THE K-COMPLETION OF E_6

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Abstract. We compute the 2-primary v_1-periodic homotopy groups of the K-completion of the exceptional Lie group E_6. This is done by computing the Bendersky-Thompson spectral sequence of E_6. We conjecture that the natural map from E_6 to its K-completion induces an isomorphism in v_1-periodic homotopy, and discuss issues related to this conjecture.

1. Introduction

The p-primary v_1-homotopy groups of a space X, denoted $v_1^{-1}\pi_*(X;p)$ and defined in [11], are a localization of the portion of the homotopy groups of X detected by p-local K-theory. In [10], the author completed the determination of the odd-primary v_1-periodic homotopy groups of all compact simple Lie groups. The groups $v_1^{-1}\pi_*(X;2)$ have been determined for $X = SU(n)$ ([3]), $Sp(n)$ ([5]), G_2 ([12]), and F_4 ([4]). Joint work of the author and Bendersky is very close to completing the computation of $v_1^{-1}\pi_*(SO(n);2)$. That will leave E_6, E_7, and E_8 to be determined, which would complete a program suggested to the author by Mimura in 1989. In this paper, we determine $v_1^{-1}\pi_*(\widehat{E}_6;2)$, where $\widehat{X}$ denotes the K-completion of X, as defined in [6]. We conjecture that the natural map $E_6 \to \widehat{E}_6$ induces an isomorphism in $v_1^{-1}\pi_*(-;2)$.

In [6], Bendersky and Thompson defined the K-completion $\widehat{X}$ of a space X to be the homotopy limit of a certain tower of spaces under X. The space X is said to satisfy the Completion Telescope Property (CTP) at the prime p if $X \to \widehat{X}$ induces an isomorphism of p-primary v_1-periodic homotopy groups. They also constructed a spectral sequence (BTSS) which, for many spaces X, converges to $v_1^{-1}\pi_*(\widehat{X};p)$. It was shown in [7] and [4] that, localized at any prime, if X is K_*-strongly spherically

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resolved (KSSR), then X satisfies the CTP. This condition means that X can be built by fibrations from spheres S^{2n_i+1} such that K_*X is built as a K_*K-coalgebra as an extension of the $K_*S^{2n_i+1}$.

In [4], it was proved that F_4 satisfies the CTP at 2, and a general result ([4, 5.8,5.11,5.15] was proved which implies that if E_6/F_4 satisfies the CTP, then so does E_6. Many standard functors of algebraic topology would lead one to expect that there is a fibration

$$S^9 \to E_6/F_4 \to S^{17}, \tag{1.1}$$

which would imply that E_6/F_4 is KSSR and then that E_6 satisfies the CTP at the prime 2. However, it was proved by Cohen and Selick ([9]) that there can be no such fibration. A fibration

$$\Omega S^9 \to \Omega(E_6/F_4) \to \Omega S^{17}$$

would also imply the CTP for E_6. It is not known whether such a fibration exists.

It was proved in [7] that, localized at an odd prime p, if X is K-algebraically spherically resolved and K-durable, then it satisfies the CTP. The first condition (KASR) means that $K_*(X)$ has the structure that it would if X were KSSR, and the second that X and its K-localization have isomorphic v_1-periodic homotopy groups. Both E_6/F_4 and E_6 are KASR and K-durable at the prime 2. However, it is not known whether, at the prime 2, this is enough to insure that the CTP is satisfied.

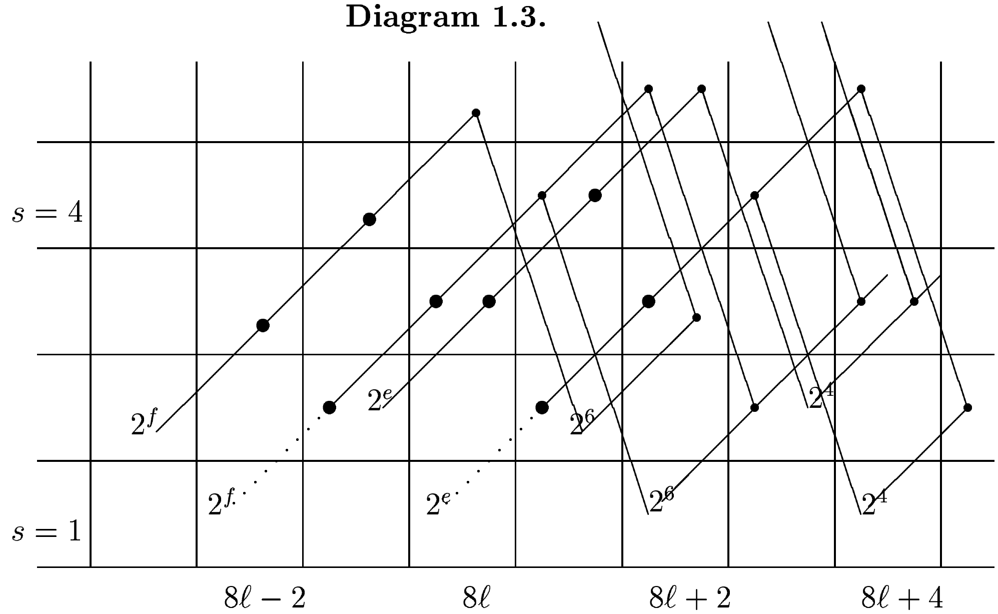
Our main result is the following determination of $v_1^{-1}\pi_*(\widehat{E}_6;2)$. As discussed above, the expectation is that this equals $v_1^{-1}\pi_*(E_6;2)$.

Theorem 1.2. *Let* $e = \min(12, \nu(\ell-18)+5)$ *and* $f = \min(12, 2\nu(\ell-3)+8)$. *Then*

$$v_1^{-1}\pi_{8\ell+d}(\widehat{E}_6;2) \approx \begin{cases} \mathbf{Z}/2^f & d=-3 \\ \mathbf{Z}/2^f \oplus \mathbf{Z}/2 & d=-2 \\ \mathbf{Z}/2^e \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 & d=-1,0 \\ \mathbf{Z}/2^5 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 & d=1 \\ \mathbf{Z}/2^5 \oplus \mathbf{Z}/2 & d=2 \\ \mathbf{Z}/2^3 & d=3,4 \end{cases}$$

Here and throughout, $\nu(-)$ denotes the exponent of 2 in an integer. The picture of the BTSS which determines these groups is given in Diagram 1.3. This is a usual sort of Adams spectral sequence type of chart, with dots representing $\mathbf{Z}/2$'s, positively sloping lines the action of the Hopf map η (or the element h_1 in the spectral sequence), and negatively sloping lines differentials in the spectral sequence, which implies that the elements which they connect do not survive to give nonzero homotopy classes. The dotted lines that look like h_1 means that h_1 is usually present, but perhaps not always.

Diagram 1.3.



The reader should observe that this chart, and the homotopy groups which it depicts, has a form very similar to the charts for $v_1^{-1}\pi_*(G_2)$ and $v_1^{-1}\pi_*(F_4)$ depicted in [4, 4.9]. The only difference is the orders of the groups on the 1- and 2-lines, and the lack of some exotic extensions.

The determination of the E_2-term of the spectral sequence is rather straightforward, given the general results of [4] and calculations of the Adams operations in $K^*(E_6)$ performed in [10]. This is performed in Section 2. The d_3-differentials are determined by showing that there are maps of spaces which relate the classes in question to classes in spaces where d_3 is known. This is performed in Sections 3 and 4. In Section 5, we expand our discussion of whether E_6 satisfies the CTP.

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2. The E_2-term

In this section we compute the E_2-term of the BTSS converging to $v_1^{-1}\pi_*(\widehat{E}_6)$. We are always using the v_1-periodic BTSS localized at 2, and all v_1-periodic homotopy groups are 2-primary.

In [4, 1.1], it was proved that for spaces whose K-homology and K-cohomology form nice exterior algebras,

$$E_2^{s,t}(X) \approx \begin{cases} \mathrm{Ext}^s_{\mathcal{A}}(QK^1(X;\mathbf{Z}_2^\wedge)/\operatorname{im}(\psi^2), QK^1(S^t;\mathbf{Z}_2^\wedge)) & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases} \tag{2.1}$$

Here $Q(-)$ denote the indecomposable quotient, and $\mathcal{A}$ is the category of stable 2-adic Adams modules.

It was proved in [4, 5.5] that simply-connected mod p finite H-spaces whose rational homology is associative satisfy the required hypothesis, and hence (2.1) applies to E_6. We remark again that although the spectral sequence is denoted as $E_2(X)$, it is only known to converge to $v_1^{-1}\pi_*(\widehat{X})$, and indeed even this is not known to be always true. It was proved in [7, 1.4] that, localized at any prime, if X is KASR, then the BTSS of X converges to $v_1^{-1}\pi_*(\widehat{X})$.

As input for the BTSS, we need the Adams module $K^*(X)$. We use $\mathbf{Z}/2$-graded K-theory. First is the general result of Hodgkin ([14]) which states that, for a simply-connected compact Lie group G, $K^*(G)$ is an exterior algebra on generators in $K^1(G)$ obtained from the fundamental representations of G. For E_6, there are six such generators. Proposition 2.2 follows from [10, 3.9], which gives ψ^k for all integers k. It can also be deduced from [18] after a lot of manipulation.

Proposition 2.2. *There is a basis for $QK^1(E_6)$ on which the matrices of ψ^{-1}, ψ^2, and ψ^3 are given by*

$$(\psi^{-1}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad (\psi^2) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 7 & 16 & 0 & 0 & 0 & 0 \\ -33 & 0 & 32 & 0 & 0 & 0 \\ 7 & 0 & -8 & 128 & 0 & 0 \\ -4 & -1 & 4 & 64 & 256 & 0 \\ 1 & 0 & -1 & -24 & 0 & 2048 \end{pmatrix},$$

$$(\psi^3)=\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0\\ 39 & 81 & 0 & 0 & 0 & 0\\ -264 & 0 & 3^5 & 0 & 0 & 0\\ 147 & 0 & -162 & 3^7 & 0 & 0\\ -87 & -27 & 81 & 3^7 & 3^8 & 0\\ 82 & 0 & -81 & -3^7 & 0 & 3^{11}\end{pmatrix}$$

Here each column expresses ψ^k of a basis element in terms of the basis elements.

By [4, 3.10],

$$\mathrm{Ext}_{\mathcal{A}}^{1,2n+1}(M)^{\#}\approx M/\operatorname{im}(\psi^{-1}-(-1)^n,\psi^3-3^n). \qquad (2.3)$$

Thus the Pontryjagin dual of $E_2^{1,2n+1}(E_6)$ is the abelian group presented by the 18×6 matrix

$$\begin{pmatrix} (\psi^2)^T\\ (\psi^{-1})^T-(-1)^n I\\ (\psi^3)^T-3^n I\end{pmatrix},$$

where $(\psi^k)^T$ refers to the transpose of the matrices of 2.2. Using `Maple`, we simplify the relations, removing generators one-at-a-time. For example, we begin by pivoting on the 1 in position (1,6), and then removing the top row, which expresses the sixth generator in terms of the others, and the last column, since that generator is no longer required. It is convenient to do separate reductions for the two parities of n.

If $n=2k$ is even, it is most efficient to express entries in terms of $R:=3^{2k}-3^{72}$. Thus 3^n is replaced by $R+3^{72}$ at the outset. Of course, it is only in hindsight that one can think to do this. We have $\nu(R)=\nu(k-36)+3$. We can eliminate all but the first column, and five nontrivial relations remain on this single generator. Omitting odd coefficients, these are

$$2^{12}+2^6R$$
$$2^{25}+2^{14}R$$
$$2^{12}+2^1R$$
$$2^{16}+2^1R+R^2$$
$$2^{13}+2^5R$$

If $\nu(R)<11$, then the relation with the smallest 2-exponent is the third (or fourth), giving $2^{\nu(R)+1}$ as the single relation, the order of the cyclic group. If $\nu(R)\geq 11$, then the smallest relation is the first, giving 2^{12} as the order of the cyclic group. These combine to give $\min(12, \nu(k-36)+4)$ as the exponent of this cyclic 2-group, and this gives the 1-line groups in Diagram 1.3 when $t-s=8\ell$ or $8\ell+4$.

Similarly, if $n=2k+1$ is odd, it turns out to be most efficient to write the entries of the matrix in terms of $Q := 3^{2k+1}-3^{11}$. Then $\nu(Q)=\nu(k-5)+3$. The matrix again reduces to one column, with relations (omitting odd coefficients)

$$\begin{aligned}&2^{14}+2^6Q\\&2^{18}+2^{14}Q\\&2^{12}+2^8Q+Q^2\\&2^{12}+2^6Q+Q^2\\&2^7Q+2^3Q^2\end{aligned}$$

One easily checks that this gives $\min(12, 2\nu(k-5)+6)$ as the minimal 2-exponent, and this yields the 1-line groups in $t-s=8\ell\pm 2$. We summarize with

Proposition 2.4. *The nonzero groups on the 1-line of the BTSS of E_6 are given by*

$$E_2^{1,2n+1}(E_6) \approx \begin{cases}\mathbf{Z}/2^{\min(12,\nu(k-36)+4)} & n=2k\\ \mathbf{Z}/2^{\min(12,2\nu(k-5)+6)} & n=2k+1.\end{cases}$$

Next, we deal with the 2-line groups. By [4, 3.10], if M is a finite stable 2-adic Adams module and $\theta_n=\psi^3-3^n$, then there is a short exact sequence

$$0\to \operatorname{coker}(\theta_n|Q_n(M)) \to \operatorname{Ext}_{\mathcal{A}}^{2,2n+1}(M)^{\#} \to \ker(\theta_n|\operatorname{coker}(\psi^{-1}-(-1)^n))|_M \to 0, \tag{2.5}$$

where the functor Q_n is defined by

$$Q_n = \frac{\ker(\psi^{-1}-(-1)^n)}{\operatorname{im}(\psi^{-1}+(-1)^n)}. \tag{2.6}$$

We evaluate the right-hand part of (2.5). A mild surprise occurred here in that the group is not always cyclic, as had been the case in most other examples.

Proposition 2.7. *For $M = QK^1(E_6)/\operatorname{im}(\psi^2)$,*

$$\ker(\theta_n|\operatorname{coker}(\psi^{-1}-(-1)^n)) \approx \begin{cases}\mathbf{Z}/2\oplus\mathbf{Z}/2^{\min(11,\nu(k-36)+3)} & n=2k\\ \mathbf{Z}/2^{\min(12,2\nu(k-5)+6)} & n=2k+1.\end{cases}$$

Proof. Since the kernel and cokernel of an endomorphism of a finite abelian group have the same order, the group has the same order as the corresponding 1-line group given in Proposition 2.4.

We begin with the case $n = 2k+1$. We argue similarly to [4, 4.4]. Let $\overline{M} := M/\operatorname{im}(\psi^{-1}+1)$, $\overline{M}_2 = \ker(\cdot 2|\overline{M})$, and $K = \ker((\psi^3 - 3^n)|\overline{M})$. We will show that $\dim(\overline{M}_2 \cap K) = 1$, which implies that K is cyclic. Using 2.2, the abelian group $\overline{M}$ can be presented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 7 & -33 & 7 & -4 & 1 \\ 0 & 16 & 0 & 0 & -1 & 0 \\ 0 & 0 & 32 & -8 & 4 & -1 \\ 0 & 0 & 0 & 128 & 64 & -24 \\ 0 & 0 & 0 & 0 & 0 & 2048 \end{pmatrix}$$

If $\{v_1, \ldots, , v_6\}$ denotes the basis on which this matrix presents relations, then a basis for $\overline{M}_2$ is given by $\psi^2(v_4)/2$ and $\psi^2(v_6)/2$, the last two rows divided by 2. Using 2.2, we have in $\overline{M}$

$$(\psi^3 - 1)(\psi^2(v_6)/2) = \psi^2(\psi^3-1)(v_6)/2 = \psi^2(\tfrac{3^{11}-1}{2}v_6) \equiv 0$$

and similarly

$$(\psi^3 - 1)(\psi^2(v_4)/2) = \psi^2(\tfrac{3^7-1}{2}v_4 + \tfrac{3^7}{2}v_5 - \tfrac{3^7}{2}v_6) \equiv \psi^2(v_6)/2 \not\equiv 0.$$

Thus $\overline{M}_2 \cap K = \langle \frac{\psi^2 v_6}{2} \rangle$.

The case $n = 2k$ is slightly more delicate. Let $\widetilde{M} = M/\operatorname{im}(\psi^{-1}-1)$; its presentation matrix can be reduced to

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 16 & 0 & -1 & -1 & 0 & 0 \\ 32 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 2^{12} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

Thus $\widetilde{M}$ is spanned by v_1 and v_3 of order 2^{12} and 2, respectively. Using 2.2 and the relations in $\widetilde{M}$, we have

$$(\psi^3 - 3^{2k})v_3 \equiv (3^5 - 3^{2k})v_3 - 162(16v_1 - v_3) + 81 \cdot 32 v_1 \equiv 0,$$

since $2v_3 = 0$. Thus K has a $\mathbf{Z}/2$-summand generated by v_3. Similarly

$(\psi^3-3^{2k})v_1 = (3-3^{2k})v_1+39\cdot 2v_1+147(16v_1-v_3)-87\cdot 32v_1 \equiv v_3-(3^{2k}+351)v_1.$

Thus v_1 is not in K, but $2v_1$ has a chance. Since $351 = 2^{11}u - 3^{72}$ with u odd,

$$(\psi^3 - 3^{2k})(2v_1) = 2(3^{72} - 3^{2k} - 2^{11}u)v_1 = 2^{3+\nu(2k-72)}u'v_1,$$

with u' odd, where $2^{12}v_1 = 0$. Thus $2^{\max(0,12-(3+\nu(2k-72)))}2v_1$ generates the other summand, which will have 2-exponent $\min(12-1, 3+\nu(2k-72)-1)$, as claimed. ■

In Section 4, we will prove that a failure of (2.5) to split compensates for the unexpected splitting in Proposition 2.7, yielding

Proposition 2.9. *The nonzero groups on the 2-line of the BTSS of E_6 are given by*

$$E_2^{2,2n+1}(E_6) \approx \mathbf{Z}/2 \oplus \begin{cases} \mathbf{Z}/2^{\min(12,\nu(k-36)+4)} & n = 2k \\ \mathbf{Z}/2^{\min(12,2\nu(k-5)+6)} & n = 2k+1. \end{cases}$$

Finally, we determine the eta-towers. This refers to elements in filtration > 2, all of which occur in families related by h_1, also known as η. ([4, 3.6]) With Q_n as defined in (2.6), we have, from [4, 3.10], for $s > 2$, a short exact sequence

$$0 \to \operatorname{coker}(\theta_n|Q_{s+n}(M)) \to \operatorname{Ext}_{\mathcal{A}}^{s,2n+1}(M)^{\#} \to \ker(\theta_n|Q_{s+n-1}(M)) \to 0. \tag{2.10}$$

We establish

Proposition 2.11. *For $M = QK^1(E_6)/\operatorname{im}(\psi^2)$, $Q_n(M) \approx \mathbf{Z}/2$, generated by v_3 if n is odd, and by $2^{17}v_1$ if n is even.*

Proof. If n is odd, $Q_n(M)$ equals $\ker(\psi^{-1}+1 : \widetilde{M} \to M)$, where $\widetilde{M}$ is presented by (2.8). Then $\widetilde{M} \approx \mathbf{Z}/2 \oplus \mathbf{Z}/2^{12}$, with generators v_3 and v_1. From 2.2, we see that $v_3 \in \ker(\psi^{-1}+1)$; however, $(\psi^{-1}+1)(2^{11}v_1) = 2^{11}v_2$, which is nonzero in M. One way to see that $2^{11}v_2 \neq 0$ is to use pivoting to obtain the following alternate presentation for M, with columns still $v_1, \dots, v_6$.

$$\begin{pmatrix} 16 & -8 & -40 & 0 & 0 & 1 \\ 0 & 16 & 0 & 0 & -1 & 0 \\ 2 & 7 & -1 & -1 & 0 & 0 \\ 2^7 5 & 2^6 27 & -2^6 17 & 0 & 0 & 0 \\ 0 & 2^{12} & 0 & 0 & 0 & 0 \\ 2^{18} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.12}$$

If n is even, we want $\ker(\psi^{-1} - 1 : \overline{M} \to M)$, where $\overline{M}$ is presented by (2.12) with the second and fifth columns omitted. We check $\psi^{-1} - 1$ on the elements of order 2. We have $2^{17} v_1 \mapsto -2^{18} v_1 + 2^{17} v_2 = 0$, while

$$2^6 5 v_1 - 2^5 17 v_3 \mapsto -2^7 5 v_1 + 2^6 5 v_2 + 2^6 17 v_3 = 2^6(5 + 27) v_2 = 2^{11} v_2,$$

which has order 2 in M. ■

If X is a topological space, let $Q_n(X) = Q_n(QK^1(X)/\operatorname{im}(\psi^2))$.

Corollary 2.13. *For $s > 2$, $E_2^{s,2n+1}(E_6)^\# \approx \mathbf{Z}/2 \oplus \mathbf{Z}/2$, with generators $2^{17} v_1$ and v_3.*

Proof. We use (2.10) and Proposition 2.11, and 2.2 to show that $\theta_n(v_3) = 0$ in $Q_{\text{od}}(E_6)$ and $\theta_n(2^{17} v_1) = 0$ in $Q_{\text{ev}}(E_6)$. Indeed, $\theta_n(v_3) = 2A v_3 - 162 v_4 + 81 v_5 - 81 v_6$, which is 0 in a group presented by (2.8), while $\theta_n(2^{17} v_1) = 0$ in a group presented by (2.12) since there $2^{18} v_1 = 0$ and $2^{17} v_i = 0$ for $2 \leq i \leq 6$. ■

This, with 2.4 and 2.9, completes the determination of the E_2-term, which is as suggested by Diagram 1.3. The h_1-extensions from the 1-line will be discussed in Section 4.

3. d_3-DIFFERENTIALS

In this section, we determine the d_3-differentials on the eta towers in the BTSS of E_6. An eta tower consists of elements in $E_2^{s,2s+i}$ for $s \geq s_0$ connected by h_1. Here $s_0 = 1$, 2, or 3. We denote by $\eta_i(X)$ the $\mathbf{Z}_2$-vector space of eta towers passing through $E_2^{s,2s+i}(X)$ for $s > 2$. Note that d_3 is a homomorphism from $\eta_i(X)$ to $\eta_{i-4}(X)$. With $E_2^{s,t}$ depicted as usual in position $(x, y) = (t - s, s)$, then $\eta_i(X)$ is a tower of elements whose position satisfies $x - y = i$. Since $\eta^4 = 0$ in homotopy, d_3-differentials must annihilate all eta towers, except for a few elements at the bottom of the target tower. By (2.10), if $QK^1(X)$ consists of elements whose dimensions are all of the same parity, then so does $\eta_*(X)$, and $\eta_i(X)$ depends only on i mod 4. What must be determined for each family of eta towers is the mod 8 value of i for which $d_3 : \eta_i(X) \to \eta_{i-4}(X)$ is nonzero.

We make heavy use of the fibration

$$F_4 \to E_6 \to EIV, \tag{3.1}$$

where EIV is the group quotient E_6/F_4. This fibration was studied quite thoroughly in [18]. It was observed there that

$$H^*(EIV; \mathbf{Z}) \approx \Lambda(x_9, x_{17}),$$

and the Serre spectral sequence of (3.1) collapses. From this, the spectral sequence

$$H^*(EIV; K^*(F_4)) \Rightarrow K^*(E_6)$$

implies that there is a short exact sequence

$$0 \to QK^1(EIV) \to QK^1(E_6) \to QK^1(F_4) \to 0, \tag{3.2}$$

with each of the three algebras $K^*(X)$ being exterior. The following key proposition implies that the Adams operations in $K(EIV)$ are as they would be if (1.1) existed.

Proposition 3.3. *There is a basis $\{y_1, y_2\}$ for $QK^1(EIV)$ on which for all k*

$$\psi^k(y_1) = k^4 y_1 + u\tfrac{k^4-k^8}{16} y_2, \tag{3.4}$$

with u a unit in $\mathbf{Z}_{(2)}$, and $\psi^k(y_2) = k^8 y_2$.

Proof. This can be deduced from [18, Thm 2], which computes the Chern character for EIV. We present a proof closer to our methods.

From (3.2), we deduce $QK^1(EIV) = \ker(QK^1(E_6) \to QK^1(F_4))$. This must be the subspace of $QK^1(E_6)$ spanned by v_2 and v_5 (in the basis used in Section 2). Perhaps the easiest way to see this is to use (ψ^{-1}) of 2.2, which shows that $\psi^{-1} = 1$ on this subspace, while it equals -1 on $QK^1(F_4)$.

From 2.2, we find that $240v_2 + v_5$ is an eigenvector for ψ^2 with eigenvalue 16. Thus, considering the rational splitting of E_6 as a product of spheres as in [10], we deduce that $\psi^k(240v_2 + v_5) = k^4(240v_2 + v_5)$ and $\psi^k(v_5) = k^8 v_5$ for all integers k. Thus $\psi^k v_2 = k^4 v_2 + \frac{k^4-k^8}{240} v_5$, from which follows the result with $y_1 = v_2$ and $y_2 = v_5$. ∎

In [4, 4.2], it was shown that there is a basis $\{x_1, x_2, x_3, x_4\}$ of $QK^1(F_4)$ on which $\psi^{-1} = -1$ and the matrix of ψ^2 is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 32 & 0 & 0 \\ 1 & -8 & 128 & 0 \\ 0 & -1 & -24 & 2048 \end{pmatrix}$$

We easily verify the following result.

Proposition 3.5. *In terms of the bases introduced, the exact sequence (3.2) is given by* $y_1 \mapsto v_2$, $y_2 \mapsto v_5$, $v_1 \mapsto -x_1 - x_2$, $v_3 \mapsto x_2$, $v_4 \mapsto x_3$, *and* $v_6 \mapsto x_4$.

Let $\overline{K}(X) := QK^1(X)/\operatorname{im}(\psi^2)$. Since ψ^2 acts injectively in our modules, the Snake Lemma applied to (3.2) yields a short exact sequence

$$0 \to \overline{K}(EIV) \to \overline{K}(E_6) \to \overline{K}(F_4) \to 0 \tag{3.6}$$

and hence by (2.1) a long exact sequence

$$0 \to E_2^{1,t}F_4 \to E_2^{1,t}E_6 \to E_2^{1,t}EIV \to E_2^{2,t}F_4 \to E_2^{2,t}E_6 \to E_2^{2,t}EIV \to E_2^{3,t}F_4 \to . \tag{3.7}$$

In [4, 4.6], it was shown that

$$E_2^{s,t}(F_4) \approx \begin{cases} \mathbf{Z}/2 & s = 1,\ t \equiv 1(4) \\ \mathbf{Z}/2^f & s = 1,\ t = 4k+3 \\ \mathbf{Z}/2^f \oplus \mathbf{Z}/2 & s = 2,\ t = 4k+3 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & s = 2,\ t \equiv 1(4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & s > 2,\ t \text{ odd} \\ 0 & t \text{ even} \end{cases}$$

with $f = \min(12, 2\nu(k-5)+6)$. Using 3.3 and the method of [4, 4.11-4.12], we obtain

$$E_2^{s,t}(EIV) \approx \begin{cases} \mathbf{Z}/2 & s = 1,\ t \equiv 3(4) \\ \mathbf{Z}/2^e & s = 1,\ t = 4k+1 \\ \mathbf{Z}/2^e \oplus \mathbf{Z}/2 & s = 2,\ t = 4k+1 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & s = 2,\ t \equiv 3(4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & s > 2,\ t \text{ odd} \\ 0 & t \text{ even} \end{cases}$$

with $e = \min(12, \nu(k - 36 - 2^8) + 3)$. By just substituting the known E_2-groups into (3.7), we can deduce quite a bit about the morphisms; in particular, the rank of every morphism beginning with $E_2^{2,t}EIV \to E_2^{3,t}F_4$ is 1. This follows by counting the alternating sum of the exponents of the orders of the groups. Thus of the two eta-towers in $E_2^{s,t}(E_6)$ for any odd value of $t-2s$, one comes from F_4 and the other maps nontrivially to EIV.

We need to know more specifically which classes map across in (3.7). For this, we use the following commutative diagram of exact sequences, where $Q_n(X)$ is the functor of (2.6) applied to $M = \overline{K}X$.

$$\begin{array}{ccccccccc}
\xrightarrow{\theta} & Q_{s+n}(EIV) & \xrightarrow{\alpha_1} & \operatorname{Ext}_{\mathcal{A}}^{s,2n+1}(\overline{K}EIV)^{\#} & \xrightarrow{\beta_1} & Q_{s+n-1}(EIV) & \xrightarrow{\theta} \\
& \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & \\
\xrightarrow{\theta} & Q_{s+n}(E_6) & \xrightarrow{\alpha_2} & \operatorname{Ext}_{\mathcal{A}}^{s,2n+1}(\overline{K}E_6)^{\#} & \xrightarrow{\beta_2} & Q_{s+n-1}(E_6) & \xrightarrow{\theta} \\
& \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & \\
\xrightarrow{\theta} & Q_{s+n}(F_4) & \xrightarrow{\alpha_3} & \operatorname{Ext}_{\mathcal{A}}^{s,2n+1}(\overline{K}F_4)^{\#} & \xrightarrow{\beta_3} & Q_{s+n-1}(F_4) & \xrightarrow{\theta}
\end{array} \tag{3.8}$$

Proposition 3.9. *In (3.8) with $s+n$ even, $g_3(v_3) = x_2$ pulls back nontrivially to g_2. If $s+n$ is odd, then $g_3(2^{17}v_1) = 2^{17}x_1$ pulls back nontrivially to g_2. If $s+n$ is even (resp. odd), there is an element $z \in \operatorname{Ext}_{\mathcal{A}}^{s,2n+1}(\overline{K}EIV)^{\#}$ such that $\beta_1(z) = 2^7y_2$ (resp. y_1) and $f_2(z) = \alpha_2(2^{17}v_1)$ (resp. $\alpha_2(v_3)$).*

Proof. The exact sequence for F_4 is given in [4, 4.5]. Actually, it is more convenient here to use the presentation

$$\begin{pmatrix} 0 & -32 & 8 & 1 \\ 2 & 3 & 1 & 0 \\ 2^75 & 2^627 & 0 & 0 \\ 2^{18} & 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

for $\overline{K}(F_4)$ instead of that of [4, 4.2]. This one can be obtained from that one by pivoting. Dividing the third and fourth rows of (3.10) by 2 gives a basis of $Q_{\text{ev}}(F_4)$, and θ sends the first element to the second. This is a computation best done in `Maple`, using the rows of (3.10) to reduce $(\psi^3-1)(2^65x_1+2^627x_2)$, obtained from [4, 4.2], to $2^{17}x_1$. As in [4, 4.5], $Q_{\text{od}}(F_4) = \langle x_1, x_2 \sim x_3\rangle$ with θ sending the first to the second. Now the g_2-part of the proposition in straightforward, using Proposition 3.5.

We have $Q_{\text{od}}(EIV) \approx \mathbf{Z}/2$ with generator 2^7y_2, and $Q_{\text{ev}}(EIV) \approx \mathbf{Z}/2$ with generator y_1. One readily checks, using 3.5 and 2.11, that $f_1 = 0$ and $f_3 = 0$ in (3.8). Thus the morphism f_2 must be as claimed because of our previous observation that it has rank 1. ■

We can deduce d_3 on half of the eta towers in E_6 from its behavior in F_4. By Corollary 2.13, each $\eta_{\mathrm{od}}(E_6)$ has basis $\{2^{17}v_1, v_3\}$.

Proposition 3.11. *The differential* $d_3 : \eta_{8\ell+1}(E_6) \to \eta_{8\ell-3}(E_6)$ *sends the* v_3*-tower to the* v_3*-tower. The differential* $d_3 : \eta_{8\ell+7}(E_6) \to \eta_{8\ell+3}(E_6)$ *sends the* $2^{17}v_1$*-tower to the* $2^{17}v_1$*-tower.*

Proof. It was shown in [4, 4.13] that in the BTSS of F_4, $d_3 : \eta_{8\ell+1}(F_4) \to \eta_{8\ell-3}(F_4)$ is the "identity map," while $d_3 : \eta_{8\ell+3}(F_4) \to \eta_{8\ell-1}(F_4)$ sends the x_1-tower to the x_1-tower, and $d_3 : \eta_{8\ell+7}(F_4) \to \eta_{8\ell+3}(F_4)$ sends the $2^{17}x_1$-tower to the $2^{17}x_1$-tower. We dualize (3.8) and use 3.9 to see that the v_3-tower is in the image of $\eta_{4*+1}(F_4) \to \eta_{4*+1}(E_6)$, and that $\eta_{4*+3}(F_4) \to \eta_{4*+3}(E_6)$ sends the $2^{17}x_1$-tower to the $2^{17}v_1$-tower. Naturality of d_3 implies the result. ■

By using the inclusion map $S^9 \to EIV$, we deduce the third family of d_3-differentials.

Proposition 3.12. *The differential* $d_3 : \eta_{8\ell+3}(E_6) \to \eta_{8\ell-1}(E_6)$ *sends the* v_3*-tower to the* v_3*-tower.*

Proof. Using the dual of f_2 in Proposition 3.9, it suffices to show that $d_3 : \eta_{8\ell+3}(EIV) \to \eta_{8\ell-1}(EIV)$ sends the y_1-tower to the y_1-tower. The map $j : S^9 \to EIV$ induces

$$QK^1 EIV \xrightarrow{j^*} QK^1 S^9$$

satisfying $j^*(y_1) = g$ and $j^*(y_2) = 0$. The generator $g \in QK^1 S^9$ gives rise to the stable eta towers in BTSS(S^9), and by [1, p.58] or [3, p.488] $d_3 : \eta_{8\ell+3}(S^9) \to \eta_{8\ell-1}(S^9)$ sends the stable eta tower to the stable eta tower. Our conclusion follows by naturality using the map j. ■

The final d_3 on eta towers requires a more elaborate argument. The conclusion is that d_3 is as it would be if the fibration (1.1) existed. Our desired result is

Proposition 3.13. *The differential* $d_3 : \eta_{8\ell+1}(E_6) \to \eta_{8\ell-3}(E_6)$ *sends the* $2^{17}v_1$*-tower to the* $2^{17}v_1$*-tower.*

Proof. By Proposition 3.9, it suffices to show $\eta_{8\ell+1}(EIV) \to \eta_{8\ell-3}(EIV)$ sends the 2^7y_2-tower to the 2^7y_2-tower. This is a consequence of the following two propositions. ■

Proposition 3.14. *There is an isomorphism of BTSSs*

$$E_r^{s,t}(\Omega EIV) \approx E_r^{s,t+1}(EIV)$$

for $r \geq 2$.

Proposition 3.15. *The differential* $d_3 : \eta_{8\ell}(\Omega EIV) \to \eta_{8\ell-4}(\Omega EIV)$ *sends the* 2^7y_2*-tower to the* 2^7y_2*-tower.*

Proof of Proposition 3.14. We need the following result.

Proposition 3.16. *The algebra* $K_*(\Omega EIV)$ *is a polynomial algebra on two classes in* $K_0(-)$.

Proof. We first note that $H_*(\Omega EIV)$ is a polynomial algebra on classes in $H_8(-)$ and $H_{16}(-)$. To see this, we first use the computations of the rings $H_*(\Omega F_4; R)$ and $H_*(\Omega E_6; R)$ for $R = \mathbf{Z}/2$ in [15, 2.3] and for $R = \mathbf{Z}/3$ in [13, Thm.1], which imply that $H_*(\Omega EIV; R)$ is polynomial on 8- and 16-dimensional generators. With coefficients $\mathbf{Q}$ or $\mathbf{Z}/p$ for $p > 3$, this is also true because the localizations of F_4 and E_6 are products of spheres whose dimensions are well known. The result with integral coefficients follows by standard methods.

Now the Atiyah-Hirzebruch spectral sequence $H_*(\Omega EIV; K_*) \Rightarrow K_*(\Omega EIV)$ implies the result for $K_*(-)$. ■

Returning to the proof of 3.14, we adapt an argument first used in [2, 6.1] to show that for the BP-based unstable Novikov spectral sequence (UNSS) $E_2^{s,t}(\Omega S^{2n+1}) \approx E_2^{s,t+1}(S^{2n+1})$. It was shown in [5, 3.3] that the same argument works if S^{2n+1} is replaced by a space Y for which $BP_*(\Omega Y)$ is the polynomial algebra on classes $\{y_{2i}\}$, and BP_*Y is isomorphic to an exterior algebra on the primitive classes σy_{2i}. It was noted in [6, 4.12] that these arguments can be adapted to the K-based BTSS. This, with Proposition 3.16, yields the proposition when $r = 2$.

To prove the result for all r, we note that the isomorphism of E_2-terms is induced by a map of towers. To see this, we use the following natural map of augmented cosimplicial spaces, where $K(X) = \Omega^\infty(K \wedge \Sigma^\infty X)$.

$$\begin{array}{ccccccccc}
\Omega X & \longrightarrow & K\Omega X & \rightrightarrows & K(K(\Omega X)) & \Rrightarrow & K^3\Omega X & \Rrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega X & \longrightarrow & \Omega K X & \rightrightarrows & \Omega K(KX) & \Rrightarrow & \Omega K^3 X & \Rrightarrow & \cdots
\end{array}$$

Applying $\pi_*(-)$ and taking homology of the alternating sum to the first row yields $E_2^{*,*}(\Omega X)$, and doing this to the second yields $E_2^{*,*-1}(X)$. The induced morphism in homology is the E_2 isomorphism observed above. But these cosimplicial spaces give rise, by the Tot construction, to the towers that define the entire spectral sequence, and so the morphism induces a morphism of spectral sequences. ■

Proof of Proposition 3.15. Let F denote the fiber of the inclusion map $S^9 \to EIV$. The Serre spectral sequence of this fibration shows that H^*F, like $H^*(\Omega S^{17})$, is a divided polynomial algebra on a 16-dimensional class. Thus $H_*F = \langle x_{16i} \rangle$ with

$$\psi(x_{16i}) = \sum \binom{i}{j} x_{16j} \otimes x_{16(i-j)}.$$

It follows that the $\mathbf{Z}/2$-graded K_*F has the same coalgebra structure, although the grading is lost. Thus there is an abstract isomorphism of coalgebras $K_*F \approx K_*(\Omega S^{17})$.

This implies that the fibration

$$\Omega S^9 \to \Omega EIV \to F$$

induces a relatively injective extension sequence

$$0 \to K_*(\Omega S^9) \to K_*(\Omega EIV) \xrightarrow{p_*} K_*(F) \to 0,$$

and hence, by [2, 4.3], a long exact sequence of BTSS, commuting with d_3

$$\to E_2^s(\Omega S^9) \to E_2^s(\Omega EIV) \xrightarrow{p_*} E_2^s(F) \to E_2^{s+1}(\Omega S^9) \to .$$

We remark here that the notion of relatively injective extension sequence was defined in [4, 5.14]. It is the notion which was intended in [2, 4.3].

The Hurewicz Theorem gives a map $f : S^{16} \to F$, and we have the map $p : \Omega EIV \to F$ from the above fibration. Both of these maps induce morphisms of

BTSS, commuting with d_3. The image under p_* of the eta towers 2^7y_2 equal the image under f_* of eta towers in S^{16} which map, in the EHP sequence, to the unstable eta towers in $\eta_{4*+1}(S^{17})$. By [1, p.58] or [3, p.488], the d_3-differential on the unstable eta towers in S^{17} is nonzero from $\eta_{8\ell+1}(S^{17})$ to $\eta_{8\ell-3}(S^{17})$. Hence it is nonzero from $\eta_{8\ell}(S^{16})$ to $\eta_{8\ell-4}(S^{16})$, hence also in F, and thence in ΩEIV. ■

4. Fine Tuning

In this section we determine the d_3-differentials from the 1-line, prove Proposition 2.9, and show that there are no exotic extensions $(\cdot 2)$ in the BTSS.

In general, the h_1-action on the 1-line of the BTSS can be delicate. See, e.g., [4, 4.15]. It is important, because one way to determine d_3 on the 1-line is to determine d_3 on the eta towers and then use the h_1-action from the 1-line to the eta towers to deduce d_3 on the 1-line. A different method was used for $Sp(n)$ in [5] and illustrated in [5, 4.3] the subtle things that can happen. In that case, the relevant 2-line group contained some classes supporting d_3-differentials and some which did not, and we argued indirectly to see whether d_3x was nonzero, which was equivalent to determining whether h_1x contained any classes of the first type.

In our situation here, the 1-line classes in E_6 map to or from classes in EIV or F_4 on which h_1 and d_3 are known, from which we deduce the following result, which will be proved along with Proposition 2.9.

Proposition 4.1. *The h_1-extensions and d_3-differentials from the 1- and 2-lines are as depicted in Diagram 1.3.*

Proof of Propositions 2.9 and 4.1. We compare the exact sequences in $(\mathrm{Ext}_{\mathcal{A}}^{*,2n+1})^{\#}$ and $(\mathrm{Ext}_{\mathcal{G}Inv}^{*,2n+1})^{\#}$ induced by (3.6). Here $\mathcal{G}Inv$ denotes the category of profinite abelian groups with involution. We will also use Inv, the category of abelian groups with involution.

The analysis is much more delicate when $n = 2k$ is even, and so we focus on this case. Some details of the proof (but not final conclusions) are slightly different when $\nu(k-36) = 8$ due to the $\nu(k-36-2^8)$ which occurs in $E_2(EIV)$, so we assume $\nu(k-36) \neq 8$ to simplify the exposition.

The exact sequences (2.5) and (2.10) are obtained in [4, §3] from a long exact sequence

$$\leftarrow \mathrm{Ext}^s_{\mathcal{A}}(M,N)^\# \leftarrow \mathrm{Ext}^s_{\mathcal{G}Inv}(M,N)^\# \xleftarrow{\psi^3_M-\psi^3_N} \mathrm{Ext}^s_{\mathcal{G}Inv}(M,N)^\# \leftarrow \mathrm{Ext}^{s+1}_{\mathcal{A}}(M,N)^\# \leftarrow$$

and isomorphisms

$$\begin{aligned}\mathrm{Ext}^{s+1,2n+1}_{\mathcal{G}Inv}(M)^\# &= \mathrm{Ext}^{s+1}_{\mathcal{G}Inv}(M,QK^1S^{2n+1})^\# \\ &\approx \mathrm{Ext}^s_{Inv}(\mathbf{Z}^{((-1)^n)}_{(2)},M^\#)^\# \approx \begin{cases} Q_{s+n+1}(M) & s>0\\ M/(\psi^{-1}-(-1)^n) & s=0.\end{cases}\end{aligned}$$

Then

$$\begin{aligned}0 \leftarrow\ & \mathrm{Ext}^{1,4k+1}_{\mathcal{G}Inv}(\overline{K}F_4)^\# \leftarrow \mathrm{Ext}^{1,4k+1}_{\mathcal{G}Inv}(\overline{K}E_6)^\# \leftarrow \mathrm{Ext}^{1,4k+1}_{\mathcal{G}Inv}(\overline{K}EIV)^\# \\ \leftarrow\ & \mathrm{Ext}^{2,4k+1}_{\mathcal{G}Inv}(\overline{K}F_4)^\# \leftarrow \mathrm{Ext}^{2,4k+1}_{\mathcal{G}Inv}(\overline{K}E_6)^\# \leftarrow \mathrm{Ext}^{2,4k+1}_{\mathcal{G}Inv}(\overline{K}EIV)^\# \leftarrow \mathrm{Ext}^{3,4k+1}_{\mathcal{G}Inv}(\overline{K}F_4)^\#\end{aligned}$$

is

$$\begin{aligned}0 \leftarrow\ & \overline{K}F_4/(\psi^{-1}-1) \leftarrow \overline{K}E_6/(\psi^{-1}-1) \leftarrow \overline{K}EIV/(\psi^{-1}-1) \\ \leftarrow\ & Q_{\mathrm{ev}}(\overline{K}F_4) \leftarrow Q_{\mathrm{ev}}(\overline{K}E_6) \leftarrow Q_{\mathrm{ev}}(\overline{K}EIV) \leftarrow Q_{\mathrm{od}}(\overline{K}F_4).\end{aligned}$$

Using results and notation from Sections 2 and 3, this sequence becomes, with $x_c = 2^6 5x_1 + 2^5 27x_2$,

$$0 \leftarrow \langle x_2\rangle \leftarrow \langle v_3, \mathbf{v}_1\rangle \leftarrow \langle \mathbf{y}_1\rangle \leftarrow \langle x_c, 2^{17}x_1\rangle \leftarrow \langle 2^{17}v_1\rangle \leftarrow \langle y_1\rangle \leftarrow \langle x_1,x_2\rangle,$$

with all elements having order 2 except $\mathbf{v}_1$ and $\mathbf{y}_1$, which have order 2^{12}. Using 3.5, the explicit morphisms, yielding an exact sequence, are

$$\begin{array}{ccccccccccc} x_2 & \hookleftarrow & v_3 & & 0 & \hookleftarrow & 2^{17}x_1 & \hookleftarrow & 2^{17}v_1 & & \\ x_1+x_2 & \hookleftarrow & \mathbf{v}_1 & & 2^{11}\mathbf{y}_1 & \hookleftarrow & x_c & & 0 & \hookleftarrow\ y_1\ \hookleftarrow & x_1 \\ & & 2\mathbf{v}_1 & \hookleftarrow & \mathbf{y}_1 & & & & & & \end{array} \qquad (4.2)$$

With $\nu = \min(12, \nu(k-36)+3)$, the nonzero occurrences of $\psi^3 - 3^{2k}$ on the groups in this sequence are $x_1 \mapsto x_2$, $\mathbf{v}_1 \mapsto 2^\nu\mathbf{v}_1 + v_3$, $\mathbf{y}_1 \mapsto 2^\nu\mathbf{y}_1$, $x_c \mapsto 2^{17}x_1$, and $x_1 \mapsto x_2$. The reader can verify that these commute with the morphisms of (4.2).

Now the short exact sequences of (2.5) and (2.10), and the isomorphism (2.3), written vertically, become as follows. We write $E_2^s(X)$ for $\mathrm{Ext}^{s,4k+1}_{\mathcal{A}}(\overline{K}X)$, and denote by $x(2^e)$ an element x of order 2^e. Elements not followed by parentheses have order

2.

$$\begin{array}{ccccccc}
x_1 & \leftarrow & \mathbf{v}_1(2^{\nu+1}) & \stackrel{\cdot 2}{\longleftarrow} & \mathbf{y}_1(2^\nu) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
E_2^1(F_4)^\# & \leftarrow & E_2^1(E_6)^\# & \leftarrow & E_2^1(EIV)^\# & \leftarrow &
\end{array}$$

$$\begin{array}{ccccccc}
x_c & & 2^{17}v_1 & & y_1 & \leftarrow & x_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_2^2(F_4)^\# & \leftarrow & E_2^2(E_6)^\# & \leftarrow & E_2^2(EIV)^\# & \leftarrow & E_2^3(F_4)^\# \\
\downarrow & & \downarrow & & \downarrow & & \\
x_2 & \leftarrow & v_3, 2^{12-\nu}\mathbf{v}_1(2^\nu) & \stackrel{2}{\longleftarrow} & 2^{12-\nu}\mathbf{y}_1(2^\nu) & &
\end{array}$$

In order for the $E_2^s(-)^\#$ sequence to be exact, $2^{12-\nu}\mathbf{v}_1$ must hit x_c in $E_2^2(F_4)^\#$, and $2^{11}\mathbf{y}_1$ must hit $2^{17}v_1$ in $E_2^2(E_6)^\#$. The latter implies the nontrivial extension in $E_2^2(E_6)^\#$, that it contains a $\mathbf{Z}/2^{\nu+1}$ with $2^{17}v_1$ the element of order 2 in this summand. This concludes the proof of 2.9.

The above analysis showed $E_2^{1,4k+1}(E_6)^\# \leftarrow E_2^{1,4k+1}(EIV)^\#$ injective; hence

$$E_2^{1,4k+1}(E_6) \to E_2^{1,4k+1}(EIV)$$

is surjective. In 3.9 and the proof of 3.12, we showed that the stable eta tower in $E_2^{*,4k+2*-1}(E_6)$ maps to the one in EIV which comes from S^9. If $\nu(k) \le 1$, $E_2^{1,4k+1}(S^9) \to E_2^{1,4k+1}(EIV)$ is an isomorphism. So the h_1-extension and d_3-differential from $E_2^{1,8\ell+5}(S^9)$ implies the same in E_6. A nonzero h_1-extension from $E_2^{1,8\ell+1}(E_6)$ is implied when ℓ is odd. It is not so important because d_3 must be 0 on $E_2^{1,8\ell+1}(E_6)$ since h_1^2 times the $\mathbf{Z}/2$ in $(8\ell+1, 2)$ is in the image of d_3. It is likely that an adaptation of [4, 3.7] to modules which do not satisfy $\psi^{-1} = -1$ would allow the determination of h_1 on $E_2^{1,8\ell+1}(E_6)$.

Since $E_2^{1,4k+3}(F_4) \to E_2^{1,4k+3}(E_6)$ is an isomorphism, the h_1-extensions and d_3-differentials from $E_2^{1,4k+3}(E_6)$ follow from those in F_4. In [4, 4.14], it was shown that h_1 on $E_2^{1,4k+3}(F_4)$ hits the stable eta tower iff $\nu(k-5) \ne 3$ and hits the unstable eta tower iff $\nu(k-5) = 3$. This implies the nonzero h_1 and d_3 from $E_2^{1,8\ell+3}(E_6)$ pictured in Diagram 1.3, and that h_1 is nonzero from $E_2^{1,8\ell-1}(E_6)$ unless $\ell \equiv 7 \bmod 8$, in which case it is 0. The latter is because the unstable eta tower in F_4 in this dimension maps to 0 in E_6. That $h_1 x$ is sometimes 0 here is mildly interesting because in [4, 3.8] it was shown that if $\psi^{-1} = -1$ then h_1 acts injectively on $E_2^1(X)/2$.

The analysis of exact sequences earlier in this proof implied that $E_2^{2,4k+1}(E_6) \to E_2^{2,4k+1}(EIV)$ sends the $\mathbf{Z}/2^e$ summand surjectively, and that the generator is dual to $2^{17}v_1$, which is the name of the unstable eta tower whose d_3-differential is described in 3.13. The h_1-extension and d_3-differential on it follow immediately.

The exact sequence in $\operatorname{Ext}_{\mathcal{A}}^{1,4k+3}(-)$ induced by (3.6) easily implies that $E_2^{2,4k+3}(F_4) \to E_2^{2,4k+3}(E_6)$ sends the $\mathbf{Z}/2^f$ bijectively. The h_1-extension (for all k) and d_3-differential when k is odd follow from those in F_4 established in [4]. ■

Finally, we have the following result about extensions in the spectral sequence.

Proposition 4.3. *There are no nontrivial extensions* $(\cdot 2)$ *from* $E_2^{s,t}(E_6)$ *to* $E_2^{s+2,t+2}(E_6)$.

Proof. For $s = 1$ or 2, the group $E_2^{s,8\ell+3}(E_6) \approx \mathbf{Z}/2^6$ is mapped isomorphically onto from $E_2^{s,8\ell+3}(F_4)$. In [4, 4.14], it is shown that when $s = 1$ (resp. $s = 2$) there is a nonzero extension into the element of $E_2^{s+2,8\ell+5}(F_4)$ which is part of the stable (resp. unstable) eta tower. In Proposition 3.9, it is proved that these eta towers map trivially to $E_2^{s+2,8\ell+5}(E_6)$, hence this extension is 0 in E_6.

In the proof of Propositions 2.9 and 4.1, it was shown that for $s = 1$ and 2, the element of order 2 in $E_2^{s,8\ell+1}(E_6)$ is in the image from $E_2^{s,8\ell+1}(F_4)$, which does not support an extension in [4, 4.9]. ■

5. The CTP for E_6

In this brief section, we mention several issues related to the CTP for E_6. In other words, how might we prove that the groups $v_1^{-1}\pi_*(\widehat{E}_6)$, which we have computed, are in fact isomorphic to $v_1^{-1}\pi_*(E_6)$, which is what we really want?

There are no known examples of spaces which do not satisfy the CTP. On the other hand, the only spaces which are known to satisfy it are certain spaces which can, in some sense, be built from spheres by fibrations.

Localized at an odd prime, the result of [7] which states that a space which is KASR and K-durable satisfies the CTP is very useful. If the 2-primary analogue were known to be true, then we would know that E_6 satisfies the CTP. However, the proof of the result of [7] relies on much delicate machinery developed at the odd primes by Bousfield, especially in [8]. One crucial difference between the odd primes and the prime 2 is that $\mathrm{Ext}^s_{\mathcal{A}}(M,N)$ vanishes for $s>2$ when p is odd. This is important for existence and uniqueness of realizations of certain morphisms of Adams modules. An adaptation to the prime 2 would be far from straightforward.

A main worry in considering convergence of spectral sequences related to $v_1^{-1}\pi_*(X)$ is the possibility that there could be elements in $v_1^{-1}\pi_*(X)$ not seen by the spectral sequence because they correspond to a family of elements in $\pi_{n_i}(X)$ for $n_i\to\infty$, related to one another by $\cdot v_1^e$ and having increasing filtrations. The algebraic $\cdot v_1$ operation in the UNSS or BTSS preserves filtration, but the homotopy-theoretic operation can increase filtration. For S^{2n+1}, the E_∞-term of the v_1-periodic BTSS yields exactly the v_1-periodic homotopy classes, which we know because they were calculated by another method in [16] ($p=2$) and [17] (p odd). Spaces built from odd spheres, or perhaps their loop spaces, by fibrations can then guarantee nonexistence of undetected v_1-periodic families by exactness properties. The proof in [7] that at the odd primes X KASR and K-durable implies the CTP for X ultimately boils down to building the localization X_K from various S_K^{2n+1}.

The isomorphism $E_2^{s,t}(S^{2n+1})\approx E_2^{s,t-1}(\Omega S^{2n+1})$ implies that if a space is nicely fibered by ΩS^{2n+1}'s, then it satisfies the CTP. For spaces fibered by Ω^2S^{2n+1}, it is not so clear.

If it could be proved that F, the fiber of $S^9\to EIV$ considered in the proof of 3.15, has the same homotopy type as ΩS^{17}, then the CTP for E_6 could be deduced. In [9, 2.1], a map $\Omega^2S^{17}\xrightarrow{f}\Omega S^9$ was constructed. If it could be shown that the composite

$$\Omega^2S^{17}\xrightarrow{f}\Omega S^9\xrightarrow{\Omega i}\Omega EIV$$

is null-homotopic, then one could deduce existence of a fibration

$$\Omega^2S^9\to\Omega^2EI\to\Omega^2S^{17}.$$

It is likely that such a fibration would imply the CTP for E_6, although a detailed argument has not been produced.

References

[1] M. Bendersky, *The v_1-periodic unstable Novikov spectral sequence*, Topology **31** (1992) 47-64.

[2] M. Bendersky, E.B. Curtis, and D.C. Ravenel, *EHP sequences in BP theory*, Topology **21** (1982) 373-391.

[3] M. Bendersky and D. M. Davis, *2-primary v_1-periodic homotopy groups of $SU(n)$*, Amer Jour Math **114** (1991) 529-544.

[4] ———, *A stable approach to an unstable homotopy spectral sequence*, to appear.

[5] M. Bendersky, D. M. Davis, and M. Mahowald, *v_1-periodic homotopy groups of $Sp(n)$*, Pac Jour Math **170** (1995) 319-378.

[6] M. Bendersky and R. D. Thompson, *The Bousfield-Kan spectral sequence for periodic homology theories*, to appear in Amer Jour Math.

[7] ———, *Some properties of the K-completion*, to appear in JAMI Conference Proceedings.

[8] A. K. Bousfield, *The K-theory localization and v_1-periodic homotopy groups of H-spaces*, Topology **38** (1999) 1239-1264.

[9] F. R. Cohen and P. S. Selick, *Splittings of two function spaces*, Quar Jour Math Oxford **41** (1990) 145-153.

[10] D. M. Davis, *From representation theory to homotopy groups*, to appear in Mem Amer Math Soc.

[11] D. M. Davis and M. Mahowald, *Some remarks on v_1-periodic homotopy groups*, London Math Society Lecture Notes **176** (1992) 293-327.

[12] ———, *Three contributions to the homotopy theory of the exceptional Lie groups G_2 and F_4*, Jour Math Soc Japan **43** (1991) 661-671.

[13] H. Hamanaka and S. Hara, *The mod 3 homology of the space of loops on the exceptional Lie groups and the adjoint action*, Jour Math Kyoto Univ **37** (1997) 441-453.

[14] L. Hodgkin, *On the K-theory of Lie groups*, Topology **6** (1967) 1-36.

[15] A. Kono and K. Kozima, *The mod 2 homology of the space of loops on the exceptional Lie group*, Proc Royal Soc Edin **112A** (1989) 187-202.

[16] M. Mahowald, *The image of J in the EHP sequence*, Annals of Math **116** (1982) 65-112.

[17] R. D. Thompson, *The v_1-periodic homotopy groups of an unstable sphere at odd primes*, Trans Amer Math Soc **319** (1990) 535-559.

[18] T. Watanabe, *The Chern character homomorphism of the compact simply-connected exceptional Lie group E_6*, Osaka Jour Math **28** (1991) 663-681.

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