1))m_1^3\Big) \\ &= c\Big(1 + m_1 + t_1 - m_1(1 + pm_1 + pt_1 + \tbinom{p}{2}(m_1^2 + t_1^2) + p(p-1)m_1t_1) \\ &\quad +pm_1^2(1 + (2p-1)(m_1 + t_1)) + (\tbinom{p}{2} - p(2p-1))m_1^3)\Big) \\ &= c\Big(1 + t_1 - pm_1t_1 + p^2m_1^2t_1 - \tbinom{p}{2}m_1t_1^2\Big). \end{aligned}$$

The desired result is now obtained using $v_1 = pm_1$ and $c(v_1) = \eta(v_1)$.

The second statement follows since the only powers of $t_1$ appearing in the expansion of $\sum b_i(\sum m_j t_k^{p^j})^{i+1}$ are $1 + t_1$. ■

This coaction formula in $SU(n)$ will be extremely important, as it determines the boundary homomorphism in the exact sequence associated to the fibration (1.1). Indeed, there is an exact sequence

$$\to E_2^{s,t}(SU(n-1)) \to E_2^{s,t}(SU(n)) \to E_2^{s,t}(S^{2n-1}) \xrightarrow{\partial} E_2^{s+1,t}(SU(n-1)) \to,$$

with

(2.6) $$\partial(y) = y \otimes \psi(x_{2n-1}).$$

This boundary formula is true since $\psi$ gives the coboundary for the unstable cobar complex of $SU(n)$. The term $1 \otimes x_{2n-1}$ will not appear in our reduced complex.

We prefer to work with the $v_1$-periodic UNSS of [1]. This has the advantage that for $X = SU(n)$ or $S^{2n+1}$,

$$E_2^{s,t}(X) = \begin{cases} v_1^{-1}\pi_{t-s}(X) & \text{if } s = 1 \text{ or } 2, \text{ and } t \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

We can still work with unstable cobar names for classes; we just view the periodic spectral sequence as a direct summand of the actual UNSS, after multiplication by sufficiently many $v_1$'s. So from now on, $E_2$ refers to the periodic spectral sequence.

We will make frequent use of the following result for the spheres. The 1-line part was proved in [4], and the 2-line part in [2]. We introduce here terminology $x \equiv y \bmod S^{2n-1}$ to mean that $x - y$ desuspends to (or is defined on) $S^{2n-1}$. For elements of $E_2^s(S^{2n+1})$, we frequently abbreviate $y\iota_{2n+1}$ as $y$.

**Theorem 2.8.** (1) *The only nonzero groups* $E_2^{s,t}(S^{2n+1})$ *are*

$$E_2^{s,2n+1+qm}(S^{2n+1}) \approx \mathbf{Z}/p^e$$

*with* $s = 1$ *or* $2$ *and* $e = \min(n, \nu(m) + 1)$.

(2) *The generator of* $E_2^{1,2n+1+qm}(S^{2n+1})$ *is* $\alpha_{m/e} = d(v_1^m)/p^e$ *and satisfies*

$$\alpha_{m/e} \equiv -v_1^{m-e}h_1^e \mod S^{2e-1}.$$

(3) *Suppose* $n \le \nu(m) + 1$. *Then for* $1 \le j \le n$, *there is an element of order* $p^j$ *in* $E_2^{2,2n+1+qm}(S^{2n+1})$ *which equals* $v_1^{m-j-1}h_1 \otimes h_1^j$ *mod* $S^{2j-1}$.

(4) *If* $\nu(m) + 1 \le n$, *then* $d(v_1^{m-n-1}h_1^{n+1})\iota_{2n+1}$ *has order* $p$ *in* $E_2^{2,2n+1+qm}(S^{2n+1})$, *and, for* $0 \le j \le \nu(m)$, $p^j$ *times a generator of* $E_2^{2,2n+1+qm}(S^{2n+1})$ *equals* $v_1^{m-n+j-1}h_1 \otimes h_1^{n-j}$ *mod* $S^{2(n-j)-1}$.

(5) *The homomorphism* $\Sigma^2 : E_2^{2,2n-1+qm}(S^{2n-1}) \to E_2^{2,2n+1+qm}(S^{2n+1})$ *is injective if* $n \le \nu(m) + 1$ *and is multiplication by* $p$ *otherwise.*

The formulas for leading terms of classes on the 2-line in parts 3 and 4 differ from those in [2], which have $v_1^e$ on the right of the first $h_1$. We write them as we have because that is the way in which they will occur in our applications. To see that $h_1 \otimes v_1^e h_1^j \equiv v_1^e h_1 \otimes h_1^j \bmod S^{2j-1}$, we use Lemma 2.3(1) to write the difference as $\sum_{i=1}^{e-1}\binom{e}{i}p^i h_1^{i+1}v_1^{e-i}h_1^j$. The $i$th term is a multiple of $p^{i-1}h_1^{i+1}v_1^{e-i}h_1^{j-1} = \sum\binom{i-1}{j}v_1^j h_1^2 v_1^{e-j}h_1^{j-1}$, which is defined on $S^{2j-1}$. The following corollary encapsulates a common feature of parts 3 and 4 of the above theorem. We use the term "leading term" to mean the term which desuspends least far.

**Corollary 2.9.** *If an element of $E_2^2(S^{2n+1})$ has order $p^f$ and leading term $h\otimes h^j\iota_{2n+1}$, then*

$$j+\nu(|E_2^2(S^{2n+1})|)=f+n.$$

We will also need the following result about the 2-line for spheres, proved in [2].

**Lemma 2.10.** *Let $E_2^s(M)=\operatorname{Ext}_\Gamma^s(BP_*,BP_*/p)$ denote the $E_2$-term of the stable Novikov spectral sequence for the mod $p$ Moore spectrum $M$. There is a homomorphism*

$$H':E_2^{s,t}(S^{2n+1})\to E_2^{s-1,t-2pn-1}(M)$$

*satisfying $H'(x)=0$ if and only if $x$ double desuspends, and if $x=z\otimes h_1^n\iota_{2n+1}$ with $z\iota_{2pn-1}$ defined, then $H'(x)=z$.*

We will need the following fact about $E_2(M)$, proved in [5, 2.11].

**Lemma 2.11.** *$v_1^kh_1\neq 0\in E_2^1(M)$ and $v_1h_1^{k(p-1)+1}=v_1^{k(p-1)+1}h_1$ in $E_2^1(M)$.*

The following result will be very useful.

**Lemma 2.12.** (1) *$h_1^{n+p-1}\otimes h_1\iota_{2n+1}\equiv -v_1^{p-1}h_1\otimes h_1^n\iota_{2n+1}$ mod $S^{2n-1}$.*
(2) *$h_1^{n+p-1}\otimes h_1\iota_{2n+1}$ double desuspends iff $n=1$.*

*Proof.* We use [5, 2.13], which says that

$$(h_1^p\otimes h_1^{K(p-1)+n}+h_1^{K(p-1)+p+n-1}\otimes h_1)v_1^K\iota_{2n+1}$$

is the sum of a boundary and a class which double desuspends, where $n\geq 1$ and $K\geq 0$. As noted in [5, 2.14], this gives immediately that $h_1^p\otimes h_1\iota_3$ double desuspends, since 2 is a unit. It also gives

$$h_1^{n+p-1}\otimes h_1\iota_{2n+1}\equiv -h_1^p\otimes h_1^n\iota_{2n+1}\quad \text{mod } S^{2n-1}.$$

Thus, if $n>1$, $H'(h_1^{n+p-1}\otimes h_1\iota_{2n+1})=-h_1^p$ by Lemma 2.10. By Lemma 2.11, this equals $-v_1^{p-1}h_1\neq 0$, so that $h_1^{n+p-1}\otimes h_1\iota_{2n+1}$ does not double desuspend. However,

$$H'(h_1^{n+p-1}\otimes h_1\iota_{2n+1}+v_1^{p-1}h_1\otimes h_1^n\iota_{2n+1})=-v_1^{p-1}h_1+v_1^{p-1}h_1=0,$$

and so the sum double desuspends. ∎

The following lemma will also be used many times. It is the one place where the number of $v_1$'s on the left is important. We begin a policy of writing $h$ for $h_1$, and $v$ for $v_1$.

**Lemma 2.13.** *For $j \geq 0$,*

$$d(v^\ell h^{n+1} v^j) \equiv -(\ell+n+1)v^\ell h \otimes h^n v^j + j v^{\ell+1} h^n \otimes v^{j-1} h \mod S^{2n-1-qj}.$$

*Proof.* We evaluate the desired boundary using Lemma 2.3, expanding

$$d(v^a) = (v - ph)^a - v^a,$$

to obtain

$$\sum_{i=1}^{\ell} \binom{\ell}{i} v^{\ell-i}(-ph)^i \otimes h^{n+1}v^j \quad - \quad \sum_{i=1}^{n} \binom{n+1}{i} v^\ell h^i \otimes h^{n+1-i} v^j$$
$$- \quad \sum_{i=1}^{j} \binom{j}{i} v^\ell h^{n+1} \otimes v^{j-i}(-ph)^i.$$

For each term in each sum, we study whether it satisfies the unstable condition on $S^{2n-1-qj}$ (number of $h$'s $\leq \frac{1}{2}$ degree of stuff to the right of it) for both the part on the left side of the $\otimes$ and the part on the right side of the $\otimes$. In the first sum, $h^i$ on the left of the $\otimes$ satisfies the unstable condition on any sphere since $p^i$ with it can be used (via $ph = v - \eta v$) to make the exponent of $h$ small. Terms with $i > 1$ can use $p^2$ to bring the $h^{n+1}$ on the right side of $\otimes$ down to $h^{n-1}$; then $h^{n-1}v^j$ is defined on $S^{2n-1-qj}$. The $(i = 1)$-term is, mod $S^{2n-1-qj}$,

$$-\ell v^{\ell-1} h \otimes (v - \eta v) h^n v^j \equiv -\ell v^{\ell-1}(v - ph) h \otimes h^n v^j \equiv -\ell v^\ell h \otimes h^n v^j.$$

All terms in the second sum with $i > 1$ are defined on $S^{2n-1-qj}$. In the third sum, terms with $i > 1$ can use $p^2$ to bring $h^{n+1}$ down to $h^{n-1}$, so that it satisfies the unstable condition. Remaining $p$'s can be used to bring $h^i$ down to $h^2$, so that it satisfies the unstable condition. The term with $i = 1$ gives the second term in the lemma since $ph^{n+1} = vh^n - h^n v$. ■

The following corollary will be useful.

**Corollary 2.14.** *If $k + n \not\equiv 0 \mod p$, and $z = (v^k h^{n+p-1} \otimes h + pw_0)\iota_{2n+1}$ is a cycle with $w_0\iota_{2n+1}$ satisfying the unstable condition 2.2, then*

$$z = d(\tfrac{1}{k+n+p} v^{k+p-1} h^{n+1} \iota_{2n+1} + w\iota_{2n+1})$$

*with $w\iota_{2n+1}$ unstable.*

*Proof.* Lemmas 2.12 and 2.13 imply that the leading term is correct. The group $E_2^{2n+1+q(k+n+p)}(S^{2n+1})$ has order $p$, and by Lemma 2.12 and Theorem 2.8(4), both of these leading terms yield nonzero classes. Since the other terms $pw_0$ are $p$ times an unstable class, they cannot cancel the leading term, and the tail terms on the second representation must be an unstable boundary. ■

Our final preliminary is to introduce the spaces which are the factors in the decomposition of [10] of the $p$-localization of $SU(n)$ and its quotients as a product of $p-1$ spaces.

**Definition 2.15.** *If $1 \le N \le p-1$, let $X_j^i(N)$ denote the direct factor space of the $p$-localization of the space $SU(N+i(p-1)+1)/SU(N+j(p-1))$ which is built up from fibrations involving $p$-local spheres $S^{2N+1+kq}$ for $j \le k \le i$.*

The UNSS of $X_j^i(N)$ has $E_2$-term $\mathrm{Ext}_{\mathcal{U}}^{s,t}(P(BP_*(X_j^i(N))))$, where $P(BP_*(X_j^i(N)))$ has $BP_*$-basis $\{x_{2N+jq+1}, x_{2N+(j+1)q+1}, \dots, x_{2N+iq+1}\}$, with coaction induced from Lemma 2.4. Because of the sparseness results for spheres given in (2.7) and Theorem 2.8, the following is immediate.

**Proposition 2.16.** (1) *For $s=1$ or $2$, $v_1^{-1}\pi_{2k+1-s}(X_j^i(N)) \approx E_2^{s,2k+1}(X_j^i(N))$, and is $0$ unless $k \equiv N \bmod p-1$.*

(2) *In the notation of Theorems 1.5 and 1.9,*

$$v_1^{-1}\pi_{2k+1-s}(SU(n)) \approx v_1^{-1}\pi_{2k+1-s}(X_0^i(N)).$$

Thus our efforts in the remainder of the paper will be to show that for $s=1$ or 2, $E_2^{s,2N+1+qm}(X_0^i(N))$ is as stated in Theorems 1.5 and 1.9. The fibrations whose exact sequence in $E_2(-)$ will be studied are of the form

$$X_k^j(N) \to X_k^i(N) \to X_{j+1}^i(N)$$

for $k \le j < i$.

## 3. The cellular spectral sequence

In this section, we sketch the proof of Theorem 1.5 by showing (modulo details postponed until Section 4) that $E_2^{1,2N+1+qm}(X_0^i(N)) \approx \mathbf{Z}/p^e$, where $e$ is as in that theorem. Here $X_0^i(N)$ is the space introduced at the end of the previous section, and all notation of Theorem 1.5 is in effect; in particular, $N < p$ and $i < p$.

Throughout the remainder of the paper, we consider $N$ and $m$ to be fixed, so that they may be omitted from some notation. The expression $E_2^s(-)$ will always mean $E_2^{s,2N+1+qm}(-)$, and we let $X_j^i = X_j^i(N)$. Note that $X_j^j = S^{2N+1+qj}$. Our goal is to determine $E_2^s(X_0^i)$ for $s = 1$ and 2.

We can organize this computation of $E_2^s(X_0^i)$ as a (cellular) spectral sequence with $\mathcal{E}_1^{s,j} = E_2^s(X_j^j)$ for $1 \le s \le 2$ and $0 \le j \le i$, and $d_r : \mathcal{E}_r^{1,j} \to \mathcal{E}_r^{2,j-r}$ induced by pulling the class in $E_2^1(X_j^j)$ back to $E_2^1(X_{j-r+1}^j)$ and then applying $\xrightarrow{\partial} E_2^2(X_{j-r}^{j-r})$. Our desired $E_2^s(X_0^i)$ is filtered with subquotients $\mathcal{E}_\infty^{s,j}$, $0 \le j \le i$. For $s = 1$, we know $E_2^1(X_0^i)$ is cyclic by Theorem 1.11, and so we just need to compute the sum

$$\nu(|E_2^1(X_0^i)|) = \sum_{j=0}^{i} \nu(|\mathcal{E}_\infty^{1,j}|). \tag{3.1}$$

Note that for $1 \le s \le 2$

$$\mathcal{E}_1^{s,j} = \begin{cases} \mathbf{Z}/p & \text{if } j \ne \hat{m} \\ \mathbf{Z}/p^{\min(N+(p-1)j,\nu(m-j)+1)} & \text{if } j = \hat{m}. \end{cases}$$

Thus every differential involves at least one group which is $\mathbf{Z}/p$ or 0, and so it suffices to tell which differentials are nonzero.

The precise form of this spectral sequence falls into three different patterns, determined by (1) $\hat{m} = 0$, (2) $N < \hat{m}$, and (3) $N \ge \hat{m} > 0$. We begin by calculating the cellular spectral sequence when $\hat{m} = 0$, this being the easiest of the three cases. It is easiest to describe the spectral sequence when $i = p - 1$, the maximal value that we are considering, and then to obtain the spectral sequence for smaller values of $i$ by restriction.

**Theorem 3.2.** *If $\hat{m} = 0$, then in the cellular spectral sequence for $X_0^{p-1}$,*

$$\mathcal{E}_\infty^{s,j} = \mathcal{E}_1^{s,j} = \begin{cases} \mathbf{Z}/p^{\min(N,\nu(m)+1)} & s = 1,\ j = 0 \\ \mathbf{Z}/p & s = 1,\ 1 \le j < N - \nu(m) \\ \mathbf{Z}/p & s = 1,\ N < j \le p - 1 \\ \mathbf{Z}/p & s = 2,\ 1 \le j \le p - 1. \end{cases}$$

*If $\max(1, N - \nu(m)) \le j \le N$, then $\mathbf{Z}/p \approx \mathcal{E}_j^{1,j} \xrightarrow{d_j} \mathcal{E}_j^{2,0}$ is nonzero. Hence $\mathcal{E}_\infty^{2,0} = 0$, and $\mathcal{E}_\infty^{1,j} = 0$ if $\max(1, N - \nu(m)) \le j \le N$.*

If $i < p - 1$, then $\mathcal{E}_\infty^{1,j}(X_0^i)$ will equal the group described in Theorem 3.2 if $j \le i$, and will equal 0 if $j > i$. The same holds for $\mathcal{E}_\infty^{2,j}(X_0^i)$ if $j > 0$ or $i \ge N$; if $i < N$,

TABLE 1. Spectral sequence when $p = 11$, $N = 6$, $\hat{m} = 0$

| | | $j$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\nu(m)$ | 1 | 2 | 1 | 1 | 1 | 1 | $\not 1$ | $\not 1$ | 1 | 1 | 1 | 1 |
| | 2 | 3 | 1 | 1 | 1 | $\not 1$ | $\not 1$ | $\not 1$ | 1 | 1 | 1 | 1 |
| | 3 | 4 | 1 | 1 | $\not 1$ | $\not 1$ | $\not 1$ | $\not 1$ | 1 | 1 | 1 | 1 |
| | 4 | 5 | 1 | $\not 1$ | $\not 1$ | $\not 1$ | $\not 1$ | $\not 1$ | 1 | 1 | 1 | 1 |
| | $\geq 5$ | 6 | $\not 1$ | $\not 1$ | $\not 1$ | $\not 1$ | $\not 1$ | $\not 1$ | 1 | 1 | 1 | 1 |

then $\mathcal{E}_\infty^{2,0}(X_0^i) = \mathbf{Z}/p^{\min(N-i,\nu(m)+1)}$. The reader can easily verify that the sum (3.1) when applied to 3.2 yields the desired result

$$\nu(|E_2^1(X_0^i)|) = \begin{cases} i & \text{if } N \leq i \\ \min(N, i+\nu(m)+1) & \text{if } i < N \end{cases}$$

for the case $\hat{m} = 0$.

The proof of Theorem 3.2 will be given in Section 4. In Table 1, we illustrate the spectral sequence in the case $p = 11$, $N = 6$. For each value of $j$ from 0 to $p-1$, we list

$$\nu(|\mathcal{E}_1^{1,j}|) = \nu(|E_2^{1,13+20m}(S^{13+20j})|).$$

A slash (/) through a number indicates that it supports a differential. In this case ($\hat{m} = 0$), all differentials hit into $\mathcal{E}^{2,0}$.

Next we describe the cellular spectral sequence when $N < \hat{m}$.

**Proposition 3.3.** *Let $\nu = \nu(m - \hat{m})$. If $N < \hat{m}$, then in the spectral sequence for $X_0^i$,*

$$\mathcal{E}_\infty^{1,j} = \begin{cases} \mathbf{Z}/p & \text{if } 0 \leq j < N \\ 0 & \text{if } j = N \\ \mathbf{Z}/p & \text{if } N < j < \hat{m} \\ \mathbf{Z}/p^{\min(\hat{m}(p-2)+N+2,\nu+1)} & \text{if } j = \hat{m} \\ \mathbf{Z}/p & \text{if } \hat{m} < j \leq \hat{m}(p-1)+N+1-\nu \\ 0 & \text{if } j > \hat{m} \text{ and } j > \hat{m}(p-1)+N+1-\nu, \end{cases}$$

*provided $j \leq i$.*

It is then an easy exercise to verify that the sum (3.1) has the desired value

$$\begin{cases} i+1 & \text{if } i<N \\ i & \text{if } N\le i<\hat m \\ \min(\hat m(p-1)+N+1, i+\nu(m-\hat m)) & \text{if } \hat m\le i. \end{cases}$$

Thus Theorem 1.5 in the case $N<\hat m$ will be proved once we have proved Proposition 3.3.

In Table 2, we illustrate the spectral sequence in a particular case, namely $p=11$, $N=2$, and $\hat m=7$. Let $\nu=\nu(m-\hat m)=\nu(m-7)$. For each value of $\nu$ and each $j$ from 0 to 10, we list

$$\nu(|\mathcal{E}_1^{1,j}|)=\nu(|E_2^{1,5+20m}(S^{5+20j})|).$$

Directly beneath this number, we list a number $k$ if there is a nonzero differential $\mathcal{E}^{1,j}\to\mathcal{E}^{2,k}$. Thus for example if $\nu=68$, there are nonzero differentials

$$\mathcal{E}_2^{1,2}\to\mathcal{E}_2^{2,0},\quad \mathcal{E}_4^{1,7}\to\mathcal{E}_4^{2,3},\quad \mathcal{E}_6^{1,7}\to\mathcal{E}_6^{2,1},\quad \mathcal{E}_4^{1,8}\to\mathcal{E}_4^{2,4},\quad \mathcal{E}_4^{1,9}\to\mathcal{E}_4^{2,5},\quad \mathcal{E}_4^{1,10}\to\mathcal{E}_4^{2,6}.$$

In this case, we have

$$\mathcal{E}_r^{1,7}\approx\begin{cases} \mathbf{Z}/p^{69} & \text{if } 1\le r\le 4 \\ \mathbf{Z}/p^{68} & \text{if } 5\le r\le 6 \\ \mathbf{Z}/p^{67} & \text{if } 7\le r\le\infty. \end{cases}$$

Proposition 3.3 is an immediate consequence of the following result, which describes the nonzero differentials in the spectral sequence when $N<\hat m$. In Section 4 we will prove this result and its analogues in the other cases. We will frequently omit writing the subscript of the differential and the $\mathcal{E}$-groups. It just equals the difference of the second superscripts.

**Theorem 3.4.** *If $N<\hat m$, the nonzero differentials in the spectral sequence converging to $E_2(X_0^i)$ are those described below on groups $\mathcal{E}^{1,j}$ with $j\le i$.*

1. $d_N\ne 0:\mathcal{E}_N^{1,N}\to\mathcal{E}_N^{2,0}$.
2. *For $t>1$, there is a nonzero differential from $x_{\hat m(p-2)+N+1+t}$ to $z_t$, where*

$$x_k = \begin{cases} \text{element of order } p^k \text{ in } \mathcal{E}_1^{1,\hat m} & \text{if } k\le\nu(|\mathcal{E}_1^{1,\hat m}|) \\ \text{generator of } \mathcal{E}_1^{1,\hat m+\Delta} & \text{if } \Delta=k-\nu(|\mathcal{E}_1^{1,\hat m}|)>0 \end{cases}$$

$$z_t = \begin{cases} \text{generator of } \mathcal{E}_1^{2,t-1} & \text{if } 1<t\le N \\ \text{generator of } \mathcal{E}_1^{2,t} & \text{if } N<t<\hat m \\ \text{element of order } p^{t-\hat m+1} \text{ in } \mathcal{E}_1^{2,\hat m} & \text{if } \hat m\le t \end{cases}$$

TABLE 2. Spectral sequence when $p = 11$, $N = 2$, $\hat{m} = 7$

| | $j$ = 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\nu \leq 63$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\nu + 1$ | 1 | 1 | 1 |
| | | | 0 | | | | | | | | |
| $\nu = 64$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 65 | 1 | 1 | 1 |
| | | | 0 | | | | | | | | 1 |
| 65 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 66 | 1 | 1 | 1 |
| | | | 0 | | | | | | | 1 | 3 |
| 66 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 67 | 1 | 1 | 1 |
| | | | 0 | | | | | | 1 | 3 | 4 |
| 67 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 68 | 1 | 1 | 1 |
| | | | 0 | | | | | 1 | 3 | 4 | 5 |
| 68 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 69 | 1 | 1 | 1 |
| | | | 0 | | | | | 1,3 | 4 | 5 | 6 |
| 69 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 70 | 1 | 1 | 1 |
| | | | 0 | | | | | 1,3,4 | 5 | 6 | 7 |
| 70 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 71 | 1 | 1 | 1 |
| | | | 0 | | | | | 1,3,4,5 | 6 | 7 | 7 |
| $\nu \geq 71$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 72 | 1 | 1 | 1 |
| | | | 0 | | | | | 1,3-6 | 7 | 7 | 7 |

We take a little liberty with notation here; although the elements $x_k$ and $z_t$ are defined as elements of $\mathcal{E}_1$, the differentials generally involve the projections of these elements in $\mathcal{E}_r$ with $r > 1$.

Next we present the analogues of Proposition 3.3 and Theorem 3.4 when $N \geq \hat{m} > 0$. The condition

$$m \equiv \hat{m} - (-1)^{\hat{m}} \hat{m} \tbinom{N}{\hat{m}} p^{\hat{m}(p-1)} \mod p^{\hat{m}(p-1)+1} \tag{3.5}$$

will be important here.

**Proposition 3.6.** *If $N \geq \hat{m} > 0$, then the nonzero groups $\mathcal{E}_\infty^{1,j}$ in the spectral sequence converging to $E_2^1(X_0^i)$ are the following groups which also satisfy $j \leq i$. Here $\nu = \nu(m - \hat{m})$.*

$$\begin{cases} \mathbf{Z}/p & j < \hat{m} \\ \mathbf{Z}/p^{\min(\nu+1, N+\hat{m}(p-2))} & j = \hat{m} \\ \mathbf{Z}/p & \nu \leq \hat{m}(p-1) \text{ and } \hat{m} < j < N \\ \mathbf{Z}/p & \hat{m}(p-1) < \nu \leq \hat{m}(p-2) + N - 2 \\ & \quad \text{and } \hat{m} < j < \hat{m}(p-1) + N - \nu \\ \mathbf{Z}/p & j = N+1 \text{ and not } (3.5) \\ \mathbf{Z}/p & j = N \text{ and } (3.5) \\ \mathbf{Z}/p & \nu < \hat{m}(p-1) \text{ and } N < j \leq \hat{m}(p-1) + N + 1 - \nu \end{cases}$$

As in the previous two cases, one can easily show that the sum (3.1) when applied to the groups of 3.6 yields the desired result, Theorem 1.5, when $N \geq \hat{m} > 0$. Also as in the previous case, Proposition 3.6 is easily derived from a listing of the differentials in the cellular spectral sequence, which we now describe. The proof of the following result will be given in Section 4.

**Theorem 3.7.** *If $N \geq \hat{m} > 0$, the nonzero differentials in the spectral sequence converging to $E_2(X_0^i)$ are those described below on groups $\mathcal{E}^{1,j}$ with $j \leq i$.*

(1) *If $\nu(m - \hat{m}) < \hat{m}(p-1)$, then $d_N \neq 0 : \mathcal{E}_N^{1,N} \to \mathcal{E}_N^{2,0}$.*
(2) *If (3.5) is satisfied, then $d_{N+1} \neq 0 : \mathcal{E}_{N+1}^{1,N+1} \to \mathcal{E}_{N+1}^{2,0}$.*
(3) *If $t \geq 1$ or if $t = 0$ and $\nu(m - \hat{m}) \geq \hat{m}(p-1)$ and (3.5) is not satisfied, then there is a nonzero differential from $x_{\hat{m}(p-2)+N+1+t}$ to $z_t$, where*

Table 3. Spectral sequence when $p = 11$, $N = 7$, $\hat{m} = 3$

| | | | $j$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\nu \le 28$ | 1 | 1 | 1 | $\nu+1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | | | 0 | | | |
| $\nu = 29$ | 1 | 1 | 1 | 30 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | | | 0 | | | 1 |
| $m \equiv 3 + 6p^{30} \mod p^{31}$ | 1 | 1 | 1 | 31 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | | | | 0 | 1 | 2 |
| $\nu = 30,\ m \not\equiv 3 + 6p^{30}$ | 1 | 1 | 1 | 31 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | | | 0 | | 1 | 2 |
| $\nu = 31$ | 1 | 1 | 1 | 32 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | | 0 | 1 | | 2 | 3 |
| $\nu = 32$ | 1 | 1 | 1 | 33 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | 0 | 1 | 2 | | 3 | 3 |
| $\nu = 33$ | 1 | 1 | 1 | 34 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | 0 | 1 | 2 | 3 | | 3 | 3 |
| $\nu = 34$ | 1 | 1 | 1 | 35 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | 0 | 1 | 2 | 3 | 3 | | 3 | 3 |
| $\nu = 35$ | 1 | 1 | 1 | 36 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | 0,1 | 2 | 3 | 3 | 3 | | 3 | 3 |
| $\nu \ge 36$ | 1 | 1 | 1 | 37 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | 0,1,2 | 3 | 3 | 3 | 3 | | 3 | 3 |

$$x_k = \begin{cases} \textit{elt of order } p^k \textit{ in } \mathcal{E}_1^{1,\hat{m}} & \textit{if } k \le \nu(|\mathcal{E}_1^{1,\hat{m}}|) \\ \textit{gen of } \mathcal{E}_1^{1,\hat{m}+\Delta} & \textit{if } \Delta = k - \nu(|\mathcal{E}_1^{1,\hat{m}}|) > 0 \textit{ and } \hat{m} + \Delta \le N \\ \textit{gen of } \mathcal{E}_1^{1,\hat{m}+\Delta+1} & \textit{if } \Delta = k - \nu(|\mathcal{E}_1^{1,\hat{m}}|) > 0 \textit{ and } \hat{m} + \Delta > N \end{cases}$$

$$z_t = \begin{cases} \textit{gen of } \mathcal{E}_1^{2,t} & \textit{if } t < \hat{m} \\ \textit{elt of order } p^{t-\hat{m}+1} \textit{ in } \mathcal{E}_1^{2,\hat{m}} & \textit{if } t \ge \hat{m} \end{cases}$$

We close this section by illustrating 3.6 and 3.7 when $p = 11$, $N = 7$, and $\hat{m} = 3$ in Table 3. The conventions are as in Table 2. In particular, if in column $j$, a number $k$ sits directly beneath a number $e$, then $\mathcal{E}_1^{1,j} \approx \mathbf{Z}/p^e$ and $d \neq 0 : \mathcal{E}^{1,j} \to \mathcal{E}^{2,k}$.

## 4. Determination of differentials in the cellular spectral sequence

In this section, we determine the differentials in the cellular spectral sequence, proving Theorems 3.2, 3.4, and 3.7, and thereby implying Theorem 1.5. We continue with the notational conventions of Section 3. In particular, $N$ and $m$ are fixed, and $E_2^s(-)$ means $E_2^{s,2N+1+qm}(-)$.

We will also take a number of liberties with notation, which we now explain. First, we omit $v_1$-powers on the left of most terms. All of our terms have degree $2N+1+qm$, and so the number of $v_1$'s can be determined. This number of $v_1$'s does not affect most formulas significantly, although it is important in applications of Lemma 2.13. Second, we frequently omit units in $\mathbf{Z}_{(p)}$ which appear as coefficients in expressions. There will be some cases, giving rise to (3.5), where a comparison of two terms will be essential, and then we will of course keep track of coefficients, but usually we only need to know whether or not they are zero mod $p$. Third, we write $x \equiv y \mod L$ to mean that $x - y$ desuspends to (or is defined on) lower spheres. "Lower than what" should be clear from the context. Sometimes we will omit writing "mod $L$." Finally, we will often write $y_j$ for $x_{2N+1+qj}$ in $BP_*(X_k^i)$ and for $\iota_{2N+1+qj}$ in $S^{2N+1+qj} = X_j^j$.

We begin with some lemmas, the first of which is an elementary fact involving binomial coefficients.

**Lemma 4.1.** *If $0 < a < p$, and $0 \le b, c < p$, then*

$$\nu\left(\binom{ap+b}{c}\right) = \begin{cases} 1 & \text{if } b < c \\ 0 & \text{if } b \ge c. \end{cases}$$

The following lemma is a basic technique that will be used often. We deal here and throughout this section with projection maps $X_a^j \to X_j^j$.

**Lemma 4.2.** *Suppose that (1) $0 \le a \le j < p$, (2) either $\hat{m} < a$ or $\hat{m} > j$, and (3) if $a = 0$, then $j < N$. Thinking of $j$ and $N$ as being fixed, let*

$$\epsilon_\ell = \begin{cases} 1 & \text{if } \ell \le N < j \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

*Then there is an element $z \in E_2^1(X_a^j)$ satisfying*

$$z \equiv \sum_{\ell=a}^{j} h^{j+1-\ell-\epsilon_\ell} y_\ell \mod L \tag{4.4}$$

*which projects to a generator of $E_2^1(X_j^j) \approx \mathbf{Z}/p$. Hence $d_r = 0 : \mathcal{E}_r^{1,j} \to \mathcal{E}_r^{2,j-r}$ for $r \le j - a$.*

*Proof.* The proof is by downward induction on $a$, and is true for $a = j$ by parts 1 and 2 of Theorem 2.8. Assume that it is true for a certain $a$ satisfying $a > N + 1$ if $N < j$, and also satisfying $a - 1 > \hat{m}$. Let $z_a$ denote the sum (4.4), which will have all $\epsilon_\ell = 0$. We will show below that

$$\partial : E_2^1(X_a^j) \to E_2^2(X_{a-1}^{a-1})$$

satisfies

$$\begin{aligned} \partial(z_a) &\equiv \sum_{\ell=a}^{j} \binom{N+(p-1)(a-1)}{\ell-a+1} h^{j+1-\ell} \otimes h^{\ell-a+1} y_{a-1} \\ &\equiv h \otimes h^{j-a+1} y_{a-1} \equiv d(h^{j-a+2} y_{a-1}) \mod L. \end{aligned} \tag{4.5}$$

Thus $z_a - h^{j-a+2} y_{a-1}$ is a cycle in $E_2^1(X_{a-1}^j)$, which extends the induction in this case $(a > N + 1)$. (Remember that we are not usually worrying about unit coefficients.)

Now we prove the first $\equiv$ in (4.5). By (2.6), Lemma 2.4, and Proposition 2.5, $\partial(h^{j+1-\ell} y_\ell) = h^{j+1-\ell} \otimes C y_{a-1}$, where $C$ is the component of

$$(1 + h - hv + hv^2 - \tbinom{p-1}{2} h^2 v)^{N+(a-1)(p-1)}$$

in degree $q(\ell - a + 1)$. We will see that terms of degree greater than $3q$ in the sum of 2.5 would not play an important role here. The term that appears in the sum of (4.5) is obtained from choosing $(\ell-a+1)$ $h$'s. If any other terms of 2.5 are chosen, the term will desuspend farther because of the $v$'s on the right. For terms in the sum of 2.5 of degree greater than $3q$, only pure powers of $h_1$ could yield terms which desuspend as far as the one already considered, and such terms were shown in Proposition 2.5 not to exist. Note that a term such as $h_2$ occurring in 2.5 desuspends no farther than $h_1$, but it uses up many more degrees. If $L y_\ell$ is a term that desuspends lower than $h^{j+1-\ell} y_\ell$, then $L$ is effectively $h^t$ for some $t < j + 1 - \ell$, and so $\partial(L y_\ell)$ will desuspend farther than $h^{j+1-\ell} \otimes h^{\ell-a+1}$.

The second $\equiv$ of (4.5) is true because the unstable condition easily implies that other terms desuspend farther than the $(\ell = j)$-term. Also we need that the binomial coefficient is a unit. This follows from Lemma 4.1 and the hypothesis that we do not have $a - 1 \le N < j$. The third $\equiv$ is true by Lemma 2.13, in which the coefficient of the first term is

$$\tfrac{1}{q}((2N + 1 + qm) - (2N + 1 + q(a - 1))) = m - (a - 1), \tag{4.6}$$

which is not a multiple of $p$, since $a - 1 > \hat{m}$.

The proof is similar when $a-1 \le N < j$, except that $\binom{N+(p-1)(a-1)}{j-a+1}$ is now $p$ times a unit, by Lemma 4.1. Thus the last terms of (4.5) become $ph \otimes h^{j-a+1}y_{a-1} \equiv h \otimes h^{j-a}y_{a-1} \equiv d(h^{j-a+1}y_{a-1})$, where we have used $ph = v - \eta(v)$ to "cancel" the $ph$, and then argued as before.

The condition that $j+1 \le N$ if $a = 0$ is required in order that all terms satisfy the unstable condition, Definition 2.2. The term $h^{j+1-a}y_a$ is closest to failing. It requires $j+1-a \le N + a(p-1)$, which is satisfied if $a > 0$ or $j+1 \le N$.

The statement about $d_r$'s being 0 follows from the definition of the spectral sequence. If an element in $E_2^1(X_j^j)$ pulls back to $E_2^1(X_a^j)$, then it survives to $\mathcal{E}_{j-a}^{1,j}$. ■

Lemma 4.2 must be modified as follows when $a = \hat{m}$.

**Lemma 4.7.** *Define $\nu = \nu(m-\hat{m})$ and $\epsilon_\ell$ as in* (4.3)*. Suppose $\hat{m} < j$ and*

$$j + 1 + \nu - \epsilon_{\hat{m}} \le N + \hat{m}p. \tag{4.8}$$

*Then, under the projection map,*

$$h^{j+1-\hat{m}-\epsilon_{\hat{m}}+\nu}y_{\hat{m}} + \sum_{\ell=\hat{m}+1}^{j} h^{j+1-\ell-\epsilon_\ell}y_\ell \in E_2^1(X_{\hat{m}}^j)$$

*maps to a generator of $E_2(X_j^j) \approx \mathbf{Z}/p$. Hence $d_r = 0 : \mathcal{E}_r^{1,j} \to \mathcal{E}_r^{1,j-r}$ for $r \le j - \hat{m}$.*

The statement of this lemma initiates another abuse of notation which we will allow ourselves. The class here is really only a cycle mod $L$. We mean here only the sort of thing that was stated explicitly in Lemma 4.2.

*Proof.* The generator of $E_2^1(X_j^j)$ pulls back to $z \in E_2^1(X_{\hat{m}+1}^j)$ as in Lemma 4.2, with

$$\partial(z) \equiv \binom{N+(p-1)\hat{m}}{j-\hat{m}} h \otimes h^{j-\hat{m}}y_{\hat{m}} \equiv h \otimes h^{j-\hat{m}-\epsilon_{\hat{m}}}y_{\hat{m}}.$$

When we try to apply Lemma 2.13 to write this as a boundary, the first coefficient is $m - \hat{m} = sp^\nu$, where $s$ is a unit is $\mathbf{Z}_{(p)}$. We don't worry about units, but the $p^\nu$ is missing from our term. To accommodate this, we note that, with $e = j - \hat{m} - \epsilon_{\hat{m}}$ and $t + 1 + e = m - \hat{m}$,

$$v^t h \otimes h^e \equiv v^{t-\nu} h \otimes v^\nu h^e \equiv v^{t-\nu} h \otimes p^\nu h^{e+\nu}$$

mod $L$. Here the first $\equiv$ follows from the remarks after Theorem 2.8, and the second follows from expanding $p^\nu h^\nu = (v-\eta(v))^\nu$, noting that terms with $\eta(v)$'s will desuspend farther than other terms. We obtain

$$\partial(z) \equiv p^\nu h \otimes h^{j-\hat m-\epsilon_{\hat m}+\nu} y_{\hat m} \equiv d(h^{j+1-\hat m-\epsilon_{\hat m}+\nu} y_{\hat m}),$$

which satisfies the unstable condition by (4.8). The argument is completed as in Lemma 4.2. ■

A similar lemma tells how classes in $\mathcal{E}^{1,\hat m}$ can be pulled back.

**Lemma 4.9.** *Let $\epsilon_\ell = 1$ if $\ell \le N < \hat m$, and $\epsilon_\ell = 0$ otherwise. Suppose $0 \le a \le \hat m$ and that $t - \epsilon_a \le N + a$. Then $h^{\hat m(p-2)+t} y_{\hat m} \in E_2^1(X_{\hat m}^{\hat m})$ pulls back, mod $L$, to a cycle $\sum_{\ell=a}^{\hat m} h^{\ell(p-2)+t-\epsilon_\ell} y_\ell \in E_2^1(X_a^{\hat m})$. Also, $d_r = 0 : \mathcal{E}_r^{1,\hat m} \to \mathcal{E}_r^{1,\hat m - r}$ for $r \le \hat m - a$.*

*Proof.* We assume that the formula is true for $a$ (with class $z_a$), and will deduce it for $a-1$. We assume at first that it is not true that $a-1 \le N < \hat m$, so that all $\epsilon$'s are 0. We claim

$$\begin{aligned} \partial(z_a) &\equiv \sum_{\ell=a}^{\hat m} \tbinom{N+(a-1)(p-1)}{\ell-a+1} h^{\ell(p-2)+t} \otimes h^{\ell-a+1} y_{a-1} \\ &\equiv h^{a(p-2)+t} \otimes h y_{a-1} \equiv h \otimes h^{(a-1)(p-2)+t-1} y_{a-1} \equiv d(h^{(a-1)(p-2)+t} y_{a-1}). \end{aligned} \tag{4.10}$$

Then $z_{a-1} \equiv z_a - h^{(a-1)(p-2)+t} y_{a-1}$ is a cycle, extending the induction. The claim about $d_r = 0$ is immediate, as in the proof of Lemma 4.2.

The first $\equiv$ in (4.10) follows by the argument used for the first $\equiv$ of (4.5). The second $\equiv$ says that the term with $\ell = a$ desuspends least far of all terms, and has coefficient a unit mod $p$. The coefficient is a unit since $a-1 \ne N$. The measure of the size of the smallest sphere on which $h^i \otimes h^j$ is defined is

$$\operatorname{exc}(h^i \otimes h^j) = \min(j, i-(p-1)j).$$

For the terms in the sum of (4.10), this is largest when $\ell = a$. The third $\equiv$ follows from Lemma 2.12, and the fourth from Lemma 2.13, where the first coefficient is, similarly to (4.6), $m-(a-1) \not\equiv 0 \mod p$.

This completes the proof of the lemma as long as it is not true that $a-1 \le N < \hat m$. If $a-1 = N$, then the binomial coefficient $\binom{N+(a-1)(p-1)}{1}$ is $p$ times a unit, and this $p$ can be used to cancel one $h$, as in the proof of Lemma 4.2; this is accommodated by

$\epsilon = 1$. If $a-1<N$, then the $t$ in the exponent has already been decreased to $t-1$, and subsequent steps will leave it there.

The condition $t-\epsilon_a \le N+a$ is necessary in order that all terms satisfy the unstable condition. ■

Now we are ready to establish the differentials claimed in Theorems 3.2, 3.4, and 3.7. The reader is encouraged to refer to Table 1 when considering this proof of Theorem 3.2.

*Proof of Theorem* 3.2. We emphasize that this is the case $\hat m = 0$. Let $1 \le j \le p-1$, $\nu = \nu(m)$, and $\epsilon_\ell$ be as in (4.3). By Lemma 4.2, a generator of $E_2^1(X_j^j)$ pulls back to $z_1 \equiv \sum_{\ell=1}^{j} h^{j+1-\ell-\epsilon_\ell} y_\ell$, and $d_r = 0 : \mathcal{E}_r^{1,j} \to \mathcal{E}_r^{2,r-j}$ for $r<j$. By Lemma 4.7, $d_j = 0 : \mathcal{E}_j^{1,j} \to \mathcal{E}_j^{2,0}$ if $j < N-\nu$. If $\max(1, N-\nu) \le j \le N$, then, as in the proof of Lemma 4.7, $\partial(z_1) \equiv h \otimes h^j y_0$, and by Theorem 2.8, this is an element of order $p^{N+1-j}$ in $\mathcal{E}_1^{2,0} = E_2^2(S^{2N+1})$. Thus $\mathcal{E}_{N+1}^{2,0} = 0$, completing the proof. ■

The reader is encouraged to refer to Table 2 in the proof of Theorem 3.4 which follows.

*Proof of Theorem* 3.4. If $j<N$, then $d_r = 0$ on $\mathcal{E}_r^{1,j}$ for all $r$ by Lemma 4.2. Also by Lemma 4.2, we have $\mathcal{E}_j^{1,j} = \mathcal{E}_1^{1,j}$ if $N \le j < \hat m$. We will show that

$$d_N \neq 0 : \mathcal{E}_N^{1,N} \to \mathcal{E}_N^{2,0}. \tag{4.11}$$

Then $\mathcal{E}_{N+1}^{2,0} = 0$ and hence $d_j$ is 0 on $\mathcal{E}_j^{1,j}$ when $j>N$, since there is nothing for it to hit. To prove (4.11), we use Lemma 4.2 to pull the generator of $\mathcal{E}_1^{1,N}$ back to

$$z_1 \equiv \sum_{\ell=1}^{N} h^{N+1-\ell} y_\ell \in E_2^1(X_1^N),$$

which satisfies $\partial(z_1) \equiv h \otimes h^N y_0$ by part of (4.5). This is nonzero by Lemmas 2.10 and 2.11.

Next we consider the differentials from $\mathcal{E}^{1,\hat m}$. We first show that elements $x = x_{\hat m(p-2)+N+1+t}$ of order $p^{\hat m(p-2)+N+1+t}$ with $N<t<\hat m$ support a nonzero differential to $\mathcal{E}^{2,t}$. By Theorem 2.8(2), $x$ is represented, mod $L$, by $h^{\hat m(p-2)+N+1+t} y_{\hat m} \in E_2^1(S^{2N+1+q\hat m})$. By Lemma 4.9, $x$ can be extended, mod $L$, to

$$x_{t+1} = \sum_{\ell=t+1}^{\hat m} h^{\ell(p-2)+N+1+t} y_\ell \in E_2^1(X_{t+1}^{\hat m}),$$

which implies that differentials from $x$ into $\mathcal{E}^{2,j}$ with $j \ge t+1$ are 0. Then

$$(4.12)\quad \partial(x_{t+1}) \equiv (N+(p-1)t)h^{(t+1)(p-2)+N+1+t} \otimes hy_t \not\equiv 0 \in E_2^2(S^{2N+1+tq}).$$

The first $\equiv$ here is obtained similarly to the first two steps of (4.10), while the last step is from Lemma 2.12.

We have now completed the proof of the differential from $x_{\hat{m}(p-2)+N+1+t}$ to $z_t$ asserted in Theorem 3.4 when $t > N$ and $\hat{m}(p-2)+N+1+t \le \nu(|\mathcal{E}_1^{1,\hat{m}}|)$. If instead we have $t \le N$, but still $\hat{m}(p-2)+N+1+t \le \nu(|\mathcal{E}_1^{1,\hat{m}}|)$, then in Lemma 4.9 $\epsilon_\ell = 1$ for $\ell \le N$, and so $x$ extends to

$$x' \equiv \sum_{\ell=N+1}^{\hat{m}} h^{\ell(p-2)+N+1+t} y_\ell + \sum_{\ell=\max(t,1)}^{N} h^{\ell(p-2)+N+t} y_\ell.$$

If $t \le 1$, this shows that $d_r(x) = 0$ for $r < \hat{m}$, and hence $x$ is a permanent cycle, since $d_{\hat{m}}^{2,0} = 0$. If $t > 1$, then, by part of (4.10), $\partial(x') \equiv h^{t(p-2)+N+t} \otimes hy_{t-1}$, which is nonzero by Lemma 2.12.

This completes the analysis of the differentials of Theorem 3.4 on $\mathcal{E}_r^{1,j}$ with $j \le \hat{m}$. Now we will establish the differential on $x = x_{\hat{m}(p-2)+N+1+t}$ when

$$(4.13)\qquad \hat{m}(p-2)+N+1+t > \nu(|\mathcal{E}_1^{1,\hat{m}}|).$$

Let $e = \nu(|\mathcal{E}_1^{1,\hat{m}}|)$. Then $x \equiv hy_{\hat{m}(p-1)+N+1+t-e}$, and can be extended by Lemma 4.2 to

$$(4.14)\qquad x' \equiv hy_{\hat{m}(p-1)+N+1+t-e} + \cdots + h^{\hat{m}(p-2)+N+1+t-e} y_{\hat{m}+1},$$

with

$$(4.15)\qquad \partial(x') \equiv h \otimes h^{\hat{m}(p-2)+N+1+t-e} y_{\hat{m}}.$$

By Corollary 2.9, this equals the element of order $p^{t-\hat{m}+1}$ on $S^{2N+1+q\hat{m}}$, establishing the claimed differential to $z_t$ when $t \ge \hat{m}$ (so that the order is greater than 1).

Now we assume $t < \hat{m}$. Let $\nu = \nu(m-\hat{m})$. Recall that

$$e = \nu(|\mathcal{E}_1^{1,\hat{m}}|) = \min(\nu+1, N+\hat{m}(p-1)).$$

Combining $t < \hat{m}$ with (4.13) yields $e < N+\hat{m}(p-1)$, and hence $e = \nu+1$. By Lemma 4.7, the class $x'$ of (4.14) extends to

$$x'' \equiv hy_{\hat{m}(p-1)+N+t-\nu} + \cdots + h^{\hat{m}(p-2)+N+t-\nu} y_{\hat{m}+1} + h^{\hat{m}(p-2)+N+t+1} y_{\hat{m}}.$$

By Lemma 4.9, $x''$ can be extended to $x'''$ satisfying

$$x''' \equiv \begin{cases} x'' + \sum_{\ell=t+1}^{\hat m-1} h^{\ell(p-2)+N+t+1} y_\ell & \text{if } t > N \\ x'' + \sum_{\ell=N+1}^{\hat m-1} h^{\ell(p-2)+N+t+1} y_\ell + \sum_{\ell=t}^{N} h^{\ell(p-2)+N+t} y_\ell & \text{if } t \le N. \end{cases}$$

Then

$$\partial(x''') \equiv \begin{cases} h^{(t+1)(p-1)+N} \otimes h y_t & \text{if } t > N \\ h^{t(p-1)+N} \otimes h y_{t-1} & \text{if } t \le N. \end{cases}$$

These are nonzero by Lemma 2.12, establishing the final differentials of Theorem 3.4, namely those from the second type of $x_k$ to the first two types of $z_t$. ■

The proof of Theorem 3.7 proceeds similarly, except that in a certain case we must keep track of unit coefficients because two terms are trying to have cancelling differentials. It is recommended that the reader consult with Table 3 while studying this proof.

*Proof of Theorem* 3.7. We divide into cases determined by the value of $j$, where the differential emanates from $\mathcal{E}^{1,j}$.

**Case 1:** $j \le \hat m$**.** If $j < \hat m$, then all differentials on $\mathcal{E}^{1,j}$ are 0 by Lemma 4.2. For $j = \hat m$, we wish to show the differential asserted in 3.7 from $x_{\hat m(p-2)+N+1+t}$ to the generator of $\mathcal{E}_1^{2,t}$. Here we must have $t < \hat m$ in order that $\mathcal{E}_1^{1,\hat m}$ has an element of the asserted order. By Lemma 4.9 and (4.10), this $x_{\hat m(p-2)+N+1+t}$ pulls back to

$$x' \equiv h^{\hat m(p-2)+N+1+t} y_{\hat m} + \cdots + h^{(t+1)(p-2)+N+1+t} y_{t+1},$$

which has $\partial(x') \equiv h^{N+(t+1)(p-1)} \otimes h y_t \not\equiv 0$, by Lemma 2.12.

**Case 2:** $\nu < \hat m(p-1)$ **and** $\hat m < j \le N$**.** Here and throughout the remainder of this proof, $\nu = \nu(m - \hat m)$. The claim here is that $\mathcal{E}_1^{1,j}$ consists of permanent cycles if $j < N$, while $d_N \ne 0 : \mathcal{E}_N^{1,N} \to \mathcal{E}_N^{2,0}$. To establish this, we begin by using Lemma 4.7 to pull the generator of $\mathcal{E}_1^{1,j}$ back to

$$x_0 \equiv h y_j + \cdots + h^{j-\hat m} y_{\hat m+1} + h^{j-\hat m+\nu+1} y_{\hat m}. \tag{4.16}$$

Let $k$ be the integer satisfying $k(p-1) \le \nu < (k+1)(p-1)$. Note that $k$ satisfies $0 \le k < \hat m$. Next we show that $x_0$ pulls back to a cycle $x'$ in $E_2^1(X_1^j)$ of the form

$$x' \equiv x_0 + \sum_{\ell=\hat m-k}^{\hat m-1} h^{j-\hat m+\nu+1-(\hat m-\ell)(p-2)} y_\ell + \sum_{\ell=1}^{\hat m-k-1} h^{j+1-\ell} y_\ell. \tag{4.17}$$

(Actually, as will be discussed in the third bullet below, it is possible that one of the terms in this sum is incorrect, but this turns out to be inconsequential.)

To obtain (4.17), we note that if $x_0$ has been pulled back to $x'' \in E_2^1(X^j_{\ell+1})$ with $\ell+1 \ge \hat m - k$, with terms like those in (4.16) and the first sum of (4.17), then $\partial(x'')$ can be evaluated by applying (4.5) to the terms of (4.16), and (4.10) to the terms of (4.17). The leading terms will be $h\otimes h^{j-\ell}y_\ell$ and $h\otimes h^{j-\hat m+\nu-(\hat m-\ell)(p-2)}y_\ell$, respectively. Now there are three possibilities.

- If $\ell \ge \hat m - k$ and it is not the case that $\ell = \hat m - k$ and $\nu = k(p-1)$, then $j-\hat m+\nu-(\hat m-\ell)(p-2) > j-\ell$, and so the leading term of $\partial(x'')$ will be the second of those listed above. Thus, mod $L$, the extension of $x''$ to $E_2^1(X^j_\ell)$ is obtained by extending the first sum of (4.17).
- If $\ell = \hat m - k - 1$, then the leading term of $\partial(x'')$ will be the first of those listed above. Thus, mod $L$, the extension of $x''$ to $E_2^1(X^j_\ell)$ contains the top term in the last sum of (4.17). From this point on, subsequent extensions will be determined as they were in (4.5), by the very top term, $hy_j$.
- If $\nu = k(p-1)$ and $\ell = \hat m - k$, these two leading terms will be equal. Actually, they may have different units as coefficients. If, when the units are taken into account, these terms do not sum to 0, then the extension of $x''$ to $E_2^1(X^j_\ell)$ is obtained by extending the first sum of (4.17), just as it was in the first case. If, when the units are taken into account, these terms sum to 0 mod $p$, then the powers of $p$ in this coefficient can be used to cancel some of the $h$'s in the $y_{\hat m-k}$-term, rendering the expression (4.17) incorrect in this term. However, at the next step, i.e., in going to $y_{\hat m-k-1}$, and in subsequent steps, the very top term will provide the leading term, just as it did in the second bullet above, and so this ambiguity in the $y_{\hat m-k}$-term will be inconsequential.

If $x'$ is as in (4.17), then the leading term in $\partial(x')$ will be $h\otimes h^j y_0 \in E_2^1(S^{2N+1})$. This is a consequence of the assumption that $\nu < \hat m(p-1)$, using the same sort of excess comparisons that occurred in the three possibilities above. This $\partial(x')$ is 0 if $j < N$, since $h\otimes h^j\iota_{2N+1}$ desuspends. Here and elsewhere, we use freely the fact that when $E_2^2(S^{2N+1}) \approx \mathbf{Z}/p$, then elements in it which desuspend are 0. If $j = N$, then $h\otimes h^N\iota_{2N+1} \ne 0$ by Lemmas 2.10 and 2.11. This completes the proof of this case.

**Case 3: $\nu = \hat m(p-1)$ and $j < N$.** Here $\mathcal{E}_1^{1,j}$ consists of permanent cycles by the argument just completed. Indeed, in this case, the first sum of (4.17) extends down to $\ell = 0$ with all terms satisfying the unstable condition.

**Case 4: $\nu = \hat m(p-1)$ and $j = N$.** This is the delicate case, requiring that we keep track of units. We will show that, under these hypotheses, $d_N : \mathcal{E}_N^{1,N} \to \mathcal{E}_N^{2,0}$ is zero if and only if (3.5) is satisfied.

As in (4.17), a generator of $\mathcal{E}_1^{1,N}$ pulls back to

$$hy_N + \cdots + h^{N-\hat{m}}y_{\hat{m}+1} + h^{N+\hat{m}(p-2)+1}y_{\hat{m}} + \cdots + h^{N+p-1}y_1, \tag{4.18}$$

but now both the first and last terms hit a nonzero element of $E_2^2(X_0^0)$ when $\partial$ is applied. We will show that if the coefficient of $hy_N$ is chosen to be 1, and if $m - \hat{m} = sp^\nu$, with $s \not\equiv 0 \mod p$, then, for $0 \le k < \hat{m}$, the coefficient of $h^{N+(\hat{m}-k)(p-2)+1}y_{\hat{m}-k}$ in (4.18) is, mod $p$,

$$u_k = (-1)^k \frac{1}{s}\binom{N - \hat{m} + k}{k}. \tag{4.19}$$

Then $E_2^1(X_1^N) \xrightarrow{\partial} E_2^2(X_0^0)$ satisfies, mod $L$,

$$\begin{aligned} \partial(hy_N + u_{\hat{m}-1}h^{N+p-1}y_1) &\equiv (h \otimes h^N + u_{\hat{m}-1}\tbinom{N}{1}h^{N+p-1} \otimes h)y_0 \\ &\equiv (1 - u_{\hat{m}-1}N)h \otimes h^N \iota_{2N+1}. \end{aligned} \tag{4.20}$$

We have used Lemma 2.12(2) at the last step.

When $\partial$ is applied to (4.18), modified so as to properly incorporate unit coefficients on all terms, the intermediate terms desuspend farther than the end terms, and the desired $d_N$ is determined by (4.20). Incorporating (4.19), we find that the image of this $d_N$ is

$$1 + (-1)^{\hat{m}}\tfrac{1}{s}N\tbinom{N-1}{\hat{m}-1} \in \mathbf{Z}/p,$$

and this is 0 if and only if (3.5), as claimed.

It remains to prove (4.19), which we do by induction on $k$. For $k = 0$, we note that, similarly to the proof of Lemma 4.7, for any numbers $c_\ell$,

$$\begin{aligned} \partial(hy_N + \sum_{\ell=2}^{N-\hat{m}} c_\ell h^\ell y_{N+1-\ell}) &\equiv \tbinom{N+\hat{m}(p-1)}{N-\hat{m}} h \otimes h^{N-\hat{m}}y_{\hat{m}} \\ &\equiv \tfrac{1}{s}(m - \hat{m})h \otimes h^{N-\hat{m}+\nu}y_{\hat{m}} \equiv -\tfrac{1}{s}d(h^{N-\hat{m}+\nu+1}y_{\hat{m}}), \end{aligned}$$

and so adding $\frac{1}{s}h^{N-\hat{m}+\nu+1}y_{\hat{m}}$ extends the cycle, with $u_0 = \frac{1}{s}$.

Assume now that $u_k$ satisfies (4.19). Let $z = hy_N + \cdots + u_k h^{N+(\hat m-k)(p-2)+1} y_{\hat m-k}$. Then, mod $L$, we have

$$\begin{aligned}\partial(z) &\equiv u_k\binom{N+(\hat m-k-1)(p-1)}{1} h^{N+(\hat m-k)(p-2)+1}\otimes hy_{\hat m-k-1}\\ &\equiv -u_k(N-\hat m+k+1)h\otimes h^{N+(\hat m-k-1)(p-2)}y_{\hat m-k-1}\\ &\equiv \tfrac{1}{m-\hat m+k+1}u_k(N-\hat m+k+1)d(h^{N+(\hat m-k-1)(p-2)+1}y_{\hat m-k-1})\\ &\equiv (-1)^k\tfrac{1}{s}\binom{N-\hat m+k+1}{k+1}d(h^{N+(\hat m-k-1)(p-2)+1}y_{\hat m-k-1}).\end{aligned}$$

Thus adding $(-1)^{k+1}\frac{1}{s}\binom{N-\hat m+k+1}{k+1}h^{N+(\hat m-k-1)(p-2)+1}y_{\hat m-k-1}$ to $z$ yields a cycle mod $L$ in $E_2^1(X^N_{\hat m-k-1})$ which projects to $z$. The coefficient here is just what we have claimed is $u_{k+1}$, and so this extends the induction.

**Case 5: $\nu > \hat m(p-1)$ and $\hat m < j \le N$.** These are the second type of elements $x_k$ in Theorem 3.7. As before, let $e = \nu(|\mathcal{E}_1^{1,\hat m}|) = \min(\nu+1, N+\hat m(p-1))$. Then, in the notation of Theorem 3.7, our $j$ becomes $\hat m(p-1)+N+1+t-e$. We show in the next paragraph that if $t \ge \hat m$, then the differential from $\mathcal{E}^{1,j}$ hits the element of order $p^{t-\hat m+1}$ in $\mathcal{E}_1^{2,\hat m}$. Of course, this hit element should really be viewed as the element of order $p$ in $\mathcal{E}^{2,\hat m}_{j-\hat m}$, elements of order less than $p^{t-\hat m+1}$ already having been hit by shorter differentials.

By the first sentence of the proof of Lemma 4.7, a generator of $E_2^1(X^j_j)$ pulls back to an element $x\in E_2^1(X^j_{\hat m+1})$ satisfying $\partial(x)\equiv h\otimes h^{j-\hat m}y_{\hat m}$. This latter element has order $p^{t-\hat m+1}$ in $E_2^2(S^{2N+1+q\hat m})$ by Corollary 2.9. Indeed, the equation of that corollary becomes

$$(\hat m(p-1)+N+1+t-e-\hat m)+e = (t-\hat m+1)+(N+\hat m(p-1)).$$

If, on the other hand, $t < \hat m$, then by Lemma 4.7 the generator of $\mathcal{E}^{1,j}$ pulls back to a cycle

$$x' \equiv \sum_{\ell=\hat m+1}^{j} h^{j+1-\ell}y_\ell + h^{\hat m(p-2)+N+t-e+\nu+2}y_{\hat m}\in E_2^1(X^j_{\hat m}).$$

We can simplify this by noting that the hypotheses imply $e=\nu+1$. (If not, then $e=\hat m(p-1)+N$, so $j=t+1$, contradicting $j>\hat m>t$.) Now by Lemma 4.9 we can pull $x'$ back to

$$x'' \equiv x' + \sum_{\ell=t+1}^{\hat m-1} h^{\ell(p-2)+N+t+1}y_\ell \in E_2^1(X^j_{t+1}).$$

As in (4.10), $\partial(x'') \equiv h^{(t+1)(p-1)+N} \otimes hy_t$, which is nonzero by Lemma 2.12. We observe that $\partial(hy_j)$ cannot interfere here, since this yields a term which desuspends unless $j = N$ and $t = 0$, and this is impossible since, if $t = 0$,

$$j = \hat{m}(p-1) + N + 1 - e = \hat{m}(p-1) + N - \nu < N.$$

**Case 6:** $j > N$**.** In the notation of the theorem, we have

$$\begin{aligned} j &= \hat{m} + \Delta + 1 = \hat{m} + k - e + 1 \\ &= \hat{m} + (\hat{m}(p-2) + N + 1 + t) - e + 1 = \hat{m}(p-1) + N + 2 + t - e. \end{aligned} \tag{4.21}$$

The generator of $E_2^1(X_j^j)$ can be pulled back as in Lemma 4.2 to a class $x \in E_2^1(X_{\hat{m}+1}^j)$ satisfying $\partial(x) \equiv h \otimes h^{j-\hat{m}-1} y_{\hat{m}}$, similarly to part of (4.5) except that $\epsilon = 1$. Using Corollary 2.9 and (4.21), we obtain that the order of $h \otimes h^{j-\hat{m}-1} \iota_{2N+1+q\hat{m}}$ is $p^{t-\hat{m}+1}$, as claimed, provided $t \geq \hat{m}$. Note that if $j = N+1$, then the class which this differential would like to hit has already been killed by a differential from $\mathcal{E}^{1,N}$.

If $t < \hat{m}$, then by Lemma 4.7 a generator of $E_2^1(X_j^j)$ pulls back to

$$x \equiv hy_j + \cdots + h^{j-\hat{m}-1} y_{\hat{m}+1} + h^{j-\hat{m}+\nu} y_{\hat{m}} \in E_2^1(X_{\hat{m}}^j).$$

By Lemma 4.9, $x$ can be extended farther, to

$$x' \equiv x + \sum_{\ell=t+1}^{\hat{m}-1} h^{\ell(p-2)+N+2+t-e+\nu} y_\ell \in E_2^1(X_{t+1}^j),$$

with $\partial(x') \equiv h^{(t+1)(p-1)+N} \otimes hy_t \neq 0$ by Lemma 2.12. We must worry here about possible cancellation from $\partial(hy_j) \equiv ch \otimes h^{j-t} y_t$. This will desuspend, and hence be 0, if $t > 0$. Thus it remains to consider the possible cancellation when $t = 0$.

If $\nu > \hat{m}(p-1)$, then $t$ must be greater than 0. This can be deduced using $j > N$, (4.21), and $e = \min(\nu+1, N + \hat{m}(p-1))$.

If $\nu \leq \hat{m}(p-1)$ and (3.5) is not satisfied, then it was proved in Cases 2 and 4 that $\mathcal{E}_{N+1}^{2,0} = 0$, and so we need not worry about the case $t = 0$ here. Finally, we must establish that $d_{N+1} \neq 0 : \mathcal{E}_{N+1}^{1,N+1} \to \mathcal{E}_{N+1}^{2,0}$ if (3.5) is satisfied. To prove this by the methods employed so far in this section is more delicate than we care to present. Instead, we deduce it using differentials already determined, together with results for cyclicity of certain $E_2$-groups which will be established in the next section.

So, we are assuming (3.5). In particular, $\nu = \hat{m}(p-1)$ and $e = \nu + 1$. Earlier in Case 6, we verified that under this hypothesis $d_{N+1} \neq 0 : \mathcal{E}_{N+1}^{1,N+2} \to \mathcal{E}_{N+1}^{2,1}$. This says that the composite $\rho\circ\partial\neq 0$ in (4.22), and that $|\operatorname{coker}(i_2)|=p$.

$$E_2^1(X_2^N)\xrightarrow{i_1}E_2^1(X_2^{N+1})\xrightarrow{i_2}E_2^1(X_2^{N+2})\xrightarrow{\partial}E_2^2(X_0^1)\xrightarrow{\rho}E_2^2(X_1^1) \quad (4.22)$$

By Proposition 5.2, $\rho$ is a surjection from $\mathbf{Z}/p^2$ to $\mathbf{Z}/p$. Thus $\partial$ is surjective, and hence $\partial\circ i_2\neq 0$, hitting $E_2^2(X_0^0)\subset E_2^2(X_0^1)$. On the other hand, the 0-differential from $\mathcal{E}_N^{1,N}$ to $\mathcal{E}_N^{2,0}$ established in Case 4 implies that the composite $\partial\circ i_2\circ i_1$ in (4.22) is 0. Thus $\partial\circ i_2$ is nonzero on an element which projects nontrivially to $E_2^1(X_{N+1}^{N+1})$, and this establishes $d_{N+1}\neq 0:\mathcal{E}_{N+1}^{1,N+1}\to\mathcal{E}_{N+1}^{2,0}$, as desired. ■

## 5. Group structure

In this section, we prove Theorem 1.9; we determine the group structure on the 2-line of the UNSS of $SU(n)$ for $n\le p^2-p+1$. We also prove a few subsidiary results about group structure of 1-line groups $E_2^1(X_a^b)$.

The conventions of the previous sections are still in force. Thus $p$, $N$, and $m$ are fixed, with $1\le N<p$. The second superscript of all $E_2$-groups is $2N+1+qm$ (unless specifically stated to the contrary), and $\hat{m}$ is the least nonnegative residue of $m$ mod $p$. Also, $X_a^b$ is a direct factor of a complex Stiefel manifold (quotient of $SU(n)$'s) built from spheres $2N+1+qj$ for $a\le j\le b$. Also, we continue the practice of omitting $v_1$'s on the left and units in $\mathbf{Z}_{(p)}$ whenever they are unimportant.

Although Theorem 1.11 implies that $E_2^1(X_0^b)$ is cyclic, it will not always be the case that $E_2^1(X_a^b)$ is cyclic. The following result presents one case in which $E_2^1(X_a^b)$ will be cyclic.

**Proposition 5.1.** *The group $E_2^1(X_a^b)$ is cyclic if there are no numbers $j$ congruent to $N$ mod $p$ satisfying $a\le j<b$.*

*Proof.* The proof is by induction on $b$. The result is true when $b=a$ by Theorem 2.8(1). We assume it is true for $X_a^b$, and will deduce it for $X_a^{b+1}$, provided $b\not\equiv N$ mod $p$. We consider the commutative diagram of exact sequences below.

$$\begin{array}{ccccccccc}
0\to & E_2^1(X_a^b) & \xrightarrow{j_*} & E_2^1(X_a^{b+1}) & \xrightarrow{\rho_*} & E_2^1(X_{b+1}^{b+1}) & \xrightarrow{\partial} & E_2^2(X_a^b)\\
 & \downarrow f_* & & \downarrow g_* & & \downarrow = & & \\
0\to & E_2^1(X_b^b) & \xrightarrow{j_{1*}} & E_2^1(X_b^{b+1}) & \longrightarrow & E_2^1(X_{b+1}^{b+1}) & &
\end{array}$$

If $j_*$ is surjective, then $E_2^1(X_a^{b+1})$ is cyclic by the induction hypothesis, and so we are done. Thus we may assume that there is an element $\beta$ such that $\rho_*(\beta)=\alpha$, where

$\alpha$ has order $p$. Since $b \not\equiv N \mod p$, the attaching map in $X_b^{b+1}$ is nontrivial, and so [5, 2.23] implies that there is a generator $\gamma$ of $E_2^1(X_b^b)$ such that $j_{1*}(\gamma) = p \cdot g_*(\beta)$. Exactness implies that there is $y$ such that $j_*(y) = p\beta$. Then $j_{1*}(f_*(y) - \gamma) = 0$, so that $f_*(y) = \gamma$, and hence $y$ is a generator of the cyclic group $E_2^1(X_a^b)$. Thus the extension is cyclic. ■

Next we prove a result that implies the first case of Theorem 1.9.

**Proposition 5.2.** *The group $E_2^2(X_a^b)$ is cyclic if either $b \le N$ or $N < a \le b < p$.*

*Proof.* The proof is by induction on $b$ and is true for $b = a$ by 2.8(1). We assume the result for $X_a^b$, and consider the exact sequence

$$E_2^2(X_a^b) \to E_2^2(X_a^{b+1}) \xrightarrow{\rho_*} E_2^2(X_{b+1}^{b+1}) \xrightarrow{\partial} 0,$$

with $b \ne N$. If

$$E_2^2(X_{b+1}^{b+1}) \approx \mathbf{Z}/p^{N+(b+1)(p-1)}, \tag{5.3}$$

then by Theorems 3.4 and 3.7, $E_2^2(X_a^b) = 0$, and so we are done in this case, since $\rho_*$ will be iso. The easiest way to believe this claim that $E_2^2(X_a^b) = 0$ is to look at the last row of Tables 2 and 3. In the first one, it is the assertion that $\mathcal{E}^{1,7}$ kills $\mathcal{E}^{2,j}$ for $3 \le j \le 6$, and in the second one, that $\mathcal{E}^{1,3}$ kills $\mathcal{E}^{2,j}$ for $0 \le j \le 2$.

Thus we may assume that (5.3) is not true, and so $d(h^{N+(b+1)(p-1)+1})y_{b+1}$ has order $p$ in $E_2^2(X_{b+1}^{b+1})$, by Theorem 2.8(4). Since $\partial$ annihilates this class, there is $w \in E_1^2(X_a^b)$ such that $z = d(h^{N+(b+1)(p-1)+1})y_{b+1} - w$ is a cycle in $E_2^2(X_a^{b+1})$. We wish to show that $pz$ is the image of a generator of $E_2^2(X_a^b)$.

We have

$$d(h^t y) = d(h^t)y + h^t d(y), \tag{5.4}$$

where $d(h^t)$ is as in Lemma 2.3(3). Note here that the minus sign which is attached to $\psi(\gamma_1)$ in (2.1) has been incorporated into $d(h^t)$ (which is really $d(v^s h^t)$) in Lemmas 2.3(3) and 2.13. Now, mod $L$,

$$pz \equiv d(h^{N+(b+1)(p-1)}y_{b+1}) - (N + b(p-1))h^{N+(b+1)(p-1)} \otimes hy_b - pw.$$

Here we have used $ph = v - \eta(v)$ and (5.4). The first term is a boundary, and so is ignored. Note that the exponent of $h$ had to be brought down to $N + (b+1)(p-1)$ in order that it be placed in front of $y_{b+1}$. Since $b \ne N$, the coefficient of the second term is a unit, and so will be ignored. Since $w$ was defined on $X_a^b$, $pw$ desuspends below $S^{N+b(p-1)}$, and so may be incorporated into $L$. Thus, by Lemma 2.12(2), $pz$ generates $E_2^2(X_b^b)$ mod $L$, and so $pz$ is the image of a generator of $E_2^2(X_a^b)$, as desired. ■

Next we prove the second part of Theorem 1.9, which we restate as follows.

**Theorem 5.5.** *Suppose $i > N$ and $\hat{m} > 0$. If $e$ is as in Theorem 1.5, then $E_2^2(X_0^i) \approx \mathbf{Z}/p^{e-1} \oplus \mathbf{Z}/p$.*

*Proof.* In the exact sequence

$$E_2^1(X_{N+1}^i) \xrightarrow{\partial} E_2^2(X_0^N) \xrightarrow{\phi} E_2^2(X_0^i) \xrightarrow{\rho} E_2^2(X_{N+1}^i) \to 0, \tag{5.6}$$

the groups $E_2^2(X_0^N)$ and $E_2^2(X_{N+1}^i) \approx \mathbf{Z}/p^a$ are cyclic by Proposition 5.2. We will show that if $x \in E_2^2(X_0^i)$ is such that $\rho(x)$ is a generator, then $p^a x = p\phi(g)$, with $g$ a generator of $E_2^2(X_0^N)$. The result is immediate from this, with $x$ and $\phi(g) - p^{a-1}x$ generating the summands.

**Case 1: $N = \hat{m}$.** The thing that distinguishes this case is that $pg$ lives on the top cell of $X_0^N$, i.e., $E_2^2(X_0^N) \to E_2^2(X_N^N)$ sends $pg$ nontrivially. Thus the extension question can be studied in the exact sequence

$$0 \to E_2^2(X_N^N) \to E_2^2(X_N^i) \to E_2^2(X_{N+1}^i) \to 0.$$

We let $x$ also equal the image of a generator of $E_2^2(X_0^i)$ in $E_2^2(X_N^i)$. As in the proof of 5.2, we can write

$$p^{a-2}x = d(h^{N+(N+2)(p-1)+1})y_{N+2} + w_0 y_{N+1} + w_1 y_N,$$

with each term unstable. Now we can use (5.4), (2.6), 2.4, and 2.5 to write

$$\begin{aligned} p^{a-1}x &= d(h^{N+(N+2)(p-1)}y_{N+2}) + pw_0 y_{N+1} + pw_1 y_N \\ &\quad -h^{N+(N+2)(p-1)} \otimes \Big((N+(N+1)(p-1))hy_{N+1} - Nphvy_N + \tbinom{Np}{2}h^2 y_N\Big). \end{aligned}$$

We ignore the boundary term and apply Corollary 2.14 to the terms on $y_{N+1}$. We also write $hv = vh - ph^2$ and ignore the $p^2 w' y_N$ term (with $w' y_N$ defined) which this yields. We obtain

$$\begin{aligned} p^{a-1}x &= -\tfrac{(N+1)p-1}{m-N-1} d(h^{N+(N+1)(p-1)+1} + w)y_{N+1} - Nph^{N+(N+2)(p-1)} \otimes (vh + uh^2)y_N \\ &\quad + pw_1 y_N, \end{aligned}$$

where $wy_{N+1}$ is defined, and $u$ is a unit. We multiply by another $p$ and ignore terms which are $p^2$ times an unstable class on $y_N$. We obtain

$$\begin{aligned} p^a x &\equiv u' d(h^{N+(N+1)(p-1)} + pw)y_{N+1} \\ &\equiv u'\Big(d((h^{N+(N+1)(p-1)} + pw)y_{N+1}) - Np(h^{N+(N+1)(p-1)} + pw) \otimes hy_N\Big) \end{aligned}$$

with $u'$ a unit. Here we have applied (5.4) again. By Lemma 2.12, this is $p$ times a generator of $E_2^2(X_N^N)$.

**Case 2:** $N > \hat{m}$ **or** $N < i < \hat{m}$ **or** $($$N < \hat{m} \le i$ **and** $i+\nu(m-\hat{m}) < \hat{m}(p-1)+2N$$)$. In these cases, $\mathcal{E}_\infty^{2,j} = \mathcal{E}_1^{2,j}$ for $j \ge N-1$, and so in the short exact sequence

$$0 \to \operatorname{coker}(\partial) \to E_2^2(X_0^i) \to E_2^2(X_{N+1}^i) \to 0, \tag{5.7}$$

the generator $g$ of $\operatorname{coker}(\partial)$ lives on $y_N$, and $pg$ lives on $y_{N-1}$. Hence (5.7) maps to a short exact sequence

$$0 \to E_2^2(X_{N-1}^N) \xrightarrow{\phi'} E_2^2(X_{N-1}^i) \to E_2^2(X_{N+1}^i) \to 0,$$

and it suffices to show $\phi'(pg) = p^a x$ in this latter sequence.

The analysis is quite similar to Case 1. We write

$$p^{a-2}x = d(h^{N+(N+2)(p-1)+1})y_{N+2} + w_0 y_{N+1} + w_1 y_N + w_2 y_{N-1}$$

with each term unstable. Then, ignoring terms which are $p$ times an unstable class on $y_{N-1}$, and letting $u$, $u'$, etc., denote units in $\mathbf{Z}_{(p)}$, we obtain

$$\begin{aligned} p^{a-1}x \equiv\ & d(h^{N+(N+2)(p-1)} y_{N+2}) + pw_0 y_{N+1} + pw_1 y_N \\ & -h^{N+(N+2)(p-1)} \otimes \Big(((N+1)p-1)hy_{N+1} - Nphvy_N + \tbinom{Np}{2} h^2 y_N \\ & \qquad + (uhv^2 + u'h^2 v)y_{N-1}\Big). \end{aligned}$$

Now we ignore boundaries and use 2.14, 2.3(1), and 2.3(2) to obtain

$$\begin{aligned} p^{a-1}x \equiv\ & -\tfrac{(N+1)p-1}{m-N-1} d(h^{N+(N+1)(p-1)+1} + w)y_{N+1} \\ & + Nph^{N+(N+1)(p-1)} \otimes hy_N + pw_1 y_N + u'' ph^{N+(N+2)(p-1)} \otimes h^2 y_N \\ & -h^{N+(N+2)(p-1)} \otimes (uhv^2 + u'h^2 v)y_{N-1}. \end{aligned}$$

Now we multiply by $p$ again, again omit $p$ times unstable classes on $y_{N-1}$, and incorporate the two surrounding terms into $w_1$, obtaining

$$p^a x \equiv u''' d(h^{N+(N+1)(p-1)} + pw)y_{N+1} + p^2 w_1' y_N.$$

Now we apply (5.4) again, and omit writing the $d(hy)$-term, obtaining

$$\begin{aligned} p^a x &\equiv -u'''\Big((h^{N+(N+1)(p-1)}+pw)\otimes\Big(Nphy_N-((N-1)p+1)hvy_{N-1} \\ &\qquad +\tbinom{(N-1)p+1}{2}h^2y_{N-1}\Big)\Big)+p^2w_1'y_N. \end{aligned}$$

We omit lots of terms which are $p$ times an unstable class on $y_{N-1}$. We omit the $-u'''$-factor, which can be considered as a factor of the entire expression. We also combine together several terms that are $p^2$ times an unstable class on $y_N$. Finally, we apply Lemma 2.3(1) to $hv$. This yields

$$p^a x\equiv Np(h^{N+(N+1)(p-1)}\otimes h+pw_1'')y_N-h^{N+(N+1)(p-1)}\otimes vhy_{N-1}.$$

Now we apply 2.14 to the first term and 2.3(2) to the second term, which effectively cancels $h^{p-1}v$. This yields

$$\begin{aligned} p^a x &\equiv \tfrac{N}{m-N}pd(h^{N+N(p-1)+1}+w_3)y_N-h^{N+N(p-1)}\otimes hy_{N-1} \\ &\equiv \tfrac{N}{m-N}d(h^{N+N(p-1)}+pw_3)y_N-h^{N+N(p-1)}\otimes hy_{N-1}, \end{aligned}$$

with $w_3y_N$ defined. We apply (5.4) once again, and omit writing the term of the form $d(hy)$, obtaining

$$\begin{aligned} p^a x &\equiv -\tfrac{N}{m-N}((N-1)p+1)(h^{N+N(p-1)}+pw_3)\otimes hy_{N-1}-h^{N+N(p-1)}\otimes hy_{N-1} \\ &\equiv -(\tfrac{N}{m-N}+1)h^{N+N(p-1)}\otimes hy_{N-1}. \end{aligned}$$

Since $\frac{N}{m-N}+1=\frac{m}{m-N}$, this is nonzero by Lemma 2.12 since $m\not\equiv 0 \mod p$ by assumption.

**Case 3:** $N<\hat{m}\le i$ **and** $i+\nu(m-\hat{m})\ge\hat{m}(p-1)+2N$. In this case, Theorem 3.7 (with Table 2 again recommended to provide insight) implies (1) $\mathcal{E}_\infty^{2,j}=0$ for $j<N$, and hence $E_2^2(X_0^N)/\operatorname{im}(\partial)\approx\mathbf{Z}/p$, and (2) $E_2^2(X_{N+1}^i)\to E_2^2(X_{N+1}^{i+1})$ is multiplication by $p$ of groups each isomorphic to $\mathbf{Z}/p^a$. To see (2), note first that the generator of $E_2^2(X_{N+1}^{i+1})$ lives on $y_{i+1}$, and second that $E_2^1(X_{i+1}^{i+1})\xrightarrow{\partial}E_2^2(X_{N+1}^i)$ hits the elements of order $p$. If $i=p-1$, then we need to extend Theorem 3.7 to include the case $i=p$. This extension is straightforward; it is only Theorem 3.2 that must be modified when $i=p$.

Now the splitting follows from the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc} 0 \to & E_2^2(X_0^N)/\operatorname{im}(\partial) & \to & E_2^2(X_0^i) & \to & E_2^2(X_{N+1}^i) & \to 0 \\ & \downarrow{=} & & \downarrow{\phi} & & \downarrow{\cdot p} & \\ 0 \to & E_2^2(X_0^N)/\operatorname{im}(\partial) & \to & E_2^2(X_0^{i+1}) & \to & E_2^2(X_{N+1}^{i+1}) & \to 0 \end{array}$$

Indeed, suppose to the contrary that $E_2^2(X_0^i) \approx \mathbf{Z}/p^{a+1}$ with generator $G$. Then the left square implies $p^a\phi(G) \neq 0$, and so we must have $E_2^2(X_0^{i+1}) \approx \mathbf{Z}/p^{a+1}$ with generator $\phi(G)$. Then the composition around the bottom of the right square is surjective, but the composite around the top is not surjective. ∎

Finally, we prove the third and fourth cases of Theorem 1.9, which we restate as follows. Again, we remind the reader that the conventions which were restated at the beginning of this section continue to be in effect.

**Theorem 5.8.** *Suppose $\hat{m} = 0$, $i > N$, and $t = \min(N, \nu(m) + 1)$. Then*

$$E_2^2(X_0^i) \approx E_2^2(X_1^i) \approx E_2^1(X_1^i) \approx \begin{cases} \mathbf{Z}/p^t \oplus \mathbf{Z}/p^{i-t} & \text{if } t \le i - N \\ \mathbf{Z}/p^N \oplus \mathbf{Z}/p^{i-N} & \text{if } t \ge i - N \end{cases}$$

*Proof.* The first isomorphism is immediate from $\mathcal{E}_\infty^{2,0} = 0$ in Theorem 3.2, and is clearly depicted in Table 1. The second and third isomorphisms could be generalized to interesting results regarding isomorphisms of 1- and 2-line groups, and evaluation of noncyclic 1-line groups of complex Stiefel manifolds, but, in the interest of expediency, we will just deal with the case at hand.

The middle isomorphism of 5.8 is the composite

$$E_2^{1,2N+1+qm}(X_1^i) \xrightarrow{\phi} E_2^{2,2N+1+qm+qL}(X_1^i) \approx E_2^{2,2N+1+qm}(X_1^i),$$

with $\nu(L)$ sufficiently large. The second isomorphism here is due to the periodicity of the groups $E_2^{2,*}(X_1^i)$, while $\phi$ is defined by

$$\phi(x) = \alpha_{L/(N+m(p-1))} \otimes x,$$

where $\alpha_{L/e}$ is as in Theorem 2.8. We need $\nu(L) \ge N + m(p-1) - 1$. Note that $\phi(x)$ is defined unstably since $\alpha_{L/e}$ is defined on $S^{2e+1}$.

We will show that $\phi$ is injective, from which its bijectivity will follow since $|E_2^2(X_1^i)| = |E_2^1(X_1^i)|$. If $x \neq 0 \in E_2^1(X_1^i)$, choose the largest integer $j$ such that the projection map $\rho$ sends $x$ to a nonzero element $\rho_*(x)$ of $E_2^1(X_j^i)$. Then there is a nonzero element

$x_1 \in E_2^1(X_j^j)$ such that $g_*(x_1) = \rho_*(x)$, where $g : X_j^j \hookrightarrow X_j^i$. Let $\alpha = \alpha_{L,N+m(p-1)}$. Then $g_*(\alpha \otimes x_1) = \rho_*(\alpha \otimes x)$. By Lemma 5.10 below, $\alpha \otimes x_1 \neq 0$. By Theorem 3.2,

$$g_* : E_2^{2,2N+1+qm+qL}(X_j^j) \to E_2^{2,2N+1+qm+qL}(X_j^i) \tag{5.9}$$

is injective, and hence $\rho_*(\alpha \otimes x) = g_*(\alpha \otimes x_1) \neq 0$. Thus $\alpha \otimes x \neq 0$, establishing the desired injectivity. The injectivity of (5.9) is the crucial hypothesis in generalizing this result. The addition of $qL$ to the second superscript doesn't matter here by periodicity. The injectivity in (5.9) is the statement that $\mathcal{E}_1^{2,j} = \mathcal{E}_\infty^{2,j}$ for $j > 0$ in Theorem 3.2, or that in Table 1 the only elements hit by differentials have $j = 0$.

Now we prove the last isomorphism of (5.8). We use the following commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc} 0 \to & E_2^1(X_0^N) & \to & E_2^1(X_0^i) & \to & E_2^1(X_{N+1}^i) & \to 0 \\ & \downarrow \phi' & & \downarrow \phi & & \downarrow = & \\ 0 \to & E_2^1(X_1^N) & \xrightarrow{\theta} & E_2^1(X_1^i) & \to & E_2^1(X_{N+1}^i) & \to 0 \end{array}$$

By Theorem 3.2, illustrated in Table 1, the diagram is

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}/p^N & \to & \mathbf{Z}/p^i & \to & \mathbf{Z}/p^{i-N} & \to 0 \\ & \downarrow \cdot p^t & & \downarrow \phi & & \downarrow = & \\ 0 \to & \mathbf{Z}/p^N & \xrightarrow{\theta} & E & \to & \mathbf{Z}/p^{i-N} & \to 0. \end{array}$$

The key observation here is that $\phi' = \cdot p^t$, which we now explain. By Theorem 3.2, there are no differentials in the cellular spectral sequence for $X_1^N$, but in the spectral sequence for $X_0^N$, there are differentials from $\mathcal{E}^{1,j}$ to $\mathcal{E}^{2,0}$ for $N - t + 1 \leq j \leq N$. Thus the generator of $E_2^1(X_0^i)$ sits on $y_{N-t}$, which is where $p^t$ times the generator of $E_2^1(X_1^i)$ lives.

Now it is a matter of simple algebra to determine the structure of the abelian group $E = E_2^1(X_1^i)$ in the diagram above. Let $g$ denote a generator of $E_2^1(X_0^i)$, and let $h$ denote a generator of $E_2^1(X_1^N)$. Then $p^{i-N}\phi(g) = p^t\theta(h)$. If $t = N$, this implies $p^{i-N}\phi(g) = 0$, and so $E \approx \mathbf{Z}/p^N \oplus \mathbf{Z}/p^{i-N}$ with generators $\theta(h)$ and $\phi(g)$. If $t < N$ and $t \leq i - N$, then $E \approx \mathbf{Z}/p^t \oplus \mathbf{Z}/p^{i-t}$ with generators $\theta(h) - p^{i-N-t}\phi(g)$ and $\phi(g)$. If $t < N$ and $t \geq i - N$, then $E \approx \mathbf{Z}/p^N \oplus \mathbf{Z}/p^{i-N}$ with generators $\theta(h)$ and $\phi(g) - p^{t-i+N}\theta(h)$. ■

We close with a lemma which was used in the preceding proof. Previous conventions for $N$ and $\hat{m}$ apply.

**Lemma 5.10.** *Suppose $\nu(L) \geq N + m(p-1) - 1$, $\hat{m} = 0$, and $1 \leq j < p$. Then*

$$\alpha_{L,N+m(p-1)}\otimes : E_2^{1,2N+1+qm}(S^{2N+1+qj}) \to E_2^{2,2N+1+qm+qL}(S^{2N+1+qj})$$

*is bijective.*

*Proof.* Since both groups have order $p$, it suffices to show $\alpha_{L/(N+m(p-1))}\otimes\alpha_{m-j}\iota_{2N+1+qj}$ is nonzero in $E_2$. Let $s = L-N-m(p-1)$. By Theorem 2.8(2), this class is equivalent, mod terms that desuspend, to

$$v^s h^{N+m(p-1)} \otimes v^{m-j-1} h\iota_{2N+1+qj}.$$

By Lemma 2.13, this is equivalent to

$$\tfrac{1}{m-j}\Big(d(v^{s-1}h^{N+m(p-1)+1}v^{m-j})\iota_{2N+1+qj} + Lv^{s-1}h \otimes h^{N+m(p-1)}v^{m-j}\iota_{2N+1+qj}\Big).$$

Since $L$ is highly $p$-divisible, the second term is 0 in $E_2$. The first term is simplified by using Lemma 2.12(2) to replace $(h^p v)^{m-j}$ by $(v^p h)^{m-j}$. Thus the desired term is a unit times $d(v^{L-N+m-jp-1}h^{N+j(p-1)+1})\iota_{2N+1+qj}$, which is nonzero by Theorem 2.8(4). ■

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