THE PRODUCT THEOREM FOR PARAMETRIZED TOPOLOGICAL REIDEMEISTER TORSION

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Abstract. The goal of this article is to prove the product formula for parametrized topological Reidemeister torsion. The theorem states that the product of the parametrized Euler characteristic of one fibration with the parametrized Reidemeister torsion class of another fibration yields the parametrized Reidemeister torsion class of the product fibration. In the process of establishing the theorem, several new products must be defined involving (derivative theories of) parametrized A-theory and a detailed description of the coassembly map for parametrized A-theory is included.

1. Introduction

Before moving into the statements of the main results, we would like to say a few words about our motivation for studying this question. The definition of topological Reidemeister torsion used here is based on that of [5]. They conjecture that their definition should become equivalent to that of [7] once their class is pushed into cohomology using the Borel regulator technique. In fact, we would eventually like to establish that our definition, the definition from [7] and the definition from [1] agree in cohomology, which would generalize the work of [12] in the non-parametrized case.

In [9], the authors give a completely algebraic analog of our product theorem. They establish the fact that the Reidemeister torsion of the tensor product $A \otimes B$ where B is contractible is equal to the Euler characteristic of A times the torsion of B. Our goal is to give a topological lifting of this result to Whitehead spaces.

Moving into specifics, all fibrations considered here will be perfect fibrations. That is, the fibers of every fibration mentioned will be finitely dominated by assumption. We will also assume the base space B comes equipped with an efficient triangulation, where each simplex contains only finitely many subcomplexes. Clearly, this allows any differentiable compact manifold as a choice of B.

The first major result is an A-theory product formula for Euler characteristics, relying upon the external parametrized A-theory product

$$\mu_{\mathrm{A}} : \mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \mathrm{A}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}.$$

Theorem 1.1. *Suppose p_1 and p_2 are perfect fibrations. Then*

$$\mu_{\mathrm{A}}(\chi_{\mathrm{A}}(p_1), \chi_{\mathrm{A}}(p_2)) = \chi_{\mathrm{A}}(p_1 \times p_2).$$

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In order to simplify notation, if a fibration $p : E \to B$ is equipped with a choice of bundle of finitely generated, free R-modules $\phi : V \to E$, it will be referred to as a fibration with flat bundle . The phrase fibration with acyclic flat bundle will then imply the additional condition that $H_*(E_b; V_b)$ is zero for each $b \in B$. For the purposes of our desired product formula for parametrized Reidemeister torsion, the following corollary is actually the key result.

Corollary 1.2. *Suppose $p_1 : E_1 \to B_1$ is a perfect fibration with flat bundle and $p_2 : E_2 \to B_2$ is a perfect fibration with acyclic flat bundle. Then*

$$\mu_{\mathrm{A^{cy}}}\left(\chi_{\mathrm{A}}\left(p_1\right), \chi_{\mathrm{A^{cy}}}\left(p_2\right)\right) = \chi_{\mathrm{A^{cy}}}\left(p_1 \times p_2\right).$$

The next result relies upon the existence of a restricted external multiplication for $\Gamma\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$:

$$\psi^{cy}_{\Gamma} : \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \Gamma^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}.$$

Theorem 1.3. *Suppose $p_1 : E_1 \to B_1$ is a perfect fibration with flat bundle and $p_2 : E_2 \to B_2$ is a perfect fibration with acyclic flat bundle. Then*

$$\psi^{cy}_{\Gamma}(\chi\left(p_1\right), \chi^{cy}\left(p_2\right)) \simeq \chi^{cy}\left(p_1 \times p_2\right)$$

where $\simeq$ means there exists a natural path connecting these points.

In section 6, we will establish the existence of a restricted product pairing

$$\nu : \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{Wh}^{R_2}_{B_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \mathrm{Wh}^{R_1 \otimes_A R_2}_{B_1 \times B_2}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

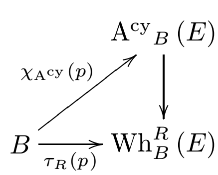
where $\mathrm{Wh}^{R_2}_{B_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix}$ represents the Whitehead space associated to p_2. Recall that the topological Reidemeister torsion $\tau_{R_2}\left(p_2\right)$ may be viewed as a point in this Whitehead space. The main result here is the product formula for torsion which generalizes a classical result of Kwun and Szczarba in [9].

Theorem 1.4 (Product Formula)**.** *Suppose $p_1 : E_1 \to B_1$ is a perfect fibration with flat bundle and $p_2 : E_2 \to B_2$ is a perfect fibration with acyclic flat bundle. Then $\tau_{R_1 \otimes_A R_2}\left(p_1 \times p_2\right)$ is defined and*

$$\nu(\chi\left(p_1\right), \tau_{R_2}\left(p_2\right)) \simeq \tau_{R_1 \otimes_A R_2}\left(p_1 \times p_2\right).$$

Remark 1.5. Note that in the informal notation of [5], $\tau_{R_1 \otimes_A R_2}\left(p_1 \times p_2\right)$ and $\nu(\chi\left(p_1\right), \tau_{R_2}\left(p_2\right))$ both correspond to maps $B_1 \times B_2 \to \mathrm{Wh}^{R_1 \otimes_A R_2}_{B_1 \times B_2}\left(E_1 \times E_2\right)$ (this is a different version of Whitehead space). The statement of the product formula then becomes that these two maps are homotopy equivalent.

The Product Theorem is deduced as a consequence of Theorem 1.3 in section 2. The basic idea, again using the informal notation of [5], is that the following diagram commutes up to homotopy.



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2. The Product Formula

This section is devoted to reducing the proofs of Theorems 1.1, 1.3, 1.4 and Corollary 1.2 to the existence of certain products, which will be defined in section 6, and several technical results which will be proven later.

The products we need will be the following.

$$\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \xrightarrow{\mu_{\mathrm{A}}} \mathrm{A}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

$$\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \xrightarrow{\mu_{\mathrm{A}^{\mathrm{cy}}}} \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

$$\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \xrightarrow{\psi_{\Gamma}} \Gamma\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

$$\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \xrightarrow{\psi_{\Gamma}^{cy}} \Gamma^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

$$\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{Wh}^{R_2}_{B_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \xrightarrow{\nu} \mathrm{Wh}^{R_1 \otimes_A R_2}_{B_1 \times B_2}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

Also in Sections 5 and 6, we will construct natural "coassembly maps"

$$\mathrm{A}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \xrightarrow{\delta} \Gamma\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$$

$$\mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \xrightarrow{\delta^{cy}} \Gamma^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$$

and

$$\Gamma^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \xrightarrow{\epsilon} \mathrm{Wh}^{R}_{B}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}.$$

We will define $\chi(p)$ as the image under the coassembly map δ of the parametrized A-theory Euler characteristic $\chi_{\mathrm{A}}(p)$ and similarly for $\chi^{cy}(p)$. At the end of Section 6, this will lead to a proof of the following.

Proposition 2.1. *Suppose $p_2 : E_2 \to B_2$ is a perfect fibration with acyclic flat bundle. Then*

$$\tau_{R_2}(p_2) = \epsilon_2(\chi^{cy}(p_2))$$

In Sections 6 and 7 and we will establish the following key technical result.

Proposition 2.2. *The following diagrams are each commutative up to a natural homotopy:*

(1)
$$\begin{array}{ccc}
\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow{\mu_{\mathrm{A}^{\mathrm{cy}}}} & \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\
\Big\downarrow \delta_1 \times \delta_2^{cy} & & \Big\downarrow \delta_{1,2}^{cy} \\
\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow[\psi_\Gamma^{cy}]{} & \Gamma^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}
\end{array}$$

and

(2)
$$\begin{array}{ccc}
\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow{\psi_\Gamma^{cy}} & \Gamma^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\
\Big\downarrow 1 \times \epsilon_2 & & \Big\downarrow \epsilon_{1,2} \\
\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{Wh}_{B_2}^{R_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow[\nu]{} & \mathrm{Wh}_{B_1 \times B_2}^{R_1 \otimes_A R_2}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}.
\end{array}$$

In fact, the first statement of the propostion follows immediately from the following theorem together with Lemma 2.5 below. Notice, we intend the product rather than smash product versions of μ_{A} and ψ_Γ for technical reasons.

Theorem 2.3. *The diagram*

$$\begin{array}{ccc}
\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow{\mu_{\mathrm{A}}} & \mathrm{A}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\
\Big\downarrow \delta_1 \times \delta_2 & & \Big\downarrow \delta_{1,2} \\
\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \Gamma\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow[\psi_\Gamma]{} & \Gamma\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}.
\end{array}$$

commutes up to a natural homotopy.

We can now establish Theorem 1.1 and Corollary 1.2, which generalize classical properties of the Euler characteristic. Recall the fold map

$$E \sqcup E \leftrightarrows E$$

defines an object of $\mathrm{Ret}^{\mathrm{fd}}(p)$, hence a point in $\mathrm{A}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ which we denote by $\chi_{\mathrm{A}}(p)$ and think of as the sphere bundle of the trivial line bundle over E. One might think of this as a parametrized version of the Euler characteristic of the fiber E_b

Since μ_{A} is defined as the map induced by the bi-exact functor (by restriction from [16]) external smash product of retractive spaces

$$\wedge_{E_1 \times E_2} : \mathrm{Ret}^{\mathrm{fd}}(p_1) \times \mathrm{Ret}^{\mathrm{fd}}(p_2) \to \mathrm{Ret}^{\mathrm{fd}}(p_1 \times p_2),$$

and the Euler characteristic is represented by

$$E \sqcup E \leftrightarrows E$$

in $\mathrm{Ret}^{\mathrm{fd}}(p)$, Theorem 1.1 is a consequence of the following simple lemma, which we prove in the next section.

Lemma 2.4. *One has the identity*

$$(E_1 \sqcup E_1) \wedge_{E_1 \times E_2} (E_2 \sqcup E_2) = (E_1 \times E_2) \sqcup (E_1 \times E_2)$$

in $\mathrm{Ret}^{fd}(p_1 \times p_2)$.

In order to prove Corollary 1.2, we need the following result on the relationship between μ_{A} and $\mu_{\mathrm{A}^{\mathrm{cy}}}$.

Lemma 2.5. (1) *The diagram*

$$\begin{array}{ccc}
\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow{\mu_{\mathrm{A}^{\mathrm{cy}}}} & \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\
\downarrow & & \downarrow \\
\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow[\mu_{\mathrm{A}}]{} & \mathrm{A}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}
\end{array}$$

commutes.

(2) *The canonical map* $\mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \mathrm{A}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ *is an injection for every fibration p.*

(3) *The canonical map* $\Gamma^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \Gamma\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ *is an injection for every fibration p.*

Proof of Corollary 1.2. The corollary follows from Theorem 1.1 and Lemma 2.5 along with the fact that $\chi_{\mathrm{A}^{\mathrm{cy}}}(p_2)$ goes to $\chi_{\mathrm{A}}(p_2)$ under the canonical inclusion $\mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix}$. □

This suffices to allow us to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. The assertion is that

$$\psi_\Gamma^{cy}(\chi(p_1), \chi^{cy}(p_2)) \simeq \chi^{cy}(p_1 \times p_2).$$

which by our definition of $\chi(p_1)$ and $\chi^{cy}(p_2)$ is equivalent to

$$\psi_\Gamma^{cy}(\delta_1(\chi_{\mathrm{A}}(p_1)), \delta_2^{cy}(\chi_{\mathrm{A}^{\mathrm{cy}}}(p_2))) \simeq \delta_{1,2}^{cy}(\chi_{\mathrm{A}^{\mathrm{cy}}}(p_1 \times p_2)).$$

By diagram 1 in Proposition 2.2, it suffices to establish

$$\mu_{\mathrm{A}^{\mathrm{cy}}}(\chi_{\mathrm{A}}(p_1), \chi_{\mathrm{A}^{\mathrm{cy}}}(p_2)) = \chi_{\mathrm{A}^{\mathrm{cy}}}(p_1 \times p_2)$$

which is the statement of Corollary 1.2. □

Proof of Theorem 1.4. The assertion is that

$$\nu(\chi(p_1), \tau_{R_2}(p_2)) \simeq \tau_{R_1 \otimes_A R_2}(p_1 \times p_2).$$

However, Proposition 2.1 implies this is equivalent to the statement

$$\nu(\chi(p_1), \epsilon_2(\chi^{cy}(p_2))) \simeq \epsilon_{1,2}(\chi^{cy}(p_1 \times p_2)).$$

Now Proposition 2.2 (2) implies this statement follows immediately from Theorem 1.3. □

A proof of Lemma 2.4 and some basic results on retractive spaces make up Section 3. Section 4 is devoted to defining all of the relevant theories, parametrized Euler characteristics and torsion. The technical details of Thomason homotopy limit problem maps necessary for constructing the coassembly maps and understanding their basic properties are handled in Section 5. The bulk of Section 6 is devoted to defining the remaining products and establishing their properties, with proofs of Proposition 2.1 and the second statement of Proposition 2.2 at the end. Finally, in Section 7 we give the proofs of Lemma 2.5 and of Theorem 2.3.

3. Functors on Retractive Spaces

In this section we would like to introduce the external smash product of retractive spaces over a fibration along with two "change of base" functors related to restriction to a subcomplex of the base space B. The external smash products induce the external product in parametrized A-theory and their interaction with the change of base functions is important to the proof of Proposition 2.2.

We begin by defining the category of retractive spaces over p, $\mathrm{Ret}^{\mathrm{fd}}(p)$, as the Waldhausen category of retractive spaces X over E where the composition $X \to E \to B$ is a (Hurewicz) fibration whose fibers are finitely dominated. (See [17].) The Waldhausen category structure has weak equivalences the homotopy equivalences which happen to be maps in this category and similarly cofibrations are the maps in this category which, as continuous maps, are closed embeddings with the homotopy extension property. Recall that a retractive space over E is a pair $X \leftrightarrows E$ where the inclusion $i : E \to X$ is a cofibration and $ri = 1_E$. In this notation, $\mathrm{Ret}^{\mathrm{fd}}(E)$ consists of finitely dominated retractive spaces over E, since it assumes the base $B = *$. In particular, notice $\mathrm{Ret}^{\mathrm{fd}}(p)$ is a sub-Waldhausen category of $\mathrm{Ret}^{\mathrm{fd}}(E)$.

Next, we introduce the external smash product.

Definition 3.1. Let $X \in \mathrm{Ret}^{\mathrm{fd}}(p_1)$ and $Y \in \mathrm{Ret}^{\mathrm{fd}}(p_2)$. Then their external smash product is a space in $\mathrm{Ret}^{\mathrm{fd}}(p_1 \times p_2)$ which we define as the following pushout

$$\begin{array}{ccc} X \times E_2 \cup E_1 \times Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ E_1 \times E_2 & \longrightarrow & X \wedge_{E_1 \times E_2} Y. \end{array}$$

Proof of Lemma 2.4. Both terms can be identified with the pushout of the following diagram.

$$\begin{array}{ccc} E_1 \times (E_2 \sqcup E_2) \cup (E_1 \sqcup E_1) \times E_2 & \longrightarrow & E_1 \times E_2 \\ \downarrow & & \downarrow \\ (E_1 \sqcup E_1) \times (E_2 \sqcup E_2) & \longrightarrow & P \end{array}$$

□

Let $\mathrm{Simp}(B)$ denote the category of simplices of the chosen triangulation of the base space B, where the morphisms are only the inclusions of subsimplices. If $\delta,\sigma \in \mathrm{Simp}(B)$, let $i^\delta_\sigma : |\delta| \to |\sigma|$ and $i^\sigma_B : |\sigma| \to B$ denote the natural inclusions. Notice that i^δ_σ is a homotopy equivalence, hence $\bar{i}^\delta_\sigma : p^{-1}(\delta) \to p^{-1}(\sigma)$ will also be a homotopy equivalence.

Definition 3.2. Given $W \in \mathrm{Ret}^{\mathrm{fd}}\left(p^{-1}(\delta)\right)$, $Y \in \mathrm{Ret}^{\mathrm{fd}}\left(p^{-1}(\sigma)\right)$ and $X \in \mathrm{Ret}^{\mathrm{fd}}(E)$ we define pullbacks and pushforwards as below.

(1) The pushforward of W along i_σ^δ is the following pushout

$$\begin{array}{ccc} p^{-1}(\delta) & \xrightarrow{\bar{i}_\sigma^\delta} & p^{-1}(\sigma) \\ \downarrow & & \downarrow \\ W & \longrightarrow & \left(i_\sigma^\delta\right)_*(W) \end{array}$$

where $\bar{i}_\sigma^\delta : p^{-1}(\delta) \to p^{-1}(\sigma)$ is the induced inclusion (which is a homotopy equivalence).

(2) Similarly, we can define $\left(i_B^\sigma\right)_*(Y)$ as the pushout of Y along the induced inclusion $\bar{i}_B^\sigma$.

(3) The pullback of Y along i_σ^δ is the following pullback

$$\begin{array}{ccc} \left(i_\sigma^\delta\right)^*(Y) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ p^{-1}(\delta) & \xrightarrow{\bar{i}_\sigma^\delta} & p^{-1}(\sigma) \end{array}$$

where $\bar{i}_\sigma^\delta : p^{-1}(\delta) \to p^{-1}(\sigma)$ is the induced inclusion (which is a homotopy equivalence).

(4) Similarly, we can define $\left(i_B^\sigma\right)^*(X)$ as the pullback of X along the induced inclusion $\bar{i}_B^\sigma$.

To see that the first two operations land in the correct categories, see [17].

For the pullback operations, it is important to note the fact that i_σ^δ is a closed embedding implies the fibers of $\left(i_\sigma^\delta\right)^*(X)$ are simply certain fibers of X itself (thereby preserving the finitely dominated condition). A result of Lück [10] implies the inclusion being a cofibration is preserved by the pullback operation as well. Finally, the fibration condition is preserved under pullbacks.

The following lemma is largely a consequence of the universal properties of pushouts and pullbacks.

Lemma 3.3. *Given $\delta \subset \sigma$, the pullback and pushforward over i_σ^δ yield an adjoint pair of functors*

$$\left(\left(i_\sigma^\delta\right)_* : \mathrm{Ret}^{fd}\left(p^{-1}(\delta)\right) \to \mathrm{Ret}^{fd}\left(p^{-1}(\sigma)\right),\left(i_\sigma^\delta\right)^* : \mathrm{Ret}^{fd}\left(p^{-1}(\sigma)\right) \to \mathrm{Ret}^{fd}\left(p^{-1}(\delta)\right)\right)$$

Hence, there is a natural homotopy equivalence $\left(i_\sigma^\delta\right)_*\left(i_\sigma^\delta\right)^*(Y) \to Y$ called the unit of adjunction and similarly for $Z \to \left(i_\sigma^\delta\right)^*\left(i_\sigma^\delta\right)_*(Z)$. The functor $\left(i_\sigma^\delta\right)_*$ is an exact functor, while the maps of spaces $\left(i_\sigma^\delta\right)^*(Y) \to Y$ and $Z \to \left(i_\sigma^\delta\right)_*(Z)$ are homotopy equivalences. Finally, the functors $\left(i_\sigma^\delta\right)^*$ and $\left(i_B^\sigma\right)^*$ preserve homotopy equivalences.

Notice, the fact that $X \to \sigma$ or $X \to B$ is assumed to be a fibration implies the pullback preserves homotopy equivalences. We now have a result about the interaction between change of base and external smash products.

Lemma 3.4. *Suppose $W_n \in \mathrm{Ret}^{fd}\left(p_n^{-1}(\delta_n)\right)$ and $X_n \in \mathrm{Ret}^{fd}\left(p_n^{-1}(\sigma_n)\right)$ with $\delta_n \subset \sigma_n$ for $n = 1{,}2$. Then there are natural homotopy equivalences*

(1)
$$\begin{array}{c}\left(i^{\delta_1}_{\sigma_1}\right)^*(X_1)\wedge_{p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2)}\left(i^{\delta_2}_{\sigma_2}\right)^*(X_2)\\ \downarrow \\ (i^{\delta_1}_{\sigma_1}\times i^{\delta_2}_{\sigma_2})^*(X_1\wedge_{p_1^{-1}(\sigma_1)\times p_2^{-1}(\sigma_2)}X_2)\end{array}$$

(2)
$$\begin{array}{c}(i^{\delta_1}_{\sigma_1}\times i^{\delta_2}_{\sigma_2})_*(W_1\wedge_{p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2)}W_2)\\ \downarrow \\ \left(i^{\delta_1}_{\sigma_1}\right)_*(W_1)\wedge_{p_1^{-1}(\sigma_1)\times p_2^{-1}(\sigma_2)}\left(i^{\delta_2}_{\sigma_2}\right)_*(W_2)\end{array}$$

Proof. We will display the second map in detail, while the first is dual. To begin, note there is a commutative diagram

$$\begin{array}{ccc}W_1\times W_2 & \longrightarrow & \left(i^{\delta_1}_{\sigma_1}\right)_*(W_1)\times\left(i^{\delta_2}_{\sigma_2}\right)_*(W_2)\\ \uparrow & & \uparrow\\ W_1\times p_2^{-1}(\delta_2)\cup p_1^{-1}(\delta_1)\times W_2 & \longrightarrow & \left(i^{\delta_1}_{\sigma_1}\right)_*(W_1)\times p_2^{-1}(\sigma_2)\cup p_1^{-1}(\sigma_1)\times\left(i^{\delta_2}_{\sigma_2}\right)_*(W_2)\\ \downarrow & & \downarrow\\ p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2) & \longrightarrow & p_1^{-1}(\sigma_1)\times p_2^{-1}(\sigma_2)\end{array}$$

where the horizontal maps are all homotopy equivalences and both maps to the top row are cofibrations. Thus, there is an induced homotopy equivalence on pushouts

$$W_1\wedge_{p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2)}W_2\to\left(i^{\delta_1}_{\sigma_1}\right)_*(W_1)\wedge_{p_1^{-1}(\sigma_1)\times p_2^{-1}(\sigma_2)}\left(i^{\delta_2}_{\sigma_2}\right)_*(W_2)$$

which factors through an induced map

$$(i^{\delta_1}_{\sigma_1}\times i^{\delta_2}_{\sigma_2})_*(W_1\wedge_{p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2)}W_2)\to\left(i^{\delta_1}_{\sigma_1}\right)_*(W_1)\wedge_{p_1^{-1}(\sigma_1)\times p_2^{-1}(\sigma_2)}\left(i^{\delta_2}_{\sigma_2}\right)_*(W_2)$$

by construction. Since

$$W_1\wedge_{p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2)}W_2\to(i^{\delta_1}_{\sigma_1}\times i^{\delta_2}_{\sigma_2})_*(W_1\wedge_{p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2)}W_2)$$

is also a homotopy equivalence, the statement follows. □

4. Parametrized Theories, Euler Characteristics and Torsion Classes

Definition 4.1. Let $\mathrm{A}\begin{pmatrix}E\\ \downarrow p\\ B\end{pmatrix}=\Omega|wS_\cdot\mathrm{Ret}^{\mathrm{fd}}(p)|$ where $\mathrm{Ret}^{\mathrm{fd}}(p)$ is the Waldhausen category of retractive spaces over E with the fibers of the composition pr finitely dominated, where $r:X\to E$ is the retraction.

Definition 4.2. Suppose p is a perfect fibration with flat bundle V. Then

$$\mathrm{A}^{\mathrm{cy}}\begin{pmatrix}E\\ \downarrow p\\ B\end{pmatrix}=\Omega|wS_\cdot\mathrm{Ret}^{\mathrm{fd,cy}}(p)|$$

where $\mathrm{Ret}^{\mathrm{fd,cy}}(p)$ is the full sub-Waldhausen category of $\mathrm{Ret}^{\mathrm{fd}}(p)$ consisting of those objects $X \leftrightarrows E$ where $r^*(V) \to X$ makes each fiber $X_b \to *$ a perfect fibration with acyclic flat bundle.

Definition 4.3. Suppose p is a perfect fibration with flat bundle V. Let

$$\mathrm{A}_R\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right) = \Omega|wS.\,\mathrm{Ret}^{\mathrm{fd}}_R(p)|$$

where $\mathrm{Ret}^{\mathrm{fd}}_R(p)$ is the category $\mathrm{Ret}^{\mathrm{fd}}(p)$ with a different Waldhausen category structure where the weak equivalences are the local homology chain equivalences with local coefficients in r^*V and cofibrations are as usual. (See [4] for details.)

Recall $\mathrm{A}^{\mathrm{cy}}\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$ is homotopy equivalent to the homotopy fiber of the map induced by the identity, (which is an exact functor in this direction)

$$\mathrm{A}\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right) \to \mathrm{A}_R\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$$

as in [4].

Definition 4.4. We set $\Gamma\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right) = \operatorname*{holim}_{\sigma \in \mathrm{Simp}(B)} \Omega|wS.\,\mathrm{Ret}^{\mathrm{fd}}\left(p^{-1}(\sigma)\right)|$.

Let

$$\begin{array}{ccc} W & \xrightarrow{\bar{f}} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

be a pullback diagram where X is equipped with the flat bundle W and Y is equipped with the flat bundle V. Then we will say $(f;\bar{f})$ is a bundle morphism from $(X;W)$ to $(Y;V)$. Let $\mathfrak{D}$ denote the category whose objects are pairs $(X;W)$ where X has flat bundle W and morphism set $\mathfrak{D}((X;W)(Y;V))$ the set of bundle maps $(f,\bar{f})$ as above.

We would like to think of A^{cy} as a functor from $\mathfrak{D}$ to Top. Thus, we need to understand the map $\mathrm{A}^{\mathrm{cy}}(X;W) \to \mathrm{A}^{\mathrm{cy}}(Y;V)$ induced by a bundle morphism $(f,\bar{f})$.

We define a functor f_* from $\mathrm{Ret}^{\mathrm{fd}}(X)$ to $\mathrm{Ret}^{\mathrm{fd}}(Y)$ by taking $Z \leftrightarrows X$ to $Z \cup_X Y \leftrightarrows Y$. We would like to know that, restricting the source to the full subcategory $\mathrm{Ret}^{\mathrm{fd,cy}}(X)_{(W)}$, the target lies in the full subcategory $\mathrm{Ret}^{\mathrm{fd,cy}}(Y)_{(V)}$. This requires a statement from homological algebra; specifically, $H_*(Z,X;W) = 0$ must imply $H_*(f_*(Z),Y;V) = 0$. However, this follows from the definition of homology with local coefficients. (See [3].)

Now we would like to know the functor f_* is exact. See Lemma 3.3 for the fact that f_* is exact in our case when considered as a functor

$$f_* : \mathrm{Ret}^{\mathrm{fd}}(X) \to \mathrm{Ret}^{\mathrm{fd}}(Y).$$

However, since both cyclic categories are sub-Waldhausen categories, the exactness in our case follows by restriction. Thus, f_* induces a map $\mathrm{A}^{\mathrm{cy}}(X) \to \mathrm{A}^{\mathrm{cy}}(Y)$.

Now we define a functor $\mathfrak{P} : \mathrm{Simp}(B) \to \mathfrak{D}$ associated to any fibration p with flat bundle V (so $(E;V) \in \mathfrak{D}$) by sending $\sigma \mapsto (p^{-1}(\sigma); V|_{p^{-1}(\sigma)})$. Any morphism $\delta \to \sigma$

in Simp (B) is the inclusion of a subcomplex. Thus, the horizontal rectangles and right squares in the following diagram are pullbacks, which implies the left squares are pullbacks as well.

$$\begin{array}{ccccc} V|_{p^{-1}(\delta)} & \longrightarrow & V|_{p^{-1}(\sigma)} & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ p^{-1}(\delta) & \longrightarrow & p^{-1}(\sigma) & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ \delta & \longrightarrow & \sigma & \longrightarrow & B \end{array}$$

Hence, we have defined a morphism

$$(p^{-1}(\delta); V|_{p^{-1}(\delta)}) \to (p^{-1}(\sigma); V|_{p^{-1}(\sigma)})$$

associated to each morphism $\delta \to \sigma$, thereby making our assignment $\mathfrak{P}$ a functor. We can now define the composite functor

$$\text{Simp}\,(B) \xrightarrow{\mathfrak{P}} \mathfrak{D} \xrightarrow{\text{A}^{\text{cy}}} Top$$

and this is the functor whose holim appears in the following definition.

Definition 4.5. We set $\Gamma^{\text{cy}} \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} = \underset{\sigma \in \text{Simp}(B)}{\text{holim}} \Omega |wS.\, \text{Ret}^{\text{fd,cy}} \left(p^{-1}(\sigma)\right)|.$

Replacing the functor $\text{A}^{\text{cy}} : \mathfrak{D} \to Top$ with the functor $\text{A}_R : \mathfrak{D} \to Top$ we may form a similar holim.

Definition 4.6. We set $\Gamma_R \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} = \underset{\sigma \in \text{Simp}(B)}{\text{holim}} \Omega |wS.\, \text{Ret}^{\text{fd}}_R \left(p^{-1}(\sigma)\right)|.$

Definition 4.7. Let $\chi_{\text{A}}\,(p) \in \text{A} \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ denote the point corresponding to

$$E \sqcup E \leftrightarrows E$$

as an object of $\text{Ret}^{\text{fd}}\,(p)$, with the fold map as retraction.

Definition 4.8. Suppose p is a fibration with acyclic flat bundle V. Let $\chi_{\text{A}^{\text{cy}}}\,(p) \in \text{A}^{\text{cy}} \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ denote the point corresponding to $E \sqcup E \leftrightarrows E$ as an object of $\text{Ret}^{\text{fd,cy}}\,(p)$.

Notice that $H_*((E \sqcup E)_b, E_b; V_b)$ is naturally isomorphic to $H_*(E_b, V_b) = 0$, so the assumption of acyclic flat bundle is simply a restatement of the fact that the sphere bundle of the trivial line bundle, $E \sqcup E \leftrightarrows E$, is an object of $\text{Ret}^{\text{fd,cy}}\,(p)$.

Now define $\chi\,(p) \in \Gamma \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ as the image of $\chi_{\text{A}}\,(p)$ under the coassembly map δ we will describe in detail in section 5. (This agrees with the definition of $\chi\,(p)$ given in [5].) Similarly, set $\chi^{cy}\,(p) = \delta^{cy}(\chi_{\text{A}^{\text{cy}}}\,(p))$.

Next, we define the parametrized Whitehead space associated to a fibration p with flat bundle V. Notice the inclusions $p^{-1}(\sigma) \to E$ will yield a map

$$\Gamma \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} = \underset{\text{Simp}(B)}{\text{holim}}\, \text{A}\left(p^{-1}(\sigma)\right) \to \underset{\text{Simp}(B)}{\text{holim}}\, \text{A}\,(E)\,.$$

Follow this by the map

$$\operatorname*{holim}_{\mathrm{Simp}(B)} \mathrm{A}(E) \to \operatorname*{holim}_{\mathrm{Simp}(B)} \mathrm{K}(R)$$

induced by the linearization map associated to V. Denote the composition

$$\Lambda_V : \Gamma\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \operatorname*{holim}_{\mathrm{Simp}(B)} \mathrm{K}(R).$$

Definition 4.9. Let $\mathrm{Wh}_B^R\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ denote the homotopy fiber of Λ_V.

For the sake of precision, we should mention our model for $\mathrm{K}(R)$ is $\Omega|wS.Ch_*(R)|$ where $Ch_*(R)$ is the category of bounded complexes of projective modules given a Waldhausen structure where injections are cofibrations and quasi-isomorphisms (homology equivalences) are the weak equivalences. Then the linearization map is induced by the functor $\mathrm{Ret}^{\mathrm{fd}}(E) \to Ch_*(R)$ sending X to the relative chain complex with local coefficients and compact supports $\mathcal{C}_*(X, E; r^*(V))$.

It will be shown at the end of the next section that there is a canonical path β in $\operatorname*{holim}_{\mathrm{Simp}(B)} \mathrm{K}(R)$ from $\Lambda_V(\chi(p))$ to the basepoint provided the flat bundle over p is assumed to be acyclic.

Definition 4.10. Suppose p is a fibration with acyclic flat bundle V. Then the Euler characteristic $\chi(p) \in \Gamma\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ together with the path β define a point in $\mathrm{Wh}_B^R\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ by definition of homotopy fiber. We will refer to this point as the torsion class $\tau_R(p)$ or the topological Reidemeister torsion.

5. Thomason Homotopy Limit Problems

We would like to understand the definition of the coassembly map

$$\delta : \mathrm{A}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \Gamma\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}.$$

This comes from a formal trick which travels under the name of a Thomason homotopy limit problem which we will describe in an abstract context. In order to be careful with the properties of the realization of a category, in this section only, we will make explicit the composition of the nerve functor and the geometric realization of a simplicial set.

Suppose $\mathcal{C}$ is a small category and $\mathcal{G} : \mathcal{C} \to \mathrm{Cat}$ is a functor. There is always another important functor $\mathcal{C} \to \mathrm{Cat}$, defined by $C \mapsto \mathcal{C}/C$ sending each object in C to the category of objects over it in $\mathcal{C}$. Clearly this is a covariant functor, defined on morphisms by postcomposition. Given two functors with the same source and target, one can define the category of natural transformations between them, which we'll denote by $Nat(\mathcal{C}/?, \mathcal{G})$.

Recall that one normally views a natural transformation as a point in a large product of mapping sets satisfying certain additional conditions. More precisely, natural transformations lie in the equalizer of the two maps induced by precomposition and postcomposition respectively. (See section IX.5 of [11] for the translation as an "end".)

Thus, it is natural to define

$$Nat\,(\mathcal{C}/?, \mathcal{G}) \subset \prod_{C \in \mathcal{C}} Fun(\mathcal{C}/C, \mathcal{G}(C))$$

as the equalizer in Cat of two functors. The first functor Ψ

$$\prod_{C \in \mathcal{C}} Fun(\mathcal{C}/C, \mathcal{G}(C)) \to \prod_{\varphi : C \to D \in \mathcal{C}} Fun(\mathcal{C}/C, \mathcal{G}(D))$$

is induced by the collection of functors

$$\mathcal{G}(\varphi)_* : Fun(\mathcal{C}/C, \mathcal{G}(C)) \to Fun(\mathcal{C}/C, \mathcal{G}(D))_\varphi.$$

The second functor Φ is defined similarly using the precomposition,

$$(\mathcal{C}/\varphi)^* : Fun(\mathcal{C}/D, \mathcal{G}(D)) \to Fun(\mathcal{C}/C, \mathcal{G}(D))_\varphi.$$

In other words, a natural transformation is a collection of functors

$$\Upsilon(C) : \mathcal{C}/C \to \mathcal{G}(C)$$

where the following diagram commutes for each morphism $\varphi : C \to D$ in $\mathcal{C}$:

$$\begin{array}{ccc} \mathcal{C}/C & \xrightarrow{\Upsilon(C)} & \mathcal{G}(C) \\ {\scriptstyle \mathcal{C}/\varphi}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(\varphi)} \\ \mathcal{C}/D & \xrightarrow[\Upsilon(D)]{} & \mathcal{G}(D). \end{array}$$

Similarly, a morphism in the category $Nat\,(\mathcal{C}/?, \mathcal{G})$ consists of a natural transformation in each factor which must be compatible in the sense that a certain cubical extension of this diagram commutes.

The nerve functor is a right adjoint, hence it preserves equalizers. Thus, applying nerve to $Nat\,(\mathcal{C}/?, \mathcal{G})$ yields an equalizer in the category of simplicial sets. Taking the geometric realization does not preserve arbitrary products, but does preserve equalizers. Hence, there is a natural isomorphism

$$|N(Nat\,(\mathcal{C}/?, \mathcal{G}))| \approx eq(|N(\Phi)|, |N(\Psi)|).$$

However, one must keep in mind that $|N(\Phi)|$ and $|N(\Psi)|$ no longer decompose as the products of $|N(?)|$ applied to the expected factors. We let $\bar{\Phi}$ and $\bar{\Psi}$ denote the same constructions where the product is taken after the realization and the components are of the form

$$|N(Fun(\mathcal{C}/C, \mathcal{G}(C)))| \quad \text{and} \quad |N(Fun(\mathcal{C}/C, \mathcal{G}(D)))|.$$

Notice there is a natural map $|N(\Phi)| \to \bar{\Phi}$ given by the universal property of the product in the definition of $\bar{\Phi}$ and similarly for Ψ.

Now suppose there is another small category $\mathcal{D}$ together with a functor

$$\mathcal{F} : \mathcal{D} \to Nat\,(\mathcal{C}/?, \mathcal{G})\,.$$

Then taking nerves yields a natural map

$$N(\mathcal{F}) : N(\mathcal{D}) \to N(Nat\,(\mathcal{C}/?, \mathcal{G}))$$

and taking geometric realization gives a natural map

$$|N(\mathcal{D})| \to |N(Nat\,(\mathcal{C}/?, \mathcal{G}))|$$

One could avoid the finiteness assumption in the following result by working throughout in a convenient category of topological spaces in the sense of Steenrod [15].

Lemma 5.1. *Suppose $\mathcal{C}$ has the property that each overcategory $\mathcal{C}/C$ is finite. Then there is a natural map*

$$|N(Nat\,(\mathcal{C}/?, \mathcal{G}))| \to \operatorname*{holim}_{\mathcal{C}} |N(\mathcal{G})|.$$

Proof. To begin, recall that $\operatorname{holim}_{\mathcal{C}} |N(\mathcal{G})|$ is itself an equalizer of two maps

$$\prod_{C \in \mathcal{C}} \operatorname{Map}(|N(\mathcal{C}/C)|, |N(\mathcal{G}(C))|) \to \prod_{\varphi: C \to D \in \mathcal{C}} \operatorname{Map}(|N(\mathcal{C}/C)|, |N(\mathcal{G}(D))|).$$

These maps are constructed as with Φ and Ψ, using the maps $|N(\mathcal{G}(\varphi))|_*$ and $|N(\mathcal{C}/C)|^*$.

Now, we will describe a collection of maps

$$|N(Fun(\mathcal{C}/C, \mathcal{G}(D)))| \to \operatorname{Map}(|N(\mathcal{C}/C)|, |N(\mathcal{G}(D))|)$$

which are natural in the sense that they send the map $\bar{\Phi}$ described above to the map induced by the maps $|N(\mathcal{C}/C)|^*$ and similarly for Ψ.

There is an evaluation functor

$$e : Fun(\mathcal{C}/C, \mathcal{G}(D)) \times \mathcal{C}/C \to \mathcal{G}(D)$$

whose nerve defines a morphism

$$N(e) : N(Fun(\mathcal{C}/C, \mathcal{G}(D))) \times N(\mathcal{C}/C) \to N(\mathcal{G}(D))$$

since the nerve functor commutes with products. Now, apply geometric realization which commutes with finite products to yield a natural map

$$|N(e)| : |N(Fun(\mathcal{C}/C, \mathcal{G}(D)))| \times |N(\mathcal{C}/C)| \to |N(\mathcal{G}(D))|$$

which is adjoint to the required map. Notice the adjoint map exists because the finiteness assumption implies $|N(\mathcal{C}/C)|$ is a compact Hausdorff space.

Since geometric realization does not commute with arbitrary products, we must be careful that the maps $|N(e)|$ described above induce a map from the equalizer of the pair $\bar{\Phi}, \bar{\Psi}$ rather than $|N(\Phi)|$ or $|N(\Psi)|$. This gives

$$eq(\bar{\Phi}, \bar{\Psi}) \to \operatorname*{holim}_{\mathcal{C}} |N(\mathcal{G})|.$$

Fortunately, there is still a natural map

$$eq(|N(\Phi)|, |N(\Psi)|) \to eq(\bar{\Phi}, \bar{\Psi})$$

given by the universal property of the products involved in defining the target. The composition is the required natural map. □

By Lemma 5.1, the composite displayed above yields a map

$$|N(\mathcal{D})| \to \operatorname*{holim}_{\mathcal{C}} |N(\mathcal{G})|.$$

This is the map usually called a Thomason homotopy limit problem map.

The case of interest for us is when $\mathcal{C} = \operatorname{Simp}(B)$ and $\mathcal{G} : \operatorname{Simp}(B) \to \operatorname{Cat}$ is defined by $\sigma \mapsto w \operatorname{Ret}^{\mathrm{fd}}\left(p^{-1}(\sigma)\right)$. In particular, notice our assumption on B from the introduction implies $\mathcal{C}$ satisfies the assumption in the statement of Lemma 5.1. Notice the natural group completion map $|N(\mathcal{G}(\sigma))| \to \mathrm{A}\left(p^{-1}(\sigma)\right)$ yields a natural map $\operatorname{holim}_{\mathcal{C}} |N(\mathcal{G})| \to \Gamma \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ as well. Precomposing with our Thomason homotopy limit problem map, this gives a map $|N(\mathcal{D})| \to \Gamma \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$. For the category $\mathcal{D} = w\,\mathrm{Ret}^{\mathrm{fd}}\,(p)$ and the functor $w\,\mathrm{Ret}^{\mathrm{fd}}\,(p) \to \mathit{Nat}\,(\mathcal{C}/?, \mathcal{G})$ discussed in detail below, this will yield a map $|N(w\,\mathrm{Ret}^{\mathrm{fd}}\,(p))| \to \Gamma \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ whose target is an infinite loop space by construction. Thus, we can extend over the natural group completion map $|N(\mathcal{D})| \to \mathrm{A} \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ to build the desired coassembly map

$$\delta : \mathrm{A} \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \Gamma \begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}.$$

In order to complete the definition of the coassembly map δ it suffices to describe the relevant functor $\mathcal{F} : w\,\mathrm{Ret}^{\mathrm{fd}}\,(p) \to \mathit{Nat}\,(\mathcal{C}/?, \mathcal{G})$. We will describe this relationship by saying δ is the group completion of the Thomason homotopy limit problem map associated to $\mathcal{F}$.

We require two other general results in this context.

Lemma 5.2. (1) *Suppose φ is a natural transformation from $\mathcal{G}$ to $\mathcal{G}'$. Then the induced maps make the following diagram commute.*

$$\begin{array}{ccc} |N(\mathit{Nat}\,(\mathcal{C}/?, \mathcal{G}))| & \longrightarrow & |N(\mathit{Nat}\,(\mathcal{C}/?, \mathcal{G}'))| \\ \downarrow & & \downarrow \\ \operatorname{holim}\limits_{\mathcal{C}} |N(\mathcal{G})| & \longrightarrow & \operatorname{holim}\limits_{\mathcal{C}} |N(\mathcal{G}')| \end{array}$$

(2) *There is a pseudo-product in $\mathit{Nat}\,(?, !)$ which makes the following diagram commute.*

$$\begin{array}{ccc} |N(\mathit{Nat}\,(\mathcal{C}_1/?, \mathcal{G}_1))| \times |N(\mathit{Nat}\,(\mathcal{C}_2/?, \mathcal{G}_2))| & \longrightarrow & |N(\mathit{Nat}\,(\mathcal{C}_1/? \times \mathcal{C}_2/?, \mathcal{G}_1 \times \mathcal{G}_2))| \\ \downarrow & & \downarrow \\ \operatorname{holim}\limits_{\mathcal{C}_1} |N(\mathcal{G}_1)| \times \operatorname{holim}\limits_{\mathcal{C}_2} |N(\mathcal{G}_2)| & \longrightarrow & \operatorname{holim}\limits_{\mathcal{C}_1 \times \mathcal{C}_2} |N(\mathcal{G}_1)| \times |N(\mathcal{G}_2)| \end{array}$$

Lemma 5.3. *Suppose $\mathcal{F} : \mathcal{D} \to \mathit{Nat}\,(\mathcal{C}/?, \mathcal{G})$ and $\mathcal{F}' : \mathcal{D} \to \mathit{Nat}\,(\mathcal{C}/?, \mathcal{G})$ are functors together with a natural transformation*

$$\psi : \mathcal{F} \to \mathcal{F}'.$$

Then the two Thomason homotopy limit problem maps

$$|N(\mathcal{D})| \to \operatorname{holim}_{\mathcal{C}} |N(\mathcal{G})|$$

are naturally homotopic, as are their group completions.

Proof. The point here is that $|N(\psi)|$ induces the necessary homotopy between the two maps $|N(\mathcal{F})|$ and $|N(\mathcal{F}')| : |N(\mathcal{D})| \to |N(\mathit{Nat}\,(\mathcal{C}/?, \mathcal{G}))|$. The remaining extensions by the composite

$$|N(\mathit{Nat}\,(\mathcal{C}/?, \mathcal{G}))| \to \mathit{eq}(\bar{\Phi}, \bar{\Psi}) \to \operatorname{holim}_{\mathcal{C}} |N(\mathcal{G})|$$

will then be homotopic as well, so the group completions will also be homotopic. □

Since $Nat(\mathcal{C}/?, \mathcal{G})$ is defined as an equalizer of Ψ and Φ in Cat, defining a functor into $Nat(\mathcal{C}/?, \mathcal{G})$ is equivalent to defining a functor into $\prod_{C\in\mathcal{C}} Fun(\mathcal{C}/C, \mathcal{G}(C))$ so that the two composite functors

$$\mathcal{D} \to \prod_{\varphi:C\to D\in\mathcal{C}} Fun(\mathcal{C}/C, \mathcal{G}(D))_{\varphi}$$

agree. As each of these target categories is a product, the universal properties imply it suffices to check on each projection. In other words, we should define a series of functors

$$\mathcal{F}_C : \mathcal{D} \to Fun(\mathcal{C}/C, \mathcal{G}(C))$$

such that each diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\mathcal{F}_C} & Fun(\mathcal{C}/C, \mathcal{G}(C)) \\ \downarrow{\scriptstyle \mathcal{F}_D} & & \downarrow{\scriptstyle \mathcal{G}(\varphi)_*} \\ Fun(\mathcal{C}/D, \mathcal{G}(D)) & \xrightarrow[(\mathcal{C}/\varphi)^*]{} & Fun(\mathcal{C}/C, \mathcal{G}(D))_{\varphi} \end{array}$$

commutes.

In our specific situation, this means we need to define functors

$$\mathcal{F}_\sigma : w\,\mathrm{Ret}^{\mathrm{fd}}(p) \to Fun(\mathrm{Simp}(B)/\sigma, w\,\mathrm{Ret}^{\mathrm{fd}}\left(p^{-1}(\sigma)\right)).$$

The functors we have in mind send $X \leftrightarrows E$ to the functor

$$\delta \mapsto \left(i_\sigma^\delta\right)_* (i_B^\delta)^*(X) \leftrightarrows p^{-1}(\sigma).$$

Functoriality of this assignment requires a natural weak equivalence

$$\left(i_\sigma^\delta\right)_* (i_B^\delta)^*(X) \to \left(i_\sigma^\tau\right)_* (i_B^\tau)^*(X)$$

whenever $\delta \subset \tau$. However, uniqueness of inclusions implies

$$\left(i_\tau^\delta\right)^* \left(i_B^\tau\right)^* (X) = \left(i_B^\delta\right)^* (X)$$

and

$$\left(i_\sigma^\tau\right)_* \left(i_\tau^\delta\right)_* = \left(i_\sigma^\delta\right)_*$$

so

$$\left(i_\sigma^\delta\right)_* (i_B^\delta)^*(X) = \left(i_\sigma^\tau\right)_* \left(i_\tau^\delta\right)_* \left(i_\tau^\delta\right)^* \left(i_B^\tau\right)^* (X).$$

Hence, $\left(i_\sigma^\tau\right)_*$ applied to the unit of adjunction gives the required natural weak equivalence. Clearly, this is natural with respect to further inclusions in either direction.

Notice that $\mathcal{C}/C$ in this case simply consists of the subcomplexes of σ, and the functors $\mathcal{C}/\varphi$ for $\varphi : \tau \to \sigma$ simply become the inclusion of the subcomplexes of τ as subcomplexes of some larger simplex σ. Hence the associated functor

$$\mathcal{G}(\varphi) : w\,\mathrm{Ret}^{\mathrm{fd}}\left(p^{-1}(\tau)\right) \to w\,\mathrm{Ret}^{\mathrm{fd}}\left(p^{-1}(\sigma)\right)$$

should be the pushforward functor $\left(i_\sigma^\tau\right)_*$.

Now the required commutative diagrams

$$\begin{array}{ccc} w\,\mathrm{Ret}^{\mathrm{fd}}\,(p) & \xrightarrow{\mathcal{F}_\tau} & \mathit{Fun}(\mathrm{Simp}\,(B)\,/\tau, w\,\mathrm{Ret}^{\mathrm{fd}}\,(p^{-1}(\tau))) \\ \downarrow{\scriptstyle \mathcal{F}_\sigma} & & \downarrow{\scriptstyle ((i^\tau_\sigma)_*)_*} \\ \mathit{Fun}(\mathrm{Simp}\,(B)\,/\sigma, w\,\mathrm{Ret}^{\mathrm{fd}}\,(p^{-1}(\sigma))) & \xrightarrow{(\mathrm{Simp}(B)/i^\tau_\sigma)^*} & \mathit{Fun}(\mathrm{Simp}\,(B)\,/\tau, w\,\mathrm{Ret}^{\mathrm{fd}}\,(p^{-1}(\sigma)))_{i^\tau_\sigma} \end{array}$$

are reduced to the statement that

$$\left(i^\tau_\sigma\right)_*\left(\left(i^\delta_\tau\right)_*\left(i^\delta_B\right)^*(X)\right) = \left(i^\delta_\sigma\right)_*\left(i^\delta_B\right)^*(X).$$

Thus, our assignment gives the required functor

$$\mathcal{F} : w\,\mathrm{Ret}^{\mathrm{fd}}\,(p) \to \mathit{Nat}\,(\mathcal{C}/?, \mathcal{G})\,.$$

If we choose a different initial functor $\mathcal{G}$, sending σ to $w\,\mathrm{Ret}^{\mathrm{fd,cy}}\left(p^{-1}(\sigma)\right)$, the same process yields the cyclic coassembly map $\delta^{cy} : \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \Gamma^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ as the group completion of the Thomason homotopy limit problem map of a modification of the functor $\mathcal{F}$ above. Similarly, one can alter $\mathcal{G}$ to send σ to $w\,\mathrm{Ret}^{\mathrm{fd}}_R\left(p^{-1}(\sigma)\right)$ and notice the functor $\mathcal{F}$ described above also yields a functor

$$\mathcal{F} : w\,\mathrm{Ret}^{\mathrm{fd}}_R\,(p) \to \mathit{Nat}\,(\mathcal{C}/?, \mathcal{G})$$

in this case. Thus, we have another coassembly map

$$\delta^R : \mathrm{A}_R\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \Gamma_R\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}.$$

In order to define the Reidemeister torsion, we needed the existence of a canonical path β from $\Lambda_V(\chi\,(p))$ to the basepoint in $\mathrm{holim}\,\mathrm{K}\,(R)$. Using the machinery of this section, it is straightforward to describe this path.

To begin, notice any object X of $\mathrm{Ret}^{\mathrm{fd,cy}}\,(p)$ sitting in $\mathrm{Ret}^{\mathrm{fd}}_R\,(p)$ has the property that the retraction is a weak equivalence by the definition of $\mathrm{Ret}^{\mathrm{fd,cy}}\,(p)$. Hence, there is a natural map from X to E in $w\,\mathrm{Ret}^{\mathrm{fd}}_R\,(p)$, which leads to a natural path from the point associated to X in $\mathrm{A}_R\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ to the basepoint.

Now suppose X represents an object in $\mathit{Nat}\left(\mathcal{C}/?, w\,\mathrm{Ret}^{\mathrm{fd}}_R\left(p^{-1}(?)\right)\right)$ which comes from an object in $\mathit{Nat}\left(\mathcal{C}/?, w\,\mathrm{Ret}^{\mathrm{fd,cy}}\left(p^{-1}(?)\right)\right)$ under the “inclusion” functor. There is another object of $\mathit{Nat}\left(\mathcal{C}/?, w\,\mathrm{Ret}^{\mathrm{fd}}_R\left(p^{-1}(?)\right)\right)$ which consists of the constant functors to $p^{-1}(?)$ and serves as the “basepoint”. Once again, the combination of all of the relevant retractions will provide a natural morphism from X to $p^{-1}(?)$. Since we will choose the image of $p^{-1}(?)$ as a point of $\Gamma_R\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ as the basepoint, this gives a natural path β' in $\Gamma_R\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ from the point associated to X to the basepoint.

Our choice for X will be the image of $E \sqcup E$ under the functor which builds the coassembly map $\mathrm{A}^{\mathrm{cy}}\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right) \to \Gamma^{\mathrm{cy}}\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$, which is only possible if the bundle involved is an acyclic flat bundle . Then β' gives a natural path in $\Gamma_R\left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$ from the image of $\chi^{cy}(p)$ to the basepoint, since $\chi^{cy}(p)$ was defined as the image of $\chi_{\mathrm{A}^{\mathrm{cy}}}(p)$ (which itself corresponds to the retractive space $E \sqcup E$) under the cyclic coassembly map. We define our natural path β in $\operatorname{holim} \mathrm{K}(R)$ from $\Lambda_V(\chi(p))$ to the basepoint as Λ_V applied to the the path β'.

Alternatively, we could define β by a construction quite similar to that of β', using $Nat\left(\mathcal{C}/?, wCh_*(R)\right)$ which corresponds to the holim of the constant functor $\mathrm{K}(R)$. Since the postcomposition by an exact functor preserves the zero map, the natural map from X to the "basepoint" will be sent to the (unique) natural map from the image of X to the constant functor on the zero chain complex, which serves as the basepoint in $Ch_*(R)$. Taking group completions then says this definition of β agrees with that given above. The interested reader may also see the path given by [5], Observation 6.4, along with Propositions 6.6 and 6.7 is yet another description of β.

6. Products and Natural Maps

The purpose of this section is to define the multiplications referred to previously in full detail.

First, recall the definition of μ_{A} as induced by the external smash product of retractive spaces defined in Section 3. In fact, this defines a bi-exact functor, hence a natural map

$$\mu_{\mathrm{A}} : \mathrm{A}\left(\begin{smallmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{smallmatrix}\right) \times \mathrm{A}\left(\begin{smallmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{smallmatrix}\right) \to \mathrm{A}\left(\begin{smallmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{smallmatrix}\right)$$

which clearly descends to the smash product by construction to give

$$\mu_{\mathrm{A}} : \mathrm{A}\left(\begin{smallmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{smallmatrix}\right) \wedge \mathrm{A}\left(\begin{smallmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{smallmatrix}\right) \to \mathrm{A}\left(\begin{smallmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{smallmatrix}\right).$$

We would like to use the external smash product again to define

$$\mu_{\mathrm{A}^{\mathrm{cy}}} : \mathrm{A}\left(\begin{smallmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{smallmatrix}\right) \wedge \mathrm{A}^{\mathrm{cy}}\left(\begin{smallmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{smallmatrix}\right) \to \mathrm{A}^{\mathrm{cy}}\left(\begin{smallmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{smallmatrix}\right).$$

However, it is not yet clear that $X \wedge_{E_1 \times E_2} Y$ lies in the subcategory $\mathrm{Ret}^{\mathrm{fd,cy}}(p_1 \times p_2)$. In other words, the construction yields a map

$$\mathrm{A}\left(\begin{smallmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{smallmatrix}\right) \wedge \mathrm{A}^{\mathrm{cy}}\left(\begin{smallmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{smallmatrix}\right) \to \mathrm{A}\left(\begin{smallmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{smallmatrix}\right)$$

which currently has the wrong target to be our $\mu_{\mathrm{A}^{\mathrm{cy}}}$. One should also keep in mind that we need flat bundles $\phi_1 : V_1 \to E_1$ and $\phi_2 : V_2 \to E_2$ in order to even define the spaces involved in $\mu_{\mathrm{A}^{\mathrm{cy}}}$. To deal with this problem we first need a technical result.

When describing relative chain complexes with local coefficients, the symbol $\mathcal{C}_*(A, B; V)$ will indicate that V is a flat bundle over B, A is retractive over B and the bundle in question over A is the pullback of V over the retraction, which remains a flat bundle. Also, given $\phi_1 : V_1 \to E_1$ and $\phi_2 : V_2 \to E_2$, the symbol

$V_1\hat{\otimes}V_2$ will denote the tensor product of the two bundles over $E_1 \times E_2$ given by pulling back each V_i over the relevant projection map.

Lemma 6.1. *Suppose $X \in \mathrm{Ret}^{fd}_{R_1}(p_1)$. Then the functor*

$$X \wedge_{E_1 \times E_2} ? : \mathrm{Ret}^{fd}_{R_2}(p_2) \to \mathrm{Ret}^{fd}_{R_1 \otimes_A R_2}(p_1 \times p_2)$$

preserves weak equivalences.

Proof. Suppose $f : Y \to Z$ is a weak equivalence in $\mathrm{Ret}^{\mathrm{fd}}_{R_2}(p_2)$. This means f induces a quasi-isomorphism

$$\mathcal{C}_*(Y, E_2, V_2) \to \mathcal{C}_*(Z, E_2, V_2)$$

with local coefficients. Thus, tensoring with $\mathcal{C}_*(X, E_1, V_1)$ yields another quasi-isomorphism

$$\mathcal{C}_*(X, E_1, V_1) \otimes \mathcal{C}_*(Y, E_2, V_2) \to \mathcal{C}_*(X, E_1, V_1) \otimes \mathcal{C}_*(Z, E_2, V_2)$$

since V_1 is a flat bundle. However, the relative Eilenberg-Zilber Theorem with local coefficients (exercise 8 on page 282 of [14] or [6]) then implies the existence of a quasi-isomorphism

$$\mathcal{C}_*\left(X \times Y, X \times E_2 \cup E_1 \times Y, r_1^*(V_1\hat{\otimes}V_2)\right) \to \mathcal{C}_*\left(X \times Z, X \times E_2 \cup E_1 \times Z, r_2^*(V_1\hat{\otimes}V_2)\right)$$

with $r_1 : X \times E_2 \cup E_1 \times Y \to E_1 \times E_2$ and similarly for r_2. Using the relative Mayer-Vietoris sequence with local coefficients and compact supports (see page 412, exercise 8 of [3]) together with the fact that

$$X \times E_2 \cup E_1 \times Y \to X \times Y$$

is a cofibration, one concludes that there is a natural quasi-isomorphism

$$\mathcal{C}_*\left(X \times Y, X \times E_2 \cup E_1 \times Y, r_1^*(V_1\hat{\otimes}V_2)\right) \approx \mathcal{C}_*\left(X \wedge_{E_1 \times E_2} Y, E_1 \times E_2, V_1\hat{\otimes}V_2\right)$$

and similarly for Z. Thus, transitivity implies the map f induces a quasi-isomorphism

$$\mathcal{C}_*\left(X \wedge_{E_1 \times E_2} Y, E_1 \times E_2, V_1\hat{\otimes}V_2\right) \to \mathcal{C}_*\left(X \wedge_{E_1 \times E_2} Z, E_1 \times E_2, V_1\hat{\otimes}V_2\right)$$

as well. □

Lemma 6.2. *Suppose $X \in \mathrm{Ret}^{fd}(p_1)$ and $Y \in \mathrm{Ret}^{fd,cy}(p_2)$. Then*

$$X \wedge_{E_1 \times E_2} Y \in \mathrm{Ret}^{fd,cy}(p_1 \times p_2).$$

Proof. The statement that $Y \in \mathrm{Ret}^{\mathrm{fd,cy}}(p_2)$ is equivalent to saying $Y \in \mathrm{Ret}^{\mathrm{fd}}_{R_2}(p_2)$ with $Y \to E_2$ a weak equivalence in this structure. Then Lemma 6.1 implies

$$X \wedge_{E_1 \times E_2} Y \to X \wedge_{E_1 \times E_2} E_2 \approx E_1 \times E_2$$

is a weak equivalence in $\mathrm{Ret}^{\mathrm{fd}}_{R_{1,2}}(p_1 \times p_2)$, or equivalently that

$$X \wedge_{E_1 \times E_2} Y \in \mathrm{Ret}^{\mathrm{fd,cy}}(p_1 \times p_2).$$

□

Lemma 6.3. (1) *There is a natural external multiplication*

$$\mu_{\mathrm{A}_R} : \mathrm{A}_{R_1}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}_{R_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \mathrm{A}_{R_1 \otimes_A R_2}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

(2) *The following diagram commutes.*

$$\begin{array}{ccc}
\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow{\mu_{\mathrm{A}}} & \mathrm{A}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\
\downarrow & & \downarrow \\
\mathrm{A}_{R_1}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{A}_{R_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow[\mu_{\mathrm{A}_R}]{} & \mathrm{A}_R\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}
\end{array}$$

Proof. The multiplication μ_{A_R} is induced by the same external smash product

$$\wedge_{E_1 \times E_2} : \mathrm{Ret}^{\mathrm{fd}}_{R_1}(p_1) \times \mathrm{Ret}^{\mathrm{fd}}_{R_2}(p_2) \to \mathrm{Ret}^{\mathrm{fd}}_{R}(p_1 \times p_2).$$

With the homology chain equivalences as weak equivalences, we must show this remains a bi-exact functor. Since we have not changed any other portion of the Waldhausen structure, we only need to check that fixing an element $X \in \mathrm{Ret}^{\mathrm{fd}}_{R_1}(p_1)$, $X \wedge_{E_1 \times E_2} ?$ preserves the new class of weak equivalences (and similarly for $? \wedge_{E_1 \times E_2} Y$). However, this follows immediately from Lemma 6.1.

Once we know the same functor induces the multiplication, the commutativity of the diagram follows from the fact that the vertical maps are induced by the identity functor at the level of retractive spaces. □

The product

$$\psi_\Gamma : \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \Gamma\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

is defined as the following composition:

$$\begin{array}{c}
\operatorname*{holim}_{\sigma_1 \in \mathrm{Simp}(B_1)} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \operatorname*{holim}_{\sigma_2 \in \mathrm{Simp}(B_2)} \mathrm{A}\left(p_2^{-1}(\sigma_2)\right) \\
\downarrow \\
\operatorname*{holim}_{\sigma_1 \times \sigma_2 \in \mathrm{Simp}(B_1 \times B_2)} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \mathrm{A}\left(p_2^{-1}(\sigma_2)\right) \\
\downarrow {\scriptstyle (\mu_{\mathrm{A}})_*} \\
\operatorname*{holim}_{\sigma_1 \times \sigma_2 \in \mathrm{Simp}(B_1 \times B_2)} \mathrm{A}\left((p_1 \times p_2)^{-1}(\sigma_1 \times \sigma_2)\right).
\end{array}$$

where the first map comes from the interaction of holim and products.

Suppose p_1 is a perfect fibration with flat bundle V_1 and p_2 is a perfect fibration with acyclic flat bundle V_2. We can now describe

$$\psi^{cy}_\Gamma : \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \Gamma^{\mathrm{cy}}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

similarly as the following composition.

$$\begin{array}{c}
\underset{\sigma_1 \in \mathrm{Simp}(B_1)}{\mathrm{holim}} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \underset{\sigma_2 \in \mathrm{Simp}(B_2)}{\mathrm{holim}} \mathrm{A}^{\mathrm{cy}}\left(p_2^{-1}(\sigma_2)\right) \\
\downarrow \\
\underset{\sigma_1 \times \sigma_2 \in \mathrm{Simp}(B_1 \times B_2)}{\mathrm{holim}} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \mathrm{A}^{\mathrm{cy}}\left(p_2^{-1}(\sigma_2)\right) \\
\downarrow{\scriptstyle (\mu_{\mathrm{A}^{\mathrm{cy}}})_*} \\
\underset{\sigma_1 \times \sigma_2 \in \mathrm{Simp}(B_1 \times B_2)}{\mathrm{holim}} \mathrm{A}^{\mathrm{cy}}\left((p_1 \times p_2)^{-1}(\sigma_1 \times \sigma_2)\right)
\end{array}$$

Next is the definition of $\epsilon : \Gamma^{\mathrm{cy}} \left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right) \to \mathrm{Wh}_B^R \left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$. We begin with a lemma from the non-parametrized case.

Lemma 6.4. *The linearization map factors through* $\mathrm{A}_R(E)$ *and its generalized linearization map. That is, the following diagram commutes.*

$$\begin{array}{ccc}
\mathrm{A}(E) & \longrightarrow & \mathrm{A}_R(E) \\
 & \searrow^{\lambda_V} & \downarrow{\scriptstyle \lambda_V} \\
 & & \mathrm{K}(R)
\end{array}$$

See [4] for details.

Now, consider the following diagram which commutes as a result of Lemma 6.4.

$$\begin{array}{ccc}
\mathrm{holim}\, \mathrm{A}\left(p^{-1}(\sigma)\right) & \xrightarrow{=} & \mathrm{holim}\, \mathrm{A}\left(p^{-1}(\sigma)\right) \\
\downarrow & & \downarrow{\scriptstyle \Lambda_V} \\
\mathrm{holim}\, \mathrm{A}_R(p^{-1}(\sigma)) & \xrightarrow[\Lambda_V]{} & \mathrm{holim}\, \mathrm{K}(R)
\end{array}$$

As mentioned previously, $\mathrm{A}^{\mathrm{cy}}\left(p^{-1}(\sigma)\right)$ is homotopy equivalent to the homotopy fiber of

$$\mathrm{A}\left(p^{-1}(\sigma)\right) \to \mathrm{A}_R\left(p^{-1}(\sigma)\right).$$

Since homotopy fibers commute with homotopy inverse limits, this implies $\Gamma^{\mathrm{cy}} \left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$ is homotopy equivalent to the homotopy fiber of the left vertical map in the previous diagram. However, by definition $\mathrm{Wh}_B^R \left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$ is the homotopy fiber of the right vertical in the previous diagram. Thus, commutativity of the diagram above induces a map $\epsilon : \Gamma^{\mathrm{cy}} \left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right) \to \mathrm{Wh}_B^R \left(\begin{smallmatrix} E \\ \downarrow p \\ B \end{smallmatrix}\right)$.

We now need a technical lemma.

Lemma 6.5. (1) *Given a diagram in Top*

$$\begin{array}{ccc} T\wedge W & \xrightarrow{f} & Y \\ {\scriptstyle 1\wedge p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ T\wedge X & \xrightarrow[g]{} & Z \end{array}$$

together with a pointed homotopy H from qf to $g(1\wedge p)$ there is a natural map φ_H making the following extension commute:

$$\begin{array}{ccc} T\wedge \operatorname{hofiber} p & \xrightarrow{\varphi_H} & \operatorname{hofiber} q \\ \big\downarrow & & \big\downarrow \\ T\wedge W & \longrightarrow & Y. \end{array}$$

(2) *Suppose one has a diagram in Top*

$$\begin{array}{ccc} T\wedge M & \xrightarrow{e} & N \\ {\scriptstyle 1\wedge i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ T\wedge W & \xrightarrow{f} & Y \\ {\scriptstyle 1\wedge p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ T\wedge X & \xrightarrow[g]{} & Z \end{array}$$

where $je = f(1\wedge i)$ and H is a pointed homotopy as above. Then one has a homotopy commutative diagram

$$\begin{array}{ccc} T\wedge \operatorname{hofiber} i & \longrightarrow & \operatorname{hofiber} j \\ \big\downarrow & & \big\downarrow \\ T\wedge \operatorname{hofiber}(pi) & \xrightarrow[\varphi_H]{} & \operatorname{hofiber}(qj). \end{array}$$

Proof. Recall, a point in hofiber p consists of a pair (α, w) where $w \in W$ and $\alpha : I \to X$ is a path from $p(w)$ to the basepoint. Define $\varphi_H(t, \alpha, w)$ as the pair $(g(t, \alpha)H(t, p(w), ?), f(t, w))$, where the first component means the concatenation of these two paths in Z.

For the second statement, define the map

$$T \wedge \operatorname{hofiber} i \to \operatorname{hofiber} j$$

by a simpler variation of φ_H, namely (t, α, m) is mapped to $(f(t, \alpha), e(t, m))$. (Notice this is homotopic to the variation of φ_H using a constant homotopy.) In order to establish the homotopy commutativity of the diagram, it suffices to show that the paths $qf(t, \alpha)$ and $g(t, p(\alpha))H(t, i(m), ?)$ are homotopic relative to the endpoints. If we use s to denote the time variables for α and H, and r as a time variable for our homotopy, the relevant formula is:

$$K(s,r) = \begin{cases} H(t, \alpha(s - rs), 2rs), & \text{if } s \le \frac{1}{2}; \\ H(t, \alpha(s - r + rs), r), & \text{if } s \ge \frac{1}{2}. \end{cases} \tag{3}$$

□

The multiplication

$$\nu : \Gamma \begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \mathrm{Wh}^{R_2}_{B_2} \begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \to \mathrm{Wh}^{R_1 \otimes_A R_2}_{B_1 \times B_2} \begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}$$

is defined by applying Lemma 6.5 to the homotopy commutative diagram given by the following proposition.

Proposition 6.6. (1) *There is a natural commutative diagram as below:*

$$\begin{array}{ccc} \Gamma \begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma \begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow{\psi_\Gamma} & \Gamma \begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\ \downarrow & & \downarrow \\ \Gamma \begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma_{R_2} \begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \longrightarrow & \Gamma_{R_1 \otimes_A R_2} \begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}. \end{array}$$

(2) *There is a natural choice of homotopy between the two compositions in the following diagram:*

$$\begin{array}{ccc} \Gamma \begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma_{R_2} \begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \longrightarrow & \Gamma_{R_1 \otimes_A R_2} \begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\ {\scriptstyle 1 \wedge \Lambda_V} \downarrow & & \downarrow {\scriptstyle \Lambda_V} \\ \Gamma \begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \operatorname{holim} \mathrm{K}(R_2) & \longrightarrow & \operatorname{holim} \mathrm{K}(R_1 \otimes_A R_2). \end{array}$$

Proof. We begin with the diagram

$$\begin{array}{ccc} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \mathrm{A}\left(p_2^{-1}(\sigma_2)\right) & \longrightarrow & \mathrm{A}\left((p_1 \times p_2)^{-1}(\sigma_1 \times \sigma_2)\right) \\ \downarrow & & \downarrow \\ \mathrm{A}_{R_1}(p_1^{-1}(\sigma_1)) \wedge \mathrm{A}_{R_2}(p_2^{-1}(\sigma_2)) & \longrightarrow & \mathrm{A}_{R_1 \otimes_A R_2}((p_1 \times p_2)^{-1}(\sigma_1 \times \sigma_2)) \end{array}$$

which commutes by Lemma 6.3. Taking holim over the category $\mathrm{Simp}\,(B_1 \times B_2)$ then implies the following diagram commutes

$$\begin{array}{ccc} \operatorname{holim}\left(\mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \mathrm{A}\left(p_2^{-1}(\sigma_2)\right)\right) & \searrow & \\ \downarrow & & \operatorname{holim} \mathrm{A}\left((p_1 \times p_2)^{-1}(\sigma_1 \times \sigma_2)\right) \\ \operatorname{holim}\left(\mathrm{A}_{R_1}(p_1^{-1}(\sigma_1)) \wedge \mathrm{A}_{R_2}(p_2^{-1}(\sigma_2))\right) & \searrow & \downarrow \\ & & \operatorname{holim} \mathrm{A}_{R_1 \otimes_A R_2}((p_1 \times p_2)^{-1}(\sigma_1 \times \sigma_2)). \end{array}$$

Of course, one also has a commutative diagram associated to holim of products

$$
\begin{array}{ccc}
\begin{array}{c}\operatorname*{holim}_{\mathrm{Simp}(B_1)} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \\ \wedge \\ \operatorname*{holim}_{\mathrm{Simp}(B_2)} \mathrm{A}\left(p_2^{-1}(\sigma_2)\right)\end{array} & \longrightarrow & \operatorname*{holim}_{\mathrm{Simp}(B_1 \times B_2)} \left(\mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \mathrm{A}\left(p_2^{-1}(\sigma_2)\right)\right) \\
\downarrow & & \downarrow \\
\begin{array}{c}\operatorname*{holim}_{\mathrm{Simp}(B_1)} \mathrm{A}_{R_1}(p_1^{-1}(\sigma_1)) \\ \wedge \\ \operatorname*{holim}_{\mathrm{Simp}(B_2)} \mathrm{A}_{R_2}(p_2^{-1}(\sigma_2))\end{array} & \longrightarrow & \operatorname*{holim}_{\mathrm{Simp}(B_1 \times B_2)} \left(\mathrm{A}_{R_1}(p_1^{-1}(\sigma_1)) \wedge \mathrm{A}_{R_2}(p_2^{-1}(\sigma_2))\right)
\end{array}
$$

which taken together yields a commutative diagram which we reinterpret as the following

$$
\begin{array}{ccc}
\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & \xrightarrow{\quad\psi_\Gamma\quad} & \Gamma\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix} \\
\downarrow & & \downarrow \\
\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma_{R_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} & & \Gamma_{R_1 \otimes_A R_2}\begin{pmatrix} E_1 \times E_2 \\ \downarrow p_1 \times p_2 \\ B_1 \times B_2 \end{pmatrix}. \\
& \searrow \qquad \nearrow & \\
& \Gamma_{R_1}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \wedge \Gamma_{R_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} &
\end{array}
$$

For the second diagram, begin with the following diagram (coming from the non-parametrized case) which commutes up to a natural homotopy

$$
\begin{array}{ccc}
\mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \wedge \mathrm{A}_{R_2}(p_2^{-1}(\sigma_2)) & \xrightarrow{\mu_{\mathrm{A}}} & \mathrm{A}_{R_1 \otimes_A R_2}((p_1 \times p_2)^{-1}(\sigma_1 \times \sigma_2)) \\
\Big\downarrow {\scriptstyle \lambda_{V_1} \wedge \lambda_{V_2}^{R_2}} & & \Big\downarrow {\scriptstyle \lambda_{V_1 \otimes V_2}^{R_1 \otimes_A R_2}} \\
\mathrm{K}(R_1) \wedge \mathrm{K}(R_2) & \xrightarrow[\eta_{\mathrm{K}}]{} & \mathrm{K}(R_1 \otimes_A R_2)
\end{array}
$$

where the bottom map is Loday's pairing on K-theory [13]. The natural homotopy in this case comes from the Eilenberg-Zilber map on chain complexes. Now take holim over the category $\mathrm{Simp}\,(B_1 \times B_2)$ and proceed as above, keeping in mind that the upper left corner in each diagram is not symmetric. The statement about naturality of the homotopy then follows from the naturality of the Eilenberg-Zilber map. □

We can now prove the second statement of Proposition 2.2.

Proposition 6.7. *The diagram*

$$\begin{array}{ccc} \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1\end{pmatrix} \times \Gamma^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2\end{pmatrix} & \xrightarrow{\psi_\Gamma^{cy}} & \Gamma^{\mathrm{cy}}\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2\end{pmatrix} \\ \downarrow 1\times\epsilon_2 & & \downarrow \epsilon_{1,2} \\ \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1\end{pmatrix} \times \mathrm{Wh}^{R_2}_{B_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2\end{pmatrix} & \xrightarrow[\nu]{} & \mathrm{Wh}^{R_1\otimes_A R_2}_{B_1\times B_2}\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2\end{pmatrix}. \end{array}$$

commutes up to a natural homotopy.

Proof. The homotopy commutative square comes from applying Lemma 6.5 to the diagrams provided by Proposition 6.6. □

We can now give a proof of Proposition 2.1.

Proof of Proposition 2.1. Consider the diagram

$$\begin{array}{ccc} \operatorname{holim} \mathrm{A}\left(p_2^{-1}(\sigma_2)\right) & \xrightarrow{=} & \operatorname{holim} \mathrm{A}\left(p_2^{-1}(\sigma_2)\right) \\ \downarrow & & \downarrow \Lambda_V \\ \operatorname{holim} \mathrm{A}_{R_2}(p_2^{-1}(\sigma_2)) & \xrightarrow[\Lambda_V]{} & \operatorname{holim} \mathrm{K}\left(R_2\right) \end{array}$$

involved in defining ϵ. By definition, $\tau_{R_2}(p_2)$ is a lift of $\chi(p_2)$ to $\mathrm{Wh}^{R_2}_{B_2}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2\end{pmatrix}$ associated to a specific choice of path β from $\Lambda_V(\chi(p_2))$ to the basepoint, discussed at the end of section 5. It is important to keep in mind this path arises as the image of a similar path to the basepoint β' in each $\mathrm{A}_{R_2}(p_2^{-1}(\sigma_2))$, given by effectively the same construction. Since the pair $(\chi^{cy}(p), \beta')$ is the image of $\chi^{cy}(p)$ in the homotopy fiber F of $\Gamma\begin{pmatrix} E \\ \downarrow p \\ B\end{pmatrix} \to \Gamma_R\begin{pmatrix} E \\ \downarrow p \\ B\end{pmatrix}$, the composite

$$\Gamma\begin{pmatrix} E \\ \downarrow p \\ B\end{pmatrix} \to F \to \Gamma_R\begin{pmatrix} E \\ \downarrow p \\ B\end{pmatrix}$$

will send $\chi^{cy}(p)$ to the point in the Whitehead space corresponding to the pair $(\chi(p), \beta)$. However, this point was our definition of the Reidemeister torsion class $\tau_{R_2}(p_2)$. □

7. The coassembly map is Multiplicative

The purpose of this section is to prove that the diagram

$$\begin{array}{ccc} \mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1\end{pmatrix} \wedge \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2\end{pmatrix} & \xrightarrow{\mu_{\mathrm{A}^{\mathrm{cy}}}} & \mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2\end{pmatrix} \\ \downarrow \delta_1\times\delta_2^{cy} & & \downarrow \delta_{1,2}^{cy} \\ \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1\end{pmatrix} \wedge \Gamma^{\mathrm{cy}}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2\end{pmatrix} & \xrightarrow[\psi_\Gamma^{cy}]{} & \Gamma^{\mathrm{cy}}\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2\end{pmatrix}. \end{array} \tag{4}$$

is commutative up to a natural homotopy. We begin with the proof of Lemma 2.5.

Proof of Lemma 2.5. Part 1 follows from the fact that the multiplications μ_{A} and $\mu_{\mathrm{A}^{\mathrm{cy}}}$ are both defined by the same bi-exact functor at the level of retractive spaces.

To see part 2, recall that $\mathrm{Ret}^{\mathrm{fd,cy}}(p) \to \mathrm{Ret}^{\mathrm{fd}}(p)$ is the inclusion of a sub-Waldhausen category. Hence,

$$wS_{.}\,\mathrm{Ret}^{\mathrm{fd,cy}}(p) \to wS_{.}\,\mathrm{Ret}^{\mathrm{fd}}(p)$$

is the inclusion of a sub-simplicial category. This implies

$$\mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} = \Omega|wS_{.}\,\mathrm{Ret}^{\mathrm{fd,cy}}(p)| \to \Omega|wS_{.}\,\mathrm{Ret}^{\mathrm{fd}}(p)| = \mathrm{A}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$$

is an injection.

Finally, part 3 follows from the nonparametrized analogue of part 2 and the standard model for the homotopy inverse limit. Since $\mathrm{A}^{\mathrm{cy}}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix} \to \mathrm{A}\begin{pmatrix} E \\ \downarrow p \\ B \end{pmatrix}$ is an injection, it should be clear that for each $\sigma \in \mathrm{Simp}(B)$ one has

$$\mathrm{Map}(|\,\mathrm{Simp}(B)/\sigma|, \mathrm{A}^{\mathrm{cy}}\left(p^{-1}(\sigma)\right)) \to \mathrm{Map}(|\,\mathrm{Simp}(B)/\sigma|, \mathrm{A}\left(p^{-1}(\sigma)\right))$$

an injection as a postcomposition by an injection. This implies the product of such maps is also an injection. However, the standard model for the homotopy inverse limit is the appropriate subspace of such a product and the restriction of an injection remains an injection. □

As a consequence of Lemma 2.5, it should be clear that the first portion of Proposition 2.2 is a corollary of Theorem 2.3, where we dealt with the product rather than smash product versions of μ_{A} and ψ_{Γ} for technical reasons.

We begin working toward this proof with another technical lemma.

Lemma 7.1. *The product map*

$$|w\,\mathrm{Ret}^{fd}(p_1)| \times |w\,\mathrm{Ret}^{fd}(p_2)| \to \mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix}$$

retains the group completion property.

Proof. The definition of the $S_{.}$ construction implies that for Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$ one has a natural isomorphism

$$S_{.}(\mathcal{C} \times \mathcal{D}) \approx S_{.}(\mathcal{C}) \times S_{.}(\mathcal{D}).$$

Of course, this assumes the Waldhausen structure on $\mathcal{C} \times \mathcal{D}$ is that coming from the product.

Clearly, the inclusion

$$|w\left(\mathrm{Ret}^{\mathrm{fd}}(p_1) \times \mathrm{Ret}^{\mathrm{fd}}(p_2)\right)| \to \Omega|wS_{.}\left(\mathrm{Ret}^{\mathrm{fd}}(p_1) \times \mathrm{Ret}^{\mathrm{fd}}(p_2)\right)|$$

has the group completion property by [16]. However, the previous paragraph leads us to conclude the target is isomorphic to

$$\Omega|wS_{.}(\mathrm{Ret}^{\mathrm{fd}}(p_1)) \times wS_{.}(\mathrm{Ret}^{\mathrm{fd}}(p_2))| \approx \mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix}.$$

On the other hand, we also have an isomorphism

$$|w\,\mathrm{Ret}^{\mathrm{fd}}(p_1)| \times |w\,\mathrm{Ret}^{\mathrm{fd}}(p_2)| \approx |w\left(\mathrm{Ret}^{\mathrm{fd}}(p_1) \times \mathrm{Ret}^{\mathrm{fd}}(p_2)\right)|.$$

Thus, it suffices to notice that the composite of these maps

$$|w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_1)\,| \times |w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_2)\,| \to \mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix}$$

is the product of the maps

$$|w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_i)\,| \to \Omega|wS.\left(\mathrm{Ret}^{\mathrm{fd}}\,(p_i)\right)| = \mathrm{A}\begin{pmatrix} E_i \\ \downarrow p_i \\ B_i \end{pmatrix}.$$

□

There are now two functors

$$w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_1) \times w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_2) \to Nat\left(\mathrm{Simp}\,(B_1 \times B_2)\,/?, w\,\mathrm{Ret}^{\mathrm{fd}}\,(((p_1 \times p_2)^{-1}(?))\right).$$

The functor $\mathcal{F}_1$ comes from following the external smash product (which induces the external product in parametrized A-theory)

$$\wedge_{E_1 \times E_2} : w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_1) \times w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_2) \to w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_1 \times p_2)$$

with the functor

$$w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_1 \times p_2) \to Nat\left(\mathrm{Simp}\,(B_1 \times B_2)\,/?, w\,\mathrm{Ret}^{\mathrm{fd}}\,((p_1 \times p_2)^{-1}(?))\right)$$

described after Lemma 5.3.

The second functor $\mathcal{F}_2$ comes from first taking the product of the functors

$$w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_i) \to Nat\left(\mathrm{Simp}\,(B_i)\,/?, w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_i^{-1}(?))\right)$$

as above, followed by a functor

$$\begin{array}{c} Nat\left(\mathrm{Simp}\,(B_1)\,/?, w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_1^{-1}(?))\right) \\ \times \\ Nat\left(\mathrm{Simp}\,(B_2)\,/?, w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_2^{-1}(?))\right) \\ \Big\downarrow \\ Nat\left(\mathrm{Simp}\,(B_1 \times B_2)\,/?, w\,\mathrm{Ret}^{\mathrm{fd}}\,((p_1 \times p_2)^{-1}(?))\right) \end{array}$$

given by the pseudo-product in $Nat\,(?, !)$ followed by the functor induced by considering the external smash product as a natural transformation.

Lemma 7.2. (1) *The composite $\delta_{1,2}(\mu_{\mathrm{A}})$ from Theorem 2.3 is the group completion of the Thomason homotopy limit problem map associated to $\mathcal{F}_1$.*
(2) *The composite $\psi_\Gamma(\delta_1 \times \delta_2)$ from Theorem 2.3 is the group completion of the Thomason homotopy limit problem map associated to $\mathcal{F}_2$.*

Proof. For the first claim, recall that $\delta_{1,2}$ is the group completion of the second functor in the definition of $\mathcal{F}_1$ and μ_{A} is the group completion of the external smash product.

To verify the second claim, it suffices by the uniqueness of group completions to establish commutativity of the following diagram

$$\begin{array}{ccc}
\left|w\,\mathrm{Ret}^{\mathrm{fd}}(p_1)\right| \times \left|w\,\mathrm{Ret}^{\mathrm{fd}}(p_2)\right| & \longrightarrow & \mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \\
\downarrow & & \downarrow{\scriptstyle \delta_1\times\delta_2} \\
\begin{array}{c} \left|Nat\left(\mathrm{Simp}(B_1)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_1^{-1}?\right)\right)\right| \\ \times \\ \left|Nat\left(\mathrm{Simp}(B_2)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_2^{-1}?\right)\right)\right| \end{array} & \longrightarrow & \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \Gamma\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \\
\downarrow & & \downarrow{\scriptstyle \psi_\Gamma} \\
\left|Nat\left(\mathrm{Simp}(B_1\times B_2)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left((p_1\times p_2)^{-1}?\right)\right)\right| & \longrightarrow & \Gamma\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2 \end{pmatrix}.
\end{array}$$

The commutativity of the top square follows from the fact that group completions commute with products by naturality. Thus, it remains only to show the bottom square commutes. However, by the construction of $\mathcal{F}_2$, this bottom square itself factors as the group completion of a pair of squares,

$$\begin{array}{ccc}
\begin{array}{c} \left|Nat\left(\mathrm{Simp}(B_1)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_1^{-1}?\right)\right)\right| \\ \times \\ \left|Nat\left(\mathrm{Simp}(B_2)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_2^{-1}?\right)\right)\right| \end{array} & \longrightarrow & \Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1 \end{pmatrix} \times \Gamma\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2 \end{pmatrix} \\
\downarrow & & \downarrow \\
\left|Nat\left(\mathrm{Simp}(B_1\times B_2)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_1^{-1}?\right)\times w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_2^{-1}?\right)\right)\right| & \searrow & \\
& & \begin{array}{c} \operatorname*{holim}_{\mathrm{Simp}(B_1\times B_2)} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \\ \times \\ \mathrm{A}\left(p_2^{-1}(\sigma_2)\right) \end{array}
\end{array}$$

and

$$\begin{array}{ccc}
\left|Nat\left(\mathrm{Simp}(B_1\times B_2)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_1^{-1}?\right)\times w\,\mathrm{Ret}^{\mathrm{fd}}\left(p_2^{-1}?\right)\right)\right| & \searrow & \\
& & \begin{array}{c} \operatorname*{holim}_{\mathrm{Simp}(B_1\times B_2)} \mathrm{A}\left(p_1^{-1}(\sigma_1)\right) \\ \times \\ \mathrm{A}\left(p_2^{-1}(\sigma_2)\right) \end{array} \\
\downarrow & & \downarrow \\
\left|Nat\left(\mathrm{Simp}(B_1\times B_2)/?, w\,\mathrm{Ret}^{\mathrm{fd}}\left((p_1\times p_2)^{-1}?\right)\right)\right| & \longrightarrow & \Gamma\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2 \end{pmatrix}.
\end{array}$$

each of which commutes by Lemma 5.2. The first comes from the pseudo-product in $Nat\,(?,!)$ and the second from the map induced by the external smash product considered as a natural transformation. □

Proposition 7.3. *There is a functor*

$$\mathcal{F}_0 : w\,\mathrm{Ret}^{fd}\,(p_1)\times w\,\mathrm{Ret}^{fd}\,(p_2) \to Nat\left(\mathrm{Simp}\,(B_1\times B_2)\,/?, w\,\mathrm{Ret}^{fd}\left((p_1\times p_2)^{-1}(?)\right)\right)$$

together with natural transformations $(\psi_n) : \mathcal{F}_0 \to \mathcal{F}_n$ *for* $n = 1{,}2$.

Proof. We return to the notation of section 5 in order to describe $\mathcal{F}_1$ and $\mathcal{F}_2$ along with their effect on an object $(X_1 \leftrightarrows E_1, X_2 \leftrightarrows E_2)$ of $w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_1)\times w\,\mathrm{Ret}^{\mathrm{fd}}\,(p_2)$. To simplify notation, let $(i_n)^* = \left(i^{\sigma_n}_{B_n}\right)^*$ and let $(j_n)_* = \left(i^{\delta_n}_{\sigma_n}\right)_*$. First, in building $\mathcal{F}_1$ one sends this pair to the functor

$$(\delta_1,\delta_2)\mapsto (j_1\times j_2)_*(i_1\times i_2)^*(X_1\wedge_{E_1\times E_2} X_2).$$

Next, in building $\mathcal{F}_2$ one sends the pair to the functor

$$(\delta_1,\delta_2)\mapsto (j_1)_*(i_1)^*(X_1)\wedge_{p_1^{-1}(\sigma_1)\times p_2^{-1}(\sigma_2)} (j_2)_*(i_2)^*(X_2).$$

The new functor $\mathcal{F}_0$ is defined by

$$(\delta_1,\delta_2)\mapsto (j_1\times j_2)_*\left((i_1)^*(X_1)\wedge_{p_1^{-1}(\delta_1)\times p_2^{-1}(\delta_2)} (i_2)^*(X_2)\right).$$

The natural transformation ψ_1 is built from the natural weak equivalences which arise by applying $(j_1\times j_2)_*$ to the natural weak equivalence of type (1) in Lemma 3.4. Similarly, the natural transformation ψ_2 is built from the natural weak equivalences of type (2) in Lemma 3.4. □

This suffices to allow us to prove Theorem 2.3.

Proof of Theorem 2.3. Lemma 5.3 together with Proposition 7.3 imply the Thomason homotopy limit problem maps associated to $\mathcal{F}_1$ and $\mathcal{F}_2$ are naturally homotopic (via the Thomason homotopy limit problem map associated to $\mathcal{F}_0$). However, Lemma 7.2 then implies

$$\begin{array}{ccc}
\mathrm{A}\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1\end{pmatrix}\times \mathrm{A}\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2\end{pmatrix} & \xrightarrow{\mu_{\mathrm{A}}} & \mathrm{A}\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2\end{pmatrix} \\
{\scriptstyle \delta_1\times\delta_2}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta_{1,2}} \\
\Gamma\begin{pmatrix} E_1 \\ \downarrow p_1 \\ B_1\end{pmatrix}\times \Gamma\begin{pmatrix} E_2 \\ \downarrow p_2 \\ B_2\end{pmatrix} & \xrightarrow[\psi_\Gamma]{} & \Gamma\begin{pmatrix} E_1\times E_2 \\ \downarrow p_1\times p_2 \\ B_1\times B_2\end{pmatrix}.
\end{array}$$

commutes up to a natural homotopy. □

8. Open Questions

There are two main directions (aside from the general goal mentioned in the introduction) in which we would like to proceed in the future. The first would be to give a definition of smooth parametrized torsion after [5]. This would lead to an appropriate definition of higher Reidemeister torsion as in [8]. Specifically, Theorem 6.6.1 of [8] and possibly conjecture 6.6.7 would follow from the generalization of Theorem 1.4 to higher torsion.

To see Theorem 6.6.1 of [8], consider the product of a perfect fibration p_1 with the constant map $p_2 : N \to *$ for a manifold N. Then Theorem 1.4 gives

$$\nu(\chi(p_2), \tau_{R_1}(p_1)) \simeq \tau_{R_1 \otimes_A R_2}(p_1 \times p_2).$$

However, from [5] we know $\chi(p_2) = \chi(N)$, the standard Euler characteristic of the manifold N.

The second logical direction is to pursue a definition of Reidemeister torsion which does not require the acyclicity assumption. A careful analysis of the results in this paper suggest the effective role of the acyclicity assumption is to establish $E \sqcup E$ as a multiplicatively natural element of $\mathrm{Ret}^{\mathrm{fd,cy}}(p)$. Without the acyclicity assumption, one might still hope to construct such a multiplicatively natural element $X(p)$ in $\mathrm{Ret}^{\mathrm{fd,cy}}(p)$. In that case, the element $X(p)$ could play the role of X in the discussion at the end of Section 5, giving an analog of the path $\beta(p)$ associated to $X(p)$. The pair $(X(p), \beta(p))$ would then be a natural choice of a point in the Whitehead space, hence a possible notion of Reidemeister torsion. With this definition, the proofs of this article should remain effective and yield an appropriate product formula without the acyclicity assumption.

Since the identity functor $\mathrm{Ret}^{\mathrm{fd}}(p) \to \mathrm{Ret}^{\mathrm{fd}}_R(p)$ induces a localization functor, one way to try to make a choice for $X(p)$ which would be natural comes from an attempt to define a functor $\mathrm{Ret}^{\mathrm{fd}}(p) \to \mathrm{Ret}^{\mathrm{fd,cy}}(p)$ analogous to the construction of the kernel of the localization map in [2]. Unfortunately, it seems optimistic to suggest such an element would be multiplicative as well, a requirement for any product theorem.

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