ALL p-LOCAL FINITE GROUPS OF RANK TWO FOR ODD PRIME p

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ABSTRACT. In this paper we give a classification of the rank two p-local finite groups for odd p. This study requires the analisis of the possible saturated fusion systems in terms of the outer automorphism group ant the proper $\mathcal{F}$-radical subgroups. Also, for each case in the classification, either we give a finite group with the corresponding fusion system or we check that it corresponds to an exotic p-local finite group, getting some new examples of these for $p = 3$.

1. Introduction

When studying the p-local homotopy theory of classifying spaces of finite groups, Broto-Levi-Oliver [11] introduced the concept of p-local finite group as a p-local analogue of the classical concept of finite group. These purely algebraic objects, whose basic properties are reviewed in Section 2, are a generalization of the classical theory of finite groups in the sense that every finite group leads to a p-local finite group, although there exist exotic p-local finite groups which are not associated to any finite group as it can be read in [11, Sect. 9], [30], [35] or Theorem 5.9 below. Besides its own interest, the systematic study of possible p-local finite groups, i.e. possible p-local structures, is meaningful when working in other research areas as transformation groups (e.g. when constructing actions on spheres [2, 1]), or modular representation theory (e.g. the study of the p-local structure of a group is a very first step when verifying conjectures like those of Alperin [3] or Dade [17]). It also provides an opportunity to enlighten one of the highest mathematical achievements in the last decades: The Classification of Finite Simple Groups [24]. A milestone in the proof of that classification is the characterization of finite simple groups of 2-rank two (e.g. [22, Chapter 1]) what is based in a deep understanding of the 2-fusion of finite simple groups of low 2-rank [21, 1.35, 4.88]. Unfortunately, almost nothing seems to be known about the p-fusion of finite groups of p-rank two for an odd prime p [19], and this work intends to remedy that lack of information by classifying all possible saturated fusion systems over finite p-groups of rank two, $p > 2$.

Theorem 1.1. *Let p be an odd prime and S be a rank two p-group. Given a saturated fusion system $(S, \mathcal{F})$, one of the following hold:*

- *$\mathcal{F}$ has no proper $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups and it corresponds to the group $S : \mathrm{Out}_{\mathcal{F}}(S)$.*

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- *$\mathcal{F}$ is the fusion system of a group G which fits in the following extension:*
$$1 \to S_0 \to G \to W \to 1$$
where S_0 is a subgroup of index p in S and W contains $\mathrm{SL}_2(p)$.
- *$\mathcal{F}$ is the fusion system of an extension of one of the following finite simple groups:*
  - *$L_3(p)$ for any p,*
  - *${}^2F_4(2)'$, J_4, $L_3^{\pm}(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$ for $p = 3$, where q is a $3'$ prime power,*
  - *Th for $p = 3$,*
  - *He, Fi_{24}', $O'N$ for $p = 7$ or*
  - *M for $p = 13$.*
- *$\mathcal{F}$ is an exotic fusion system characterized by the following data:*
  - *$S = 7_+^{1+2}$ and all the rank two elementary abelian subgroups are $\mathcal{F}$-radical,*
  - *$S = G(3, 2k; 0, \gamma, 0)$ and the only proper $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroups are one or two S-conjugacy classes of rank two elementary abelian subgroups,*
  - *$S = G(3, 2k + 1; 0, 0, 0)$ and the proper $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups are either one or two S-conjugacy classes of subgroups isomorphic to 3_+^{1+2}, either one S-conjugacy class of rank two elementary abelian subgroups and one subgroup isomorphic to $\mathbb{Z}/3^k + \mathbb{Z}/3^k$.*

Proof. For an odd prime p, the isomorphism type of a rank two p-group, namely S, is described in Theorem A.1, hence the proof is done by studying the different cases of S.

For $p > 3$, Theorems 4.1, 4.2 and 4.3 show that any saturated fusion system $(S, \mathcal{F})$ is induced by $S : \mathrm{Out}_{\mathcal{F}}(S)$, i.e. S is resistant (see Definition 3.1), unless $S \cong p_+^{1+2}$, hence [35] completes the proof for the $p > 3$ case.

For $p = 3$, Theorems 4.1, 4.2, 5.1 and 5.8 describe all rank two 3-groups which are resistant. The saturated fusion system over non resistant rank two 3-groups are then obtained in Theorems 4.8 and 5.9 when $S \ncong 3_+^{1+2}$, what completes the information in [35] and finishes the proof. □

It is worth noticing that along the proof of the theorem above some interesting contributions are made:

- The Appendix provides a neat compendium of the group theoretical properties of rank two p-groups, p odd, including a description of their automorphism groups, and p-centric subgroups. It does not only collect the related results in the literature [6, 7, 18, 19, 26], but extends them.
- Theorems 4.1, 4.2, 4.3, 5.1 and 5.8 identify a large family of resistant groups complementing the results in [37], and extending those in [32].
- Theorem 5.9 provides infinite families of exotic rank two p-local finite groups with arbitrary large Sylow p-subgroup. Unlikely the other known infinite families of exotic p-local finite groups [13, 30], some of these new families cannot be constructed as the homotopy fixed points of automorphism of a p-compact group. Nevertheless, it is still possible to construct an "ascending" chain of exotic p-local finite groups whose colimit, we conjecture, should provide an example of an exotic p-local compact group [12, Section 6].

Organization of the paper: Along Section 2 we quickly review the basics on p-local finite groups. In Section 3 we define the concept of resistant p-group, similar to that of Swan group, and develop some machinery to identify resistant groups. In Section 4 we study the fusion systems over non maximal nilpotency class rank two p-groups, while the study of fusion systems over maximal nilpotency class rank two p-groups is done in Section 5. We finish this paper with an Appendix collecting the group theoretical background on rank two p-groups which is needed along the classification.

Notation: By p we always denote an odd prime, and S a p-group of size p^r. The group theoretical notation used along this paper is that described in the Atlas [15, 5.2]. For a group G, and $g \in G$, we denote by c_g the conjugation morphism $x \mapsto g^{-1}xg$. If $P, Q \leq G$, the set of G-conjugation morphisms from P to Q is denoted by $\mathrm{Hom}_G(P, Q)$, so if $P = Q$ then $\mathrm{Aut}_G(P) = \mathrm{Hom}_G(P, P)$. Notice that $\mathrm{Aut}_G(G)$ is then $\mathrm{Inn}(G)$, the group of inner automorphisms of G. Given P and Q groups, Inj(P, Q) denotes the set of injective homomorphisms.

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2. p-LOCAL FINITE GROUPS

At the beginning of this section we quickly review the concept of p-local finite group introduced in [11] that builds on a previous unpublished work of L. Puig, where the axioms for fusion systems are already established. See [12] for a survey on this subject.

After that we study some particular cases where a saturated fusion system is controlled by the normalizer of the Sylow p-subgroup.

We end this section with some tools which allow us to study the exoticism of an abstract saturated fusion system.

Definition 2.1. A fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following two conditions:

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all P and Q subgroups of S.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

We say that two subgroups $P, Q \leq S$ are *$\mathcal{F}$-conjugate* if there is an isomorphism between them in $\mathcal{F}$. As all the morphisms are injective by condition (b), we denote by $\mathrm{Aut}_{\mathcal{F}}(P)$ the group $\mathrm{Hom}_{\mathcal{F}}(P, P)$. We denote by $\mathrm{Out}_{\mathcal{F}}(P)$ the quotient group $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_P(P)$.

The fusion systems that we consider are saturated, so we need the following definitions:

Definition 2.2. Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P.
- A subgroup $P \leq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P.
- $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:
  (I) Every fully normalized subgroup $P \leq S$ is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that φP is fully centralized, and if we set
$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

Remark 2.3. From the definition of fully normalized and the condition (I) of saturated fusion system we get that if $\mathcal{F}$ is a saturated fusion system over a p-group S then p cannot divide the order of the outer automorphism group $\mathrm{Out}_{\mathcal{F}}(S)$.

As expected, every finite group G gives rise to a saturated fusion system [11, Proposition 1.3], which provides valuable information about $BG^\wedge_p$ [33]. Some classical results for finite groups can be generalized to saturated fusion systems, as for example, Alperin's fusion theorem for saturated fusion systems [11, Theorem A.10]:

Definition 2.4. Let $\mathcal{F}$ be any fusion system over a p-group S. A subgroup $P \leq S$ is:

- *$\mathcal{F}$-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers.
- *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced, that is, if $\mathrm{Out}_{\mathcal{F}}(P)$ has no nontrivial normal p-subgroups.

Theorem 2.5 (Alperin's fusion theorem for saturated fusion systems)**.** *Let $\mathcal{F}$ be a saturated fusion system over S. Then for each morphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(P,P')$, there exists a sequence of subgroups of S*
$$P = P_0, P_1, \ldots, P_k = P' \quad \textit{and} \quad Q_1, Q_2, \ldots, Q_k,$$
and morphisms $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

- *Q_i is fully normalized in $\mathcal{F}$, $\mathcal{F}$-radical and $\mathcal{F}$-centric for each i;*
- *$P_{i-1}, P_i \leq Q_i$ and $\psi_i(P_{i-1}) = P_i$ for each i; and*
- *$\psi = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1$.*

The subgroups Q_i's in the theorem above determine the structure of $\mathcal{F}$, so they deserve a name:

Definition 2.6. Let $\mathcal{F}$ be any fusion system over a p-group S. We say that a subgroup $Q \leq S$ is *$\mathcal{F}$-Alperin* if it is fully normalized in $\mathcal{F}$, $\mathcal{F}$-radical and $\mathcal{F}$-centric.

The definition of p-local finite group still requires one more new concept.

Let $\mathcal{F}^c$ denote the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S.

Definition 2.7. Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor
$$\pi \colon \mathcal{L} \longrightarrow \mathcal{F}^c$$
and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions:

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection
$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.
(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$, the equality $\delta_Q(\pi(f)(g)) \circ f = f \circ \delta_P(g)$ holds in $\mathcal{L}$

Finally, the definition of p-local finite group is:

Definition 2.8. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a p-group, $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|^\wedge_p$.

Given a fusion system $\mathcal{F}$ over the p-group S, there exists an obstruction theory for the existence and uniqueness of a centric linking system, i.e. a p-local finite group, associated to $\mathcal{F}$. The question is solved for p-groups of small rank by the following result [11, Theorem E]:

Theorem 2.9. *Let $\mathcal{F}$ be any saturated fusion system over a p-group S. If $\mathrm{rk}_p(S) < p^3$, then there exists a centric linking system associated to $\mathcal{F}$. And if $\mathrm{rk}_p(S) < p^2$, then there exists a unique centric linking system associated to $\mathcal{F}$.*

As all p-local finite groups studied in this work are over rank two p-groups S, we obtain:

Corollary 2.10. *Let p be an odd prime. Then the set of p-local finite groups over a rank two p-group S is in bijective correspondence with the set of saturated fusion systems over S.*

In [11, Section 2] is defined the "centralizer" fusion system of a given fully centralized subgroup:

Definition 2.11. Let $\mathcal{F}$ be a fusion system over S and $P \leq S$ a fully centralized subgroup in $\mathcal{F}$. The *centralizer fusion system of P in $\mathcal{F}$, $C_{\mathcal{F}}(P)$* is the fusion system over $C_S(P)$ with objects $Q \leq C_S(P)$ and morphisms,

$$\begin{aligned}&\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q,Q') \\ &\quad = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,Q') | \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(QP,Q'P), \psi|_Q = \varphi, \psi|_P = \mathrm{Id}_P\}.\end{aligned}$$

Remark 2.12. If we consider the fusion system corresponding to a finite group G with Sylow p-subgroup S and $P \leq S$ such that $C_S(P) \in \mathrm{Syl}_p(C_G(P))$, i.e. P is fully centralized in $\mathcal{F}_S(G)$, then the fusion system $(C_S(P), C_{\mathcal{F}}(P))$ is the fusion system $(C_S(P), \mathcal{F}_{C_S(P)}(C_G(P)))$, so it is again the fusion system of a finite group.

In [11, Section 6] the "normalizer" fusion system of a given fully normalized subgroup is defined:

Definition 2.13. Let $\mathcal{F}$ be a fusion system over S and $P \leq S$ a fully normalized subgroup in $\mathcal{F}$. The *normalizer fusion system of P in $\mathcal{F}$, $N_{\mathcal{F}}(P)$* is the fusion system over $N_S(P)$ with objects $Q \leq N_S(P)$ and morphisms,

$$\begin{aligned}&\mathrm{Hom}_{N_{\mathcal{F}}(P)}(Q,Q') \\ &\quad = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,Q') | \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(QP,Q'P), \psi|_Q = \varphi, \psi|_P \in \mathrm{Aut}(P)\}.\end{aligned}$$

Also in [11, Section 6] it is proved that if $\mathcal{F}$ is a saturated fusion system over S and P is a fully normalized subgroup then $N_{\mathcal{F}}(P)$ is a saturated fusion system over $N_S(P)$.

Remark 2.14. For a fusion system $\mathcal{F}$ over S, when considering the normalizer fusion system over the own Sylow S, it turns out that $N_{\mathcal{F}}(S) = \mathcal{F}$ if and only if every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, Q')$ extends to $\psi \in \mathrm{Aut}_{\mathcal{F}}(S)$ for each $Q, Q' \leq S$.

Moreover, we have the following two characterizations of $\mathcal{F}$ reducing to the normalizer of the Sylow:

Lemma 2.15. *Let $\mathcal{F}$ be a saturated fusion system over the p-group S. Then $\mathcal{F} = N_{\mathcal{F}}(S)$ if and only if S itself is the only $\mathcal{F}$-Alperin subgroup of S.*

Proof. If S is the unique $\mathcal{F}$-Alperin subgroup then the assertion follows from Alperin's fusion theorem for saturated fusion systems (Theorem 2.5). Assume then that $\mathcal{F} = N_{\mathcal{F}}(S)$ and choose $P \leq S$ $\mathcal{F}$-Alperin. Using that $\mathcal{F} = N_{\mathcal{F}}(S)$, it is straightforward that $\mathrm{Out}_S(P)$ is normal in $\mathrm{Out}_{\mathcal{F}}(P)$ and, as P is $\mathcal{F}$-radical, it must be trivial. Then $P = N_S(P)$, and as S is a p-group, P must be equal to S. □

Lemma 2.16. *Let $\mathcal{F}$ be a saturated fusion system over the p-group S. Then $\mathcal{F} = N_{\mathcal{F}}(S)$ if and only if $N_{\mathcal{F}}(P) = N_{N_{\mathcal{F}}(P)}(N_S(P))$ for every $P \leq S$ fully normalized in $\mathcal{F}$.*

Proof. Assume first that $\mathcal{F} = N_{\mathcal{F}}(S)$ and $P \leq S$ is fully normalized in $\mathcal{F}$. In general we have that $N_{\mathcal{F}}(P) \supset N_{N_{\mathcal{F}}(P)}(N_S(P))$. The other inclusion follows if all $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(P)}(Q, Q')$ which extend to a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(PQ, PQ')$ such that $\psi|_Q = \varphi$ and $\psi|_P \in \mathrm{Aut}_{\mathcal{F}}(P)$, extend to an element of $\mathrm{Aut}(N_S(P))$. But using Remark 2.14 we get that ψ extends to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$, which restricts to an element of $\mathrm{Aut}_{\mathcal{F}}(N_S(P))$ because ψ restricts to an element of $\mathrm{Aut}(P)$.

Assume now that for every $P \leq S$ fully normalized in $\mathcal{F}$ we have $N_{\mathcal{F}}(P) = N_{N_{\mathcal{F}}(P)}(N_S(P))$. According to Lemma 2.15 we have to check that S does not contain any proper $\mathcal{F}$-Alperin subgroup. Let P be a $\mathcal{F}$-Alperin subgroup, then it is $N_{\mathcal{F}}(P)$-Alperin too. But, applying Lemma 2.15 to $N_{\mathcal{F}}(P)$, we get that $S = P$. □

Finally in this section we give some results which allow us to determine in some special cases the existence of a finite group with a fixed saturated fusion system.

We begin with a definition which does only depend on the p-group S:

Definition 2.17. Let S be a p-group. A subgroup $P \leq S$ is *p-centric* in S if $C_S(P) = Z(P)$.

Remark 2.18. If $\mathcal{F}$ is any fusion system over the p-group S, then $\mathcal{F}$-centric subgroups are p-centric subgroups too.

The following result is a generalization of [11, Proposition 9.2] which apply to some of our cases. Recall that given a fusion system $(S, \mathcal{F})$, a subgroup $P \leq S$ is called *strongly closed in $\mathcal{F}$* if no element of P if $\mathcal{F}$-conjugate to any element of $S \setminus P$.

Proposition 2.19. *Let $(S, \mathcal{F})$ be a saturated fusion system such that every non trivial strongly closed subgroup $P \leq S$ is non elementary abelian, p-centric and does not factorize as a product of two or more subgroups which are permuted transitively by $\mathrm{Aut}_{\mathcal{F}}(P)$. Then if $\mathcal{F}$ is the fusion system of a finite group, it is the fusion system of a finite almost simple group.*

Proof. Suppose $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group G. Assume also that $\#|G|$ is minimal with this property. Consider $H \lhd G$ a minimal non trivial normal subgroup in G. Then $H \cap S$ is a strongly closed subgroup of $(S, \mathcal{F})$. If $H \cap S = 1$ then $\mathcal{F}$ is also the fusion system of G/H, which contradicts the assumption of minimality on G. Now $P \overset{\text{def}}{=} H \cap S$ is a non trivial normal strongly closed subgroup in $(S, \mathcal{F})$ and, as H is normal, P is the Sylow p-subgroup in H. By [20, Theorem 2.1.5] H must be either elementary abelian or a product of non abelian simple groups isomorphic one to each other which must be permuted transitively by $N_G(H) = G$ (now by minimality of H). Notice that H cannot be elementary abelian as that would imply that P is so while this is not possible by hypothesis. Therefore H is a product of non abelian simple groups. If H is not simple (so there is more than one factor) this would break P into two ore more factors which would be permuted transitively, so H must be simple. Now, as P is p-centric, $C_G(H) \cap S \subset P \subset H$, so $C_G(H) \cap S \subset C_G(H) \cap H = 1$ and $C_G(H) = 1$ ($C_G(H)$ is a normal subgroup in G of order prime to p, so if $C_G(H) \neq 1$ taking $G/C_G(H)$ gives again a contradiction with the minimality of $\#|G|$). This tells us that $H \lhd G \leq \mathrm{Aut}(H)$, so G is almost simple. □

Remark 2.20. In fact [11, Proposition 9.2] proves that if the only non trivial strongly closed p-subgroup in $(S, \mathcal{F})$ is S, and moreover S is non abelian and it does not factorize as a product of two or more subgroups which are permuted transitively by $\mathrm{Aut}_{\mathcal{F}}(S)$, then if $(S, \mathcal{F})$ is the fusion system of a finite group, it is the fusion system of an extension of a simple group by outer automorphisms of order prime to p.

We finish this section with the following result, which can be found in [9, Corollary 6.17]:

Lemma 2.21. *Let $(S, \mathcal{F})$ be a saturated fusion system, and assume there is a nontrivial subgroup $A \leq Z(S)$ which is central in $\mathcal{F}$ (i.e. $C_{\mathcal{F}}(A) = \mathcal{F}$). Then $\mathcal{F}$ is the fusion system of a finite group if and only if $\mathcal{F}/A$ is so.*

3. Resistant p-groups

In this section we recall the notion of Swan group and introduce its generalization for fusion systems, as well as we discuss some related results. Some of these results were considered independently by Stancu in [36].

For a fixed prime p we recall that a subgroup $H \leq G$ is said to *control (strong) p-fusion* in G if H contains a Sylow p-subgroup of G and whenever P, $g^{-1}Pg \leq H$ for P a p-subgroup of G then $g = hc$ where $h \in H$ and $c \in C_G(P)$.

If we focus on a p-group S we may wonder if $N_G(S)$ *controls p-fusion* in G whenever $S \in \mathrm{Syl}_p(G)$. If this is always the case, then S is called a *Swan* group. The equivalent concept in the setting of p-local finite groups is:

Definition 3.1. A p-group S is called *resistant* if whenever $\mathcal{F}$ is a saturated fusion system over S then $N_{\mathcal{F}}(S) = \mathcal{F}$.

Remark 3.2. Considering the saturated fusion system associated to $S \in \mathrm{Syl}_p(G)$ it is clear that every resistant group is a Swan group. In the opposite way, up to date there is no known Swan group that is not a resistant group.

Following [32, Section 2] we look for conditions on a p-group S for being resistant. If the p-group S is resistant, when treating with a fusion system $\mathcal{F}$ over S all morphisms in $\mathcal{F}$ are restrictions of $\mathcal{F}$-automorphisms of S. In the general case, we must pay attention to possible $\mathcal{F}$-Alperin subgroups to understand the whole category $\mathcal{F}$. The first step towards this objective is to examine p-centric subgroups.

Theorem 3.3. *Let S be a p-group. If every proper p-centric subgroup $P \lneq S$ verifies*

$$\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) \neq 1\,,$$

then S is a resistant group.

Proof. Let $\mathcal{F}$ be a saturated fusion system over S. According to Lemma 2.15, it is enough to prove that S is the only $\mathcal{F}$-Alperin subgroup. Let $P \lneq S$ be a proper $\mathcal{F}$-Alperin subgroup, hence p-centric by Remark 2.18. As $\mathrm{Out}_S(P) \leq \mathrm{Out}_\mathcal{F}(P)$, we have that

$$1 \neq \mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) \leq \mathrm{Out}_\mathcal{F}(P) \cap O_p(\mathrm{Out}(P)) \trianglelefteq O_p(\mathrm{Out}_\mathcal{F}(P))\,.$$

Hence P cannot be $\mathcal{F}$-radical. □

We obtain a family of resistant groups:

Corollary 3.4. *Abelian p-groups are resistant groups.*

Also is meaningful determining whether or not a p-group can be $\mathcal{F}$-Alperin for some saturated fusion system $\mathcal{F}$:

Lemma 3.5. *Let P be a p-group such that $O_p(\mathrm{Out}(P))$ is the Sylow p-subgroup of* $\mathrm{Out}(P)$. *Then P is not $\mathcal{F}$-Alperin for any saturated fusion system $\mathcal{F}$ over S with $P \lneq S$.*

Proof. Let S be a p-group with $P \lneq S$ and $\mathcal{F}$ be a saturated fusion system over S. If P is $\mathcal{F}$-radical then the normal p-subgroup $O_p(\mathrm{Out}(P)) \cap \mathrm{Out}_\mathcal{F}(P)$ of $\mathrm{Out}_\mathcal{F}(P)$ must be trivial. Being $O_p(\mathrm{Out}(P))$ a Sylow p-subgroup and normal this implies that $\mathrm{Out}_\mathcal{F}(P)$ is a p'-group and so, by definition, $\mathrm{Aut}_P(P) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}(P))$. If in addition P is fully normalized then we know that $\mathrm{Aut}_S(P)$ is another Sylow p-subgroup of $\mathrm{Aut}_\mathcal{F}(P)$. So they both must be same size. Finally, if P is $\mathcal{F}$-centric then $Z(P) = C_S(P)$ and $P = N_S(P)$, which is false for p-groups unless P is equal to S. □

We obtain some useful corollaries:

Corollary 3.6. *$\mathbb{Z}/p^n$ is not $\mathcal{F}$-Alperin in any saturated fusion system $\mathcal{F}$ over S where $\mathbb{Z}/p^n \lneq S$.*

Proof. An easy verification shows that $\mathrm{Aut}(\mathbb{Z}/p^n)$ is equal to $(\mathbb{Z}/p^n)^*$, which is abelian. Now just apply Lemma 3.5. □

Corollary 3.7. *If n and m are two different non zero positive integers, then $\mathbb{Z}/p^n \times \mathbb{Z}/p^m$ is not $\mathcal{F}$-Alperin in any saturated fusion system $\mathcal{F}$ over S where $\mathbb{Z}/p^n \times \mathbb{Z}/p^m \lneq S$.*

Proof. Suppose $n < m$ and take $f \in \mathrm{End}(\mathbb{Z}/p^n \times \mathbb{Z}/p^m)$ with $f(\bar{1},\bar{0}) = (\bar{a},\bar{b})$ and $f(\bar{0},\bar{1}) = (\bar{c},\bar{d})$. It must hold that $b \equiv 0 \bmod p^{m-n}$. Moreover, f is an automorphism if and only if $\mathrm{order}(\bar{a},\bar{b}) = p^n$, $\mathrm{order}(\bar{c},\bar{d}) = p^m$ and $\langle(\bar{a},\bar{b}),(\bar{c},\bar{d})\rangle = \mathbb{Z}/p^n \times \mathbb{Z}/p^m$. It can be checked that the first condition is equivalent to $a \neq 0 \bmod p$, the second to $d \neq 0 \bmod p$, and the third is consequence of the previous ones. Counting elements it turns out that $\mathrm{Aut}(\mathbb{Z}/p^n \times \mathbb{Z}/p^m)$ has order $p^{3n+m-2}(p-1)^2$. Finally, the subgroup $\{(\bar{a},\bar{b}),(\bar{c},\bar{d})$ with $a = 1 \bmod p$ and $d = 1 \bmod p\}$ is normal and has order p^{3n+m-2}. □

Corollary 3.8. *If p is odd then non abelian metacyclic p-groups M are not $\mathcal{F}$-Alperin in any saturated fusion system $\mathcal{F}$ over S with $M \lneq S$.*

Proof. According to [18, Section 3], for p odd and M a non abelian metacyclic p-group, $O_p(\mathrm{Out}(M))$ is the Sylow p-subgroup of $\mathrm{Out}(M)$, so the result follows using Lemma 3.5. □

Recall the notation in Theorem A.1 for the families of p-rank two p-groups and Theorem A.2 for the maximal nilpotency class 3-rank two 3-groups.

Corollary 3.9. *If p is odd then $G(p,r;\epsilon)$ cannot be $\mathcal{F}$-Alperin in any saturated fusion system over S, with $G(p,r;\epsilon) \lneq S$.*

Proof. By [19, Proposition 1.6] $O_p(\mathrm{Out}(G(p,r;\epsilon)))$ is the Sylow p-subgroup of $\mathrm{Out}(G(p,r;\epsilon))$, so applying Lemma 3.5 $G(p,r;\epsilon)$ could not be $\mathcal{F}$-Alperin. □

Corollary 3.10. *$B(3,r;\beta,\gamma,\delta)$ is not $\mathcal{F}$-Alperin in any saturated fusion system $\mathcal{F}$ over S where $B(3,r;\beta,\gamma,\delta) \lneq S$.*

Proof. From [6] we have that the Frattini subgroup of $B(3,r;\beta,\gamma,\delta)$, $\Phi(B(3,r;\beta,\gamma,\delta))$, is $\langle s_2, s_3, \ldots, s_{r-1}\rangle = \langle s_2, s_3\rangle$. Consider the Frattini map

$$B(3,r;\beta,\gamma,\delta)) \to B(3,r;\beta,\gamma,\delta)/\Phi(B(3,r;\beta,\gamma,\delta)) \simeq \langle \overline{s}, \overline{s_1}\rangle$$

and the induced map

$$\rho : \mathrm{Out}(G) \to \mathrm{Aut}(B(3,r;\beta,\gamma,\delta)/\Phi(B(3,r;\beta,\gamma,\delta))) \simeq \mathrm{GL}_2(3)$$

whose kernel is a 3-group. As γ_1 is characteristic in $B(3,r;\beta,\gamma,\delta)$, for every class of morphisms φ in $\mathrm{Out}(G)$ it holds that $\rho(\varphi)(\langle\overline{s_1}\rangle) \leq \langle\overline{s_1}\rangle$, and so the image of ρ is contained in the lower triangular matrices.

The subgroup generated by $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ is normal and is the Sylow 3-subgroup of the lower triangular matrices of $\mathrm{GL}_2(3)$. Its preimage by ρ is normal in $\mathrm{Out}(G)$ and, as the kernel of ρ is a 3-group, it is the Sylow 3-subgroup of $\mathrm{Out}(G)$ too. To finish the proof apply Lemma 3.5. □

The following result related to Corollary 3.7 is needed in successive sections, but before we recall here [35, Lemma 4.1] in a clearer form:

Lemma 3.11. *Let G be a p-reduced subgroup (that is, G has no nontrivial normal p-subgroup) of $\mathrm{GL}_2(p)$, $p \geq 3$. If p divides the order of G then $\mathrm{SL}_2(p) \leq G$.*

Proof. If p divides the order of the group G, then there is an element of order p. As G is p-reduced it cannot be the only one, so we have a subgroup of $\mathrm{GL}_2(p)$ with more than one nontrivial p-subgroup. Observe that the only nontrivial p-subroups in $\mathrm{GL}_2(p)$ are the Sylow p-subgroups, so G has more than one Sylow p-subgroup. Using the Third Sylow Theorem there are at least $p+1$ different Sylow p-subgroups in $G \leq \mathrm{GL}_2(p)$, but there are exactly $p+1$ Sylow p-subgroups in $\mathrm{GL}_2(p)$, so G contains all the Sylow p-subgroups in $\mathrm{GL}_2(p)$ and the subgroup they generate, thus $\mathrm{SL}_2(p) \leq G$. □

Proposition 3.12. *For a prime $p > 3$ and an integer $n > 1$, $\mathbb{Z}/p^n \times \mathbb{Z}/p^n$ is not $\mathcal{F}$-Alperin in any saturated fusion system $\mathcal{F}$ over a p-group S where $\mathbb{Z}/p^n \times \mathbb{Z}/p^n \lneq S$.*

Proof. Consider S a p-group and $\mathbb{Z}/p^n \times \mathbb{Z}/p^n \lneq S$. If we assume that $\mathbb{Z}/p^n \times \mathbb{Z}/p^n$ is $\mathcal{F}$-radical then $G \stackrel{\text{def}}{=} \mathrm{Aut}_{\mathcal{F}}(\mathbb{Z}/p^n \times \mathbb{Z}/p^n)$ must be p-reduced, and if $\mathbb{Z}/p^n \times \mathbb{Z}/p^n$ is $\mathcal{F}$-centric it is self-centralizing in S, so taking the conjugation by an element in $S \setminus \mathbb{Z}/p^n \times \mathbb{Z}/p^n$ we get that there exist an element of order p in G.

We can consider $\mathrm{Aut}(\mathbb{Z}/p^n \times \mathbb{Z}/p^n)$ as 2×2 matrices with coefficients in $\mathbb{Z}/p^n$ and with determinant non divisible by p. In that case the reduction modulo p induces a short exact sequence:

$$\{1\} \to P \to \mathrm{Aut}(\mathbb{Z}/p^n \times \mathbb{Z}/p^n) \xrightarrow{\rho} \mathrm{GL}_2(p) \to \{1\}\,,$$

with P a p-group. If the intersection $P \cap G$ is not trivial, then we have a non trivial normal p-subgroup in G and G would not be p-reduced, so $G \cap P = \{1\}$ and ρ restricts to an injection of G in $\mathrm{GL}_2(p)$.

Using that $\rho(G)$ has an element of order p, and it is a p-reduced subgroup of $\mathrm{GL}_2(p)$ and applying the Lemma 3.11 we get that $\rho(G)$ contains $\mathrm{SL}_2(p)$. In particular we get that the matrix $\overline{A} \stackrel{\text{def}}{=} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is in $\rho(G)$, so we have an element in $A \in G \subset \mathrm{Aut}(\mathbb{Z}/p^n \times \mathbb{Z}/p^n)$ which reduction modulo p is $\overline{A}$. A must be matrix of the form $A = \left(\begin{smallmatrix} 1+\xi p & 1+\eta p \\ \lambda p & 1+\mu p \end{smallmatrix}\right)$, and an easy computation tell us that

$$A^m \equiv \begin{pmatrix} 1+m\xi p+\binom{m}{2}\lambda p & m+\binom{m}{2}\xi p+m\eta p+\binom{m}{3}\lambda p+\binom{m}{2}\mu p \\ m\lambda p & 1+\binom{m}{2}\lambda p+m\mu p \end{pmatrix} \mod p^2.$$

So, for $p > 3$, we get $A^p \equiv \left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right) \mod p^2$ and as $n > 1$, the order of A is bigger than p, which contradicts the fact that ρ is injective. □

If p equals 3 or n equals 1 then the thesis of the previous lemma is false, that is, $\mathbb{Z}/p^n \times \mathbb{Z}/p^n$ could be $\mathcal{F}$-Alperin when $p = 3$ or $n = 1$. But then we can sharp our result in the following way:

Lemma 3.13. *If $p = 3$ or $n = 1$ and if $\mathbb{Z}/p^n \times \mathbb{Z}/p^n$ is $\mathcal{F}$-Alperin in a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ with $\mathbb{Z}/p^n \times \mathbb{Z}/p^n \lneq S$, then $p_+^{1+2} \leq S$.*

Proof. Let $\mathcal{F}$ be a saturated fusion system over S, and suppose that $P \stackrel{\text{def}}{=} \mathbb{Z}/p^n \times \mathbb{Z}/p^n \lneq S$. As in the proof of Proposition 3.12, if P is $\mathcal{F}$-radical and $\mathcal{F}$-centric then $\mathrm{Aut}_{\mathcal{F}}(P)$ contains $\mathrm{SL}_2(p)$.

Let $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ be the centric linking system, and take the short exact sequence of groups induced by π:

$$1 \to P \to \mathrm{Aut}_{\mathcal{L}}(P) \xrightarrow{\pi} \mathrm{Aut}_{\mathcal{F}}(P) \to 1.$$

Because $\mathrm{Aut}_{\mathcal{F}}(P)$ contains the special matrices over $\mathbb{F}_p$ we have another short exact sequence:

$$1 \to P \to M \xrightarrow{\pi} \mathrm{SL}_2(p) \to 1.$$

To see that there are no more extensions that the split one, consider the central subgroup V of $\mathrm{SL}_2(p)$ generated by the involution $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since $p \geq 3$, multiplication by $|V| = 2$ is invertible in P and so $H^k(V, P) = 0$ for all $k > 0$, and because V acts on P without fixed points also $H^0(V, P) = 0$. Now the Hochschild-Serre spectral sequence corresponding to the normal subgroup $V \leq \mathrm{SL}_2(p)$ give us that $H^k(\mathrm{SL}_2(p), P) = 0$ for all $k \geq 0$.

Now, as $H^2(\mathrm{SL}_2(p), P) = 0$, the middle term M of the short exact sequence above equals P: $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{L}}(P)$.

As we are assuming in addition that the abelian p-group P is fully normalized then, as $\mathcal{F}$ is saturated, we obtain from Definitions 2.2 and 2.7(A) that the Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{L}}(P)$ is $N_S(P)$. So then we have that $p_+^{1+2} \leq P : \mathbb{Z}/p \leq N_S(P) \leq S$. □

Lemma 3.11 also applies for giving some restrictions to the family $C(p, r)$:

Lemma 3.14. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S with $p \geq 3$. If $C(p, r) \lneq S$ (with $r \geq 3$) is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(C(p, r)) \leq \mathrm{GL}_2(p)$.*

Proof. If $C(p, r)$ is $\mathcal{F}$-radical, then $\mathrm{Out}_{\mathcal{F}}(C(p, r))$ must be p-reduced. If we consider the projection ρ from Lemma A.5 we have that it must restrict to a monomorphism in $\mathrm{Out}_{\mathcal{F}}(C(p, r))$ (otherwise we would have a non-trivial normal p-subgroup) so we can consider $\mathrm{Out}_{\mathcal{F}}(C(p, r)) \leq \mathrm{GL}_2(p)$.

Now as $C(p, r)$ is $\mathcal{F}$-centric and different from S, we have an element of order p in $\mathrm{Out}_{\mathcal{F}}(C(p, r))$. Now, using again that $\mathrm{Out}_{\mathcal{F}}(C(p, r))$ must be p reduced, the result follows from Lemma 3.11. □

4. Non-maximal class rank two p-groups

In this section we consider the non-maximal class rank two p-groups for odd p, which are listed in the classification in Theorem A.1.

We begin with metacyclic groups:

Theorem 4.1. *Metacyclic p-groups are resistant for $p \geq 3$.*

Proof. We prove that if S is a metacyclic group then it cannot contain any proper $\mathcal{F}$-Alperin subgroup.

Let P be a proper subgroup of S, then it must be again metacyclic.

If P is not abelian, we can use Corollary 3.8 to deduce that it cannot be $\mathcal{F}$-Alperin.

If P is abelian, using Corollaries 3.6 and 3.7 it cannot be $\mathcal{F}$-Alperin unless $P \cong \mathbb{Z}/p^n \times \mathbb{Z}/p^n$, but in this case p_+^{1+2} should be a subgroup of $M(p, r)$ by Lemma 3.13, which is impossible because p_+^{1+2} is not metacyclic. □

In the study of $C(p, r)$ in this section we assume $r \geq 4$: for $r = 3$ we have that $C(p, 3) \cong p_+^{1+2}$, which it is a maximal nilpotency class p-group and the fusion systems over that group are studied in [35].

Theorem 4.2. *If $r > 3$ and $p \geq 3$ then $C(p, r)$ is resistant.*

Proof. Let $\mathcal{F}$ be a saturated fusion system over $C(p, r)$. The possible proper $\mathcal{F}$-Alperin subgroups are proper p-centric subgroups, and using Lemma A.6 these are isomorphic to $\mathbb{Z}/p^{r-2} \times \mathbb{Z}/p$. But as $r > 3$, $\mathbb{Z}/p^{r-2} \times \mathbb{Z}/p$ cannot be $\mathcal{F}$-radical by Corollary 3.7. □

It remains to study the non-maximal nilpotency class groups of type $G(p, r; \epsilon)$. Remark A.3 tells us that in this section we have to consider all of them but $G(3, 4; \pm 1)$.

The study of groups of type $G(p,r;\epsilon)$ is divided in two cases: for $p \geq 5$ these are resistant groups, while for $p=3$ we obtain saturated fusion systems with proper $\mathcal{F}$-Alperin subgroups.

Theorem 4.3. *If $p>3$ and $r \geq 4$, $G(p,r;\epsilon)$ is resistant.*

The proof needs the following lemma:

Lemma 4.4. *Let $\mathcal{F}$ be a saturated fusion system over $G(p,r;\epsilon)$ with $p>3$ and $r\geq 4$. Then $C(p,r-1) < G(p,r;\epsilon)$ is not $\mathcal{F}$-radical. Moreover, $\mathrm{Aut}_{\mathcal{F}}(C(p,r-1))$ is a subgroup of the lower triangular matrices with first diagonal entry ± 1.*

Proof. Consider $p\geq 5$ and assume that $C(p,r-1)$ is $\mathcal{F}$-radical, then by Lemma 3.14 $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(C(p,r-1))$ and therefore the matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)$, where x is a primitive $(p-1)$-th root of the unity in $\mathbb{F}_p$, is the image of some $\varphi \in \mathrm{Aut}_{\mathcal{F}}(C(p,r-1))$ by the composition

$$\mathrm{Aut}_{\mathcal{F}}(C(p,r-1)) \xrightarrow{\pi} \mathrm{Out}_{\mathcal{F}}(C(p,r-1)) \xrightarrow{\rho} \mathrm{GL}_2(p)$$

of the projection and ρ from Lemma A.5. By Definition 2.2 this morphism φ must lift to $\mathrm{Aut}_{\mathcal{F}}(G(p,r;\epsilon))$ as a morphism that maps a to a^x (up to $b^n c^{mp}$ multiplication). But automorphisms of $G(p,r;\epsilon)$ map a to $a^{\pm 1}$ (up to $b^n c^{mp}$ multiplication), and -1 is not a primitive $(p-1)$-th root of the unity in $\mathbb{F}_p$ for $p\geq 5$. Therefore $C(p,r-1)$ is not $\mathcal{F}$-radical.

Finally, as $C(p,r-1)$ is not $\mathcal{F}$-radical, then $\pi(\mathrm{Aut}_{\mathcal{F}}(C(p,r-1))) < N_{\mathrm{GL}_2(p)}(V)$ where V is the group generated by $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1\end{smallmatrix}\right)$, that is, the projection $\pi(\mathrm{Aut}_{\mathcal{F}}(C(p,r-1)))$ is a subgroup of the lower triangular matrices. Then all morphisms in $\mathrm{Aut}_{\mathcal{F}}(C(p,r-1))$ must lift to $\mathrm{Aut}_{\mathcal{F}}(G(p,r;\epsilon))$, and again only those with ± 1 in the first diagonal entry are allowed, what proves the second part of the lemma. □

Proof of Theorem 4.3. If $r>4$ we have just to consider the case of $C(p,r-1)$ being $\mathcal{F}$-radical, and Lemma 4.4 shows that it cannot be.

If $r=4$ it remains to check what happens with the rank two elementary abelian subgroups in $G(p,4;\epsilon)$. According to Lemma A.8 there are exactly p of these subgroups, namely, $V_i \stackrel{\mathrm{def}}{=} \langle ab^i, c^p\rangle$ for $i=0,\dots,p-1$, and all of them lie inside $C(p,3) \cong p^{1+2}_+$. Notice that conjugation by c permutes all of them cyclically and so they are $\mathcal{F}$-conjugate in any saturated fusion system over $G(p,4;\varepsilon)$. Thus if any of them is $\mathcal{F}$-radical then all of them are $\mathcal{F}$-radical.

If this is the case, fix x a primitive $(p-1)$-th root of the unity in $\mathbb{F}_p$ and let V_i be one of these rank two p-subgroups with $\mathbb{F}_p$ basis $\langle ab^i, c^{-ep}\rangle$. Then $\mathrm{SL}_2(p) \leq \mathrm{Aut}_{\mathcal{F}}(V_i)$ by Lemma 3.11. Now, the element $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1}\end{smallmatrix}\right) \in \mathrm{Aut}_{\mathcal{F}}(V)$ must lift to an automorphism of p^{1+2}_+ by Definition 2.2. The image of this extension in $\mathrm{Out}_{\mathcal{F}}(p^{1+2}_+) = \mathrm{GL}_2(p)$ is a matrix L_i with x as eigenvalue and with determinant x^{-1}, so it has x^{-2} as the other eigenvalue. Notice that each V_i gives a different matrix $L_i \in \mathrm{GL}_2(p)$, so different elements of order $p-1$ in $\mathrm{Out}_{\mathcal{F}}(p^{1+2}_+)$. But the description of $\mathrm{Aut}_{\mathcal{F}}(p^{1+2}_+)$ in Lemma 4.4 shows us that there are not such matrices L_i if $p>5$ and at most one when $p=5$ (cf. [35, Section 4]). □

So it remains to check the cases with $p=3$, and as this section only copes with the non maximal nilpotency class groups, we consider $r\geq 5$.

Lemma 4.5. *Fix $r \geq 4$ and let W be either $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$. Given a faithful representation of $\mathbb{Z}/3$ in $\mathrm{Out}(C(p,r-1))$ and any group S fitting in the exact sequence*

$$1 \longrightarrow C(3,r-1) \longrightarrow S \longrightarrow \mathbb{Z}/3 \longrightarrow 1$$

that induces the given representation of $\mathbb{Z}/3$, there is a finite group G which fits in the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C(3,r-1) & \longrightarrow & S & \longrightarrow & \mathbb{Z}/3 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C(3,r-1) & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \end{array}$$

where the last column is an inclusion of the Sylow 3-subgroup in W.

Proof. Fix a faithful representation of $\mathbb{Z}/3$ in $\mathrm{Out}(C(p,r-1))$.

Consider first $r=4$ or $W=\mathrm{GL}_2(3)$. Using the study of the group $\mathrm{Aut}(C(3,r-1))$ in Lemma A.5 we have that there is only one faithful representation of W in $\mathrm{Out}(C(3,r-1))$ up to conjugation.

To compute the possible equivalence class of extensions in the exact sequence $1 \to C(3,r-1) \to G \to W \to 1$ with the fixed representation of W in $\mathrm{Out}(C(3,r-1))$ we have to compute $H^2(W; Z(C(3,r-1)))$ [8, Theorem IV.6.6], getting $H^2(W;\mathbb{Z}/3^{r-3}) \cong \mathbb{Z}/3$.

Consider now $\mathbb{Z}/3$, the Sylow 3-subgroup of W, and consider the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C(3,r-1) & \longrightarrow & S & \longrightarrow & \mathbb{Z}/3 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C(3,r-1) & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \end{array}$$

with exact rows and do the same computation for the first row in cohomology. This gives us that there are three possible equivalence classes of extensions S fitting in that exact sequence. Moreover the map induced in cohomology $H^2(W;\mathbb{Z}/3^{r-3}) \to H^2(\mathbb{Z}/3;\mathbb{Z}/3^{r-3})$ is an isomorphism (use a transfer argument), so we have that all three S appear as the Sylow 3-subgroup of G.

Consider now the case $r \geq 5$ and $W = \mathrm{SL}_2(3)$. Now by Lemma A.5 we have two possible faithful representations of W in $\mathrm{Out}(C(p,r-1))$ up to conjugation. Now we do the same computations and we use the same argument as before for each action, getting 6 possible equivalence classes of extensions which appear as the Sylow 3-subgroup of the 6 possible G's. □

Remark 4.6. The previous assertion fails for $p > 3$ because in the cohomology groups computation we get that $H^2(\mathrm{SL}_2(p); Z(C(p,r-1)))$ and $H^2(\mathrm{GL}_2(p); Z(C(p,r-1)))$ are trivial, so the only extensions are the split ones. So in both cases the Sylow p-subgroup has p-rank three.

Remark 4.7. We are interested in the possible extensions $1 \to C(3,r-1) \to S \to \mathbb{Z}/3 \to 1$ such that the 3-rank of S is two. Looking at the groups appearing in the classification in Theorem A.1 we get that the only possible ones are $C(3,r)$, $G(3,r;1)$ and $G(3,r;-1)$.

Fix the notation $C(3,r-1)\cdot_\epsilon W$ for the groups in Lemma 4.5 with Sylow 3-subgroup isomorphic to $G(3,r;\epsilon)$.

Theorem 4.8. *The* 3*-local finite groups over* $G(3,r;\epsilon)$ *with* $r\geq 5$ *and at least one proper* $\mathcal{F}$*-Alperin subgroup are classified by the following parameters:*

$\mathrm{Out}_\mathcal{F}(G(3,r;\epsilon))$	$\mathrm{Out}_\mathcal{F}(C(3,r-1))$	*Group*
$\mathbb{Z}/2$	$\mathrm{SL}_2(3)$	$C(3,r-1)\cdot_\epsilon \mathrm{SL}_2(3)$
$\mathbb{Z}/2\times\mathbb{Z}/2$	$\mathrm{GL}_2(3)$	$C(3,r-1)\cdot_\epsilon \mathrm{GL}_2(3)$

TABLE 1. s.f.s. over $G(3,r;\epsilon)$ for $r\geq 5$.

where the second column gives the outer automorphism group over the only proper $\mathcal{F}$*-Alperin subgroup, which is isomorphic to* $C(3,r-1)$*. In the last column we refer to non split extensions such that the Sylow* 3*-subgroup is isomorphic to* $G(3,r;\epsilon)$ *for* $\epsilon=\pm1$ *and it induces the desired fusion system.*

Proof. Assume now that $C(3,r-1)\leq G(3,r;\epsilon)$ is $\mathcal{F}$-Alperin. Then $\mathrm{Out}_\mathcal{F}(C(3,r-1))\cong \mathrm{SL}_2(3)$ or $\mathrm{Out}_\mathcal{F}(C(3,r-1))\cong \mathrm{GL}_2(3)$.

If we look at the possible $\mathrm{Out}_\mathcal{F}(G(3,r;\epsilon))$ we get that it is $\mathbb{Z}/2$, generated by a matrix which induces $-\mathrm{Id}$ in the identification $\mathrm{Out}_\mathcal{F}(C(3,r-1))\leq \mathrm{GL}_2(3)$ or $\mathbb{Z}/2\times\mathbb{Z}/2$, generated by $-\mathrm{Id}$ and a matrix of determinant -1 in $\mathrm{Out}_\mathcal{F}(C(3,r-1))\leq \mathrm{GL}_2(3)$. From here we deduce that if $\mathrm{Out}_\mathcal{F}(G(3,r;\epsilon))\cong\mathbb{Z}/2$ and $C(3,r-1)$ is $\mathcal{F}$-radical then $\mathrm{Out}_\mathcal{F}(C(3,r-1))\cong \mathrm{SL}_2(3)$, while if $\mathrm{Out}_\mathcal{F}(G(3,r;\epsilon))\cong\mathbb{Z}/2\times\mathbb{Z}/2$ then $\mathrm{Out}_\mathcal{F}(C(3,r-1))\cong \mathrm{GL}_2(3)$ what completes the table.

All of them are saturated because are the fusion systems of the finite groups described in Lemma 4.5. □

5. MAXIMAL NILPOTENCY CLASS RANK TWO p-GROUPS

In this section we classify the p-local finite groups over p-groups of maximal nilpotency class and p-rank two. Recall that by Corollary 2.10 we have just to classify the saturated fusion systems over these groups.

Consider S a p-rank two maximal nilpotency class p-group of order p^r. For $r=2$ then $S\cong\mathbb{Z}/p\times\mathbb{Z}/p$, which is resistant by Corollary 3.4. If $r=3$ then $S\cong p_+^{1+2}$ and this case has been studied in [35]. For $r\geq 4$ all the p-rank two maximal nilpotency class groups appear only at $p=3$, and we use the description and properties given in Appendix A.

The description of the maximal nilpotency class 3-groups of order bigger than 3^3 depends on three parameters β, γ and δ, and we use the notation given in Theorem A.2, so we call the groups $B(3,r;\beta,\gamma,\delta)$ and $\{s,s_1,s_2,\ldots,s_{r-1}\}$ the generators.

First we consider the non split case, that is, $\delta>0$.

Theorem 5.1. *Every group of type* $B(3,r;\beta,\gamma,\delta)$*,* $\delta>0$*, is resistant.*

Proof. Let $\mathcal{F}$ be a saturated fusion system over $B(3,r;\beta,\gamma,\delta)$. Using Alperin's fusion theorem for saturated fusion systems (Theorem 2.5) it is enough to see whether $B(3,r;\beta,\gamma,\delta)$ is the only $\mathcal{F}$-Alperin subgroup. So let P be a proper subgroup.

If P has 3-rank one then, it is cyclic and we can apply Corollary 3.6 to obtain that P cannot be $\mathcal{F}$-Alperin.

Now assume that P has 3-rank two. Then, by Theorem A.1, P is one of the following:

- $M(3,r)$ non abelian: according to Corollary 3.8 P can not be $\mathcal{F}$-Alperin.
- $G(3,r;\epsilon)$ group: it cannot be $\mathcal{F}$-Alperin by Corollary 3.9.
- $B(3,m;\beta,\gamma,\delta)$ with $m<r$: by Lemma 3.10 P cannot be $\mathcal{F}$-Alperin.
- $C(3,r)$ group: as $3_+^{1+2}=C(3,3)$ is contained in $C(3,r)$ we obtain that $3_+^{1+2}\leq B(3,r;\beta,\gamma,\delta)$.
- Abelian: say $P=\mathbb{Z}/3^m\times\mathbb{Z}/3^n$. If $m\neq n$ then P cannot be $\mathcal{F}$-Alperin by Corollary 3.7, and if $m=n$ then again $3_+^{1+2}\leq B(3,r;\beta,\gamma,\delta)$ by Lemma 3.13.

So if P is $\mathcal{F}$-Alperin then $B(3,r;\beta,\gamma,\delta)$ must contain 3_+^{1+2}.

We finish the proof by showing that $3_+^{1+2}\not\leq B(3,r;\beta,\gamma,\delta)$ since we are in the non split case ($\delta>0$). Consider the short exact sequence:

$$1\to\gamma_1\to B(3,r;\beta,\gamma,\delta)\xrightarrow{\pi}\mathbb{Z}/3\to 1.$$

If $\pi(3_+^{1+2})$ is trivial then $3_+^{1+2}\leq\gamma_1$. But by [26, Satz III.§14.17] γ_1 is metacyclic, and consequently all its subgroups are metacyclic too. Thus γ_1 cannot contain the $C(3,3)$ group. We obtain then the short exact sequence:

$$1\to\mathbb{Z}/3\times\mathbb{Z}/3\to 3_+^{1+2}\xrightarrow{\pi}\mathbb{Z}/3\to 1.$$

As this sequence splits, the same would holds for the exact sequence involving $B(3,r;\beta,\gamma,\delta)$, and this is not the case. □

We now consider the split case, that is $\delta=0$. We prove that for $\beta=1$ these maximal nilpotency class groups are resistant, while for $\beta=0$ they are not. We use the information about $B(3,r;\beta,\gamma,0)$ contained in the appendix. According to Alperin's fusion theorem for saturated fusion systems (Theorem 2.5) we must focus on the $\mathcal{F}$-Alperin subgroups. The next two lemmas list the subgroups candidates for being $\mathcal{F}$-Alperin in a saturated fusion system over $B(3,r;0,\gamma,0)$ and $B(3,r;1,0,0)$:

Lemma 5.2. *Let $\mathcal{F}$ be a saturated fusion system over $B(3,r;0,\gamma,0)$ and let P be a proper $\mathcal{F}$-Alperin subgroup. Then P is one of the following table:*

Isomorphism type	*Subgroup (up to conjugation)*	*Conditions*
$\mathbb{Z}/3^k\times\mathbb{Z}/3^k$	$\gamma_1=\langle s_1,s_2\rangle$	$r=2k+1$.
3_+^{1+2}	$E_i\stackrel{def}{=}\langle\zeta,\zeta',ss_1{}^i\rangle$	$\zeta=s_2{}^{3^{k-1}},\zeta'=s_1{}^{3^{k-1}}$ *for* $r=2k+1$, $\zeta=s_1{}^{3^{k-1}},\zeta'=s_2{}^{-3^{k-2}}$ *for* $r=2k$, $i\in\{-1,0,1\}$ *if* $\gamma=0$ *and* $i=0$ *if* $\gamma=1,2$.
$\mathbb{Z}/3\times\mathbb{Z}/3$	$V_i\stackrel{def}{=}\langle\zeta,ss_1{}^i\rangle$	

Moreover, for the subgroups in the table, γ_1 and any E_i are always $\mathcal{F}$-centric, while any V_i is $\mathcal{F}$-centric only if it is not $\mathcal{F}$-conjugate to $\langle\zeta,\zeta'\rangle\cong\mathbb{Z}/3\times\mathbb{Z}/3$.

Proof. If P is $\mathcal{F}$-Alperin then arguing as in Theorem 5.1 we obtain that $P \cong \mathbb{Z}/3^n \times \mathbb{Z}/3^n$ or $P \cong C(3,n)$. Then, using Lemma A.15 we reach the subgroups in the statement.

To check that indeed γ_1 is $\mathcal{F}$-centric it is enough to notice that it is self-centralizing and that γ_1 is $\mathcal{F}$-conjugate just to itself in any fusion system $\mathcal{F}$ (γ_1 is strongly characteristic in $B(3,r;0,\gamma,0)$). It is clear that the copies of 3^{1+2}_+ are $\mathcal{F}$-centric because these are the only copies lying in $B(3,r;0,\gamma,0)$, and the center of all of them is $\langle\zeta\rangle$. To conclude the lemma, notice that the only copies of $\mathbb{Z}/3\times\mathbb{Z}/3$ in $B(3,r;0,\gamma,0)$ are the V_i's and $\langle\zeta,\zeta'\rangle$, and that the former are self-centralizing while the centralizer of the latter is γ_1. □

Lemma 5.3. *Let $\mathcal{F}$ be a saturated fusion system over $B(3,r;1,0,0)$ and P be a proper $\mathcal{F}$-Alperin subgroup. Then P is one of the following table:*

Isomorphism type	*Subgroup (up to conjugation)*	*Conditions*
$\mathbb{Z}/3^{k-1}\times\mathbb{Z}/3^{k-1}$	$\gamma_2 = \langle s_2, s_3\rangle$	$r = 2k$.
3^{1+2}_+	$E_0 \overset{def}{=} \langle\zeta,\zeta',s\rangle$	$\zeta = {s_2}^{3^{k-1}}$, $\zeta' = {s_3}^{-3^{k-2}}$ *for* $r = 2k+1$, $\zeta = {s_3}^{3^{k-2}}$, $\zeta' = {s_2}^{3^{k-2}}$ *for* $r = 2k$.
$\mathbb{Z}/3\times\mathbb{Z}/3$	$V_0 \overset{def}{=} \langle\zeta, s\rangle$	

Moreover, for the subgroups in the table, γ_2 and E_0 are always $\mathcal{F}$-centric, and V_0 is $\mathcal{F}$-centric only if it is not $\mathcal{F}$-conjugate to $\langle\zeta,\zeta'\rangle \cong \mathbb{Z}/3\times\mathbb{Z}/3$.

Proof. The reasoning is totally analogous to that of the previous lemma using Lemma A.16. □

Remark 5.4. The isomorphism between 3^{1+2}_+ and E_i is given by $a \mapsto \zeta'$, $b \mapsto s{s_1}^i$ and $c \mapsto \zeta$, where a, b and c are the generators of $3^{1+2}_+ = C(3,3)$ given in Theorem A.1. Therefore the morphisms in $\mathrm{Out}(E_i)$ are described as in [35, Lemma 3.1] by means of the mentioned isomorphism, and for $\mathrm{Out}(V_i)$ we choose the ordered basis $\{\zeta, s{s_1}^i\}$.

When studying a saturated fusion system $\mathcal{F}$ over $B(3,r;0,\gamma,0)$ or $B(3,r;1,0,0)$ it is enough to study the representatives subgroups given in the tables of Lemmas A.15 and A.16 because $\mathcal{F}$-properties are invariant under conjugation.

As a last step before the classification itself, we work out some information on lifts and restrictions of automorphism of some subgroups of $B(3,r;\beta,\gamma,0)$. Given a saturated fusion system $\mathcal{F}$ over $B(3,r;\beta,\gamma,0)$ subsequently it will be advantageous to consider for every subgroup of rank two $P \leq B(3,r;\beta,\gamma,0)$ the Frattini map $\mathrm{Out}(P) \xrightarrow{\rho} \mathrm{GL}_2(3)$, which kernel is a 3 group, and its restriction

$$\mathrm{Out}_{\mathcal{F}}(P) \xrightarrow{\rho} \mathrm{GL}_2(3)\,.$$

Notice that by Remark 2.3 this restriction is a monomorphism for $P = B(3,r;\beta,\gamma,0)$. For $P = \gamma_1$ or γ_2 it is a monomorphism if P is $\mathcal{F}$-Alperin as $\mathrm{Out}_{\mathcal{F}}(P)$ is 3-reduced. For $P = E_i$, V_i they are inclusions by [35, Lemma 3.1] and by definition respectively. In the case this restriction is a monomorphism we identify $\mathrm{Out}_{\mathcal{F}}(P)$ with its image in $\mathrm{GL}_2(3)$ without explicit mention. We divide the results in three lemmas, the first gives a description of $\mathrm{Out}_{\mathcal{F}}(P)$ for P an $\mathcal{F}$-Alperin subgroup, the second copes with restrictions and the third with lifts.

Lemma 5.5. *Let $\mathcal{F}$ be a saturated fusion system over $B(3,r;\beta,\gamma,0)$. Then:*
(a) $\rho(\mathrm{Out}_{\mathcal{F}}(B(3,r;\beta,\gamma,0)))$ *is a subgroup of the lower triangular matrices,*
(b) $\mathrm{Out}_{\mathcal{F}}(B(3,2k;0,\gamma,0)) \leq \mathbb{Z}/2\times\mathbb{Z}/2$,
(c) $\mathrm{Out}_{\mathcal{F}}(B(3,2k+1;0,1,0)) \leq \mathbb{Z}/2$,
(d) $\mathrm{Out}_{\mathcal{F}}(B(3,r;1,0,0)) \leq \mathbb{Z}/2$ *and*
(e) *if $P=\gamma_1$, γ_2, E_i or V_i is $\mathcal{F}$-Alperin then* $\mathrm{Out}_{\mathcal{F}}(P)=\mathrm{SL}_2(3)$ *or* $\mathrm{GL}_2(3)$.

Proof. For the first claim notice that from the order of $\mathrm{Aut}(B(3,r;\beta,\gamma,0))$ we deduce that a $3'$ element φ should have order 2 or 4. Now, as the Frattini subgroup of $B(3,r;\beta,\gamma,0)$ is $\langle s_2,s_3\rangle$, the projection on the Frattini quotient becomes

$$\mathrm{Out}_{\mathcal{F}}(B(3,r;\beta,\gamma,0)) \xrightarrow{\rho} \mathrm{GL}_2(3)$$

where if $\overline{\varphi}$ maps s to $s^e {s_1}^{e'} {s_2}^{e''}$ and s_1 to $s_1^{f'} s_2^{f''}$ then $\rho(\overline{\varphi}) = \left(\begin{smallmatrix} e & 0\\ e' & f'\end{smallmatrix}\right)$. So we obtain lower triangular matrices. Checking cases leads to obtain that the order of φ is two, and that $\varphi\in\{\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&-1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\1&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\-1&-1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\-1&1\end{smallmatrix}\right)\}$. For $\beta=0$, $\gamma=1$ and odd r or $\beta=1$ Lemma A.14 tells us that e must be equal to 1, and then it is easily deduced that $\mathrm{Out}_{\mathcal{F}}(B(3,r;\beta,\gamma,0))$ must have order two.

For the last point just use Lemma 3.11 and that $[\mathrm{GL}_2(3):\mathrm{SL}_2(3)]=2$. □

Now we focus on the restrictions. For the study of the possible saturated fusion systems $\mathcal{F}$ it is enough to consider diagonal matrices of $\mathrm{Out}_{\mathcal{F}}(B(3,r;\beta,\gamma,0))$ instead of lower triangular ones. This is so because every $\mathbb{Z}/2$ and $\mathbb{Z}/2\times\mathbb{Z}/2$ in $\mathrm{Out}_{\mathcal{F}}(B(3,r;\beta,\gamma,0))$ is $B(3,r;\beta,\gamma,0)$-conjugate to a diagonal one.

Lemma 5.6. *Let $\mathcal{F}$ be a saturated fusion system over $B(3,r;\beta,\gamma,0)$. Then:*
(a) *If $\beta=0$ the restrictions of the elements $\overline{\varphi}\in\mathrm{Out}_{\mathcal{F}}(B(3,r;0,\gamma,0))$ to $\mathrm{Out}_{\mathcal{F}}(\gamma_1)$ are given by the following table, where it is also described the permutation of the E_i's induced by $\overline{\varphi}$, and the restrictions to $\mathrm{Out}_{\mathcal{F}}(E_{i_0})$ for $i_0\in\{-1,0,1\}$ such that E_{i_0} is fixed by the permutation.*

$\mathrm{Out}_{\mathcal{F}}(B(3,r;0,\gamma,0))$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$
$\mathrm{Out}_{\mathcal{F}}(\gamma_1)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$
$\mathcal{F}$-conjugation	$E_1\leftrightarrow E_{-1}$	$E_1\leftrightarrow E_{-1}$	$-$
$\mathrm{Out}_{\mathcal{F}}(E_{i_0})$, *$r$ odd*	$\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$
$\mathrm{Out}_{\mathcal{F}}(E_{i_0})$, *$r$ even*	$\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$

(b) *If $\beta=1$ (thus $\gamma=0$) the restrictions to $\mathrm{Out}_{\mathcal{F}}(\gamma_2)$ and to $\mathrm{Out}_{\mathcal{F}}(E_0)$ of outer automorphisms $\overline{\varphi}\in\mathrm{Out}_{\mathcal{F}}(B(3,r;1,0,0))$ are given by the following table.*

$\mathrm{Out}_{\mathcal{F}}(B(3,r;1,0,0))$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$
$\mathrm{Out}_{\mathcal{F}}(\gamma_2)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$
$\mathrm{Out}_{\mathcal{F}}(E_0)$	$\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$

(c) *For the outer automorphism groups of E_i and V_i we have the following restrictions:*

$\mathrm{Out}_{\mathcal{F}}(E_i)$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$
$\mathrm{Out}_{\mathcal{F}}(V_i)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$

Proof. These are the only possible elements by Lemma 5.5, and the restrictions are computed directly using the explicit form of the subgroups given in Lemmas 5.2 and 5.3 using the basis described in Remark 5.4. □

Finally we reach the lemma about lifts:

Lemma 5.7. *Let $\mathcal{F}$ be a saturated fusion system over $B(3,r;\beta,\gamma,0)$, and P be one of the proper subgroups appearing in the tables of Lemmas 5.2 or 5.3. Then every diagonal outer automorphism of P in $\mathcal{F}$ (like those appearing in Lemma 5.6) can be lifted to the whole $B(3,r;\beta,\gamma,0)$. In particular, every admissible diagonal outer automorphism of P in $\mathcal{F}$ is listed in the tables of restrictions of Lemma 5.6.*

Proof. We study the two cases $\beta = 0$ and $\beta = 1$ separately.

If $\beta = 0$ we begin with the morphisms from $B(3,r;0,\gamma,0)$ restricted to γ_1. If γ_1 is not $\mathcal{F}$-Alperin then every morphism in $\mathrm{Aut}_{\mathcal{F}}(\gamma_1)$ can be lifted by Theorem 2.5. Suppose then that γ_1 is $\mathcal{F}$-Alperin. Take $\varphi \in \mathrm{Out}_{\mathcal{F}}(\gamma_1)$ appearing in the table of Lemma 5.6 and consider the images $c_s, c_{s^2} \in \mathrm{Out}_{\mathcal{F}}(\gamma_1)$ of the restrictions to γ_1 of conjugation by s and s^2. Then it can be checked that $\varphi c_s \varphi^{-1}$ equals c_s or c_{s^2}. Now apply (II) from Definition 2.2.

For $E_i \leq B(3,r;0,\gamma,0)$ there is a little bit more work to do. If E_i is not $\mathcal{F}$-Alperin apply Theorem 2.5 again. So suppose E_i is $\mathcal{F}$-Alperin, take $\overline{\varphi} \in \mathrm{Out}_{\mathcal{F}}(E_i)$ and compute N_φ from Definition 2.2. The normalizer of E_i is $\langle \zeta, \zeta'', sa\rangle \cong (\mathbb{Z}/9 \times \mathbb{Z}/3) : \mathbb{Z}/3$ where the order of ζ'' equals 9, and $\varphi c_{\zeta''} \varphi^{-1}$ equals $c_{\zeta''}$ or $c_{\zeta''^2}$. So φ can be lifted to the normalizer (Definition 2.2) and, as this subgroup cannot be $\mathcal{F}$-Alperin by Lemma 5.2, we can extend again to the whole $B(3,r;0,\gamma,0)$ by Theorem 2.5. The last case is to consider $V_i \leq E_i$. The details are analogous with $c_{\zeta'}$.

If $\beta = 1$, then P can be identified with a centric subgroup of $B(3,r-1;0,0,0) < B(3,r;1,0,0)$ (see proof of Lemma A.16) and then use the arguments above to extend the morphisms to $B(3,r-1;0,0,0)$. We then use saturateness to extend the morphism to $B(3,r;1,0,0)$. □

Theorem 5.8. *Every rank two 3-group isomorphic to $B(3,r;1,0,0)$ is resistant.*

Proof. Using Lemma 5.3, the only possible $\mathcal{F}$-Alperin proper subgroups of $B(3,r;1,0,0)$ are E_0 and V_0 if r is odd and γ_2, E_0 and V_0 if r is even. In both cases, according to Lemmas 5.7 and 5.6, if E_0 (respectively V_0) is $\mathcal{F}$-Alperin, then $\mathrm{Out}_{\mathcal{F}}(E_0)$ contains the diagonal matrix $-\mathrm{Id}$ (respectively the matrix $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$) which is not admissible by Lemma 5.7. Therefore we are left with the case of $r = 2k$ when γ_2 is the only $\mathcal{F}$-Alperin proper subgroup. But $\gamma_2 \cong \mathbb{Z}/3^{k-1} \times \mathbb{Z}/3^{k-1}$ is normal in $B(3,r;1,0,0)$ hence the Sylow 3-subgroup of $\mathrm{Out}_{\mathcal{F}}(\gamma_2)$ has size 9, but that contradicts $\mathrm{Out}_{\mathcal{F}}(\gamma_2) \leq \mathrm{GL}_2(3)$. □

It remains to study the case $B(3,r;0,\gamma,0)$. In this case we obtain saturated fusion systems with proper $\mathcal{F}$-Alperin subgroups.

Notation. In what follows we consider the following notation:

- Fix the following elements in $\mathrm{Aut}(B(3,r;\beta,\gamma,0))$:
  - η an element of order two which fixes E_0 and permutes E_1 with E_{-1} and such that projects to $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ in $\mathrm{Out}(B(3,r;\beta,\gamma,0))$.
  - ω an element of order two which commutes with η, which projects to $-\mathrm{Id}$ in $\mathrm{Out}(B(3,r;\beta,\gamma,0))$, and such that fixes E_i for $i\in\{-1,0,1\}$.
- By $N\cdot_\gamma W$ we denote an extension of type $N\cdot W$ such that its Sylow 3-subgroup is isomorphic to $B(3,r;0,\gamma,0)$.

With all that information now we get the tables with the possible fusion systems over $B(3,r;0,\gamma,0)$ which are not the normalizer of the Sylow 3-subgroup:

Theorem 5.9. *Let B be a rank two 3-group of maximal nilpotency class of order at least 3^4, and let $(B,\mathcal{F})$ be a saturated fusion system with at least one proper $\mathcal{F}$-Alperin subgroup. Then it must correspond to one of the cases listed in the following tables.*

- *If $B\cong B(3,4;0,0,0)$ then the outer automorphism group of the $\mathcal{F}$-Alperin subgroups are in the following table:*

<table>
<tr><th>B</th><th>E_0</th><th>E_1</th><th>E_{-1}</th><th>V_0</th><th>V_1</th><th>V_{-1}</th><th>p-lfg</th></tr>
<tr><td rowspan="3">$\langle\omega\rangle$</td><td rowspan="3">-</td><td rowspan="3">-</td><td rowspan="3">-</td><td>$\mathrm{SL}_2(3)$</td><td>-</td><td>-</td><td>$\mathcal{F}(3^4,1)$</td></tr>
<tr><td>-</td><td>$\mathrm{SL}_2(3)$</td><td>$\mathrm{SL}_2(3)$</td><td>$\mathcal{F}(3^4,2)$</td></tr>
<tr><td>$\mathrm{SL}_2(3)$</td><td>$\mathrm{SL}_2(3)$</td><td>$\mathrm{SL}_2(3)$</td><td>$L_3^{\pm}(q_1)$</td></tr>
<tr><td>$\langle\eta\omega\rangle$</td><td>$\mathrm{SL}_2(3)$</td><td colspan="2">-</td><td>-</td><td colspan="2">-</td><td>$E_0\cdot_0\mathrm{SL}_2(3)$</td></tr>
<tr><td rowspan="5">$\langle\eta,\omega\rangle$</td><td rowspan="3">-</td><td colspan="2" rowspan="3">-</td><td>-</td><td colspan="2">$\mathrm{SL}_2(3)$</td><td>$\mathcal{F}(3^4,2).2$</td></tr>
<tr><td rowspan="2">$\mathrm{GL}_2(3)$</td><td colspan="2">-</td><td>$\mathcal{F}(3^4,1).2$</td></tr>
<tr><td colspan="2">$\mathrm{SL}_2(3)$</td><td>$L_3^{\pm}(q_1):2$</td></tr>
<tr><td rowspan="2">$\mathrm{GL}_2(3)$</td><td colspan="2" rowspan="2">-</td><td rowspan="2">-</td><td colspan="2">-</td><td>$E_0\cdot_0\mathrm{GL}_2(3)$</td></tr>
<tr><td colspan="2">$\mathrm{SL}_2(3)$</td><td>${}^3D_4(q_2)$</td></tr>
</table>

TABLE 2. s.f.s. over $B(3,4;0,0,0)$.

Where q_1 and q_2 are prime powers such that $\nu_3(q_1\mp 1)=2$ and $\nu_3(q_2^2-1)=1$.

- *If $B\cong B(3,4;0,2,0)$ then the outer automorphism group of the $\mathcal{F}$-Alperin subgroups are in the following table:*

B	E_0	V_0	p-*lfg*
$\langle\omega\rangle$	–	$\mathrm{SL}_2(3)$	$\mathcal{F}(3^4,3)$
$\langle\eta\omega\rangle$	$\mathrm{SL}_2(3)$	–	$E_0 \cdot_2 \mathrm{SL}_2(3)$
$\langle\eta,\omega\rangle$	–	$\mathrm{GL}_2(3)$	$\mathcal{F}(3^4,3).2$
	$\mathrm{GL}_2(3)$	–	$E_0 \cdot_2 \mathrm{GL}_2(3)$

TABLE 3. s.f.s. over $B(3,4;0,2,0)$.

- *If* $B \cong B(3,2k;0,0,0)$ *with* $k \geq 3$ *then the outer automorphism group of the* $\mathcal{F}$*-Alperin subgroups are in the following table:*

B	E_0	E_1	E_{-1}	V_0	V_1	V_{-1}	p-*lfg*
$\langle\omega\rangle$	-	-	-	$\mathrm{SL}_2(3)$	-	-	$\mathcal{F}(3^{2k},1)$
				-	$\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$	$\mathcal{F}(3^{2k},2)$
				$\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$	$L_3^{\pm}(q_1)$
$\langle\eta\omega\rangle$	$\mathrm{SL}_2(3)$	-		-	-		$3 \cdot_0 \mathrm{PGL}_3(q_2)$
$\langle\eta,\omega\rangle$	-	-		-	$\mathrm{SL}_2(3)$		$\mathcal{F}(3^{2k},2).2$
				$\mathrm{GL}_2(3)$	-		$\mathcal{F}(3^{2k},1).2$
					$\mathrm{SL}_2(3)$		$L_3^{\pm}(q_1):2$
	$\mathrm{GL}_2(3)$	-		-	-		$3\cdot_0\mathrm{PGL}_3(q_2)\cdot 2$
					$\mathrm{SL}_2(3)$		${}^3D_4(q_3)$

TABLE 4. s.f.s. over $B(3,2k;0,0,0)$ with $k \geq 3$.

Where q_i *are prime powers such that* $\nu_3(q_1 \mp 1) = k$, $\nu_3(q_2 - 1) = k-1$ *and* $\nu_3(q_3^2 - 1) = k-1$.

- *If* $B \cong B(3,2k;0,\gamma,0)$ *with* $k \geq 3$ *and* $\gamma = 1,2$, *then the outer automorphism group of the* $\mathcal{F}$*-Alperin subgroups are in the following table:*

B	E_0	V_0	p-*lfg*
$\langle\omega\rangle$	–	$\mathrm{SL}_2(3)$	$\mathcal{F}(3^{2k},2+\gamma)$
$\langle\eta\omega\rangle$	$\mathrm{SL}_2(3)$	–	$3 \cdot_\gamma \mathrm{PGL}_3(q)$
$\langle\eta,\omega\rangle$	–	$\mathrm{GL}_2(3)$	$\mathcal{F}(3^{2k},2+\gamma).2$
	$\mathrm{GL}_2(3)$	–	$3 \cdot_\gamma \mathrm{PGL}_3(q) \cdot 2$

TABLE 5. s.f.s. over $B(3,2k;0,\gamma,0)$ with $\gamma = 1,2$ and $k \geq 3$.

Where q *is a prime power such that* $\nu_3(q-1) = k-1$.

- *If $B \cong B(3, 2k+1; 0, 0, 0)$ with $k \geq 2$ then the outer automorphism group of the $\mathcal{F}$-Alperin subgroups are in the following table:*

B	V_0	V_1	V_{-1}	E_0	E_1	E_{-1}	γ_1	p-*lfg*
$\langle \omega \rangle$	-	-	-	$\mathrm{SL}_2(3)$	-	-	-	$3.\mathcal{F}(3^{2k}, 1)$
				-	$\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$	-	$3.\mathcal{F}(3^{2k}, 2)$
				$\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$	$\mathrm{SL}_2(3)$	-	$3 \cdot L_3^{\pm}(q_1)$
$\langle \eta \rangle$	-	-		-	-		$\mathrm{SL}_2(3)$	$\gamma_1 : \mathrm{SL}_2(3)$
$\langle \eta\omega \rangle$	$\mathrm{SL}_2(3)$	-		-	-		-	$\mathrm{PGL}_3(q_2)$
$\langle \eta, \omega \rangle$	-	-		-	-		$\mathrm{GL}_2(3)$	$\gamma_1 : \mathrm{GL}_2(3)$
					$\mathrm{SL}_2(3)$		-	$3.\mathcal{F}(3^{2k}, 2).2$
							$\mathrm{GL}_2(3)$	$\mathcal{F}(3^{2k+1}, 1)$
				$\mathrm{GL}_2(3)$	-		-	$3.\mathcal{F}(3^{2k}, 1).2$
							$\mathrm{GL}_2(3)$	${}^2F_4(q_3)$
					$\mathrm{SL}_2(3)$		-	$3 \cdot L_3^{\pm}(q_1) : 2$
							$\mathrm{GL}_2(3)$	$\mathcal{F}(3^{2k+1}, 2)$
	$\mathrm{GL}_2(3)$	-		-	-		-	$\mathrm{PGL}_3(q_2) \cdot 2$
							$\mathrm{GL}_2(3)$	$\mathcal{F}(3^{2k+1}, 3)$
					$\mathrm{SL}_2(3)$		-	$3 \cdot {}^3D_4(q_4)$
							$\mathrm{GL}_2(3)$	$\mathcal{F}(3^{2k+1}, 4)$

TABLE 6. s.f.s. over $B(3, 2k+1; 0, 0, 0)$ with $k \geq 2$.

Where q_i are prime powers such that $\nu_3(q_1 \mp 1) = k$, $\nu_3(q_2 - 1) = k$, $\nu_3(q_3^2 - 1) = k$ and $\nu_3(q_4^2 - 1) = k - 1$.

- *If $B \cong B(3, 2k+1; 0, 1, 0)$ with $k \geq 2$ then $\mathcal{F}$ is the saturated fusion system associated to the group $\gamma_1 : \mathrm{SL}_2(3)$.*

The last column of all this tables gives either the information about the groups which have the corresponding fusion system, either a name encoded as $\mathcal{F}(3^r, i)$ to refer to an exotic 3-local finite group or the expression of that 3-local finite group as extension of another 3-local finite group.

Remark 5.10. As particular cases of the classification we find the exotic fusion systems over 3-groups of order 3^4 which were announced previously by Broto-Levi-Oliver in [12, Section 5].

Proof. We divide the proof in three parts:
Classification: In this part we describe the different possibilities for the saturated fusion system $(B, \mathcal{F})$ by means of the $\mathcal{F}$-Alperin subgroups and their outer automorphisms groups $\mathrm{Out}_{\mathcal{F}}(P)$. Because $\mathrm{Aut}_P(P) \leq \mathrm{Aut}_{\mathcal{F}}(P)$ by Definition 2.1, $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_P(P)$ also determines $\mathrm{Aut}_{\mathcal{F}}(P)$, and by Theorem 2.5, these subgroups of automorphisms describe completely the category $\mathcal{F}$.

By hypothesis we have a proper $\mathcal{F}$-Alperin subgroup in $B(3,r;\beta,\gamma,\delta)$, so by Lemma 5.1 $\delta=0$, and using now Lemma 5.8 also $\beta=0$. So we just have to cope with $B(3,r;0,\gamma,0)$.

First of all obverve that fixed $i\in\{-1,0,1\}$, E_i and V_i cannot be at the same time $\mathcal{F}$-Alperin subgroups: if E_i is $\mathcal{F}$-Alperin then V_i is $\mathcal{F}$-conjugate to $\langle\zeta,\zeta'\rangle$, so V_i is not $\mathcal{F}$-centric.

Notice that a saturated fusion systems with $\mathrm{Out}_{\mathcal{F}}(B(3,r;0,\gamma,0))=1$ cannot contain any proper $\mathcal{F}$-Alperin subgroup: if it had a proper $\mathcal{F}$-Alperin subgroup, by Lemma 5.7 we would have a nontrivial morphism in $\mathrm{Out}_{\mathcal{F}}(B(3,r;0,\gamma,0))$.

We now begin the analysis depending on the parity of r.

Case $r=2k$. Suppose that $\gamma=0$. If $\mathrm{Out}_{\mathcal{F}}(B)=\langle\omega\rangle$ then it is immediate from Lemmas 5.6 and 5.7 that V_i may be $\mathcal{F}$-Alperin but not E_i. The reason is that $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\in\mathrm{SL}_2(3)$ lifts to $\omega\in\mathrm{Out}_{\mathcal{F}}(B)=\langle\omega\rangle$ from $\mathrm{Out}_{\mathcal{F}}(V_i)$ (it is admissible and the lifting does exist), while it would lift to $\eta\omega\notin\mathrm{Out}_{\mathcal{F}}(B)$ from $\mathrm{Out}_{\mathcal{F}}(E_0)$ (it is admissible but the lifting does not exist), and it does not lift for E_i with $i=-1,1$ (it is not admissible). It is also deduced from Lemmas 5.6 and 5.7 that no V_i can be $\mathcal{F}$-conjugate to V_j for $i\neq j$ because inner conjugation in B does not move $\{V_{-1},V_0,V_1\}$ and outer conjugation is induced just by η and $\eta\omega$. If V_i is $\mathcal{F}$-Alperin then $\mathrm{Out}_{\mathcal{F}}(V_i)$ must equal $\mathrm{SL}_2(3)$ because otherwise we would have a nontrivial element in $\mathrm{Out}_{\mathcal{F}}(B)$ different from ω for V_0, or we would have a non admissible map in E_{-1} or E_1 for V_{-1} or V_1 respectively. So one, two or three of the V_i's can be $\mathcal{F}$-Alperin. A symmetry argument, obtained by conjugation by B (see remarks after Lemma 5.5), yields the first three rows of tables 2 and 4. Now assume $\mathrm{Out}_{\mathcal{F}}(B)=\langle\eta\omega\rangle$. Looking at Lemmas 5.5, 5.6 and 5.7 we obtain that only E_0 can be $\mathcal{F}$-Alperin and moreover, $\mathrm{Out}_{\mathcal{F}}(E_0)=\mathrm{SL}_2(3)$ because otherwise $\mathrm{Out}_{\mathcal{F}}(B)$ would be $\mathbb{Z}/2\times\mathbb{Z}/2$. This is the fourth row of tables 2 and 4. For $\mathrm{Out}_{\mathcal{F}}(B)=\langle\eta\rangle$ there is no chance for E_i or V_i to be $\mathcal{F}$-Alperin. It remains to cope with the case $\mathrm{Out}_{\mathcal{F}}(B)=\langle\eta,\omega\rangle$. First, from the argument above E_1 and E_{-1} cannot be $\mathcal{F}$-Alperin. Second, recall from the beginning of this proof that, for i fixed, E_i and V_i cannot be $\mathcal{F}$-Alperin simultaneously. Third, notice that η (and $\eta\omega$) swaps V_1 and V_{-1}, so they are $\mathcal{F}$-conjugate. Lastly, from Lemma 5.5, the only possibility for the outer automorphism groups of E_0 and V_0 is $\mathrm{GL}_2(3)$ in case they are $\mathcal{F}$-Alperin. Analogously if V_i is $\mathcal{F}$-Alperin, for $i=\pm1$, then $\mathrm{Out}_{\mathcal{F}}(V_i)$ must be $\mathrm{SL}_2(3)$. Now a case by case checking yields the last five entries of tables 2 and 4. If $r>4$ and $\gamma=1,2$ (or $r=4$ and $\gamma=2$) recall from the conditions in the table of Lemma 5.2 that only E_0 and V_0 are allowed to be $\mathcal{F}$-Alperin. Similar arguments to those above lead us to tables 3 and 5.

Case $r=2k+1$. For $\gamma=0$ the fusion systems in the table 6 are obtained by similar arguments to those of the preceding cases, bearing γ_1 may be $\mathcal{F}$-Alperin too in mind. To fill in this table, notice that if some E_i or V_i is $\mathcal{F}$-Alperin then $-\mathrm{Id}$ is an outer automorphism of this group that must lift, following Lemma 5.6, to ω or $\eta\omega$ for E_i and V_i respectively. But ω and $\eta\omega$ restrict in $\mathrm{Out}_{\mathcal{F}}(\gamma_1)$ to automorphisms of determinant -1, so in case γ_1 is $\mathcal{F}$-Alperin, when some E_i or V_i is so, its outer automorphism group must be $\mathrm{Out}_{\mathcal{F}}(\gamma_1)=\mathrm{GL}_2(3)$. Notice that when γ_1 is $\mathcal{F}$-Alperin, $\mathrm{Out}_{\mathcal{F}}(B)$ must contain η by Lemma 5.6 and so E_{-1} and E_1 are $\mathcal{F}$-conjugate. In case of $\gamma=1$, Lemmas 5.5 and A.14 imply that only $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ can be in $\mathrm{Out}_{\mathcal{F}}(B)$. If γ_1, E_0 or V_0 were $\mathcal{F}$-Alperin then $\mathrm{Out}_{\mathcal{F}}(B)$ would contain η, ω or $\eta\omega$ respectively. So the only chance is that γ_1 is the only $\mathcal{F}$-Alperin subgroup. Notice also that $\mathrm{Out}_{\mathcal{F}}(\gamma_1)$ must equal $\mathrm{SL}_2(3)$, because in other case there would be a non-trivial element in $\mathrm{Out}_{\mathcal{F}}(B)$ different from η.

Saturation: Now we prove that all the fusion systems obtained in the classification part of this proof are saturated by means of [12, Proposition 5.3]. To show that a certain fusion system $(B,\mathcal{F})$ appearing in the tables is saturated the method consists in setting $G \overset{\text{def}}{=} B : \mathrm{Out}_{\mathcal{F}}(B)$, where $\mathrm{Out}_{\mathcal{F}}(B)$ is the entry for B in the table, and for each G-conjugacy class of $\mathcal{F}$-Alperin subgroups choosing a representative $P \leq B$ and setting $K_P \overset{\text{def}}{=} \mathrm{Out}_G(P)$ (K_P is determined by Lemma 5.6) and $\Delta_P \overset{\text{def}}{=} \mathrm{Out}_{\mathcal{F}}(P)$, where $\mathrm{Out}_{\mathcal{F}}(P)$ is the entry for P in the table. In order to obtain that the fusion system under consideration is saturated, for each chosen $\mathcal{F}$-Alperin subgroup $P \leq B$ it must be checked that:

(1) P does not contain any proper $\mathcal{F}$-centric subgroup.
(2) $p \nmid [\Delta_P : K_P]$ and for each $\alpha \in \Delta_P \setminus K_P$, $K_P \cap \alpha^{-1} K_P \alpha$ has order prime to p.

On the one hand, by Lemma 5.2 the first condition is fulfilled by all the fusion systems obtained in the classification part of this proof. On the other hand, it is verified that $\mathrm{Out}_B(P)$ equals $\langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \rangle$, $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ and $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ for γ_1, E_i and V_i respectively. Denoting by μ_P this order 3 outer automorphism it is an easy check that for all the fusion systems in the tables:

- P is $\mathcal{F}$-Alperin with $\Delta_P = \mathrm{SL}_2(3)$ just in case $K_P = \langle \mu_P, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$,
- P is $\mathcal{F}$-Alperin with $\Delta_P = \mathrm{GL}_2(3)$ just in case $K_P = \langle \mu_P, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \rangle$.

Now the two pairs (K_P, Δ_P) above verify the condition (2).

Notice that the group $G = B : \mathrm{Out}_{\mathcal{F}}(B)$ defined earlier can be constructed because 3 does not divide the order of $\mathrm{Out}_{\mathcal{F}}(B)$ (see Remark 2.3) and the projection $\mathrm{Aut}(B) \twoheadrightarrow \mathrm{Out}(B)$ has kernel a 3-group, and thus there is a lifting of $\mathrm{Out}_{\mathcal{F}}(B)$ to $\mathrm{Aut}(B)$. A more delicate point in the proofs of classification and saturation is that (recall the remarks after the Lemma 5.3) the outer automorphisms groups $\mathrm{Out}_{\mathcal{F}}(P)$ for $P \leq B$ $\mathcal{F}$-Alperin (P can be the whole B) appearing in the tables are described as subgroups of $\mathrm{GL}_2(3)$, and that while for $P = E_i, V_i$ the Frattini maps $\mathrm{Out}(P) \xrightarrow{\rho} \mathrm{GL}_2(3)$ are isomorphisms, for γ_1 and B they are not.

The choice of $\mathrm{SL}_2(3)$ and $\mathrm{GL}_2(3)$ lying in $\mathrm{Out}(\gamma_1) = \mathrm{Aut}(\gamma_1)$ and of $\mathbb{Z}/2$ and $\mathbb{Z}/2 \times \mathbb{Z}/2 \leq \mathrm{Out}(B)$ are not totally arbitrary. In fact, the choice of $\mathrm{Aut}_{\mathcal{F}}(B)$ must go by the restriction map $\mathrm{Aut}(B) \to \mathrm{Aut}(\gamma_1)$ to the choice of $\mathrm{Aut}_{\mathcal{F}}(\gamma_1)$. Moreover, as γ_1 is characteristic in B and diagonal automorphisms in $\mathrm{Aut}_{\mathcal{F}}(\gamma_1) = \mathrm{Out}_{\mathcal{F}}(\gamma_1)$ must lift to $\mathrm{Aut}_{\mathcal{F}}(B)$ (Lemma 5.7), one can check that the choice of $\mathrm{Aut}_{\mathcal{F}}(\gamma_1)$ determines completely the choice of $\mathrm{Out}_{\mathcal{F}}(B)$.

Now we prove that different choices gives isomorphic saturated fusions systems. Let $(B,\mathcal{F})$ and $(B,\mathcal{F}')$ be saturated fusion systems that correspond to the same row in some table of the classification. Suppose first that γ_1 is not $\mathcal{F}$-Alperin: the two semidirect products $H = B : \mathrm{Out}_{\mathcal{F}}(B)$ and $H' = B : \mathrm{Out}_{\mathcal{F}'}(B)$ are isomorphic because the lifts of $\mathrm{Out}_{\mathcal{F}}(B)$ and $\mathrm{Out}_{\mathcal{F}'}(B)$ to $\mathrm{Aut}(B)$ are $\mathrm{Aut}(B)$-conjugate as the projection $\mathrm{Aut}(B) \twoheadrightarrow \mathrm{GL}_2(3)$ has kernel a 3-group. Then we have an isomorphism of categories $\mathcal{F}_B(H) \cong \mathcal{F}_B(H')$ which can be extended to an isomorphism $\mathcal{F} \cong \mathcal{F}'$.

If γ_1 is $\mathcal{F}$-Alperin then (recall that $\mathrm{Aut}_{\mathcal{F}}(\gamma_1)$ determines $\mathrm{Out}_{\mathcal{F}}(B)$) we build the semidirect products $H = \gamma_1 : \mathrm{Out}_{\mathcal{F}}(\gamma_1)$ and $H' = \gamma_1 : \mathrm{Out}_{\mathcal{F}'}(\gamma_1)$. These groups are isomorphic by Corollary A.19 and have Sylow 3-subgroup B. Then we have an isomorphism of categories $\mathcal{F}_B(H) \cong \mathcal{F}_B(H')$ which can be extended to an isomorphism $\mathcal{F} \cong \mathcal{F}'$.

Exoticism: To justify the values in the last column we have to cope with the possible finite groups with the fusion systems described there.

Consider first all the fusion systems in the tables such that they have at least one $\mathcal{F}$-Alperin rank two elementary abelian subgroup (respectively at least one $\mathcal{F}$-Alperin subgroup isomorphic to 3^{1+2}_+ and also γ_1 is $\mathcal{F}$-Alperin). Consider $N \lneq B(3,r;0,\gamma,0)$ a nontrivial proper normal subgroup which is strongly closed in $\mathcal{F}$. By Lemma A.10, N must contain the center of B, and as there is an $\mathcal{F}$-Alperin rank two elementary subgroup N must also contain s (respectively, if γ_1 is $\mathcal{F}$-radical N must also contain γ_{r-2}, and as there is an $\mathcal{F}$-Alperin subgroup isomorphic to 3^{1+2}_+, N must contain s too). Again by Lemma A.10 N must be isomorphic to 3^{1+2}_+ if $r = 4$ or $B(3, r-1; 0, 0, 0)$ if $r > 4$.

In all these cases we can apply Proposition 2.19, getting that if they are the fusion system of a group G, then G can be choosen to be almost simple. Moreover the 3-rank of G and the simple group of which G is an extension must be two, so we have to look at the list of all the simple groups of 3-rank two:

a) The information about the sporadic simple groups can be deduced from [25, Tables 5.3 & 5.6.1], getting that all of the groups in that family which 3-rank equals two have Sylow 3-subgroup of order at most 3^3, and there are not outer automorphisms of order 3.
b) The p-rank over the Lie type simple groups in a field of characteristic p are in [25, Table 3.3.1], where taking $p = 3$ and the possibilities of the groups of 3-rank two one gets that the order of the Sylow 3-subgroup is at most 3^3, and again there are not outer automorphism of order 3.
c) The Lie type simple groups in characteristic prime to 3 have a unique elementary 3-subgroup of maximal rank, out of $L_3(q)$ with $3|q-1$, $L_3^-(q)$ with $3|q+1$, $G_2(q)$, ${}^3D_4(q)$ or ${}^2F_4(q)$ by [23, 10-2]. So, as 3^{1+2}_+ and $B(3,r;0,\gamma,0)$ do not have a unique elementary abelian 3-subgroup of maximal rank, we have to look at the fusion systems of this small list.

   The fusion systems induced by $L_3^+(q)$, when $3|(q-1)$, and by $L_3^-(q)$, when $3|(q+1)$, are the same, and can be deduced from [13, Example 3.6] and [5] using that there is a bijection between radical subgroups in $\mathrm{SL}_3(q)$ and radical subgroups in $\mathrm{PSL}_3(q)$, so obtaining the result in Table 4. The extensions of $L_3^{\pm}(q)$ by a group of order prime to 3 must be also considered, getting fusion systems over the same 3-group. Finally, $\mathrm{Out}(L_3^{\pm}(q))$ has 3-torsion, so we must consider the possible extensions, getting the group $\mathrm{PGL}_3(q)$ and an extension $\mathrm{PGL}_3(q).2$. The study of the proper radical subgroups in this case is done in [5], getting that the only proper $\mathcal{F}$-radical is V_0.

   The fusion system of $G_2(q)$ is studied in [28] and [16], getting that it corresponds to the fusion system labeled as $3L_3^+(q):2$.

   The fusion system of ${}^3D_4(q)$ can be deduced from [27], getting the desired result.

   Finally the fusion system of ${}^2F_4(q)$ has been studied in [13, Example 9.7].

This classification tells us that all the other cases where there is an $\mathcal{F}$-Alperin rank two elementary abelian 3-subgroup, and also the ones such that γ_1 and one subgroup isomorphic to 3^{1+2}_+ are $\mathcal{F}$-radical, must correspond to exotic p-local finite groups.

Consider now the cases where the only proper $\mathcal{F}$-Alperin subgroup is γ_1. In those cases it is straightforward to check that they correspond to the groups $\gamma_1 : \mathrm{SL}_2(3)$ and $\gamma_1 : \mathrm{GL}_2(3)$, where the actions are described in Lemma A.17.

In all of the remaining cases, the ones where all the proper $\mathcal{F}$-Alperin subgroups are isomorphic to 3^{1+2}_+, the normalizer of the center of $B(3, r; 0, \gamma, 0)$ in $\mathcal{F}$ is the whole fusion system $\mathcal{F}$ (i.e. $Z(B(3, r; 0, \gamma, 0))$ is normal in $\mathcal{F}$).

Consider first the ones where $Z(B(3, r; 0, \gamma, 0))$ is central in $\mathcal{F}$, i.e. the ones where for all E_i proper $\mathcal{F}$-radical subgroup isomorphic to 3^{1+2}_+ we have $\mathrm{Out}_{\mathcal{F}}(E_i) = \mathrm{SL}_2(3)$. Using Lemma 2.21, we get that they correspond to groups if and only if the quotient by the center corresponds also to a group, getting again the results in the tables.

Finally it remains to justify that $3\mathcal{F}(3^{2k}, 1).2$ and $3\mathcal{F}(3^{2k}, 2).2$ are not the fusion system of finite groups. Consider G a finite group with one of those fusion systems, and consider $Z(B)$ the center of $B(3, 2k + 1; 0, 0, 0)$. Consider now the fusion system constructed as the centralizer of $Z(B)$ (Definition 2.11) then, by Remark 2.12 $3\mathcal{F}(3^{2k}, 1)$ or $3\mathcal{F}(3^{2k}, 2)$ would be also the fusion system of the group $C_{Z(B)}(G)$, and we know that these are exotic. □

Appendix A. Rank two p-groups

In this appendix we recall all the information and properties of p-rank two p-groups that we need to classify the saturated fusion systems over these groups.

The classification of the rank two p-groups, $p > 2$, traces back to Blackburn (e.g. see [29, Theorem 3.1]):

Theorem A.1. *Let p be an odd prime. Then the p-groups of p-rank two are the ones listed here:*

(i) *The non-cyclic metacyclic p-groups, which we denote $M(p, r)$.*

(ii) *The groups $C(p, r)$, $r \geq 3$ defined by the following presentation:*

$$C(p, r) \stackrel{def}{=} \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = 1, [a, b] = c^{p^{r-3}}, c \in Z(C(p, r))\rangle.$$

(iii) *The groups $G(p, r; \epsilon)$, where $r \geq 4$ and ϵ is either 1 or a quadratic non-residue modulo p defined by the following presentation:*

$$G(p, r; \epsilon) \stackrel{def}{=} \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = [b, c] = 1, [a, b^{-1}] = c^{\epsilon p^{r-3}}, [a, c] = b\rangle.$$

(iv) *If $p = 3$ the 3-groups of maximal nilpotency class, except the cyclic groups and the wreath product of $\mathbb{Z}/3$ by itself.*

Where $[x, y] = x^{-1}y^{-1}xy$.

Proof. According to [20, Theorem 5.4.15], the class of rank two p-groups (p odd) agrees with the class of p-groups in which every maximal normal abelian subgroup has rank two, or equivalently every maximal normal elementary abelian subgroup has rank two. As the only group of order p^3 which requires at least three generators is $(\mathbb{Z}/p)^3$, the class of rank two p-groups (p odd) agrees with the class of p-groups in which every normal subgroup of size p^3 is generated by at most two elements. The latter class of p-groups is described in [7, Theorem 4.1] when the order of the group is p^n for $n \geq 5$ (and $p > 2$) while the case $n \leq 4$ can be deduced for the classification of p-groups of size at most p^4 [14, p. 145–146]. □

To complete the classification above we also need a description of maximal nilpotency class 3-groups, which is given in [6, last paragraph p. 88]:

Theorem A.2. *The non cyclic* 3*-groups of maximal nilpotency class and order greater than* 3^3 *are the groups* $B(3,r;\beta,\gamma,\delta)$ *with* (β,γ,δ) *taking the values:*

- *For any* $r \geq 5$*,* $(\beta,\gamma,\delta)=(1,0,\delta)$*, with* $\delta\in\{0,1,2\}$*.*
- *For even* $r\geq 4$*,* $(\beta,\gamma,\delta)\in\{(0,\gamma,0),(0,0,\delta)\}$*, with* $\gamma\in\{1,2\}$ *and* $\delta\in\{0,1\}$*.*
- *For odd* $r\geq 5$*,* $(\beta,\gamma,\delta)\in\{(0,1,0),(0,0,\delta)\}$*, with* $\delta\in\{0,1\}$*.*

With these parameters, $B(3,r;\beta,\gamma,\delta)$ *is the group of order* 3^r *defined by the set of generators* $\{s,s_1,s_2,\ldots,s_{r-1}\}$ *and relations*

$$s_i=[s_{i-1},s]\,\text{for } i\in\{2,3,\ldots,r-1\}, \tag{1}$$
$$[s_1,s_2]=s_{r-1}^{\beta}\,, \tag{2}$$
$$[s_1,s_i]=1\,\text{for } i\in\{3,4,\ldots,r-1\}, \tag{3}$$
$$s^3=s_{r-1}^{\delta}\,, \tag{4}$$
$$s_1^3s_2^3s_3=s_{r-1}^{\gamma}\,, \tag{5}$$
$$s_i^3s_{i+1}^3s_{i+2}=1\,\text{for } i\in\{2,3,\ldots,r-1\},\ \text{and assuming } s_r=s_{r+1}=1. \tag{6}$$

Remark A.3. For $p=3$ and $r=4$ we have that $B(3,4;0,0,0)\cong G(3,4;1)$, $B(3,4;0,2,0)\cong G(3,4;-1)$ and $B(3,4;0,1,0)$ is the wreath product $3\wr 3$ that has 3-rank three.

Remark A.4. In [6] the classification of the p-groups of maximal rank depends on four parameters α, β, γ and δ, but for $p=3$ we have $\alpha=0$. In all the paper, with the notation $B(3,r;\beta,\gamma,\delta)$ we assume that the parameters (β,γ,δ) correspond to the stated in Theorem A.2 for rank two 3-groups.

What follows is a description of the group theoretical properties of the groups listed in Theorems A.1 and A.2, that are used along the paper.

We begin with the family $C(p,r)$:

Lemma A.5. *Consider* $C(p,r)$ *as in Theorem A.1, with the same notation for the generators:*

(a) *The center is* $\langle c\rangle\cong\mathbb{Z}/p^{r-2}$*.*
(b) *The commutators are determined by* $[a^ib^j,a^sb^t]=c^{(it-sj)p^{r-3}}$*.*
(c) $C(p,r)=Z(C(p,r))\Omega_1(C(p,r))$ *and therefore* $C(p,r)$ *is isomorphic to the central product* $\mathbb{Z}/p^{r-2}\circ C(p,3)$*.*
(d) *The restriction of the elements in* $\mathrm{Aut}(C(p,r))$ *to* $Z(C(p,r))$ *and* $\Omega_1(C(p,r))$ *provides an isomorphism*

$$\mathrm{Aut}(C(p,r))\cong\big(\mathbb{Z}/p^{r-3}\times \mathrm{ASL}_2(p)\big):(p-1)<\mathrm{Aut}(\mathbb{Z}/p^{r-2})\times\mathrm{Aut}(C(p,3))$$

that gives rise to a group epimorphism $\rho\colon\ \mathrm{Out}(C(p,r))\to\mathrm{GL}_2(p)$ *which maps a morphism of type* $a\mapsto a^ib^jc^k$*,* $b\mapsto a^{i'}b^{j'}c^{k'}$ *to the matrix* $\left(\begin{smallmatrix} i & i'\\ j & j'\end{smallmatrix}\right)$*.*
(e) $\mathrm{Aut}(C(p,r))=\mathrm{Inn}(C(p,r))\rtimes\mathrm{Out}(C(p,r))\cong(\mathbb{Z}/p\times\mathbb{Z}/p)\rtimes(\mathbb{Z}/p^{r-3}\times\mathrm{GL}_2(p))$*.*
(f) *For* $G=\mathrm{GL}_2(p)$ *or* $\mathrm{SL}_2(p)$*, the group* $\mathrm{Out}(C(p,r))$ *contains just one subgroup isomorphic to* G *up to conjugation, but in the case* $G=\mathrm{SL}_2(3)$ *and* $r>3$*. The group* $\mathrm{Out}(C(3,r))$*,* $r>3$*, contains two different conjugacy classes of subgroups isomorphic to* $\mathrm{SL}_2(3)$*.*

Proof. The statement (a) follows from the presentation of $C(p,r)$ while (b) can be read from [19, Lemma 1.1]. The central product given in (c) is obtained by identifying $\mathbb{Z}/p^{r-2}$ with $\langle c\rangle = Z(C(p,r))$ and $C(p,3)$ with $\langle a,b\rangle = \Omega_1(C(p,r))$, so their intersection is $\langle c^{p^{r-3}}\rangle = Z(\langle a,b\rangle)$. Moreover, $Z(C(p,r))$ and $\Omega_1(C(p,r))$ are characteristic subgroups of $C(p,r)$, and therefore every element in $\mathrm{Aut}(C(p,r))$ maps each of these subgroups to itself. Then (c) implies that

$$\mathrm{Aut}(C(p,r)) = \{(f,g) \in \mathrm{Aut}(\langle c\rangle) \times \mathrm{Aut}(\langle a,b\rangle) | \quad f(c^{p^{r-3}}) = g(c^{p^{r-3}})\}$$

providing the description of $\mathrm{Aut}(C(p,r))$ in (d) while the morphism ρ is obtained by considering outer automorphisms and projecting on the $C(p,3)$ factor (see [35, Lemma 3.1]). As $\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \mathrm{Out}(G)$ for $G = \mathbb{Z}/p^{r-2}, C(p,3)$, (d) implies (e). Finally, notice that if $G = \mathrm{GL}_2(p)$ or $\mathrm{SL}_2(p)$, $\mathrm{GL}_2(p)$ contains just one copy of G up to conjugacy, and therefore, the number of subgroups of $\mathrm{Out}(C(p,r)) \cong \mathbb{Z}/p^{r-3} \times \mathrm{GL}_2(p)$ isomorphic to G up to conjugation depends on $\mathrm{Hom}(G, \mathbb{Z}/p^{r-3})$. Since G is p-perfect for $p > 3$, the latter set contains just one element unless $p = 3$, $r > 3$ and $G = \mathrm{SL}_2(3)$. Finally, $\mathrm{Hom}(\mathrm{SL}_2(3), \mathbb{Z}/3^{r-3})$ contains three elements that give rise to three subgroups of type $\mathrm{SL}_2(3)$ in $\mathrm{Out}(C(3,r))$, determined by their Sylow 3-subgroups $S_i \stackrel{\text{def}}{=} \langle(3^{r-4}i, \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right))\rangle$ for $i = 0,1,2$ (notice that $\mathrm{SL}_2(p)$ is generated by elements of order p [20, Theorem 2.8.4]). But S_1 and S_2 are conjugate in $\mathrm{Out}(C(3,r))$ and therefore there are just two conjugacy classes of $\mathrm{SL}_2(3)$ in $\mathrm{Out}(C(3,r))$ if $r > 3$. □

Lemma A.6. *Let H be a p-centric subgroup of $C(p,r)$, then H is either the total or $H \cong \mathbb{Z}/p \times \mathbb{Z}/p^{r-2}$.*

Proof. Assume H is a p-centric centric subgroup of $C(p,r)$. So H must contain the center $\langle c\rangle$ as a proper subgroup. Let $a^ib^j \in H \setminus \langle c\rangle$, then we have that $\langle a^ib^j, c\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p^{r-2}$ is a self-centralizing maximal subgroup in $C(p,r)$, so then H is either the total or $\langle a^ib^j, c\rangle$. □

The following properties of $G(p,r;\epsilon)$ can be deduced directly from [19, Section 1]

Lemma A.7. *Consider $G(p,r;\epsilon)$ as in Theorem A.1, with the same notation for the generators:*

(a) *The commutators are determined by the formula $[a^ib^jc^k, a^sb^tc^u] = b^{iu-sk}c^n$ where $n = \epsilon p^{r-3}(u\frac{i(i-1)}{2} + js - it - k\frac{s(s-1)}{2})$.*
(b) *The center of $G(p,r;\epsilon)$ is the group generated by $\langle c^p\rangle$, and, as $r \geq 4$, it contains $\langle c^{p^{r-3}}\rangle$.*
(c) *There is a unique automorphism in $G(p,r;\epsilon)$ which maps $\rho(a) = a^ib^jc^{lp^{r-3}}$ and $\rho(c) = b^tc^u$ for any $i,j,l,t,u \in \{\pm 1\} \times \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p \times (\mathbb{Z}/p^{r-2})^*$.*

Lemma A.8. *If $(p, r; \epsilon) \neq (3, 4; 1)$, the p-centric subgroups of $G(p, r; \epsilon)$ are the ones in the following table:*

Isomorphism type	*Subgroup*
$G(p, r; \epsilon)$	$\langle a, b, c\rangle$
$\mathbb{Z}/p \times \mathbb{Z}/p^{r-2}$	$\langle b, c\rangle$
$\mathbb{Z}/p^{r-2}$	$\langle ab^i c^j\rangle$ *with* $i \in \mathbb{Z}/p$ *and* $j \in (\mathbb{Z}/p^{r-2})^*$
$M(p, r-1)$	$\langle ac^j, b\rangle$ *with* $j \in (\mathbb{Z}/p^{r-2})^*$
$\mathbb{Z}/p \times \mathbb{Z}/p^{r-3}$	$\langle ab^i, c^p\rangle$
$C(p, r-1)$	$\langle a, b, c^p\rangle$

Proof. It is clear that the total is a p-centric subgroup, so let $H < G(p, r; \epsilon)$ be a p-centric subgroup different from the total. As it must contain its centralizer in $G(p, r; \epsilon)$ we have that $\langle c^p\rangle < H$.

We divide the proof in different cases:

Case $H \leq \langle b, c\rangle$: then, as $\langle b, c\rangle$ is commutative, we have $H = \langle b, c\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p^{r-2}$, and using the commutator rules of this group one can check that it is p-centric.

So in the following cases there is an element of the form $\alpha = ab^i c^j$.

Case $p \nmid j$ and H cyclic: as $p \nmid j$ we can construct an automorphism of $G(p, r; \epsilon)$ sending $a \mapsto ab^i$ and $c \mapsto c^j$, so we can compute the order of α computing the order of ac. Now one can check the following formula:

$$(ac)^n = a^n b^{-\binom{n}{2}} c^{n-\binom{n}{3}\epsilon p^{r-3}}.$$

So if $(p; r, \epsilon) \neq (3; 4, 1)$ we get that $(ac)^p = c^{\pm p}$, so ac has order p^{r-2} and it is self-centralizing, so it is p-centric. In this case we have $H = \langle\alpha\rangle \cong \mathbb{Z}/p^{r-2}$.

Case $p \nmid j$ and H not cyclic: we can assume that H has two generators, and one of them is α: if we consider an element β of $H \setminus \langle\alpha\rangle$ then the order of $\langle\alpha, \beta\rangle$ is at least p^{r-1}, so if we add another element we would have the total. As we are considering that H is not the total, we can assume $H = \langle\alpha, \beta\rangle$. We can also assume that $\beta = b^k c^l$ (if the generator a would appear in the expression of β we could take a power of β and multiply it by α^{-1} to change the generators). So consider $H = \langle\alpha, \beta\rangle$ with H not cyclic and different from the total, then we prove that it is metacyclic: the order of H is p^{r-1} and, as H is not cyclic, then $\beta \notin \langle\alpha\rangle$, so either $k \neq 0$ or $p \nmid l$. If $k = 0$ then, as $p \nmid l$ we have $H = \langle ab^i, c\rangle$, and as $[ab^i, c] = b$, H is the total, which is not considered. So $k \neq 0$ and now we have to distinguish between two cases: $p|l$ and $p \nmid l$. If $p \nmid l$ we can consider the inverse of the automorphism ρ in $G(p, r; \epsilon)$ with $\rho(a) = a$ and $\rho(c) = b^k c^l$ and we have $\rho^{-1}(H) = \langle ab^s c^t, c\rangle = G(p, r; \epsilon)$, so $H = G(p, r; \epsilon)$, that implies that $p|l$ and we can consider the group generated by $\langle ab^i c^j, bc^{pf}\rangle$. One more reduction is cancellation of b^i by means of a multiplication by $(bc^{pf})^{-i}$, so the generators are $\alpha = ac^j$ and $\beta = bc^{pf}$. We can still simplify the generators using that $\langle\alpha^p\rangle = \langle c^p\rangle$, getting $\langle ac^j, b\rangle$. To see that it is metacyclic just check that $[H, H] = \langle c^{p^{r-3}}\rangle \cong \mathbb{Z}/p$, and that the class of α in the quotient $H/[H, H]$ has order p^{r-3}, the same order as $H/[H, H]$, so it is cyclic, and $H \cong M(p, r-1)$.

Case $p \mid j$: we can assume that all the elements in H are of the form $a^k b^l c^{pm}$, because if there were an element in H with an exponent in c which is not multiple of p then we would be in one of the previously studied cases. As c^p must be in H, then we have that $\langle ab^i, c^p\rangle \leq H$. One can check by means of the commutator formula that the group $\langle ab^i, c^p\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p^{r-3}$ is self-centralizing, so p-centric. Moreover if $H \setminus \langle ab^i, c^p\rangle \neq \emptyset$, all the other restrictions of this case imply that there is an element of the form $a^j b^k$ in $H \setminus \langle ab^i, c^p\rangle$, and an easy calculation gives us that $\langle ab^i, a^j b^k, c^p\rangle = \langle a, b, c^p\rangle \cong C(p, r-1)$. This also proves that there is only one p-centric subgroup in $G(p, r; \epsilon)$ isomorphic to $C(p, r-1)$. □

It remains to study maximal nilpotency class 3-groups, beginning with the following properties that can be read in [6] and [26, III.§14]:

Proposition A.9. *Consider $B(3, r; \beta, \gamma, \delta)$ as defined in Theorem A.2 with the same notation for the generators. Then the following hold:*

(a) *From relations* (1) *to* (6) *we get:*

$$(s^{\pm 1} s_1^{\zeta_1} \cdots s_{r-1}^{\zeta_{r-1}})^3 = s_{r-1}^{\pm\delta + \gamma\zeta_1 \pm \beta\zeta_1^2} . \tag{7}$$

(b) $\gamma_i(B(3, r; \beta, \gamma, \delta)) \stackrel{def}{=} \langle s_i, s_{i+1}, \ldots, s_{r-1}\rangle$ *are characteristic subgroups of order 3^{r-i} generated by s_i and s_{i+1} for $i = 1, .., r-1$ (assuming $s_r = 1$).*

(c) $\gamma_1(B(3, r; \beta, \gamma, \delta))$ *is a metacyclic subgroup.*

(d) $\gamma_1(B(3, r; \beta, \gamma, \delta))$ *is abelian if and only if $\beta = 0$.*

(e) *The extension*

$$1 \to \gamma_1 \to B(3, r; \beta, \gamma, \delta) \xrightarrow{\pi} \mathbb{Z}/3 \to 1$$

is split if and only if $\delta = 0$.

(f) $Z(B(3, r; \beta, \gamma, \delta)) = \gamma_{r-1}(B(3, r; \beta, \gamma, \delta)) = \langle s_{r-1}\rangle$.

The following lemma is useful when studying the exoticism of the fusion systems constructed in Section 5.

Lemma A.10. *Let N be a nontrivial proper normal subgroup in $B(3, r; 0, \gamma, 0)$. Then*

(a) *N contains $Z(B(3, r; 0, \gamma, 0))$.*

(b) *If N contains s then $N \cong 3^{1+2}_+$ if $r = 4$ or $N \cong B(3, r-1; 0, 0, 0)$ if $r > 4$.*

Proof. According to [4, Theorem 8.1] N must intersect $Z(B(3, r; 0, \gamma, 0))$ in a nontrivial subgroup. As in our case the order of the center is p, we obtain (a).

From [6, Lemma 2.2] we deduce that if the index of N in $B(3, r; 0, \gamma, 0)$ is 3^l with $l \geq 2$ then $N = \gamma_l$, and it does not contain s. So a proper normal subgroup N which contains s must be of index 3 in $B(3, r; 0, \gamma, 0)$, so $B(3, r; 0, \gamma, 0)/N$ is abelian, and the quotient morphism $B(3, r; 0, \gamma, 0) \to B(3, r; 0, \gamma, 0)/N$ factors through $B(3, r; 0, \gamma, 0)/\gamma_2(B(3, r; 0, \gamma, 0)) \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ (the commutator of $B(3, r; 0, \gamma, 0)$ is $\gamma_2(B(3, r; 0, \gamma, 0))$), generated by the classes $\overline{s}$ and $\overline{s_1}$. Now we have to take the inverse image of the proper subgroups in $\mathbb{Z}/3 \times \mathbb{Z}/3$ of order 3, getting that N must be γ_1, $\langle s, s_2\rangle$, $\langle ss_1, s_2\rangle$ or $\langle ss_1^{-1}, s_2\rangle$. Only the second contains s, getting the second part of the result. □

In what follows we restrict to $r \geq 5$ and the parameters $(\beta, \gamma, \delta) \in \{(0, \gamma, 0), (1, 0, 0)\}$, with $\gamma \in \{0, 1, 2\}$. This includes all the possibilities for the parameters described in Theorem A.2 except the cases $\delta = 1$ and $\{(0, 0, 0), (0, 2, 0)\}$ for $r = 4$. We fix k the integer such that $r = 2k$ if r is even and $r = 2k + 1$ if r is odd.

Lemma A.11. *For the groups $B(3, r; \beta, \gamma, 0)$ it holds that:*
(a) *$\gamma_2(B(3, r; \beta, \gamma, 0))$ is abelian.*
(b) *The orders of s_1, s_2 and s_3 are 3^k, 3^k and 3^{k-1} if $r = 2k + 1$ and 3^k, 3^{k-1} and 3^{k-1} if $r = 2k$.*

Proof. We check that $\gamma_2(B(3, r; \beta, \gamma, 0))$ is abelian. If $\beta = 0$ then γ_1 is abelian and $\gamma_2 < \gamma_1$, and if $\beta = 1$ (which implies $\gamma = 0$) we have to see that s_2 and s_3 commute. We have the following equalities:

$$[s_3, s_2] = s_3^{-1} s_2^{-1} s_3 s_2 \overset{(5)}{=} s_1^3 s_2^3 s_2^{-1} s_2^{-3} s_1^{-3} s_2 = s_1^3 s_2^{-1} s_1^{-3} s_2 \,.$$

So we have reduced to check that s_2 and s_1^3 commute.

We use equation (2) to deduce $s_2^{-1} s_1 s_2 = s_1 s_{r-1}$, and raise it to the cubic power to get $s_2^{-1} s_1^3 s_2 = s_1^3 s_{r-1}^3$. Equation (6) for $i = r - 1$ tells us that the order of s_{r-1} is 3, so s_2 and s_1^3 commute.

We now compute the orders of s_1, s_2 and s_3. Begin using that s_{r-1} has order 3. Equation (6) for $i = r - 2$ yields $s_{r-2}^3 s_{r-1}^3 = 1$, from which s_{r-2} has order 3 too. For $i = r - 3$ the equation becomes $s_{r-3}^3 s_{r-2}^3 s_{r-1} = 1$, and so s_{r-3} has order 9. An induction procedure, taking care of the parity of r, provides us with the desired result. ☐

The next lemma describes the conjugation by s action on γ_1:

Lemma A.12. *For $B(3, r; \beta, \gamma, 0)$ the conjugation by s on the characteristic subgroup $\gamma_1(B(3, r; \beta, \gamma, 0))$ is given by:*

	$\beta = 0,\ r = 2k + 1$	$\beta = 0,\ r = 2k$	$\beta = 1$
$\gamma = 0$	$s_1^s = s_1 s_2$ $s_2^s = s_1^{-3} s_2^{-2}$	$s_1^s = s_1 s_2$ $s_2^s = s_1^{-3} s_2^{-2}$	$({s_1}^f)^s = {s_1}^f {s_2}^f {s_{r-1}}^{f(1-f)/2}$ $({s_2}^f)^s = {s_1}^{-3f} {s_2}^{-2f}$
$\gamma = 1$	$s_1^s = s_1 s_2$ $s_2^s = s_1^{-3} s_2^{(-3)^{k-1} - 2}$	$s_1^s = s_1 s_2$ $s_2^s = s_1^{-3((-3)^{k-2}+1)} s_2^{-2}$	–
$\gamma = 2$	–	$s_1^s = s_1 s_2$ $s_2^s = s_1^{3((-3)^{k-2}-1)} s_2^{-2}$	–

Proof. We begin first with the case of $\beta = 0$. To find the expression for the conjugation by s action on γ_1 notice that ${s_1}^s = s_1[s_1, s] = s_1 s_2$ by equation (1) and that analogously ${s_2}^s = s_2[s_2, s] = s_2 s_3$. So we need to express s_3 as a product of powers of s_1 and s_2. We begin writing s_{r-1} as a product of powers of s_2 and s_3. Bearing this objective in mind we use the same equation (6) as before, but beginning with $i = 2$. In this case we obtain $s_4 = {s_2}^{-3} {s_3}^{-3}$. For $i = 3$ the relation is

$$s_5 = {s_3}^{-3} {s_4}^{-3} = {s_3}^{-3} ({s_2}^{-3} {s_3}^{-3})^{-3} = {s_2}^9 {s_3}^6 .$$

If $i \geq 4$ and we got in an earlier stage

$$s_i = {s_2}^{a_i}{s_3}^{b_i},\ s_{i+1} = {s_2}^{a_{i+1}}{s_3}^{b_{i+1}}$$

then equation (6) reads as

$$s_{i+2} = {s_i}^{-3}{s_{i+1}}^{-3} = ({s_2}^{a_i}{s_3}^{b_i})^{-3}({s_2}^{a_{i+1}}{s_3}^{b_{i+1}})^{-3} = {s_2}^{-3a_i-3a_{i+1}}{s_3}^{-3b_i-3b_{i+1}}.$$

So $s_{r-1} = {s_2}^{a_{r-1}}{s_3}^{b_{r-1}}$, where a_{r-1} and b_{r-1} are obtained from the recursive sequences $a_4 = -3$, $a_5 = 9$, $a_{i+2} = -3a_i - 3a_{i+1}$ and $b_4 = -3$, $b_5 = 6$, $b_{i+2} = -3b_i - 3b_{i+1}$ for $i \geq 4$.

Substituting this last result in equation (5) in Theorem A.2 we reach $s_3 = {s_1}^{3b'}{s_2}^{(3-\gamma a_{r-1})b'}$, where $b'(\gamma b_{r-1} - 1) = 1$ modulus the order of s_3 and ${s_2}^s = {s_1}^{3b'}{s_2}^{1+(3-\gamma a_{r-1})b'}$. Finally, some further calculus using the recursive sequences $\{a_i\}$ and $\{b_i\}$ and taking into account separately the three possible values of γ finishes the proof for $\beta = 0$.

For the case $\beta = 1$ use the relations in the presentation of Theorem A.2 to find the commutator rules in $B(3, r; 1, 0, 0)$. □

Remark A.13. In the cases with $\beta = 0$ we have that $\gamma_1(B(3, r; 0, \gamma, 0))$ is a rank two abelian subgroup generated by $\{s_1, s_2\}$ so we can identify the conjugations by s with a matrix $M_s^{r,\gamma}$, obtaining:

$$M_s^{r,0} = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}, \quad M_s^{2k+1,1} = \begin{pmatrix} 1 & -3 \\ 1 & (-3)^{k-1}-2 \end{pmatrix},$$

$$M_s^{2k,1} = \begin{pmatrix} 1 & -3((-3)^{k-2}+1) \\ 1 & -2 \end{pmatrix} \text{ and } M_s^{2k,2} = \begin{pmatrix} 1 & 3((-3)^{k-2}-1) \\ 1 & -2 \end{pmatrix}.$$

Next we determine the automorphism groups of $B(3, r; 0, \gamma, 0)$ and $B(3, r; 1, 0, 0)$.

Lemma A.14. *The automorphism group of $B(3, r; \beta, \gamma, 0)$ consists of the homomorphisms that send*

$$s \mapsto s^e{s_1}^{e'}{s_2}^{e''}$$

$$s_1 \mapsto s_1^{f'}s_2^{f''}$$

where the parameters verify the following conditions:

	$\beta = 0$, r *odd*	$\beta = 0$, r *even*	$\beta = 1$
$\gamma = 0$	$e = \pm 1, 3 \nmid f'$	$e = \pm 1, 3 \nmid f'$	$e = 1, 3 \mid e', 3 \nmid f'$
$\gamma = 1$	$e = 1, 3 \mid e', 3 \nmid f'$	$e = \pm 1, 3 \mid e', 3 \nmid f'$	–
$\gamma = 2$	–	$e = \pm 1, 3 \mid e', 3 \nmid f'$	–

Proof. We begin with the case of $\beta = 0$. As $\gamma_1 = \langle s_1, s_2 \rangle$ is characteristic and $B(3, r; 0, \gamma, 0)$ is generated by s and s_1, every homomorphism $\varphi \in B(3, r; 0, \gamma, 0)$ is determined by

$$s \mapsto s^e{s_1}^{e'}{s_2}^{e''}, s_1 \mapsto s_1^{f'}s_2^{f''}$$

for some integers e, e', e'', f', f''. In the other way, given such a set of parameters the equations (1)-(6) from Theorem A.2 give us which conditions must verify these parameters in order to get a homomorphism φ. In the study of these equations denote by M_s the matrix of conjugation by s on γ_1 described in Remark A.13.

- Equation (1): it is straightforward to check that these conditions are equivalent to $\varphi(s_i) = {s_1}^{f'_i} {s_2}^{f''_i}$ for every $2 \leq i \leq n-1$ with $\binom{f'_i}{f''_i} = M_e \binom{f'_{i-1}}{f''_{i-1}}$, where $M_e = {M_s}^e - \mathrm{Id}$, $f'_1 = f'$ and $f''_1 = f''$. In particular we obtain the value of $\varphi(s_2)$, and so we get the restriction of φ to $\gamma_1 = \langle s_1, s_2 \rangle$.
- Equations (2) and (3): as $\beta = 0$ γ_1 is abelian and these equations are satisfied if so are the conditions for equation (1).
- Equation (4): using equation (7) from Proposition A.9 we find that if $\gamma = 0$ there are no additional conditions and if $\gamma = 1, 2$ it must be verified that $3 \mid e'$.
- Equation (5): the condition is:

$$(3 + 3M_e + M_e^2 - \gamma {M_e}^{r-2}) \binom{f'}{f''} = 0.$$

For $\gamma = 0$ it holds that $3 + 3M_e + M_e^2 = 0$. So there is not more conditions on the parameters. If $\gamma = 1, 2$, using the integer characteristic polynomial of M_e to compute ${M_e}^{r-2}$ we produce two recursive sequences, similar to the ones in Lemma A.12, which $(r-1)$-th term equals $(3 + 3M_e + M_e^2 - \gamma {M_e}^{r-2}) \binom{f'}{f''}$. Checking details we obtain just the condition $3 \mid f'$ if r is odd, $\gamma = 1$ and $e = -1$.

- Equation (6): it is easy to check that, as $r \geq 5$, the imposed conditions for $i = 2, \ldots, r-1$ are equivalent to:

$$M_e(3 + 3M_e + M_e^2) \binom{f'}{f''} = 0.$$

Because $M_s^3 = \mathrm{Id}$, in fact it holds that $M_e(3 + 3M_e + M_e^2) = 0$. So there are not additional conditions on the parameters.

So we have just got the conditions on the parameters so that there exists a homomorphism φ such that the images on s and s_1 are predetermined by these parameters e, e', e'', f', f''. It just remains to look out the conditions on φ so it is an automorphism. A quick check shows that the conditions are $3 \nmid e$ and the determinant of $\varphi|\gamma_1$ is invertible modulus 3, which is equivalent to $3 \nmid f'$.

For $B(3, r; 1, 0, 0)$ the arguments are similar using the commutator rules for powers of s, s_1 and s_2. □

In the next two lemmas we find out which copies of $\mathbb{Z}/3^n \times \mathbb{Z}/3^n$ and $C(3, n)$ are in $B(3, r; \beta, \gamma, 0)$ and how they lie inside this group.

Lemma A.15. *Let P be a proper p-centric subgroup of $B(3, r; 0, \gamma, 0)$ isomorphic to $\mathbb{Z}/3^n \times \mathbb{Z}/3^n$ or $C(3, n)$ for some n. Then P is determined, up to conjugation, by the following table:*

Isomorphism type	*Subgroup (up to conjugation)*	*Conditions*
$\mathbb{Z}/3^k \times \mathbb{Z}/3^k$	$\gamma_1 = \langle s_1, s_2 \rangle$	$r = 2k + 1$.
3^{1+2}_+	$E_i \overset{def}{=} \langle \zeta, \zeta', s{s_1}^i \rangle$	$\zeta = {s_2}^{3^{k-1}}, \zeta' = {s_1}^{3^{k-1}}$ *for* $r = 2k+1$, $\zeta = {s_1}^{3^{k-1}}, \zeta' = {s_2}^{-3^{k-2}}$ *for* $r = 2k$, $i \in \{-1, 0, 1\}$ *if* $\gamma = 0$ *and* $i = 0$ *if* $\gamma = 1, 2$.
$\mathbb{Z}/3 \times \mathbb{Z}/3$	$V_i \overset{def}{=} \langle \zeta, s{s_1}^i \rangle$	

Proof. Firstly, suppose that P is contained in γ_1. As γ_1 is abelian and P must contain its centralizer as it is p-centric, P must equal γ_1. Recalling the orders of s_1 and s_2 we check that only the case r odd is allowed.

Suppose now that P is not contained in γ_1. Then P fits in the short exact sequence:

$$1 \to K \to P \xrightarrow{\pi} \mathbb{Z}/3 \to 1,$$

with $K \leq \gamma_1$. If $K = \mathbb{Z}/3^m$ then, as $P \cong \mathbb{Z}/3^n \times \mathbb{Z}/3^n$ or $P \cong C(3,n)$, we get that the only possibility is $m = 1$ and $P \cong \mathbb{Z}/3 \times \mathbb{Z}/3$. Suppose then that $K = \mathbb{Z}/3^m \times \mathbb{Z}/3^{m'}$. Now, checking cases again, the only chance for P is to be $C(3,n)$. An easy calculation shows that $C_{B(3,r;0,\gamma,0)}(P) \cong \mathbb{Z}/3$. As this centralizer must contain the center of $P \cong C(3,n)$, which from Lemma A.5 is $Z(C(3,n)) \cong \mathbb{Z}/3^{n-2}$, n must equal 3 and $P \cong 3^{1+2}_+$. Now we determine precisely all the p-centric subgroups isomorphic to 3^{1+2}_+ or $\mathbb{Z}/3 \times \mathbb{Z}/3$, and their orbits under $B(3,r;0,\gamma,0)$-conjugation.

- We begin with the **p-centric subgroups isomorphic to** 3^{1+2}_+. As they are p-centric they must contain the center of $B(3,r;0,\gamma,0)$ which is equal to $\langle\zeta\rangle$. So, from the earlier discussion, they are of the form $3^{1+2}_{+a} \stackrel{\text{def}}{=} \langle\zeta,\zeta'\rangle : \langle sa\rangle$, where ζ' is as in the statement of the lemma and $a \in \gamma_1$ is, in principle, arbitrary. Now, sa having order 3 is seen to be equivalent to $N(a) = 0$ where N is the norm operator $N = 1 + s + s^2$. On the other hand, from the description of M_s in Remark A.13, if $a = {s_1}^{a_1}{s_2}^{a_2}$ and $a' = {s_1}^{a'_1}{s_2}^{a'_2}$ then 3^{1+2}_{+a} and $3^{1+2}_{+a'}$ are $B(3,r;0,\gamma,0)$-conjugate if and only if a_1 and a'_1 are congruent modulus 3. Because $N = 0$ for $\gamma = 0$ and $a_1 \equiv 0 \bmod 3$ for every $a = {s_1}^{a_1}{s_2}^{a_2} \in \mathrm{Ker}(N)$ for $\gamma = 1, 2$, we obtain the desired results.
- The argument to obtain all the **p-centric subgroups isomorphic to** $\mathbb{Z}/3 \times \mathbb{Z}/3$ is similar. First, from the earlier discussion, they must be of the form $\langle\zeta, sa\rangle$ for ζ as in the statement and some $a \in \gamma_1$. As before, the condition for $\langle sa\rangle$ to have order 3 is $N(a) = 0$, and $\langle\zeta, sa\rangle$ and $\langle\zeta, sa'\rangle$ are $B(3,r;0,\gamma,0)$-conjugate if and only if a_1 and a'_1 are in the same class modulus 3.

□

Lemma A.16. *Let P be a proper p-centric subgroup of $B(3,r;1,0,0)$ isomorphic to $\mathbb{Z}/3^n \times \mathbb{Z}/3^n$ or $C(3,n)$ for some n. Then P is determined, up to conjugation, by the following table:*

Isomorphism type	*Subgroup (up to conjugation)*	*Conditions*
$\mathbb{Z}/3^{k-1}\times\mathbb{Z}/3^{k-1}$	$\gamma_2 = \langle s_2, s_3\rangle$	$r = 2k$.
3^{1+2}_+	$E_0 \stackrel{def}{=} \langle\zeta,\zeta',s\rangle$	$\zeta = {s_2}^{3^{k-1}}, \zeta' = {s_3}^{-3^{k-2}}$ *for* $r = 2k+1$, $\zeta = {s_3}^{3^{k-2}}, \zeta' = {s_2}^{3^{k-2}}$ *for* $r = 2k$.
$\mathbb{Z}/3 \times \mathbb{Z}/3$	$V_0 \stackrel{def}{=} \langle\zeta, s\rangle$	

Proof. Consider the following short exact sequence induced by the abelian characteristic subgroup $\gamma_2 = \langle s_2, s_3\rangle$ of $B(3,r;1,0,0)$:

$$1 \to \gamma_2 \to B(3,r;1,0,0) \xrightarrow{\pi} \mathbb{Z}/3 \times \mathbb{Z}/3 \cong \langle\overline{s},\overline{s_1}\rangle \to 1.$$

If P is contained in γ_2 then, as γ_2 is abelian and P must contains its centralizer, P must equal $\gamma_2 = \langle s_2, s_3\rangle$.

Recalling the orders of s_2 and s_3 we check that only the case r even is allowed.

Suppose now that P is not contained in γ_2 and consider the non-trivial subgroup $\pi(P)$:

Case $\pi(P) = \langle \overline{s_1}\rangle$,$\langle\overline{s}\overline{s_1}\rangle$ or $\langle \overline{s}^{-1}\overline{s_1}\rangle$: then P fits in a non-split short exact sequence:

$$1 \to K \to P \xrightarrow{\pi} \mathbb{Z}/3 \to 1,$$

with $K \leq \gamma_2$. Checking cases for K and P we obtain that in any case this short exact sequence would split, which is a contradiction.

Case $\pi(P) = \langle\overline{s}\rangle$: then P is a subgroup of $\tau = \langle s_2, s_3, s\rangle \cong B(3, r-1; 0, 0, 0)$. Now apply Lemma A.15 and notice that conjugation by s_1 conjugates the three copies of 3^{1+2}_{+} and $\mathbb{Z}/3 \times \mathbb{Z}/3$.

Case $\pi(P) = \mathbb{Z}/3\times\mathbb{Z}/3$: If $P \neq B(3, r; 1, 0, 0)$ then there exists a maximal proper subgroup $H < B$ containing P. As γ_2 is the Frattini subgroup of $B(3, r; 1, 0, 0)$, that is, the intersection of the maximal subgroups, then $\pi(H) = \mathbb{Z}/3$. This is a contradiction with $\pi(P) = \mathbb{Z}/3\times\mathbb{Z}/3$, and thus P equals $B(3, r; 1, 0, 0)$. □

In the proof of Theorem 5.9 we use implicitly some particular copies of $\mathrm{SL}_2(3)$ and $\mathrm{GL}_2(3)$ lying in $\mathrm{Aut}(\gamma_1)$. These are characterized by containing a fixed matrix. In the next lemma we show when they do exist:

Lemma A.17. *Consider $P \cong \mathbb{Z}/3^k \times \mathbb{Z}/3^k$ and $M_s^{2k+1,\gamma}$ the matrix defined in Remark A.13 for the case $B(3, 2k+1; 0, \gamma, 0)$. Then:*

- *For $\gamma = 0$ there is, up to conjugacy, one copy of $\mathrm{SL}_2(3)$ (respectively $\mathrm{GL}_2(3)$) in $\mathrm{Aut}(P)$ containing $M_s^{2k+1,0}$.*
- *For $\gamma = 1$ there is, up to conjugacy, one copy of $\mathrm{SL}_2(3)$ (respectively none of $\mathrm{GL}_2(3)$) in $\mathrm{Aut}(P)$ containing $M_s^{2k+1,1}$.*

Proof. As $\mathrm{SL}_2(3)$ and $\mathrm{GL}_2(3)$ are 3-reduced any copy of these groups lying in $\mathrm{Aut}(P)$ is a lift of a subgroup of $\mathrm{GL}_2(3)$ by the Frattini map $\mathrm{Aut}(P) \xrightarrow{\rho} \mathrm{GL}_2(3)$.

It can be checked that for $k = 2$ the statements of the Lemma are true. Now, call P' to the Frattini subgroup of P, $P' \overset{\text{def}}{=} \mathbb{Z}/3^{k-1} \times \mathbb{Z}/3^{k-1}$, and consider the restriction map with abelian kernel:

$$1 \to (\mathbb{Z}/3)^4 \to \mathrm{Aut}(P) \xrightarrow{\pi} \mathrm{Aut}(P') \to 1.$$

Let A denote $(\mathbb{Z}/3)^4$. We use this exact sequence to prove the statements by induction on k. We suppose the lemma is true for $k-1$ and prove it for $k \geq 3$. We take $G = \mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ and $\gamma = 0$ or 1. Call $L^{k,\gamma} \overset{\text{def}}{=} M_s^{2k+1,\gamma}$, where $M_s^{2k+1,\gamma}$ is the matrix defined in Remark A.13. It is straightforward that $\pi(L^{k,0}) = L^{k-1,0}$ and $\pi(L^{k,1}) = L^{k-1,0}$. We prove the lemma in three steps:

Existence: We show the existence of the three stated copies. By hypothesis there exists a lift $G \xrightarrow{\sigma} \mathrm{Aut}(P')$ such that $\rho\sigma = \mathrm{Id}_G$ and with $\mu \overset{\text{def}}{=} L^{k-1,0} = \left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right) \in \sigma(G)$. Write

$H \stackrel{\text{def}}{=} \langle \mu \rangle \in \mathrm{Syl}_3(\sigma(G))$ and take the pullback twice:

$$\begin{array}{ccccc}
A & \hookrightarrow & \mathrm{Aut}(P) & \stackrel{\pi}{\twoheadrightarrow} & \mathrm{Aut}(P') \\
\| & & \uparrow & & \uparrow \\
A & \hookrightarrow & \pi^{-1}(\sigma(G)) & \stackrel{\pi}{\twoheadrightarrow} & \sigma(G) \\
\| & & \uparrow & & \uparrow \\
A & \hookrightarrow & \pi^{-1}(H) & \stackrel{\pi}{\twoheadrightarrow} & H
\end{array}$$

As $L^{k,0} = \left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)$ lies in $\pi^{-1}(H)$ the bottom short exact sequence splits and its middle term can be identified with $A : L^{k,0}$. Notice also that $L^{k,1}$ lies in $A : L^{k,0}$, there are several lifts of H and the A-conjugacy classes of these lifts are in $1-1$ correspondence with $H^1(H;A)$. In fact, each section $H \to \pi^{-1}(H)$, corresponds, by the earlier identification, to a derivation $d\colon H \to A$, such that $d(\mu) = a\mu$ and $d(\mu^2) = b\mu^2$ (notice that the μ's inside and outside the d lie in different automorphism groups).

Recall that we are interested in building lifts $G \to \mathrm{Aut}(P)$ containing $L^{k,\gamma}$, for which is enough to give sections of the middle short exact sequence in the diagram above which images contains $L^{k,\gamma}$. If this sequence splits then the A-conjugacy classes of its sections are in $1-1$ correspondence with $H^1(G;A)$ (for clarity we do not write $H^1(\sigma(G);A)$), and the sections which image contains $L^{k,\gamma}$ are precisely those which goes by the restriction map $\mathrm{res}^G_H\colon H^1(G;A) \to H^1(H;A)$ to the class of the section induced by $L^{k,\gamma}$ in the bottom short exact sequence.

In fact, that the middle sequence splits is due to $[G:H]$ being invertible in A and applying a transfer argument we find that $\mathrm{res}^G_H\colon H^*(G;A) \to H^*(H;A)$ is a monomorphism (and $\mathrm{cor}^H_G\colon H^*(H;A) \to H^*(G;A)$ is an epimorphism), so the class of the middle sequence goes by $\mathrm{res}^G_H\colon H^2(G;A) \to H^2(H;A)$ to the class of the bottom sequence, which is zero, and must be the zero class too, that is, the split one.

Finally, the sections $\sigma^0, \sigma^1\colon H \to \pi^{-1}(H)$ which take μ to $L^{k,0}$ and $L^{k,1}$ correspond to the identically zero derivation and to $a = \left(\begin{smallmatrix} 1 & 0 \\ \mp 3^{k-1} & 1 \pm 3^{k-1} \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \mp 3^{k-1} \end{smallmatrix}\right)$ respectively. These sections are in the image of the restriction map $\mathrm{res}^G_H\colon H^1(G;A) \to H^1(H;A)$ if and only if they are in the G-invariants in $H^1(H;A)$. And easy check shows that $z \in H^1(H;A)$ is a G-invariant if and only if $gz = z$ for every $g \in N_{\sigma(G)}(H)$. Once computed the derivations $\mathrm{Der}(H,A)$ and the principal derivations $P(H,A)$, we apply the action of g on σ^0 and σ^1 at the cochain level, and check that the class of σ^0 is always G-invariant and that the class of σ^1 is G-invariant just for $G = \mathrm{SL}_2(3)$ in $H^1(H;A) \cong \mathrm{Der}(H,A)/P(H,A)$.

Uniqueness: Now we show that the three found copies are unique up to $\mathrm{Aut}(P)$-conjugation, as claimed. We use induction and the same tools as in the first step. Take two lifts $\sigma_1, \sigma_2 : G \to \mathrm{Aut}(P)$ containing $L^{k,\gamma}$. Composing with π we obtain two maps from G to $\mathrm{Aut}(P')$ containing $L^{k-1,0}$. They are lifts if they are injective, that is, if $A \cap \sigma_i(G)$ is trivial for $i = 1,2$. As $\mathrm{GL}_2(3)$ and $\mathrm{SL}_2(3)$ are 3-reduced, and as A is a normal 3-group, these groups are indeed trivial. So, by the induction hypothesis, the two lifts arriving at $\mathrm{Aut}(P')$ must be conjugated by some $g' \in \mathrm{Aut}(P')$ which centralizes $L^{k-1,0}$.

It is a straightforward calculation that the order of the centralizers $C_{\mathrm{Aut}(P)}(L^{k,\gamma})$ for $\gamma = 0, 1$ is $2 \cdot 3^{2k-1}$, and that of $A \cap C_{\mathrm{Aut}(P)}(L^{k,\gamma})$ for $\gamma = 0, 1$ is 9 (for every $k \geq 2$). Because π maps $C_{\mathrm{Aut}(P)}(L^{k,\gamma})$ to $C_{\mathrm{Aut}(P')}(L^{k-1,0})$, a element counting argument shows that in fact $\pi(C_{\mathrm{Aut}(P)}(L^{k,\gamma})) = C_{\mathrm{Aut}(P')}(L^{k-1,0})$.

Thus, there exists $g \in \mathrm{Aut}(P)$ with $\pi(g) = g'$ and such that g centralizes $L^{k,\gamma}$. Therefore the images of σ_1 and $\sigma_2' \overset{\text{def}}{=} c_g \circ \sigma_2$ contain $L^{k,\gamma}$ and have the same image by π, that is, they both lie in $A : \sigma_1(G) = \pi^{-1}(\pi\sigma_1(G))$.

Choosing the Sylow 3-subgroup $H = \langle \mu \rangle$ of $\pi\sigma_1(G)$ we can construct a three rows short exact sequences diagram as before, and argue using the injectivity of the restriction map $\mathrm{res}^G_H \colon H^1(G; A) \to H^1(H; A)$ to obtain the uniqueness. More precisely, as the two sections $\pi^{-1} \colon \pi\sigma_1(G) \to \sigma_1(G)$ and $\pi^{-1} : \pi\sigma_1(G) \to \sigma_2'(G)$ of the middle sequence of the diagram induce the same section in the bottom row, that is, the one which maps μ to $L^{k,\gamma}$, they must be in the same class in $H^1(G; A)$, which means that they are A-conjugate. Therefore σ_1 and σ_2 are $\mathrm{Aut}(P)$-conjugate.

Non existence: The arguments of the two preceding parts prove also the non existence of sections $\mathrm{GL}_2(3) \to \mathrm{Aut}(P)$ containing $L^{k,1}$. □

Remark A.18. A cohomology-free proof of the non existence of copies (lifts) of $\mathrm{GL}_2(3)$ in $\mathrm{Aut}(\mathbb{Z}/3^k \times \mathbb{Z}/3^k)$ containing $L^{k,1} = M_s^{2k+1,1}$ runs as follows: if this were the case, then, as the elements of order 3 form a single conjugacy class in $\mathrm{GL}_2(3)$, we would obtain that $L^{k,1}$ and its square are conjugate, and so would have same determinant and trace. But one can checks that this is not the case.

If H and K are two groups and H acts on K by $\varphi : H \to \mathrm{Aut}(K)$ then we can construct the semidirect product $K :_\varphi H$. In fact, if $\psi : H \to \mathrm{Aut}(K)$ is another action conjugate to φ, that is, exists $\alpha \in \mathrm{Aut}(K)$ such that $\psi(h) = \alpha^{-1} \circ \varphi(h) \circ \alpha$ for every $h \in H$, then $K :_\varphi H \cong K :_\psi H$. The lemma above implies:

Corollary A.19. *There exists the groups $\gamma_1 : \mathrm{SL}_2(3)$ and $\gamma_1 : \mathrm{GL}_2(3)$ where the actions maps $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to $M_s^{2k+1,\gamma}$, with $\gamma = 0, 1$ for $\mathrm{SL}_2(3)$ and $\gamma = 0$ for $\mathrm{GL}_2(3)$. Moreover, these semidirect products with actions as stated are unique up to isomorphism.*

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